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CONVERGENCE AND ALMOST SURE PROPERTIES IN HARDY SPACES OF DIRICHLET SERIES

FRÉDÉRIC BAYART

ABSTRACT. Given a frequency λ , we study general Dirichlet series $\sum a_n e^{-\lambda_n s}$. First, we give a new condition on λ which ensures that a somewhere convergent Dirichlet series defining a bounded holomorphic function in the right half-plane converges uniformly in this half-plane, improving classical results of Bohr and Landau. Then, following recent works of Defant and Schoolmann, we investigate Hardy spaces of these Dirichlet series. We get general results on almost sure convergence which have an harmonic analysis flavour. Nevertheless, we also exhibit examples showing that it seems hard to get general results on these spaces as spaces of holomorphic functions.

1. INTRODUCTION

1.1. λ -Dirichlet series and their convergence. A general Dirichlet series is a series $\sum_n a_n e^{-\lambda_n s}$ where $(a_n) \subset \mathbb{C}^{\mathbb{N}}$, $s \in \mathbb{C}$ and $\lambda = (\lambda_n)$ is an increasing sequence of nonnegative real numbers tending to $+\infty$, called a *frequency*. We shall denote by $\mathcal{D}(\lambda)$ the space of all formal λ -Dirichlet series. The two most natural examples are $(\lambda_n) = (n)$ which gives rise to power series and $(\lambda_n) = (\log n)$, the case of ordinary Dirichlet series. A prominent problem at the beginning of the twentieth century was the study of the convergence of these series, starting from the following theorem of Bohr [5] on ordinary Dirichlet series: let $D = \sum_n a_n n^{-s}$ be a somewhere convergent ordinary Dirichlet series having a holomorphic and bounded extension to the half-plane \mathbb{C}_0 . Then D converges uniformly on each half-plane \mathbb{C}_ε for all $\varepsilon > 0$. Here, \mathbb{C}_θ means the right half-plane $[\text{Re} > \theta]$.

An important effort was done to extend this result to general frequencies. Two sufficient conditions were isolated firstly by Bohr [5] and then by Landau [19]. Following the terminology of [21], we say that a frequency satisfies (BC) provided

$$(BC) \quad \exists \ell > 0, \exists C > 0, \forall n \in \mathbb{N}, \lambda_{n+1} - \lambda_n \geq C e^{-\ell \lambda_n}.$$

A frequency λ satisfies (LC) provided

$$(LC) \quad \forall \delta > 0, \exists C > 0, \forall n \in \mathbb{N}, \lambda_{n+1} - \lambda_n \geq C e^{-e^{\delta \lambda_n}}.$$

Of course, (BC) is a stronger condition than (LC), and Landau has shown that any frequency satisfying (LC) verifies Bohr's theorem: all Dirichlet series $D = \sum_n a_n e^{-\lambda_n s}$

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belonging to $\mathcal{D}_\infty^{\text{ext}}(\lambda)$, the space of somewhere convergent λ -Dirichlet series which allow a holomorphic and bounded extension to \mathbb{C}_0 , converge uniformly to \mathbb{C}_ε , for all $\varepsilon > 0$.

This problem was studied again recently in [7] and [21] in connection with Banach spaces of Dirichlet series and quantitative estimates. Define by $\mathcal{D}_\infty(\lambda)$ the subspace of $\mathcal{D}_\infty^{\text{ext}}(\lambda)$ of Dirichlet series which converge on \mathbb{C}_0 (in general, $\mathcal{D}_\infty(\lambda)$ can be a proper subspace of $\mathcal{D}_\infty^{\text{ext}}(\lambda)$ if λ does not satisfy Bohr's theorem) and define $S_N : \mathcal{D}_\infty^{\text{ext}}(\lambda) \rightarrow \mathcal{D}_\infty(\lambda)$, $\sum_1^{+\infty} a_n e^{-\lambda_n s} \mapsto \sum_1^N a_n e^{-\lambda_n s}$ the N -th partial sum operator. In [21], a thorough study of the norm of S_N is done (the case of ordinary Dirichlet series was settled in [1]), leading to consequences on the existence of a Montel's theorem in $\mathcal{D}_\infty^{\text{ext}}(\lambda)$ or on the completeness of $\mathcal{D}_\infty(\lambda)$.

Our first main result is an extension of the results of [5, 19, 21]: we provide a new class of frequencies, that we will call (NC) (see Definition 2.2) such that Bohr's theorem, and all its consequences, are true. Like (BC) or (LC), this class of frequencies quantifies how fast λ goes to $+\infty$ and how close its terms are, but in a less demanding way since (NC) is strictly weaker than (LC).

Our method of proof also differs from that of [21]. In [21], the estimation of $\|S_N\|$ is based on Riesz means of λ -Dirichlet polynomials: recall that for a sequence of complex numbers (c_n) and for $k > 0$, the finite sum

$$R_x^{\lambda,k}(\sum_n c_n) := \sum_{\lambda_n < x} \left(1 - \frac{\lambda_n}{x}\right)^k c_n$$

is called the first (λ, k) -Riesz mean of $\sum_n c_n$ of length $x > 0$. I. Schoolmann uses approximation of D by $R_x^{\lambda,k}(D)$ for a suitable choice of k to deduce its result on $\|S_N\|$. Our alternate approach is based on mollifiers and on a formula of convolution due to Saksman for ordinary Dirichlet series.

1.2. Hardy spaces of λ -Dirichlet series: Banach spaces and harmonic analysis.

Our second approach deals with Hardy spaces of Dirichlet series. For ordinary Dirichlet series, they have been introduced and studied in [15] (see also [1]) and this has caused an important renew of interest for this subject. The general case has been introduced and investigated very recently in [10, 12, 11]. For $p \in [1, +\infty)$, the Hardy space $\mathcal{H}_p(\lambda)$ may be defined as follows: given a λ -Dirichlet polynomial $D = \sum_{n=1}^N a_n e^{-\lambda_n s}$, define its \mathcal{H}_p -norm by

$$\|D\|_p^p := \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T |D(it)|^p dt.$$

Then $\mathcal{H}_p(\lambda)$ is the completion of the set of λ -Dirichlet polynomials for this norm. However, this internal description is often not sufficient to get the main properties of $\mathcal{H}_p(\lambda)$ and we need a group approach. Let G be a compact abelian group and let $\beta : (\mathbb{R}, +) \rightarrow G$ be a continuous homomorphism with dense range. Then we say that (G, β) is a λ -Dirichlet group provided, for all $n \in \mathbb{N}$, there exists $h_{\lambda_n} \in \hat{G}$ such that $h_{\lambda_n} \circ \beta = e^{-i\lambda_n \cdot}$. The space $H_p^\lambda(G)$, $p \in [1, +\infty]$ is then defined as the subspace of $L^p(G)$ of functions f such that

$\text{supp}(\hat{f}) \subset \{h_{\lambda_n} : n \in \mathbb{N}\}$. Now define the Bohr map \mathcal{B} by

$$\begin{aligned} \mathcal{B} : H_p^\lambda(G) &\rightarrow \mathcal{D}(\lambda) \\ f &\mapsto \sum \hat{f}(h_{\lambda_n})e^{-\lambda_n s} \end{aligned}$$

and set $\mathcal{H}_p(\lambda) = \mathcal{B}(H_p^\lambda(G))$ with $\|\mathcal{B}f\|_p := \|f\|_p$. Then it has been shown in [10] that

- given a frequency λ , there always exists a λ -Dirichlet group (G, β) ;
- the Hardy space $\mathcal{H}_p(\lambda)$ does not depend on the chosen λ -Dirichlet group;
- when $p \neq +\infty$, it coincides with $\mathcal{H}_p(\lambda)$ defined internally.

Our second aim in this paper is solve some of the problems on the spaces $\mathcal{H}_p(\lambda)$ raised in [9] and to exhibit new properties of them. In particular, we investigate properties of $\mathcal{H}_p(\lambda)$ coming from functional analysis and harmonic analysis.

As an example, let us discuss a famous theorem of Helson [17] which ensures, in our terminology, that if λ satisfies (BC) and $D = \sum_n a_n e^{-\lambda_n s}$ belongs to $\mathcal{H}_2(\lambda)$, then for almost all homomorphisms $\omega : (\mathbb{R}, +) \rightarrow \mathbb{T}$, the Dirichlet series $\sum_n a_n \omega(\lambda_n) e^{-\lambda_n s}$ converges on \mathbb{C}_0 . This has been extended to the Hardy spaces $\mathcal{H}_p((\log n))$ for $p \geq 1$ by Bayart in [1] and this result is at the heart of many further investigations on these spaces (e.g. composition operators, Volterra operators). Therefore, it is a challenge to put it in the general framework of $\mathcal{H}_p(\lambda)$ or, equivalently - via the Bohr transform - of $H_p^\lambda(G)$. When λ satisfies (BC), this has been done in [12], adding moreover the maximal inequality.

Theorem A (Defant-Schoolmann). *Let λ satisfy (BC), let (G, β) be a λ -Dirichlet group. For every $u > 0$, there exists a constant $C := C(u, \lambda)$ such that, for all $1 \leq p \leq +\infty$ and for all $f \in H_p^\lambda(G)$,*

$$(1) \quad \left\| \sup_{\sigma \geq u} \sup_N \left\| \sum_{n=1}^N \hat{f}(h_{\lambda_n}) e^{-\sigma \lambda_n} h_{\lambda_n} \right\| \right\|_p \leq C \|f\|_p.$$

In particular, for every $u > 0$, $\sum_1^{+\infty} \hat{f}(h_{\lambda_n}) e^{-u \lambda_n} h_{\lambda_n}$ converges almost everywhere on G .

When λ satisfies (LC), the almost everywhere statement is known to be true, as well as the maximal inequality for $p > 1$ with a constant now depending on p . When $p = 1$, it is valid if we replace the $L_1(G)$ -norm by the weak $L_1(G)$ -norm. We shall prove that inequality (1) remains true even on the weaker assumption that λ satisfies (NC), even for $p = 1$, and with a constant independent of p . Our approach, which seems less technical than that of [12], is based again on a version of Saksman's convolution formula and on a Carleson-Hunt type maximal inequality of independent interest.

1.3. $\mathcal{H}_p(\lambda)$ as a Banach space of holomorphic functions. The results announced in the previous section indicate that the spaces $\mathcal{H}_p(\lambda)$ seem to behave well if we look at their almost sure properties. The classical case $\mathcal{H}_p((\log n))$ was also investigated as a Banach space of holomorphic function. Even in that case, it is a nontrivial problem to determine the optimal half-plane of convergence of elements in $\mathcal{H}_p(\lambda)$, namely to compute

$$\sigma_{\mathcal{H}_p(\lambda)} := \inf \{ \sigma \in \mathbb{R} : \sigma_c(D) \leq \sigma \text{ for all } D \in \mathcal{H}_p(\lambda) \}$$

where, for a Dirichlet series $D \in \mathcal{D}(\lambda)$, $\sigma_c(D) := \inf\{\sigma \in \mathbb{R} : D \text{ converges on } \mathbb{C}_\sigma\}$. This has been settled in [1], using that $\sigma_{\mathcal{H}_2((\log n))} = 1/2$ (easy by the Cauchy-Schwarz inequality) and that $T_\sigma(\sum_n a_n e^{-\lambda s}) = \sum_n a_n e^{-\sigma \lambda_n} e^{-\lambda_n s}$ maps $\mathcal{H}_p((\log n))$ into $\mathcal{H}_q((\log n))$ for all $p, q \in [1, +\infty)$ and all $\sigma > 0$. The argument is based on multiplicativity (namely on the fact that the natural λ -Dirichlet group for $(\log n)$ is the infinite polytorus \mathbb{T}^∞) and on a hypercontractive estimate for the Poisson kernel acting on the Hardy spaces $H^p(\mathbb{T})$ of the disk.

