An arbitrary-order discrete de Rham complex on polyhedral meshes. Part II: Consistency

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Abstract

In this paper we prove a complete panel of consistency results for the discrete de Rham (DDR) complex introduced in the companion paper [10], including primal and adjoint consistency for the discrete vector calculus operators, and consistency of the corresponding potentials. The theoretical results are showcased by performing a full convergence analysis for a DDR approximation of a magnetostatics model. Numerical results on three-dimensional polyhedral meshes complete the exposition.

Key words. Discrete de Rham complex, compatible discretisations, polyhedral methods, mixed methods

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1 Introduction

We prove complete consistency results for the discrete de Rham (DDR) complex introduced in the companion paper [10]. Specifically, the first set of results concerns *primal consistency* of the local discrete vector calculus operators introduced in [10, Section 3.3] and of the corresponding potentials defined in Section 3.1 below. The second set of results concerns *adjoint consistency*, that relates to the ability to approximate formal adjoint operators, and therefore requires to estimate the residuals of global integration by parts formulas.

For specific space dimensions, polynomial degrees, and operators, consistency results that bear relations to ours can be found in the literature on polytopal methods.

Starting from low-order methods, consistency results for Compatible Discrete Operator approximations of the Poisson problem based on nodal unknowns can be found in [5]; see in particular the proof of Theorem 3.3 therein, which contains an adjoint consistency result for a gradient reconstructed from vertex values. In the same framework, an adjoint consistency estimate for a discrete curl constructed from edge values can be found in [6, Lemma 2.3]. A rather complete set of consistency results for Mimetic Finite Difference operators can be found in [4], where they appear as intermediate steps in the error analyses of Chapters 5–7. A notable exception is provided by the adjoint consistency of the curl operator, which is not needed in the error estimate of [4, Theorem 7.3] since the authors consider an approximation of the current density based on the knowledge of a vector potential.

Moving to arbitrary-order methods, error estimates that involve the adjoint consistency of a gradient and the consistency of the corresponding potential have been recently derived in [7] in the framework of the H^1-conforming Virtual Element method. The same method is considered in [9, Section 3.2], where

a different analysis is proposed based on the third Strang lemma. The estimate of the consistency error in [9, Theorem 19] involves, in particular, the adjoint consistency of a discrete gradient reconstructed as the gradient of a scalar polynomial rather than a vector-valued polynomial. We note, in passing, that the concept of adjoint consistency for (discrete) gradients is directly related to the notion of limit-conformity in the Gradient Discretisation Method [15], a generic framework which encompasses several polytopal methods. Primal and dual consistency estimates for a discrete divergence and the corresponding vector potential similar (but not identical) to the ones considered here have been established in [14] in the framework of Mixed High-Order methods. Note that these methods, the H^1-conforming Virtual Element method, and the Mixed High-Order method, do not lead to a discrete de Rham complex. In the framework of arbitrary-order compatible discretisations, on the other hand, primal consistency results for the curl appear as intermediate results in [3], where an error analysis for a Virtual Element approximation of magnetostatics is carried out assuming interpolation estimates for three-dimensional vector valued virtual spaces; see Remark 4.4 therein. However, [3] does not establish any adjoint consistency property of the discrete curl (the formulation of magnetostatics considered in this reference does not require this).

The results presented in this paper are, to the best of our knowledge, the first ones to span the full set of discrete vector calculus operators for an arbitrary-order discrete de Rham complex on polyhedral meshes. The key ingredients to establish primal consistency are the polynomial consistency of discrete vector calculus operators along with the corresponding potentials, and their boundedness when applied to the interpolates of smooth functions. The proofs of adjoint consistency, on the other hand, rely on operator-specific techniques, all grounded in discrete integration by parts formulas for the corresponding potential reconstructions (see (3.1) along with Remark 3 for the gradient, (3.6) for the curl, and (3.10) for the divergence). Specifically, the key point for the adjoint consistency of the gradient are estimates for local H^1-like seminorms of the scalar potentials. The adjoint consistency of curl requires, on the other hand, the construction of liftings of the discrete face potentials that satisfy an orthogonality and a boundedness condition. These reconstructions are inspired by the minimal reconstruction operators of [4, Chapter 3], with a key novelty provided by a curl correction which ensures the well-posedness of the reconstruction inside mesh elements.

In order to showcase the theoretical results derived here and in the companion paper [10], we carry out a full convergence analysis for a DDR approximation of magnetostatics. This is, to the best of our knowledge, the first full theoretical result of this kind for arbitrary-order polytopal methods.

The rest of this paper is organised as follows. In Section 2 we briefly recall the key elements of the setting introduced in [10]. Section 3 contains the statement of the primal and adjoint consistency results, whose proofs are given in Section 4. The application of the theoretical tools to the error analysis of a DDR approximation of magnetostatics is considered in Section 5, where numerical evidence supporting the error estimates is also provided. Finally, Appendix A contains an in-depth and novel study of the div–**curl** problems defining the curl liftings on polytopal elements: well-posedness, orthogonality and boundedness properties.

2 Setting

We briefly recall here the setting introduced in the companion paper [10], to which we refer for a more detailed description of the following notions.

2.1 Mesh and orientation

Let $\mathcal{H} \subset \mathbb{R}_+^*$ be a countable set with 0 as its unique accumulation point. Let $\Omega \subset \mathbb{R}^3$ denote an open connected polyhedral set and $(\mathcal{M}_h)_{h\in\mathcal{H}}$ a family of meshes indexed by their size h. We write $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ with $\mathcal{T}_h$ the set of elements T, $\mathcal{F}_h$ the set of faces F, $\mathcal{E}_h$ the set of edges E, and $\mathcal{V}_h$ the set of vertices V. We additionally denote by $\mathcal{F}_h^{\mathrm{b}}$ the subset of $\mathcal{F}_h$ collecting the faces that lie on the boundary $\partial\Omega$ of Ω. It is assumed that $(\mathcal{T}_h, \mathcal{F}_h)_{h\in\mathcal{H}}$ matches the regularity conditions in [12,

Definition 1.9] (with $\rho \in (0,1)$ denoting the mesh regularity parameter), and that elements and faces are simply connected with Lipschitz continuous boundary. For $T \in \mathcal{T}_h$, we set $\mathcal{F}_T \coloneqq \{F \in \mathcal{F}_h \,:\, F \subset \partial T\}$ and, for $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, $\mathcal{E}_Y \coloneqq \{E \in \mathcal{E}_h \,:\, E \subset \partial Y\}$. The real number h_Y denotes the diameter of a mesh element, face, or edge $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$.

Each face $F \in \mathcal{F}_h$ is equipped with a unit normal vector $\boldsymbol{n}_F$, and each edge $E \in \mathcal{E}_h$ with a unit tangent vector $\boldsymbol{t}_E$. Given $F \in \mathcal{F}_h$ and $E \in \mathcal{E}_F$, we also denote by $\boldsymbol{n}_{FE}$ the unit vector normal to E lying in the plane of F. The families of numbers $\{\omega_{TF} \in \{-1,1\} \,:\, T \in \mathcal{T}_h, F \in \mathcal{F}_T\}$ and $\{\omega_{FE} \in \{-1,1\} \,:\, F \in \mathcal{F}_h, E \in \mathcal{E}_F\}$ collect relative orientations selected so that: for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, $\omega_{TF}\boldsymbol{n}_F$ points out of T and, for all $F \in \mathcal{F}_h$ and all $E \in \mathcal{E}_F$, $\omega_{FE}\boldsymbol{n}_{FE}$ points out of F. Given $F \in \mathcal{F}_h$, the tangent gradient, divergence, two-dimensional vector and scalar curl operators are denoted by $\mathbf{grad}_F$, div_F, $\mathbf{rot}_F$ and rot_F, respectively.

2.2 Polynomial spaces

Let $\ell \geq -1$ be an integer. For $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, with n the dimension of Y, we denote by $\mathcal{P}^\ell(Y)$ the space of polynomial functions over Y of total degree $\leq \ell$, and we set $\boldsymbol{\mathcal{P}}^\ell(Y) = \mathcal{P}^\ell(Y)^n$. The L^2-orthogonal projector on $\mathcal{P}^\ell(Y)$ is $\pi^\ell_{\mathcal{P},Y}$, and $\boldsymbol{\pi}^\ell_{\mathcal{P},Y} : \mathbf{L}^2(Y) \to \boldsymbol{\mathcal{P}}^\ell(Y)$ is its vector-valued counterpart. The set $\mathcal{P}^\ell_{\mathrm{c}}(\mathcal{E}_h)$ is made of all continuous functions over the mesh skeleton $\bigcup_{E\in\mathcal{E}_h} \overline{E}$ that are polynomial of total degree $\leq \ell$ on each $E \in \mathcal{E}_h$.