We will show that there is no hope to get such a result for general frequencies λ even if they satisfy (BC). For instance, if we will be able to prove that for all frequencies $\sigma_{\mathcal{H}_1(\lambda)} \leq 2\sigma_{\mathcal{H}_2(\lambda)}$, we will nevertheless point out that, even if we assume (BC), this is optimal and in particular that we may have $\sigma_{\mathcal{H}_1(\lambda)} > \sigma_{\mathcal{H}_2(\lambda)}$. We will also exhibit a sequence λ , which still satisfies (BC), such that T_σ maps boundedly $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ if and only if $\sigma \geq (k-1)/2k$. In particular, it seems very hard to compute $\sigma_{\mathcal{H}_p(\lambda)}$ in the general case and the behaviour of $\mathcal{H}_p(\lambda)$ as a space of holomorphic function seems more difficult to predict if we assume only growth and separation conditions on λ .

1.4. Notations. Throughout this work, we shall use the following notations. For λ a frequency and $D \in \mathcal{D}(\lambda)$, the abscissa of absolute convergence of D and the abscissa of uniform convergence of D are defined by

$$\begin{aligned}\sigma_a(D) &:= \inf\{\sigma \in \mathbb{R} : D \text{ converges absolutely on } \mathbb{C}_\sigma\} \\ \sigma_u(D) &:= \inf\{\sigma \in \mathbb{R} : D \text{ converges uniformly on } \mathbb{C}_\sigma\}.\end{aligned}$$

We set

$$\begin{aligned}L(\lambda) &:= \limsup_{N \rightarrow +\infty} \frac{\log N}{\lambda_N} \\ &= \sup_{D \in \mathcal{D}(\lambda)} \sigma_a(D) - \sigma_c(D).\end{aligned}$$

Given (G, β) a λ -Dirichlet group we shall denote by $\text{Pol}_\lambda(G)$ the set of polynomials with spectrum in λ , namely finite sums $\sum_{k=1}^n a_k h_{\lambda_k}$ with $\lambda_k \in G$ for each $k = 1, \dots, n$. We shall also use the following result: for all $f : G \rightarrow \mathbb{C}$ measurable, for almost all $\omega \in G$, the function $f_\omega := f(\omega\beta(\cdot)) : \mathbb{R} \rightarrow \mathbb{C}$ is measurable. If additionally $f \in L_\infty(G)$, then for almost all $\omega \in G$, $f_\omega \in L_\infty(\mathbb{R})$ with $\|f_\omega\|_\infty \leq \|f\|_\infty$. Moreover, if $f \in L_1(G)$, then f_ω is locally integrable for almost all $\omega \in G$, and for $g \in L_1(\mathbb{R})$, the convolution

$$g \star f_\omega(t) := \int_{\mathbb{R}} f(\omega\beta(t-y))g(y)dy$$

is almost everywhere defined on \mathbb{R} and measurable (see [10, Lemma 3.11]).

2. PRELIMINARIES

2.1. A new class of frequencies. We introduce our new condition, more general than (LC), under which most of our results will be satisfied. We first reformulate (LC).

Lemma 2.1. *A frequency λ satisfies (LC) if and only if there exists $C > 0$ such that, for all $\delta > 0$, for all $n \in \mathbb{N}$,*

$$\log \left(\frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} \right) \leq C e^{\delta \lambda_n}.$$

Proof. Assume first that λ satisfies (LC) and let $\delta > 0$. Then there exists $C > 0$ such that, for all $n \in \mathbb{N}$, $\lambda_{n+1} - \lambda_n \geq Ce^{-e^{\frac{\delta}{2}\lambda_n}}$. Let $n \in \mathbb{N}$ and set $\xi_n = \lambda_n + Ce^{-e^{\frac{\delta}{2}\lambda_n}}$. Since the function $x \mapsto (x + \lambda_n)/(x - \lambda_n)$ is decreasing on $(\lambda_n, +\infty)$, one gets

$$\begin{aligned} \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} &\leq \frac{\xi_n + \lambda_n}{\xi_n - \lambda_n} \\ &\leq C^{-1} \left(2\lambda_n + Ce^{-e^{\frac{\delta}{2}\lambda_n}} \right) e^{e^{\frac{\delta}{2}\lambda_n}} \\ &\leq C' e^{e^{\delta\lambda_n}}. \end{aligned}$$

The converse implication is easier and left to the reader. \square

The main idea to introduce (NC) is to allow to compare the position of λ_n with λ_m for some $m > n$ and not only with λ_{n+1} .

Definition 2.2. We say that a frequency λ satisfies (NC) if, for all $\delta > 0$, there exists $C > 0$ such that, for all $n \geq 1$, there exists $m > n$ such that

$$(NC) \quad \log \left(\frac{\lambda_m + \lambda_n}{\lambda_m - \lambda_n} \right) + (m - n) \leq Ce^{\delta\lambda_n}.$$

Condition (NC) provides a nontrivial extension of (LC).

Example 2.3. Let λ be defined by $\lambda_{2^n+k} = n^2 + ke^{-e^{n^2}}$ for $k = 0, \dots, 2^n - 1$. Then $L(\lambda) = +\infty$, λ satisfies (NC) and λ is not the finite union of frequencies satisfying (LC).

Proof. Let $\delta > 0$, $n \in \mathbb{N}$, $k \in \{0, \dots, 2^n - 1\}$, then provided n is large enough

$$\begin{aligned} \log \left(\frac{\lambda_{2^{n+1}} + \lambda_{2^n+k}}{\lambda_{2^{n+1}} - \lambda_{2^n+k}} \right) + (2^{n+1} - 2^n - k) &\leq \log(2(n+1)^2) + 2^n \\ &\leq Ce^{\delta n^2} \\ &\leq Ce^{\delta\lambda_{2^n+k}} \end{aligned}$$

for some $C > 0$. Moreover, if λ was the finite union of $\lambda^1, \dots, \lambda^p$, each λ^j satisfying (LC), then at least one of the λ^j , say λ^1 , will contain an infinite number of consecutive terms $\lambda_m^1 = \lambda_{2^n+k}$, $\lambda_{m+1}^1 = \lambda_{2^n+k'}$ with $1 \leq k' - k \leq p$ and $k, k' \in \{0, \dots, 2^n - 1\}$. For these m ,

$$\log \left(\frac{\lambda_{m+1}^1 + \lambda_m^1}{\lambda_{m+1}^1 - \lambda_m^1} \right) \geq e^{n^2} - \log p \geq Ce^{\lambda_m^1/2}$$

contradicting that λ^1 satisfies (LC). \square

2.2. Saksman's vertical convolution formula. Saksman's vertical convolution formula was introduced to express weighted sums of ordinary Dirichlet series using an integral. It says essentially that if $D = \sum_n a_n n^{-s}$ is an ordinary Dirichlet series and ψ is in L^1 with $\hat{\psi}$ compactly supported, then

$$\sum_{n=1}^{+\infty} a_n \hat{\psi}(\log n) n^{-s} = \int_{\mathbb{R}} D(s+it) \psi(t) dt$$

with a sense that has to be made precise. It was used in [4] for Dirichlet series in \mathcal{H}_1 and in [20] for Dirichlet series in \mathcal{H}_∞ . We shall extend it to general Dirichlet series and we will use it as a much more flexible substitute of Perron's formula.

Theorem 2.4. *Let $\psi \in L^1(\mathbb{R})$ be such that $\hat{\psi}$ is compactly supported and let λ be a frequency with (G, β) an associated λ -Dirichlet group.*

(a) *Let $D = \sum_n a_n e^{-\lambda_n s} \in \mathcal{D}_\infty^{\text{ext}}(\lambda)$ with bounded and holomorphic extension to \mathbb{C}_0 denoted by f . Then for all $s \in \mathbb{C}$ with $\Re(s) > 0$*

$$\sum_{n=1}^{+\infty} a_n \hat{\psi}(\lambda_n) e^{-\lambda_n s} = \int_{\mathbb{R}} f(s + it) \psi(t) dt.$$

(b) *Let $f = \sum_n a_n h_{\lambda_n} \in H_1^\lambda(G)$. Then for almost all $\omega \in G$,*

$$\sum_{n=1}^{+\infty} a_n \hat{\psi}(\lambda_n) h_{\lambda_n}(\omega) = \int_{\mathbb{R}} f_\omega(t) \psi(t) dt.$$

In the sequel, for $D = \sum_n a_n e^{-\lambda_n s} \in \mathcal{D}_\infty^{\text{ext}}(\lambda)$, respectively for $f = \sum_n a_n h_{\lambda_n}$ in $H_1^\lambda(G)$, and for $\psi \in L^1(\mathbb{R})$ compactly supported, we shall denote

$$\begin{aligned} R_\psi(D) &:= \sum_n a_n \hat{\psi}(\lambda_n) e^{-\lambda_n s} \\ R_\psi(f) &:= \sum_n a_n \hat{\psi}(\lambda_n) h_{\lambda_n}. \end{aligned}$$

Proof. (a) Observe first that the equality is true provided D is a Dirichlet polynomial and that the two members of the equality define an analytic function on \mathbb{C}_0 . Assume first that $L(\lambda) < +\infty$. Then $\sigma_a(D) < +\infty$ and for $s > \sigma_a(D)$, the formula is true just by exchanging the sum and the integral. We conclude by analytic continuation.

When $L(\lambda) = +\infty$, the proof is more difficult. We use (see [14, Theorem 41 p. 53] or [18, Theorem 22]) that there exist a half-plane \mathbb{C}_θ , $\theta > 0$, and a sequence of λ -Dirichlet polynomials $D_j = \sum_{n=1}^{+\infty} a_n^j e^{-\lambda_n s}$ such that (a_n^j) tends to a_n as j tends to $+\infty$ for any n and (D_j) converges uniformly to f on \mathbb{C}_θ . Since each D_j is a Dirichlet polynomial, we know that for all $s \in \mathbb{C}_\theta$ and all $j \in \mathbb{N}$,

$$\sum_{n=1}^{+\infty} a_n^j \hat{\psi}(\lambda_n) e^{-\lambda_n s} = \int_{\mathbb{R}} D_j(s + it) \psi(t) dt.$$

Letting j to $+\infty$ in the previous inequality for a fixed $s \in \mathbb{C}_\theta$, since the sum on the left handside is finite ($\hat{\psi}$ has compact support), and by uniform convergence, we get the result on \mathbb{C}_θ . We conclude again by analytic continuation.

(b) When $f \in \text{Pol}_\lambda(G)$, the equality follows immediately by interverting a finite sum and

an integral, and the definition of the objects that come into play:

$$\begin{aligned} R_\psi(f)(\omega) &= \int_{\mathbb{R}} \sum_n a_n e^{-it\lambda_n} \psi(t) h_{\lambda_n}(\omega) dt \\ &= \int_{\mathbb{R}} \sum_n a_n h_{\lambda_n}(\beta(t)) h_{\lambda_n}(\omega) \psi(t) dt \\ &= \int_{\mathbb{R}} f_\omega(t) \psi(t) dt \end{aligned}$$

(here the equality is valid for all $\omega \in G$). Let now $f \in H_1^\lambda(G)$. Then

$$\int_G \int_{\mathbb{R}} |f_\omega(t) \psi(t)| dt \leq \|f\|_1 \|\psi\|_1.$$

Therefore, for almost all $\omega \in G$, the function $t \mapsto f_\omega(t) \psi(t)$ belongs to $L_1(\mathbb{R})$ and the operator $S_\psi : H_1^\lambda(G) \rightarrow L_1(G, L_1(\mathbb{R}))$, $f \mapsto [\omega \mapsto f_\omega(\cdot) \psi(\cdot)]$ is continuous. If (f_n) is a sequence in $\text{Pol}_\lambda(G)$ tending to $f \in H_1^\lambda(G)$, then there exists a sequence (n_k) such that, for a.e. $\omega \in G$,

$$\begin{aligned} S_\psi(f_{n_k})(\omega) &\rightarrow S_\psi(f)(\omega) \text{ in } L_1(\mathbb{R}) \\ R_\psi(f_{n_k})(\omega) &\rightarrow R_\psi(f)(\omega) \end{aligned}$$

(recall that the sum defining R_ψ is finite). Since $R_\psi(f_{n_k})(\omega) = \int_{\mathbb{R}} S_\psi(f_{n_k})(\omega)$ for all k and all $\omega \in G$, we get the conclusion by taking the limit. \square

Remark 2.5. The statement of Theorem 2.4 remains true provided ψ is not compactly supported but still satisfies $\sum_n |\hat{\psi}(\lambda_n)| < +\infty$.