For all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, denote by $\boldsymbol{x}_Y$ a point inside Y such that Y contains a ball centered at $\boldsymbol{x}_Y$ and of diameter ρh_Y. For any mesh face $F \in \mathcal{F}_h$, any mesh element $T \in \mathcal{T}_h$, and any integer $\ell \geq -1$, we define

$$\begin{aligned}
\boldsymbol{\mathcal{G}}^\ell(F) &\coloneqq \mathbf{grad}_F\, \mathcal{P}^{\ell+1}(F), & \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(F) &\coloneqq (\boldsymbol{x}-\boldsymbol{x}_F)^\perp \mathcal{P}^{\ell-1}(F),\\
\boldsymbol{\mathcal{R}}^\ell(F) &\coloneqq \mathbf{rot}_F\, \mathcal{P}^{\ell+1}(F), & \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(F) &\coloneqq (\boldsymbol{x}-\boldsymbol{x}_F)\mathcal{P}^{\ell-1}(F),\\
\boldsymbol{\mathcal{G}}^\ell(T) &\coloneqq \mathbf{grad}\, \mathcal{P}^{\ell+1}(T), & \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(T) &\coloneqq (\boldsymbol{x}-\boldsymbol{x}_T)\times \boldsymbol{\mathcal{P}}^{\ell-1}(T),\\
\boldsymbol{\mathcal{R}}^\ell(T) &\coloneqq \mathbf{curl}\, \boldsymbol{\mathcal{P}}^{\ell+1}(T), & \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(T) &\coloneqq (\boldsymbol{x}-\boldsymbol{x}_T)\mathcal{P}^{\ell-1}(T)
\end{aligned} \tag{2.1}$$

where $(\boldsymbol{x}-\boldsymbol{x}_F)^\perp$ denotes the vector $\boldsymbol{x}-\boldsymbol{x}_F$ rotated by an angle $-\pi/2$ in the plane spanned by F and oriented by $\boldsymbol{n}_F$. If $Y = F$ or $Y = T$, the following direct (but not necessarily orthogonal) decompositions hold:

$$\boldsymbol{\mathcal{P}}^\ell(Y) = \boldsymbol{\mathcal{G}}^\ell(Y) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(Y) = \boldsymbol{\mathcal{R}}^\ell(Y) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(Y). \tag{2.2}$$

With obvious notations, the L^2-orthogonal projectors on the subspaces appearing in these decompositions are denoted by $\boldsymbol{\pi}^\ell_{\mathcal{G},Y}$, $\boldsymbol{\pi}^{\mathrm{c},\ell}_{\mathcal{G},Y}$, $\boldsymbol{\pi}^\ell_{\mathcal{R},Y}$, and $\boldsymbol{\pi}^{\mathrm{c},\ell}_{\mathcal{R},Y}$. The local Nédélec and Raviart–Thomas spaces over Y are denoted by

$$\boldsymbol{\mathcal{N}}^\ell(Y) \coloneqq \boldsymbol{\mathcal{G}}^{\ell-1}(Y) \oplus \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(Y), \qquad \boldsymbol{\mathcal{RT}}^\ell(Y) \coloneqq \boldsymbol{\mathcal{R}}^{\ell-1}(Y) \oplus \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(Y). \tag{2.3}$$

As detailed in [10, Lemma 4], the knowledge of the L^2-projections of a polynomial $z \in \mathcal{P}^\ell(Y)$ on each element of the space pairs $(\boldsymbol{\mathcal{G}}^\ell(Y), \boldsymbol{\mathcal{G}}^{\mathrm{c},\ell}(Y))$ or $(\boldsymbol{\mathcal{R}}^\ell(Y), \boldsymbol{\mathcal{R}}^{\mathrm{c},\ell}(Y))$ appearing in (2.2) enables the recovery of z. Specifically, for $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, $\boldsymbol{\mathcal{X}} \in \{\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{R}}\}$, and $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{\mathcal{X}}^\ell(Y) \times \boldsymbol{\mathcal{X}}^{\mathrm{c},\ell}(Y)$, letting

$$\mathfrak{R}^\ell_{\boldsymbol{\mathcal{X}},Y}(\boldsymbol{v},\boldsymbol{w}) \coloneqq (\mathrm{Id} - \boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{X}},Y}\boldsymbol{\pi}^{\mathrm{c},\ell}_{\boldsymbol{\mathcal{X}},Y})^{-1}(\boldsymbol{v} - \boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{X}},Y}\boldsymbol{w}) + (\mathrm{Id} - \boldsymbol{\pi}^{\mathrm{c},\ell}_{\boldsymbol{\mathcal{X}},Y}\boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{X}},Y})^{-1}(\boldsymbol{v} - \boldsymbol{\pi}^{\mathrm{c},\ell}_{\boldsymbol{\mathcal{X}},Y}\boldsymbol{w}) \tag{2.4}$$

we have

$$\boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{X}},Y}\big(\mathfrak{R}^\ell_{\boldsymbol{\mathcal{X}},Y}(\boldsymbol{v},\boldsymbol{w})\big) = \boldsymbol{v} \quad \text{and} \quad \boldsymbol{\pi}^{\mathrm{c},\ell}_{\boldsymbol{\mathcal{X}},Y}\big(\mathfrak{R}^\ell_{\boldsymbol{\mathcal{X}},Y}(\boldsymbol{v},\boldsymbol{w})\big) = \boldsymbol{w} \qquad \forall (\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{\mathcal{X}}^\ell(Y) \times \boldsymbol{\mathcal{X}}^{\mathrm{c},\ell}(Y), \tag{2.5}$$