Remark 2.6. To obtain Theorem 2.4, in both cases, we use the density of polynomials for a suitable topology. In $H_1^\lambda(G)$, this is trivial which is not the case in $\mathcal{D}_{\text{ext}}^\infty(\lambda)$. More specifically we intend to use Theorem 2.4 to obtain results that do not seem easily reachable using Riesz means. Therefore it is interesting to obtain a proof of Theorem 2.4 that do not use Riesz means. This is the case if we use [18, Theorem 22]. We thank A. Defant and I. Schoolmann for pointing out to me this reference.

Remark 2.7. Part (b) of the vertical convolution formula is more precised than the statement established and used in [4]. The equivalent statement in this context would be that, for all $f \in H_1^\lambda(G)$,

$$\sum_{n=1}^{+\infty} a_n \hat{\psi}(n) h_{\lambda_n} = \int_{\mathbb{R}} T_t f \psi(t) dt,$$

where $T_t : H_1^\lambda(G) \rightarrow H_1^\lambda(G)$, $f \mapsto f(\beta(t)\cdot)$ is an onto isometry of $H_1^\lambda(G)$ and the right handside denotes a vector-valued integral in $H_1^\lambda(G)$. We will need a pointwise statement in order to obtain maximal inequalities.

2.3. Riesz means and Saksman's vertical convolution formula. We now show how the results on (λ, k) -Riesz means will follow from our results coming from Saksman's vertical convolution formula. This is a consequence of the following easy proposition.

Proposition 2.8. *Let $\alpha > 0$. Then there exists an $L^1(\mathbb{R})$ -function ψ such that, for all $t \in \mathbb{R}$, $\hat{\psi}(x) = (1 - |x|)^\alpha$ provided $|x| < 1$, $\hat{\psi}(x) = 0$ otherwise.*

Proof. Define $u(x) = (1 - |x|)^\alpha \mathbf{1}_{[-1,1]}(x)$. Then u is piecewise \mathcal{C}^1 , its derivative $u'(x) = \pm\alpha(1 - |x|)^{\alpha-1} \mathbf{1}_{[-1,1]}(x)$ belongs to $L^1(\mathbb{R})$ and thus we know that, for all $t \neq 0$, $\hat{u}(t) = \frac{1}{it} \widehat{u}'(t)$. Now, it is easy to see that u' belongs to $L^{1+\varepsilon}(\mathbb{R}) \cap L^1(\mathbb{R})$ for some $\varepsilon > 0$. Hence, \widehat{u}' belongs to $L^q(\mathbb{R})$ for some $q < +\infty$. In particular, by Hölder's inequality, \hat{u} belongs to L^1 , so that we may apply the inverse Fourier transform to get the statement. \square

In the sequel, for λ a frequency, $\alpha > 0$, (G, β) a λ -Dirichlet group and $N > 0$, we shall use the following notations:

$$R_N^{\lambda, \alpha}(f) = \sum_{\lambda_n \leq N} \widehat{f}(h_{\lambda_n}) \left(1 - \frac{\lambda_n}{N}\right)^\alpha h_{\lambda_n}$$

$$R_N^{\lambda, \alpha}(D) = \sum_{\lambda_n \leq N} a_n \left(1 - \frac{\lambda_n}{N}\right)^\alpha e^{-\lambda_n s}$$

where $f \in H_1^\lambda(G)$ and $D = \sum_n a_n e^{-\lambda_n s} \in \mathcal{D}(\lambda)$. Many of the results of [21, 12, 11] are based on a detailed study of these operators $R_N^{\lambda, \alpha}$. We shall extend them via the convolution formula to other operators R_ψ , allowing better results with a different choice of ψ .

3. BOHR'S THEOREM UNDER (NC)

3.1. The case of $\mathcal{D}_\infty^{\text{ext}}(\lambda)$. In his study of Bohr's theorem [21], I. Schoolmann used that, for all $D = \sum_n a_n e^{-\lambda_n s} \in \mathcal{D}_\infty^{\text{ext}}(\lambda)$ with extension f , the sequence of its Riesz means of order k

$$R_x^k(D) = \sum_{\lambda_n < x} a_n \left(1 - \frac{\lambda_n}{x}\right)^k e^{-\lambda_n s}$$

converge uniformly to f on each halfplane \mathbb{C}_ε , for all $\varepsilon > 0$, as $x \rightarrow +\infty$. We now show that we may replace the function ψ such that $\hat{\psi}(t) = (1 - |t|)^k \mathbf{1}_{[-1,1]}(t)$ by any function L^1 -function ψ such that $\hat{\psi}$ has compact support.

Lemma 3.1. *Let λ be a frequency, let $\psi \in L^1(\mathbb{R})$ be such that $\hat{\psi}$ has compact support and let $D = \sum_n a_n e^{-\lambda_n s} \in \mathcal{D}_\infty^{\text{ext}}(\lambda)$ with extension f . Then*

$$\|R_\psi(D)\|_\infty \leq \|\psi\|_1 \|f\|_\infty.$$

Moreover, if $\int_{\mathbb{R}} \psi = 1$, denoting by $\psi_N(\cdot) = N\psi(N\cdot)$, the sequence of λ -Dirichlet polynomials $(R_{\psi_N}(D))$ converges uniformly to f on each half-plane \mathbb{C}_ε , for all $\varepsilon > 0$.

Proof. The inequality follows immediately from Theorem 2.4. The statement on uniform convergence follows as well from this formula and from standard results on mollifiers, provided we know that f is uniformly continuous on \mathbb{C}_ε . Again, this can be deduced from the fact that on this half-plane, f is the uniform limit of λ -Dirichlet polynomials, which are themselves uniformly continuous. \square

We shall now apply this to a suitable choice of ψ in order to get good estimates of the norm of the projection S_N .

Theorem 3.2. *Let λ be a frequency. There exists $C > 0$ such that, for all $M > N \geq 1$,*

$$\|S_N\|_{\mathcal{D}_{\infty}^{ext}(\lambda) \rightarrow \mathcal{D}_{\infty}(\lambda)} \leq C \left(\log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) + (M - N - 1) \right).$$

Proof. We set $h = \frac{\lambda_M - \lambda_N}{2}$. Let u be the function equal to 1 on $[-\lambda_N, \lambda_N]$, to 0 on $\mathbb{R} \setminus (-\lambda_M, \lambda_M)$, and which is affine on $(-\lambda_M, \lambda_N)$ and on (λ_N, λ_M) . The function u may be written

$$u = \mathbf{1}_{[-\lambda_N - h, \lambda_N + h]} \star \left(\frac{1}{2h} \mathbf{1}_{[-h, h]} \right).$$

This formula allows us to compute the Fourier transform of u which is equal to

$$\widehat{u}(t) = 2 \frac{\sin((\lambda_N + h)t)}{t} \cdot \frac{\sin(ht)}{ht}$$

which is an L^1 function. Moreover

$$\begin{aligned} \|\widehat{u}\|_1 &\leq 4 \int_0^{+\infty} \left| \frac{\sin((\lambda_N + h)t)}{t} \right| \times \left| \frac{\sin(ht)}{ht} \right| dt \\ &\leq 4 \int_0^{+\infty} \left| \frac{\sin\left(\frac{\lambda_N + h}{h}x\right)}{x} \right| \times \left| \frac{\sin(x)}{x} \right| dx \\ &\leq 4 \int_0^1 \left| \frac{\sin\left(\frac{\lambda_N + h}{h}x\right)}{x} \right| dx + 4 \int_1^{+\infty} \frac{1}{x^2} dx \\ &\leq C \log \left(\frac{\lambda_N + h}{h} \right) + 4 = C \log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) + 4 \end{aligned}$$

where we have used well-known estimates of the L^1 -norm of the sinus cardinal function. We then applied Lemma 3.1 to $\psi \in L^1$ defined by $\widehat{\psi} = u$. By the Fourier inverse formula,

$$\left\| \sum_{n=1}^{M-1} a_n \widehat{\psi}(\lambda_n) e^{-\lambda_n s} \right\|_{\infty} \leq \left(C \log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) + 4 \right) \|f\|_{\infty}.$$

We get the conclusion by writing

$$\sum_{n=1}^N a_n e^{-\lambda_n s} = \sum_{n=1}^{M-1} a_n \widehat{\psi}(\lambda_n) e^{-\lambda_n s} - \sum_{n=N+1}^{M-1} a_n \widehat{\psi}(\lambda_n) e^{-\lambda_n s}$$

and by using that $|a_n| \leq \|f\|_{\infty}$ (see [21, Corollary 3.9]) and $\|\widehat{\psi}\|_{\infty} \leq 1$. \square

From the Bohr-Cahen formula to compute the abscissa of uniform convergence of a λ -Dirichlet series,

$$\sigma_u(D) \leq \limsup_N \frac{\log \left(\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n e^{-\lambda_n it} \right| \right)}{\lambda_N}$$

we get the following corollary.

Corollary 3.3. *Let λ be a frequency satisfying (NC). Then λ satisfies Bohr's theorem.*

Let us now compare Theorem 3.2 with the results of [21]. There it is shown that, for all $N \geq 1$ and all $k \in (0, 1]$,

$$\|S_N\|_{\mathcal{D}_\infty^{\text{ext}}(\lambda) \rightarrow \mathcal{D}_\infty(\lambda)} \leq \frac{C}{k} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{1/k}.$$

The right hand side is optimal for $k = \frac{1}{\log\left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N}\right)}$ which implies that

$$\|S_N\|_{\mathcal{D}_\infty^{\text{ext}}(\lambda) \rightarrow \mathcal{D}_\infty(\lambda)} \leq C \log \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right).$$

Hence, we get the case $M = N + 1$ of Theorem 3.2.

3.2. The case of $\mathcal{H}_\infty(\lambda)$. So far, we have defined three spaces which are candidates for being the H_∞ -space of λ -Dirichlet series: $\mathcal{D}_\infty(\lambda)$, $\mathcal{D}_\infty^{\text{ext}}(\lambda)$, and $\mathcal{H}_\infty(\lambda)$. We know that we always have the canonical inclusion $\mathcal{D}_\infty(\lambda) \subset \mathcal{D}_\infty^{\text{ext}}(\lambda) \subset \mathcal{H}_\infty(\lambda)$ (see [11, Theorem 2.17]) and that, when λ satisfies Bohr's theorem, the three spaces are equal. Observe also that $\mathcal{H}_\infty(\lambda)$ is the only space that is always complete.

Thus, Theorem 3.2 does not always provide an answer for estimating the norm of S_N as an operator on $\mathcal{H}_\infty(\lambda)$. Fortunately, the proof extends easily using the second (and easiest part) of Theorem 2.4.

Theorem 3.4. *Let λ be a frequency. There exists $C > 0$ such that, for all $M > N \geq 1$,*

$$\|S_N\|_{\mathcal{H}_\infty(\lambda) \rightarrow \mathcal{D}_\infty(\lambda)} \leq C \left(\log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) + (M - N - 1) \right).$$

Proof. We do the proof in $H_\infty^\lambda(G)$. Let $f = \sum_n a_n h_{\lambda_n} \in H_\infty^\lambda(G)$. We pick the same function ψ and observe, that for almost all $\omega \in G$,

$$\begin{aligned} \left| \sum_{n=1}^{+\infty} a_n \widehat{\psi}(\lambda_n) h_{\lambda_n}(\omega) \right| &\leq \int_{\mathbb{R}} |f_\omega(it) \psi(t)| dt \\ &\leq \left(C \log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) + 4 \right) \|f\|_\infty \end{aligned}$$

and we conclude as above. \square

4. MAXIMAL INEQUALITIES IN $H_p^\lambda(G)$

4.1. Helson's theorem under (NC). In this section, we prove the following theorem, which improves the main result of [12] and answers an open question of [9].

Theorem 4.1. *Let λ satisfy (NC), let (G, β) be a λ -Dirichlet group. For every $u > 0$, there exists a constant $C := C(u, \lambda)$ such that, for all $1 \leq p \leq +\infty$ and for all $f \in H_p^\lambda(G)$,*

$$\left\| \sup_{\sigma \geq u} \sup_N \left\| \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) e^{-\sigma \lambda_n} h_{\lambda_n} \right\| \right\|_p \leq C \|f\|_p.$$

In particular, for every $u > 0$, $\sum_1^{+\infty} \widehat{f}(h_{\lambda_n}) e^{-u \lambda_n} h_{\lambda_n}$ converges almost everywhere on G .

Let us explain the strategy for the proof. When $p > 1$, the almost everywhere convergence is known to hold without any assumption on λ . This is a consequence of the Carleson-Hunt type result proved in [12]: for all frequencies λ , for all (G, β) a λ -Dirichlet group, for all $p \in (1, +\infty)$, there exists $C(p) > 0$ such that for all $f \in H_p^\lambda(G)$,

$$(2) \quad \left(\int_G \sup_n |S_n f(\omega)|^p d\omega \right)^{1/p} \leq C(p) \|f\|_p$$

where $S_n(f) = \sum_{k=1}^n \widehat{f}(h_{\lambda_k}) h_{\lambda_k}$ is the partial sum operator (the constant $C(p)$ does not even depend on λ). We shall prove a variant of (2) under (NC), namely

$$(3) \quad \left(\int_G \sup_n e^{-\delta \lambda_n} |S_n f(\omega)|^p d\omega \right)^{1/p} \leq C(\lambda, \delta) \|f\|_p$$

valid for all $p \geq 1$, all $\delta > 0$ and all $f \in H_p^\lambda(G)$, with a constant $C(\lambda, \delta)$ independent of p . The proof of (3) will be done for $p = 1$ and for $p = +\infty$ and will be finished by interpolation. Unfortunately, it is in general false that $[H_{p_0}^\lambda(G), H_{p_1}^\lambda(G)]_\theta = H_{p_\theta}^\lambda(G)$ (see [3]) and we will use an auxiliary operator defined on the whole $L_1(G)$.

We begin by establishing several lemmas. First, we shall prove that we may require additional properties on a sequence satisfying (NC).

Lemma 4.2. *Let λ be a frequency satisfying (NC). Then there exists a frequency λ' such that $\lambda \subset \lambda'$ and, for all $\delta > 0$, there exists $C > 0$ such that, for all $n \in \mathbb{N}$, there exists $m > n$ with*

$$(4) \quad \log(\lambda'_m + \lambda'_n) \leq C e^{\delta \lambda'_n}$$

$$(5) \quad -\log(\lambda'_m - \lambda'_n) \leq C e^{\delta \lambda'_n}$$

$$(6) \quad m - n \leq C e^{\delta \lambda'_n}.$$

Proof. We construct inductively λ' as follows. We set $\lambda'_1 = \lambda_1$. Assume that the sequence λ' has been built until step n , namely that we have constructed $\lambda'_1, \dots, \lambda'_{k_n}$ with $\lambda'_{k_n} = \lambda_n$. If $\lambda_{n+1} \leq \lambda_n + 1$, then we set $\lambda'_{k_n+1} = \lambda_{n+1}$ and $k_{n+1} = k_n + 1$. Otherwise, we include as many terms $\lambda'_{k_n+1}, \dots, \lambda'_{k_{n+1}}$ as necessary so that, for all $j = k_n + 1, \dots, k_{n+1} - 1$, $1/2 \leq \lambda'_{j+1} - \lambda'_j \leq 1$ and $\lambda'_{k_{n+1}} = \lambda_{n+1}$. Namely, we add terms in the sequence λ when there is a gap greater than 1 between two successive terms, and the difference between two consecutive terms of λ' is now less than 1.

Let us show that the sequence λ' satisfies the above conclusion. Let λ'_n be any term of the sequence λ' . If λ'_n does not belong to λ , then we have just to consider $m = n + 1$. Otherwise, if $\lambda'_n = \lambda_k$ for some $k \leq n$, there exists $l > k$ such that

$$(7) \quad \log \left(\frac{\lambda_l + \lambda_k}{\lambda_l - \lambda_k} \right) \leq C e^{\delta \lambda_k}$$

$$l - k \leq C e^{\delta \lambda_k}.$$

Set $m = n + (l - k)$ and observe that we have

$$\log(\lambda'_m + \lambda'_n) \leq \log \left(C e^{\delta \lambda'_n} + 2\lambda'_n \right) \leq C' e^{\delta \lambda'_n}.$$

If there is no gap between λ_k and λ_l , then $\lambda'_m = \lambda_l$ and (7) implies

$$-\log(\lambda'_m - \lambda'_n) = -\log(\lambda_l - \lambda_k) \leq Ce^{\delta\lambda_k} = Ce^{\delta\lambda_n}.$$

If there is a gap between λ_k and λ_l , then $\lambda'_m - \lambda'_n \geq 1/2$, and (5) holds trivially. \square

In the sequel, when we will pick a frequency λ satisfying (NC), we will in fact assume that it satisfies the stronger properties given by Lemma 4.2.

For $a > 0$ and $h > 0$, we shall denote by $\psi_{a,h}$ the function defined by

$$\psi_{a,h}(t) = \frac{\sin((a+h)t)}{t} \times \frac{\sin(ht)}{ht}.$$

The estimation of the L^1 -norm of $\psi_{a,h}$ was a crucial point in order to apply Saksman's convolution formula during the proof of Theorem 3.2. In order to obtain our maximal estimates, we will need a similar inequality allowing now a and h to vary.

Lemma 4.3. *Let $a : \mathbb{R} \rightarrow (0, +\infty)$ and $h : \mathbb{R} \rightarrow (0, +\infty)$ be two measurable functions. Assume that there exists $\kappa > 0$ such that $a(t) + h(t) \leq \kappa$ and $h(t) \geq \kappa^{-1}$ for all $t \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}} |\psi_{a(t),h(t)}(t)| dt \leq 4 + 4 \log \kappa.$$

Proof. It suffices to observe that

- when $0 < |t| \leq \kappa^{-1}$, then

$$|\psi_{a(t),h(t)}(t)| \leq |a(t) + h(t)| \times 1 \leq \kappa.$$

- when $\kappa^{-1} \leq |t| \leq \kappa$, then

$$|\psi_{a(t),h(t)}(t)| \leq \frac{1}{|t|} \times 1 = \frac{1}{|t|}.$$

- when $|t| \geq \kappa$, then

$$|\psi_{a(t),h(t)}(t)| \leq \frac{1}{h(t)t^2} \leq \frac{\kappa}{t^2}.$$

\square

We now fix a frequency λ satisfying (NC) and $\delta > 0$. Let $C > 0$ and $m : \mathbb{N} \rightarrow \mathbb{N}$ be such that $m(n) > n$ for all $n \in \mathbb{N}$ and (4), (5), (6) are satisfied for $m = m(n)$. For $n \in \mathbb{N}$, we shall denote by $h_n = (\lambda_{m(n)} - \lambda_n)/2$ and by ϕ_n the function $\phi_n = \psi_{\lambda_n, h_n}$. Let us recall that R_{ϕ_n} is defined on $H_1^\lambda(G)$ by

$$(8) \quad R_{\phi_n}(f) = \sum_k \widehat{f}(h_{\lambda_k}) \widehat{\phi_n}(\lambda_k) h_{\lambda_k}.$$

By the vertical convolution formula, we also know that R_{ϕ_n} is given by, for a.e. $\omega \in G$,

$$(9) \quad R_{\phi_n}(f)(\omega) = \int_{\mathbb{R}} f(\omega\beta(t)) \phi_n(t) dt.$$

Now the right hand side of the previous equality is well-defined for all functions in $L_1(G)$. Thus we will think at R_{ϕ_n} as the operator on $L_1(G)$ defined by (9), keeping in mind that it also verifies (8) for $f \in H_1^\lambda(G)$. In this context, we shall prove the following maximal inequality on R_{ϕ_n} :

Lemma 4.4. *For all $\delta > 0$, there exists $C > 0$ such that, for all $p \in [1, +\infty]$, for all $N \in \mathbb{N}$, for all $f \in L_p(G)$,*

$$\left(\int_G \sup_{n \leq N} |R_{\phi_n} f(\omega)|^p d\omega \right)^{1/p} \leq C e^{\delta \lambda_N} \|f\|_p.$$

Proof. We start with the case $p = 1$. It is enough to prove it for $f \in \mathcal{C}(G)$. Define $n : G \rightarrow \{1, \dots, N\}$, $\omega \mapsto n(\omega)$ by

$$n(\omega) = \inf \left\{ l \in \{1, \dots, N\} : |R_{\phi_l} f(\omega)| = \sup_{n \leq N} |R_{\phi_n} f(\omega)| \right\}.$$

The function n is measurable and

$$\begin{aligned} \int_G \sup_{n \leq N} |R_{\phi_n}(f)(\omega)| d\omega &= \int_G |R_{\phi_{n(\omega)}}(f)(\omega)| d\omega \\ &\leq \int_{\mathbb{R}} \int_G |f(\omega\beta(t))| \cdot |\psi_{\lambda_{n(\omega)}, h_{n(\omega)}}(t)| d\omega dt. \end{aligned}$$

In the inner integral we do the change of variables $\omega' = \omega\beta(t)$ so that

$$\begin{aligned} \int_G \sup_{n \leq N} |R_{\phi_n}(f)(\omega)| d\omega &\leq \int_{\mathbb{R}} \int_G |f(\omega')| \cdot |\psi_{\lambda_{n(\omega' \cdot \beta(t))^{-1}}, h_{n(\omega' \cdot \beta(t))^{-1}}}(t)| d\omega dt \\ &\leq \int_G |f(\omega')| \int_{\mathbb{R}} |\psi_{\lambda_{n(\omega' \cdot \beta(t))^{-1}}, h_{n(\omega' \cdot \beta(t))^{-1}}}(t)| dt d\omega. \end{aligned}$$

We now use Lemma 4.3 together with (4) and (5). This yields

$$\int_G \sup_{n \leq N} |R_{\phi_n}(f)(\omega)| d\omega \leq C \int_G |f(\omega')| e^{\delta \lambda_N} d\omega' = C e^{\delta \lambda_N} \|f\|_1.$$

We then do the case $p = +\infty$. Let $f \in L_\infty(G)$. Then

$$\begin{aligned} \sup_{\omega \in G} \sup_{n \leq N} |R_{\phi_n} f(\omega)| &= \sup_{n \leq N} \sup_{\omega \in G} \int_{\mathbb{R}} |f(\omega\beta(t))| \cdot |\psi_{\lambda_n, h_n}(t)| dt \\ &\leq \sup_{n \leq N} \|\psi_{\lambda_n, h_n}\|_1 \|f\|_\infty \\ &\leq C e^{\delta \lambda_N} \|f\|_\infty. \end{aligned}$$

We then conclude by interpolation. □

We deduce from the above work a weighted Carleson-Hunt maximal inequality for $H_1^\lambda(G)$ -functions, which seems interesting for itself when $p = 1$ (for $p \in (1, +\infty)$, an unweighted Carleson-Hunt inequality is true, the point here is that the constant does not depend on p). This statement was inspired by [2] where a similar result in the (much easier) case of $H_1(\mathbb{T})$ was essential to do a multifractal analysis of the divergence of Fourier series of functions of $H_1(\mathbb{T})$.

Theorem 4.5. *Let λ satisfying (NC). For all $\delta > 0$ there exists $C > 0$ such that, for all $N \in \mathbb{N}$, for all $p \geq 1$, for all $f \in H_p^\lambda(G)$,*

$$\left(\int_G \sup_{n \leq N} |S_n f(\omega)|^p d\omega \right)^{1/p} \leq C e^{\delta \lambda_N}.$$

Proof. We argue as in the proof of Theorem 3.2, namely we write for a fixed $n \in \mathbb{N}$,

$$|S_n f(\omega)| \leq |R_{\phi_n} f(\omega)| + m(n) - n \leq |R_{\phi(n)} f(\omega)| + C e^{\delta \lambda_n}.$$

Therefore,

$$\sup_{n \leq N} |S_n f(\omega)| \leq \sup_{n \leq N} |R_{\phi_n}(f)(\omega)| + C e^{\delta \lambda_N}$$

and we conclude by taking the $L_p(G)$ -norm. \square

We are now ready for the proof of Theorem 4.1.

Proof of Theorem 4.1. We first proceed with the case $p \in [1, +\infty)$. We may assume that $f \in \text{Pol}_\lambda(G)$. Let $\delta = u/3$. For $\sigma \geq u$, using Lemma 3.4 of [12], we have

$$\left| \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) e^{-\sigma \lambda_n} h_{\lambda_n} \right|^p \leq C(u)^p \sup_{n \leq N} \left| e^{-2\delta \lambda_n} \sum_{k=1}^n \widehat{f}(h_{\lambda_k}) h_{\lambda_k} \right|^p.$$

Hence,

$$\sup_{\sigma > u} \sup_N \left| \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) e^{-\sigma \lambda_n} h_{\lambda_n} \right|^p \leq C(u)^p \sup_N \left| e^{-2\delta \lambda_N} \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right|^p.$$

For $\omega \in G$, we define

$$n(\omega) = \inf \left\{ l \geq 0 : \left| e^{-2\delta \lambda_l} \sum_{n=1}^l \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right| = \sup_N \left| e^{-2\delta \lambda_N} \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right| \right\}$$

which is measurable. For $k \geq 0$, we set

$$A_k = \{n : \lambda_n \in [k, k+1)\}, \quad G_k = \{\omega \in G : n(\omega) \in A_k\},$$

$$I(\sigma) = \int_G \sup_{\sigma > u} \sup_N \left| \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) e^{-\sigma \lambda_n} h_{\lambda_n} \right|^p d\omega.$$

We can write

$$\begin{aligned}
I(\sigma) &\leq C(u)^p \sum_{k \geq 0} \int_{G_k} \sup_N \left| e^{-2\delta\lambda_N} \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right|^p d\omega \\
&\leq C(u)^p \sum_{k \geq 0} \int_{G_k} \sup_{N \in A_k} \left| e^{-2\delta\lambda_N} \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right|^p d\omega \\
&\leq C(u)^p \sum_{k \geq 0} \int_{G_k} e^{-2\delta pk} \sup_{N \in A_k} \left| \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right|^p d\omega \\
&\leq C(u, \lambda)^p \sum_{k \geq 0} e^{-2\delta pk} e^{\delta p(k+1)} \|f\|_p^p \\
&\leq C(u, \lambda)^p \|f\|_p^p.
\end{aligned}$$

As for the proof of Lemma 4.4, the proof is easier for $p = \infty$ and is left to the reader. \square

If we analyze the previous proof carefully, we observe that we have obtained the following (slightly stronger) variant of Theorem 4.5.

Corollary 4.6. *Let λ satisfying (NC). For all $\delta > 0$, there exists $C > 0$ such that, for all $p \geq 1$, for all $f \in H_p^\lambda(G)$,*

$$\left(\int_G \sup_N \left| \frac{S_N f(\omega)}{e^{\delta\lambda_N}} \right|^p d\omega \right)^{1/p} \leq C(\delta) \|f\|_p.$$

When λ satisfies (BC), it is possible to improve this inequality.

Proposition 4.7. *Let λ satisfy (BC). For all $\alpha > 1$, there exists $C > 1$ such that, for all $p \geq 1$, for all $f \in H_p^\lambda(G)$,*

$$\left(\int_G \sup_N \left| \frac{S_N f(\omega)}{\lambda_N^\alpha} \right|^p d\omega \right)^{1/p} \leq C(\alpha) \|f\|_p.$$

Proof. We just sketch the proof. If λ satisfies (BC), then we know that there exists $C > 0$ such that, for all $n \in \mathbb{N}$, $\log(\lambda_{n+1} - \lambda_n) \geq -C\lambda_n$. Adding terms if necessary, we can also assume that $\log(\lambda_{n+1} + \lambda_n) \leq C\lambda_n$. Arguing exactly as in the proof of Theorem 4.5, we can prove the existence of $C > 0$ such that, for all $f \in H_p^\lambda(G)$, for all $n \in \mathbb{N}$,

$$\int_G \sup_{n \leq N} |S_n f(\omega)|^p d\omega \leq C\lambda_N^p.$$

Let now $\alpha > 1$, fix $f \in \text{Pol}_\lambda(G)$ and define, for $\omega \in G$,

$$\begin{aligned}
n(\omega) &= \inf \left\{ l \geq 0 : \left| \lambda_l^{-\alpha} \sum_{n=1}^l \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right| = \sup_N \left| \lambda_N^{-\alpha} \sum_{n=1}^N \widehat{f}(h_{\lambda_n}) h_{\lambda_n} \right| \right\} \\
A_k &= \{n : \lambda_n \in [2^k, 2^{k+1})\} \\
G_k &= \{\omega : n(\omega) \in A_k\}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_G \sup_N \left| \frac{S_N f(\omega)}{\lambda_N^\alpha} \right|^p d\omega &= \sum_k \int_{G_k} \sup_N \left| \frac{S_N f(\omega)}{\lambda_N^\alpha} \right|^p d\omega \\
&= \sum_k \int_{G_k} \sup_{N \in A_k} \left| \frac{S_N f(\omega)}{\lambda_N^\alpha} \right|^p d\omega \\
&\leq \sum_k 2^{-pk\alpha} \int_G \sup_{\lambda_N \leq 2^{k+1}} |S_N f(\omega)|^p d\omega \\
&\leq C \sum_k 2^{-pk\alpha} 2^{p(k+1)} \|f\|_p^p.
\end{aligned}$$

□

Question 4.8. We know that λ satisfies Bohr's theorem if and only if for all $\delta > 0$, there exists $C > 0$ such that, for all $f \in H_\infty^\lambda(G)$, for all $N \geq 1$,

$$(10) \quad \left\| \sup_{n \leq N} \left| \sum_{k=1}^n \hat{f}(h_{\lambda_k}) h_{\lambda_k}(\omega) \right| \right\|_{L_\infty(G)} \leq C e^{\delta \lambda_N} \|f\|_\infty.$$

We have shown that if λ satisfies (NC), then it satisfies the previous inequality. To prove that Helson's theorem is satisfied (and even to prove that the relevant maximal inequality holds true), it is sufficient to prove that, for all $\delta > 0$, there exists $C > 0$ such that, for all $f \in H_1^\lambda(G)$, for all $N \geq 1$,

$$(11) \quad \left\| \sup_{n \leq N} \left| \sum_{k=1}^n \hat{f}(h_{\lambda_k}) h_{\lambda_k}(\omega) \right| \right\|_{L_1(G)} \leq C e^{\delta \lambda_N} \|f\|_1.$$

Again we have shown that if λ satisfies (NC), then (11) is true. It seems natural to ask whether (11) always follows from (10) or, equivalently, if any frequency λ satisfying Bohr's theorem also satisfies Helson's theorem. Inequalities in $\mathcal{H}_1(\lambda)$ have already been deduced for their vector-valued counterpart in $\mathcal{H}_\infty(\lambda)$ in [10]. At first glance, it seems that this argument cannot be applied here.

4.2. Failure of Helson's theorem for $p = 1$. Since for $p > 1$, for any frequency λ , for any (G, β) a λ -Dirichlet group, for any $g \in H_p^\lambda(G)$, the series $\sum_{n=1}^{+\infty} \hat{f}(h_{\lambda_n}) h_{\lambda_n}$ converges almost everywhere on G (this follows from the Carleson-Hunt theorem of [12]), it is natural to ask whether Theorem 4.1 remains true without any assumption on λ . We show that this is not the case.

Theorem 4.9. *There exists a frequency λ , a λ -Dirichlet group (G, β) and $f \in H_1^\lambda(G)$ such that, for all $u > 0$, the series $\sum_{n=1}^{+\infty} \hat{f}(h_{\lambda_n}) e^{-u\lambda_n} h_{\lambda_n}$ diverges almost everywhere on G .*

As we might guess, the proof will use the results of Kolmogorov on a.e. divergent Fourier series in $L_1(\mathbb{T})$ (see for instance [22]).

Lemma 4.10. *Let $A, \delta > 0$. There exists $P \in H_1(\mathbb{T})$ a polynomial and $E \subset \mathbb{T}$ measurable such that*

- $\|P\|_1 \leq \delta$.
- $m_{\mathbb{T}}(E) \geq 1 - \delta$ (here, $m_{\mathbb{T}}$ denotes the Lebesgue measure on \mathbb{T}).
- for all $z \in E$, there exists $n(z) \in \mathbb{N}$ such that $|S_{n(z)}P(z)| \geq A$.

Proof. By induction on $j \geq 1$, we construct a sequence of holomorphic polynomials (P_j) with $\deg(P_j) = d_j$, two sequences of positive real numbers (μ_j) and (ε_j) and a sequence (E_j) of measurable subsets of \mathbb{T} such that the following properties are true for each j :

- $m_{\mathbb{T}}(E_j) \geq 1 - 2^{-j}$
- $\|P_j\|_1 \leq 2^{-j}$
- $\mu_j > \mu_{j-1} + d_{j-1}\varepsilon_{j-1}$
- the real numbers $2\pi, \mu_1, \dots, \mu_j, \varepsilon_1, \dots, \varepsilon_j$ are \mathbb{Q} -independent
- for each $z \in E_j$, there exists an integer $n_j(z)$ such that, for all $u \in [0, j]$,

$$\left| \sum_{k=0}^{n_j(z)} \widehat{P}_j(k) e^{-uk\varepsilon_j} z^k \right| \geq j e^{j\mu_j}.$$

Let us proceed with the construction. We choose for μ_j any real number such that $\mu_j > \mu_{j-1} + d_{j-1}\varepsilon_{j-1}$ and the real numbers $2\pi, \mu_1, \dots, \mu_j, \varepsilon_1, \dots, \varepsilon_{j-1}$ are independent over \mathbb{Q} (when $j = 1$, we simply choose $\mu_1 > 2\pi$ with $(2\pi, \mu_1)$ independent over \mathbb{Q}). We then apply Lemma 4.10 with $A = (j+1)e^{j\mu_j}$ and $\delta = 2^{-j}$ to get a polynomial P_j with degree d_j and a subset $E_j \subset \mathbb{T}$ satisfying (a) and (b). Since the functions $(u, z) \mapsto \sum_{k=0}^n \widehat{P}_j(k) e^{-uk\varepsilon_j} z^k$, for $0 \leq n \leq d_j$, converge uniformly on $[0, j] \times \mathbb{T}$ to $(u, z) \mapsto S_n P(z)$ as $\varepsilon \rightarrow 0$, we may choose ε_j a sufficiently small positive real number such that (d) and (e) are satisfied.

Define now $\lambda = \{\mu_j + k\varepsilon_j : j \geq 1, 0 \leq k \leq d_j\}$, $G = \prod_{j=1}^{+\infty} \mathbb{T}^2$ endowed with the canonical product structure and define $\beta : (\mathbb{R}, +) \rightarrow G$, $t \mapsto ((e^{-it\mu_j}, e^{-it\varepsilon_j}))_j$. By (d) and Kronecker's theorem, the homomorphism β has dense range. Moreover, let $\lambda_n \in \lambda$. Then $\lambda_n = \mu_j + k\varepsilon_j$ for some $j \geq 1$ and some $0 \leq k \leq d_j$. Write an element $\omega \in G$ as $\prod_{l=1}^{+\infty} (w_l, z_l)$ and define $h_{\lambda_n}(\omega) = w_j z_j^k$. Then

$$h_{\lambda_n} \circ \beta(t) = e^{-it(\mu_j + k\varepsilon_j)} = e^{-i\lambda_n t}$$

so that (G, β) is a λ -Dirichlet group. Now, define

$$f_j(\omega) = w_j P_j(z_j) = \sum_{k=0}^{d_j} \widehat{P}_j(k) h_{\mu_j + k\varepsilon_j}(\omega).$$

We get $\|f_j\|_{H_1^\lambda(G)} = \|P_j\|_{H_1(\mathbb{T})}$ so that the series $f = \sum_{j \geq 1} f_j$ converges in $H_1^\lambda(G)$. Let us also define

$$F_j = \{\omega \in G : z_j \in E_j\}.$$

Then $m_G(F_j) = m_{\mathbb{T}}(E_j) \geq 1 - 2^{-j}$ (here, m_G denotes the Haar measure on G). Thus, if we set $F = \bigcap_{j_0 \geq 1} \bigcup_{j \geq j_0} F_j$, then $m_G(F) = 1$. Pick now $\omega \in F$. We may find j as large as we want such that $\omega \in F_j$. The construction of P_j ensures that there exists $0 \leq n_j(z_j) \leq d_j$ such that, for all $u \in [0, j]$,

$$\left| \sum_{k=0}^{n_j(z_j)} \widehat{P}_j(k) e^{-u(\mu_j + k\varepsilon_j)} z_j^k \right| \geq j.$$

Setting N , resp. M , such that $\lambda_N = \mu_{j-1} + d_{j-1}\varepsilon_{j-1}$, resp. $\lambda_M = \mu_j + n_j(z_j)\varepsilon_j$, the previous inequality translates into

$$\left| \sum_{n=N+1}^M \hat{f}(h_{\lambda_n}) e^{-u\lambda_n} h_{\lambda_n}(\omega) \right| \geq j.$$

This easily yields the a.e. divergence of $\sum_{n=1}^{+\infty} \hat{f}(h_{\lambda_n}) e^{-u\lambda_n} h_{\lambda_n}$. \square

4.3. Maximal inequalities for mollifiers. Since $(S_n f)$ does not necessarily converge pointwise or even in norm for all functions in $H_1^\lambda(G)$, Defant and Schoolmann looked in [11] for a substitute by changing the summation method. They succeeded by choosing Riesz means. Precisely they showed (see [11, Theorem 2.1]), through a maximal inequality, that for all frequencies λ , for all $f \in H_1^\lambda(G)$, for all $\alpha > 0$, the sequence $(R_N^{\lambda, \alpha}(f)(\omega))$ converges to $f(\omega)$ for almost all $\omega \in G$. We extend this to a large class of mollifiers.

Theorem 4.11. *Let λ be a frequency, (G, β) a λ -Dirichlet group. Let $\psi \in L_1(\mathbb{R})$ be a continuous function (except at a finite number of points) such that $\hat{\psi}$ has compact support and there exists a nonincreasing function $g \in L_1(0, +\infty)$ such that $|\psi(x)| \leq g(|x|)$ for all $x \in \mathbb{R}$. For $N \geq 1$, define $\psi_N(\cdot) = \psi(\cdot/N)$. Then*

$$R_{\max, \psi}(f) := \sup_N |R_{\psi_N}(f)|$$

defines a bounded sublinear operator from $H_1^\lambda(G)$ into $L_{1, \infty}(G)$. Moreover, if $\int \psi = 1$, then for all $f \in H_1^\lambda(G)$, $R_{\psi_N}(f)(\omega)$ converges for almost every $\omega \in G$ to $f(\omega)$.

Proof. Again, the key point is the vertical convolution formula. Indeed, we know that for a.e. $\omega \in G$,

$$R_{\psi_N}(f)(\omega) = f_\omega \star \psi_N(0).$$

For those ω , using [13, Theorem 2.1.10 and Remark 2.1.11],

$$\begin{aligned} \sup_N |R_{\psi_N}(f)(\omega)| &\leq \sup_N |f_\omega| \star \psi_N(0) \\ &\leq 2\|g\|_1 \overline{M}f(\omega) \end{aligned}$$

where $\overline{M}(f)(\omega) = \sup_{T>0} \frac{1}{2T} \int_{-T}^T |f_\omega(t)| dt$ is the appropriate Hardy-Littlewood maximal operator. Since \overline{M} maps $H_1^\lambda(G)$ into $L_{1, \infty}(G)$ by [11, Theorem 2.10], we can conclude about the first assertion of the theorem. The result on a.e. convergence is then a standard corollary of it, using that it is clearly true for polynomials since $\hat{\psi}(0) = 1$. \square

Remark 4.12. We can replace the assumption that ψ is compactly supported by the assumption that, for all $N \geq 1$, $\sum_n |\hat{\psi}(\lambda_n/N)| < +\infty$.

This last theorem covers many examples. For instance, for all $0 \leq a < b$, we may choose the function $\psi \in L_1(\mathbb{R})$ such that $\hat{\psi} = 1$ on $[-a, a]$, $\hat{\psi} = 0$ on $(-\infty, -b) \cup (b, +\infty)$ and $\hat{\psi}$ is affine on $(-b, -a)$ and on (a, b) . As already observed during the proof of Theorem 3.2, the function ψ is given by

$$\psi(x) = C(a, b) \frac{\sin\left(\frac{a+b}{2}x\right) \sin\left(\frac{b-a}{2}x\right)}{x^2}$$

which clearly satisfies the assumptions of Theorem 4.11. This is also the case for $\psi(x) = e^{-|x|}$ or $\psi(x) = e^{-x^2}$, provided the frequency λ satisfies $\sum |\widehat{\psi}(\lambda_n/N)| < +\infty$ for all $N \geq 1$. To show that our result covers Theorem 2.1 of [11], we also have to show that for $\alpha > 0$ the function $\psi \in L_1(\mathbb{R})$ that satisfies

$$\widehat{\psi}(t) = (1 - |t|)^\alpha \mathbf{1}_{[-1,1]}(t)$$

verifies the assumptions of Theorem 4.11. Let $x > 0$. We already have observed that

$$\psi(x) = \frac{1}{ix} \mathcal{F}(\pm \alpha (1 - |t|)^{\alpha-1} \mathbf{1}_{[-1,1]})(x).$$

Fix $\beta > 0$ such that $|\beta(\alpha - 1)| < 1$ and let $x \geq 1$. Then

$$\begin{aligned} |\psi(x)| &\leq \frac{2\alpha}{x} \left| \int_0^1 (1-t)^{\alpha-1} e^{itx} dt \right| \\ &\leq \frac{2\alpha}{x} \left| \int_0^1 u^{\alpha-1} e^{-iux} du \right|. \end{aligned}$$

We split the integral into two parts. First,

$$\left| \int_0^{x^{-\beta}} u^{\alpha-1} e^{-iux} du \right| \leq \frac{1}{\alpha} x^{-\alpha\beta}.$$

Second, integrating by parts,

$$\int_{x^{-\beta}}^1 u^{\alpha-1} e^{-iux} du = \frac{-1}{ix} [u^{\alpha-1} e^{-iux}]_{x^{-\beta}}^1 + \frac{\alpha-1}{ix} \int_{x^{-\beta}}^1 u^{\alpha-2} e^{-iux} du$$

so that

$$\left| \int_{x^{-\beta}}^1 u^{\alpha-1} e^{-iux} du \right| \leq C \left(\frac{1}{x} + \frac{1}{x^{1+(\alpha-1)\beta}} \right).$$

Our choice of β guarantees that there is $\delta > 0$ and $C > 0$ such that, for $x \geq 1$,

$$|\psi(x)| \leq \frac{C}{x^{1+\delta}}.$$

This shows that the assumptions of Theorem 4.11 are satisfied with

$$g(x) = \begin{cases} \frac{C}{x^{1+\delta}}, & x \geq 1 \\ \max(\|\psi\|_\infty, C), & x \in [0, 1). \end{cases}$$

Remark 4.13. Lemma 4.4 and Theorem 4.11 are of course very close. The latter one is true for all frequencies λ , but we start from a fixed function ψ and it does not cover fully the case $p = 1$. Lemma 4.4 adapts at each step the L^1 -function to the frequency λ and to the function f . The price to pay is that we lose some factor $e^{\delta\lambda_N}$ and that we cannot use general results on the Hardy-Littlewood maximal function.

5. HORIZONTAL TRANSLATIONS

In this section, we investigate the boundedness from $\mathcal{H}_p(\lambda)$ into $\mathcal{H}_q(\lambda)$, for $q > p$, of the horizontal translation map $T_\sigma(\sum_n a_n e^{-\lambda_n s}) = \sum_n a_n e^{-\sigma \lambda_n} e^{-\lambda_n s}$. We are interested in this map to determine the exact value of

$$\sigma_{\mathcal{H}_p(\lambda)} = \inf\{\sigma \in \mathbb{R} : \sigma_c(D) \leq \sigma \text{ for all } D \in \mathcal{H}_p(\lambda)\}$$

since it is easy to prove, using the Cauchy-Schwarz inequality, that $\sigma_{\mathcal{H}_2(\lambda)} = L(\lambda)/2$. Recall that, when $\lambda = (\log n)$, $\sigma_{\mathcal{H}_p(\lambda)} = 1/2$ for all $p \in [1, +\infty)$. In the general case, it is always possible to majorize $\sigma_{\mathcal{H}_1(\lambda)}$ if we know $\sigma_{\mathcal{H}_2(\lambda)}$.

Proposition 5.1. *Let λ be a frequency. Then $\sigma_{\mathcal{H}_1(\lambda)} \leq 2\sigma_{\mathcal{H}_2(\lambda)}$.*

Proof. Let $\varepsilon > 0$. It is sufficient to prove that, for all $f = \sum_j a_j e^{-\lambda_j s}$ belonging to $\mathcal{H}_1(\lambda)$, for all $\sigma > 2\sigma_{\mathcal{H}_2(\lambda)} + \varepsilon = L(\lambda) + \varepsilon$,

$$\sum_{j=1}^{+\infty} |a_j| e^{-\lambda_j \sigma} < +\infty.$$

Let $J \geq 1$ be such that, for all $j \geq J$, $\log(j)/\lambda_j \leq L(\lambda) + \varepsilon$. Then

$$\sum_j |a_j| e^{-\lambda_j \sigma} \leq (J-1) \|f\|_1 + \sum_{j=J}^{+\infty} \|f\|_1 e^{-\frac{\sigma}{L(\lambda)+\varepsilon} \log(j)} < +\infty$$

by our assumption on σ . □

It turns out that, even if we put strong growth and separation conditions on λ , we cannot go further.

Theorem 5.2. *There exists a frequency λ satisfying (BC) such that $\sigma_{\mathcal{H}_1(\lambda)} = 2\sigma_{\mathcal{H}_2(\lambda)}$ and $\sigma_{\mathcal{H}_2(\lambda)} > 0$.*

Proof. For $n \geq 2$, let $\delta_n \in (2^{-n-1}, 2^{-n}]$ such that $(2\pi, n, \delta_n)$ are \mathbb{Z} -independent. We set

$$\lambda_{2^n+k} = n + k\delta_n \text{ for } n \geq 1, k = 0, \dots, 2^n - 1.$$

It is easy to check that $L(\lambda) = \log(2)$ so that $\sigma_{\mathcal{H}_2(\lambda)} = (\log 2)/2$. Moreover, it is also easy to check that λ satisfies (BC). Indeed, for $n \geq 2$ and $k = 0, \dots, 2^n - 2$,

$$\lambda_{2^{n+k+1}} - \lambda_{2^{n+k}} = \delta_n \geq \frac{1}{2} 2^{-n} \geq C e^{-(\log 2)\lambda_{2^{n+k}}}$$

and similarly

$$\lambda_{2^{n+1}} - \lambda_{2^{n+1}-1} \geq 2^{-n} \geq C e^{-(\log 2)\lambda_{2^{n+1}-1}}.$$

Pick now any $\sigma > \sigma_{\mathcal{H}_1(\lambda)}$. By the principle of uniform boundedness, there exists $C_0 > 0$ such that, for all $D = \sum_j a_j e^{-\lambda_j s}$ belonging to $\mathcal{H}_1(\lambda)$, for all $N \geq 2$,

$$(12) \quad \left| \sum_{j=2}^N a_j e^{-\lambda_j \sigma} \right| \leq C_0 \|D\|_1$$

Let $n \geq 2$ and choose $D = \sum_{k=0}^{2^n-1} e^{-\lambda_{2^n+k}s}$. Then

$$(13) \quad \sum_{k=0}^{2^n-1} e^{-\lambda_{2^n+k}\sigma} \geq 2^n e^{-\sigma(n+1)} \geq C_1 e^{(\log 2 - \sigma)n}.$$

On the other hand, set $\lambda' = \{n + k\delta_n : k \geq 0\}$ and observe that, using the internal description of the norm of \mathcal{H}_1 ,

$$\|D\|_{\mathcal{H}_1(\lambda)} = \|D\|_{\mathcal{H}_1(\lambda')}.$$

We shall compute $\|D\|_{\mathcal{H}_1(\lambda')}$ using Fourier analysis. Indeed, since $(2\pi, n, \delta_n)$ are \mathbb{Z} -independent, the map $\beta : \mathbb{R} \rightarrow \mathbb{T}^2$, $t \mapsto (e^{-itn}, e^{-it\delta_n})$ has dense range, so that (\mathbb{T}^2, β) is a λ' -Dirichlet group. Therefore,

$$(14) \quad \|D\|_{\mathcal{H}_1(\lambda')} = \int_{\mathbb{T}^2} \left| \sum_{k=0}^{2^n-1} z_1 z_2^k \right| dz_1 dz_2 \leq C_2 n$$

by the classical estimate of the norm of the Dirichlet kernel. Hence, (12), (13) and (14) imply that, for all $\sigma > \sigma_{\mathcal{H}_1(\lambda)}$, $\sigma \geq \log(2)$. This yields $\sigma_{\mathcal{H}_1(\lambda)} \geq 2\sigma_{\mathcal{H}_2(\lambda)}$. \square

In view of the previous results, it seems natural to study how arithmetical properties of the frequency λ can influence the values of σ for which $T_\sigma : \mathcal{H}_p(\lambda) \rightarrow \mathcal{H}_q(\lambda)$, $p < q$, is bounded. We concentrate on the case $p = 2$ and $q = 2k$, $k \geq 1$, because we can compute the norms using the coefficients. We define $\lambda * \lambda$ as $\{\lambda_l + \lambda_k : l, k \geq 0\}$ and $\lambda^{*k} = \lambda * \dots * \lambda$ (with k factors).

Definition 5.3. Let λ be a frequency and $k \geq 1$. Write $\lambda^{*k} = (\mu_l)$ where the sequence (μ_l) is increasing. We set

$$A(\lambda, k) = \limsup_{l \rightarrow +\infty} \frac{\log(\text{card}\{(n_1, \dots, n_k) : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu_l\})}{2\mu_l}.$$

Proposition 5.4. Let λ be a frequency and $k \geq 1$. Then for $\sigma > A(\lambda, k)$, T_σ maps boundedly $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$.

Proof. We shall prove a slightly stronger statement : if $\sigma > 0$ is such that there exists $C > 0$ such that for any $\mu > 0$,

$$e^{-2\mu\sigma} \text{card}\{(n_1, \dots, n_k) : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu\} \leq C,$$

then T_σ maps boundedly $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$. Indeed, let $D = \sum_n a_n e^{-\lambda_n s}$ belonging to $\mathcal{H}_2(\lambda)$. We write $T_\sigma(D)^k = \sum_l b_{\mu_l} e^{-\mu_l s}$ where

$$b_{\mu_l} = \sum_{\lambda_{n_1} + \dots + \lambda_{n_k} = \mu_l} a_{n_1} \dots a_{n_k} e^{-\mu_l \sigma}.$$

We just need to prove that the sequence (b_{μ_l}) is square summable, namely that for all square summable sequences (c_{μ_l}) , $\sum_l b_{\mu_l} c_{\mu_l}$ is convergent, namely that

$$\sum_{n_1, \dots, n_k} a_{n_1} \dots a_{n_k} e^{-(\lambda_{n_1} + \dots + \lambda_{n_k})\sigma} c_{\lambda_{n_1} + \dots + \lambda_{n_k}} \text{ is convergent.}$$

By the Cauchy-Schwarz inequality, since (a_n) is square summable, it is sufficient to prove that

$$\sum_{n_1, \dots, n_k} e^{-2\lambda_{n_1}\sigma} \dots e^{-2\lambda_{n_k}\sigma} |c_{\lambda_{n_1} + \dots + \lambda_{n_k}}|^2 < +\infty.$$

Rewriting this

$$\sum_l |c_{\mu_l}|^2 e^{-2\mu_l\sigma} \text{card}\{(n_1, \dots, n_k) : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu_l\} < +\infty$$

this follows from the assumption. \square

Corollary 5.5. *Let λ be a frequency, $k \geq 1$ and $\sigma > (k-1)L(\lambda)/2$. Then T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$.*

Proof. Let $\varepsilon > 0$. There exists $N \geq 1$ such that, for all $n \geq N$, $\log(n)/\lambda_n \leq L(\lambda) + \varepsilon$. Let $\mu \in (\lambda^*)^k$ and n_1, \dots, n_k be such that $\lambda_{n_1} + \dots + \lambda_{n_k} = \mu$. Then each λ_{n_i} is smaller than μ so that either $n_i \leq N$ or $n_i \leq \exp(\mu(L(\lambda) + \varepsilon))$. Since the knowledge of n_1, \dots, n_{k-1} determines the value of n_k , we have

$$\text{card}\{(n_1, \dots, n_k) : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu\} \leq N^{k-1} \exp((k-1)\mu(L(\lambda) + \varepsilon)).$$

Taking the logarithm and letting μ to $+\infty$, we find $A(\lambda, k) \leq \frac{(k-1)(L(\lambda) + \varepsilon)}{2}$, hence the inequality $A(\lambda, k) \leq \frac{(k-1)L(\lambda)}{2}$ since ε is arbitrary. \square

Corollary 5.6. *Let λ be a frequency such that $L(\lambda) = 0$. Then T_σ maps boundedly $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_q(\lambda)$ for all $q \geq 2$.*

Question 5.7. *Let $p \geq 2$ and let λ be a frequency. Does T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_p(\lambda)$ as soon as $\sigma > \frac{(p-2)L(\lambda)}{4}$?*

Example 5.8. Let $\lambda = (\log n)$. Then for all $k \geq 1$, $A(\lambda, k) = 0$.

Proof. We first observe that $(\lambda^*)^k = \lambda$. Pick now $\log n \in \lambda$. We want to know the cardinal number of $\{(n_1, \dots, n_k) \in \mathbb{N} : n_1 \times \dots \times n_k = n\}$. Decompose n into a product of prime numbers, $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Then each n_k writes $p_1^{\alpha_1^{(k)}} \dots p_r^{\alpha_r^{(k)}}$ with $\alpha_j(1) + \dots + \alpha_j(r) = \alpha_j$, $1 \leq j \leq r$. Hence, $(\alpha_j(1), \dots, \alpha_j(r))$ is a weak composition of α_j into k parts which can be done in $\binom{\alpha_j + k - 1}{k - 1}$ ways. In total, there are

$$\prod_{i=1}^r \binom{\alpha_i + k - 1}{k - 1} \leq \prod_{i=1}^r (\alpha_i + k)^k$$

ways to write n as a product of k factors. Thus,

$$A(\lambda, k) \leq \limsup_{n = \prod_{i=1}^r p_i^{\alpha_i} \rightarrow +\infty} \frac{\sum_{i=1}^r k \log(\alpha_i + k)}{2 \sum_{i=1}^r \alpha_i \log(p_i)} = 0.$$

\square

We finish this section by exhibiting a frequency λ satisfying (BC) and such that, for all $k \geq 1$, T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ if and only if $\sigma \geq A(\lambda, k) = \frac{k-1}{2k}$. We begin with two combinatorial lemmas.

Lemma 5.9. *Let $b, c > 0$, let $n \in \mathbb{N}$ and let $\lambda_j = b + jc$, $j \geq 0$. For all $k \in \mathbb{N}$, there exist $\gamma_k \in (0, 1]$ and $\delta_k > 0$ such that, for all $n \geq 2^k$, for all $\ell \in [(k - \gamma_k)n, (k + \gamma_k)n] \cap \mathbb{N}_0$,*

$$\text{card} \left\{ (j_1, \dots, j_k) \in \{0, \dots, 2n\}^k : \lambda_{j_1} + \dots + \lambda_{j_k} = kb + \ell c \right\} \geq \delta_k n^{k-1}.$$

Proof. We define the sequences (γ_k) and (δ_k) by $\gamma_k = 2^{-(k-1)}$ and $\delta_1 = 1$, $\delta_{k+1} = \delta_k \cdot \gamma_{k+1}$. We proceed by induction over k . The case $k = 1$ is trivial. Assume that the result has been proved for k and let us prove it for $k + 1$. Let $n \geq 2^{k+1}$. Choose j_{k+1} any integer in $[(1 - \gamma_{k+1})n, (1 + \gamma_{k+1})n]$ and $\ell \in [(k + 1 - \gamma_{k+1})n, (k + 1 + \gamma_{k+1})n] \cap \mathbb{N}_0$. Then

$$(15) \quad \lambda_{j_1} + \dots + \lambda_{j_{k+1}} = (k + 1)b + \ell c \iff \lambda_{j_1} + \dots + \lambda_{j_k} = kb + (\ell - j_{k+1})c.$$

Now,

$$|\ell - j_{k+1} - kn| \leq 2\gamma_{k+1}n = \gamma_k n$$

so that there exist at least $\delta_k n^{k-1}$ choices of (j_1, \dots, j_k) such that (15) is true, j_{k+1} being fixed. Now, there are $2[\gamma_{k+1}n] + 1$ choices of j_{k+1} and since $\gamma_{k+1}n - 1 \geq \gamma_{k+1}n/2$ because $\gamma_{k+1}n \geq 2$, we get the result. \square

Lemma 5.10. *Let (b_n) and (c_n) be two sequences of positive real numbers such that the sequences $(b_1, \dots, b_N, c_1, \dots, c_N)$ are \mathbb{Z} -independent for all $N \geq 1$, $2n + 1 \leq \exp(b_n)$ and $nc_n \leq 1$ for each $n \in \mathbb{N}$. Define a sequence (λ_n) by $\lambda_{m^2+j} = b_m + jc_m$, $m \geq 1$, $j = 0, \dots, 2m$. Then for all $k > 0$ there exists $C_k > 0$ such that, for all $\mu > 0$,*

$$\text{card} \left\{ (n_1, \dots, n_k) \in \mathbb{N}^k : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu \right\} \leq C_k \exp \left(\frac{(k-1)\mu}{k} \right).$$

Proof. If μ can be written $\mu = \lambda_{n_1} + \dots + \lambda_{n_k}$ for some sequence (n_1, \dots, n_k) , it can be uniquely written

$$(16) \quad \mu = \alpha_1 b_{r_1} + \beta_1 c_{r_1} + \dots + \alpha_l b_{r_l} + \beta_l c_{r_l}$$

with $1 \leq l \leq k$, $r_1 < r_2 < \dots < r_l$, $\alpha_i \geq 1$, $\alpha_1 + \dots + \alpha_l = k$ and $0 \leq \beta_i \leq 2\alpha_i r_i$. We will first estimate $\text{card} \left\{ (n_1, \dots, n_k) \in \mathbb{N}^k : \lambda_{n_1} + \dots + \lambda_{n_k} = \mu \right\}$ by a quantity depending on k, l, α_i, r_i and β_i . In view of the definition of the sequence λ and of (16), we are reduced to estimate the number of $2k$ -tuples $(m_1, \dots, m_k, j_1, \dots, j_k)$ such that for all $s = 1, \dots, k$, $0 \leq j_s \leq 2m_s$ and, for all $i = 1, \dots, l$,

- there are α_i elements in m_1, \dots, m_k which are equal to r_i ;
- if $\phi_i(1), \dots, \phi_i(\alpha_i)$ are the indices of these elements, then

$$(17) \quad j_{\phi_i(1)} + \dots + j_{\phi_i(\alpha_i)} = \beta_i.$$

We first choose the values of m_1, \dots, m_k . We choose the α_1 indices in $\{1, \dots, k\}$ such that the corresponding m_i are equal to r_1 . We then do the same for the α_2 elements equal to r_2 and so on until $k - 1$ (the remaining m_i are fixed and equal to r_l). Thus the number of choices for m_1, \dots, m_k is equal to

$$\binom{k}{\alpha_1} \times \binom{k - \alpha_1}{\alpha_2} \times \dots \times \binom{k - (\alpha_1 + \dots + \alpha_{l-2})}{\alpha_{l-1}}.$$

Because $l \leq k$ and $\alpha_1 + \dots + \alpha_l = k$, this number can be bounded from above by some number depending only on k . The integers m_1, \dots, m_k having been fixed, we now choose the integers j_1, \dots, j_k . For each $i \in \{1, \dots, l\}$, (17) implies that there are at most

$(2r_i + 1)^{\alpha_i - 1}$ choices for the values of $j_{\phi_i(1)}, \dots, j_{\phi_i(\alpha_i)}$: indeed, each $j_{\phi_i(t)}$ belongs to $\{0, \dots, 2r_i\}$ and the last one is fixed when we know the values of the first $\alpha_i - 1$ ones). Finally, we have found that

$$\begin{aligned} \text{card} \left\{ (n_1, \dots, n_k) \in \mathbb{N}^k : \mu = \lambda_{n_1} + \dots + \lambda_{n_k} \right\} &\leq C_k \prod_{i=1}^l (2r_i + 1)^{\alpha_i - 1} \\ &\leq C_k \exp \left(\sum_{i=1}^l (\alpha_i - 1) b_{r_i} \right) \end{aligned}$$

where the last inequality follows from the assumption $2n + 1 \leq \exp(b_n)$ for all $n \in \mathbb{N}$. Now we have $\sum_{i=1}^l \alpha_i b_{r_i} \leq \mu$ and

$$\begin{aligned} k \sum_{i=1}^l b_{r_i} &\geq \sum_{i=1}^l \alpha_i b_{r_i} = \mu - \sum_{i=1}^l \beta_i c_{r_i} \\ &\geq \mu - 2 \sum_{i=1}^l \alpha_i r_i c_{r_i} \geq \mu - 2k \sum_{i=1}^l r_i c_{r_i} \\ &\geq \mu - 2k^2 \end{aligned}$$

since $n c_n \leq 1$ and $l \leq k$. This implies that

$$\sum_{i=1}^l (\alpha_i - 1) b_{r_i} \leq \mu - \frac{\mu}{k} + 2k = \frac{(k-1)\mu}{k} + 2k,$$

hence the result. \square

Theorem 5.11. *There exists a frequency (λ_n) satisfying (BC) such that for all $k \geq 1$, T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ if and only if $\sigma \geq \frac{k-1}{2k} = A(\lambda, k)$.*

Proof. Let (b_n) and (c_n) be two sequences of positive real numbers such that

- for all $n \geq 1$, $\log(2n + 1) \leq b_n \leq \log(2n + 2)$;
- for all $n \geq 1$, $(b_{n+1} - b_n)/8n \leq c_n \leq (b_{n+1} - b_n)/4n$;
- for all $N \geq 1$, the sequences $(b_1, \dots, b_N, c_1, \dots, c_N)$ are \mathbb{Z} -independent.

We then define λ by $\lambda_{m^2+j} = b_m + j c_m$, $m \geq 1$, $j = 0, \dots, 2m$. We may argue as in the proof of Theorem 5.2 to show that the frequency λ satisfies (BC). Using Proposition 5.4 (look at the first sentence of the proof) and Lemma 5.10, we get easily that T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ for $\sigma \geq \frac{k-1}{2k}$ and also that $A(\lambda, k) \leq \frac{k-1}{2k}$.

Conversely, assume that T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ (boundedness is automatic by the closed graph theorem). Let us consider $D^{[m]} = \sum_{j=0}^{2m} e^{-\lambda_{m^2+j} s}$ for $m \geq 1$, so that $\|D^{[m]}\|_2 = \sqrt{2m+1}$. Write λ^{*k} as the increasing sequence (μ_l) and observe that

$$(T_\sigma D^{[m]})^k = \sum_l a_{\mu_l} e^{-\mu_l \sigma} e^{-\mu_l s}$$

where $a_{\mu_l} = \text{card} \{ (j_1, \dots, j_k) \in \{0, \dots, 2m\}^k : \mu_l = k b_m + (j_1 + \dots + j_k) c_m \}$. Lemma 5.9 tells us that, for m sufficiently large, there is at least $\gamma_k m$ terms of the sequence (μ_l) so that $a_{\mu_l} \geq \delta_k m^{k-1}$. Furthermore, for those μ_l ,

$$\mu_l \leq k(b_m + 2m c_m) \leq k \log m + A_k.$$

In particular,

$$\frac{\log a_{\mu_l}}{2\mu_l} \geq \frac{(k-1)\log m + \log \delta_k}{2(k\log m + A_k)}$$

which shows that $A(\lambda, k) \geq \frac{k-1}{2k}$. Furthermore,

$$\begin{aligned} \|T_\sigma D^{[m]}\|_{2k} &= \|(T_\sigma D^{[m]})^k\|_2^{1/k} \geq C_k \left(m \cdot m^{2(k-1)} \cdot m^{-2\sigma k} \right)^{1/2k} \\ &\geq C_k m^{\frac{1}{2k} + \frac{k-1}{k} - \sigma}. \end{aligned}$$

Therefore, the boundedness of T_σ from $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_{2k}(\lambda)$ implies that

$$m^{\frac{1}{2k} + \frac{k-1}{k} - \sigma} \leq C'_k \sqrt{2m+1}$$

for all sufficiently large m , which itself yields $\sigma \geq \frac{k-1}{2k}$. \square

Question 5.12. *Let $p \geq 2$. Is it true that, for the previous sequence (λ_n) , T_σ maps $\mathcal{H}_2(\lambda)$ into $\mathcal{H}_p(\lambda)$ if and only if $\sigma \geq \frac{p-2}{2p}$?*

6. OTHER RESULTS

6.1. Norm of the projection in $\mathcal{H}_1(\lambda)$. In [10], Defant and Schoolmann have shown, using a vector-valued argument, that for all frequencies λ and for all $N \geq 1$, $\|S_N\|_{\mathcal{H}_1(\lambda) \rightarrow \mathcal{H}_1(\lambda)} \leq \|S_N\|_{\mathcal{H}_\infty(\lambda) \rightarrow \mathcal{H}_\infty(\lambda)}$. We provide a different approach to estimate $\|S_N\|_{\mathcal{H}_1(\lambda) \rightarrow \mathcal{H}_1(\lambda)}$, inspired by [6].

Proposition 6.1. *Let λ be a frequency. Then, for all $N \geq 1$, $\|S_N\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1} \leq C \log(\Lambda_N)$ where*

$$\Lambda_N = \sup \left\{ \frac{\left\| \sum_{n=1}^N a_n e^{-\lambda_n s} \right\|_2}{\left\| \sum_{n=1}^N a_n e^{-\lambda_n s} \right\|_1} : a_1, \dots, a_N \in \mathbb{C} \right\}.$$

Proof. We work in $H_1^\lambda(G)$ where (G, β) is a λ -Dirichlet group. Let $g \in L_1(G)$. Then, for $\varepsilon \in (0, 1)$,

$$\begin{aligned} \int |g| &= \int |g|^{\frac{2\varepsilon}{1+\varepsilon}} |g|^{\frac{1-\varepsilon}{1+\varepsilon}} \\ &\leq \left(\int |g|^2 \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int |g|^{1-\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \end{aligned}$$

where we have applied Hölder's inequality for $(1+\varepsilon)/\varepsilon$ and $1+\varepsilon$. Applying this to $S_N f$, where $f \in H_1^\lambda(G)$, we get

$$\|S_N f\|_1 \leq \|S_N f\|_2^{\frac{2\varepsilon}{1+\varepsilon}} \|S_N f\|_{1-\varepsilon}^{\frac{1-\varepsilon}{1+\varepsilon}}.$$

We now use a result of Helson [16], saying that

$$\|S_N f\|_{1-\varepsilon} \leq \frac{A}{\varepsilon} \|f\|_1$$

where the constant A is absolute. Therefore, assuming $\|f\|_1 = 1$, after some simplifications, we get

$$\|S_N f\|_1 \leq \left(\frac{A}{\varepsilon} \right) \exp \left(\frac{2\varepsilon}{1-\varepsilon} \log(\Lambda_N) \right).$$

We conclude by choosing $\varepsilon = 1/\log(\Lambda_N)$. \square

Corollary 6.2. *There exists $C > 0$ such that, for all frequency λ , $\|S_N\|_{\mathcal{H}_1(\lambda) \rightarrow \mathcal{H}_1(\lambda)} \leq C \log(N)$.*

Proof. We get immediately that $\Lambda_N \leq \sqrt{N}$ by using the Cauchy-Schwarz inequality and the fact that $|a_n| \leq \|D\|_1$ for all $D = \sum_{n=1}^N a_n e^{-\lambda_n s}$. \square

This last corollary has an interest provided we are unable to prove that $\|S_N\|_{\mathcal{H}_\infty(\lambda) \rightarrow \mathcal{H}_\infty(\lambda)}$ is less than $C \log(N)$. The best known estimation on $\|S_N\|_{\mathcal{H}_\infty(\lambda) \rightarrow \mathcal{H}_\infty(\lambda)}$ is given by Theorem 3.4 and indeed it provides worst estimations for some sequences λ . Indeed, pick the sequence λ defined in Example 2.3. Let $N = 2^n$ for some n and pick $M > N$. Then, if $M < 2^{n+1}$, then

$$\log \left(\frac{\lambda_M + \lambda_N}{\lambda_M - \lambda_N} \right) \geq \frac{1}{2} e^{n^2}$$

whereas, if $M \geq 2^{n+1}$, then

$$M - N - 1 \geq 2^n - 1.$$

A variant of Proposition 6.1 was already used in the classical case $\lambda = (\log n)$ to prove that $\|S_N\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1} \leq C \frac{\log N}{\log \log N}$. A precise solution to the problem on how large can be Λ_N in this case can be found in [8].

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