$$z = \mathfrak{R}^\ell_{\boldsymbol{\mathcal{X}},Y}(\boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{X}},Y} z, \boldsymbol{\pi}^{\mathrm{c},\ell}_{\boldsymbol{\mathcal{X}},Y} z) \qquad \forall z \in \boldsymbol{\mathcal{P}}^\ell(Y), \tag{2.6}$$

and

$$\|\mathfrak{R}^\ell_{\boldsymbol{\mathcal{X}},Y}(\boldsymbol{v},\boldsymbol{w})\|_{\mathbf{L}^2(Y)} \simeq \|\boldsymbol{v}\|_{\mathbf{L}^2(Y)} + \|\boldsymbol{w}\|_{\mathbf{L}^2(Y)} \qquad \forall (\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{\mathcal{X}}^\ell(Y) \times \boldsymbol{\mathcal{X}}^{\mathrm{c},\ell}(Y). \tag{2.7}$$

Above, writing $a \lesssim b$ in place of $a \le Cb$ with C depending only on Ω, the mesh regularity parameter ρ of [12, Definition 1.9], and the considered polynomial degree, we have used $a \simeq b$ with the meaning of "$a \lesssim b$ and $b \lesssim a$". Both shorthand notations $\lesssim$ and $\simeq$ will be used throughout the paper.

2.3 Discrete spaces

The discrete counterpart of the space $\mathrm{H}^1(\Omega)$ in the DDR sequence is

$$\underline{X}^k_{\mathbf{grad},h} \coloneqq \Big\{ \underline{q}_h = \big((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}, q_{\mathcal{E}_h}\big) \,:\, q_T \in \mathcal{P}^{k-1}(T) \text{ for all } T\in\mathcal{T}_h,\, q_F\in\mathcal{P}^{k-1}(F) \text{ for all } F\in\mathcal{F}_h, \text{ and } q_{\mathcal{E}_h}\in\mathcal{P}^{k+1}_{\mathrm{c}}(\mathcal{E}_h)\Big\},$$

and the corresponding interpolator $\underline{I}^k_{\mathbf{grad},h} : \mathrm{C}^0(\overline{\Omega}) \to \underline{X}^k_{\mathbf{grad},h}$ is such that, for all $q\in\mathrm{C}^0(\overline{\Omega})$,

$$\begin{gathered}\underline{I}^k_{\mathbf{grad},h} q \coloneqq \big((\pi^{k-1}_{\mathcal{P},T} q_{|T})_{T\in\mathcal{T}_h}, (\pi^{k-1}_{\mathcal{P},F} q_{|F})_{F\in\mathcal{F}_h}, q_{\mathcal{E}_h}\big) \in \underline{X}^k_{\mathbf{grad},h},\\ \text{where } \pi^{k-1}_{\mathcal{P},E}(q_{\mathcal{E}_h})_{|E} = \pi^{k-1}_{\mathcal{P},E} q_{|E} \text{ for all } E\in\mathcal{E}_h \text{ and } q_{\mathcal{E}_h}(\boldsymbol{x}_V) = q(\boldsymbol{x}_V) \text{ for all } V\in\mathcal{V}_h,\end{gathered} \tag{2.8}$$

with $\boldsymbol{x}_V$ denoting the coordinates vector of the vertex V. The discrete $\mathbf{H}(\mathbf{curl};\Omega)$ space is

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \coloneqq \Big\{ \underline{\boldsymbol{v}}_T = \big((\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T})_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_h}, (v_E)_{E\in\mathcal{E}_h}\big) \,:\, &(\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T}) \in \boldsymbol{\mathcal{R}}^{k-1}(T)\times\boldsymbol{\mathcal{R}}^{\mathrm{c},k}(T) \text{ for all } T\in\mathcal{T}_h,\\ &(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}) \in \boldsymbol{\mathcal{R}}^{k-1}(F)\times\boldsymbol{\mathcal{R}}^{\mathrm{c},k}(F) \text{ for all } F\in\mathcal{F}_h, \text{ and } v_E\in\mathcal{P}^k(E) \text{ for all } E\in\mathcal{E}_h\Big\},\end{aligned}$$

with interpolator $\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} : \mathbf{C}^0(\overline{\Omega}) \to \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ such that, for all $\boldsymbol{v}\in\mathbf{C}^0(\overline{\Omega})$,

$$\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} \coloneqq \big((\boldsymbol{\pi}^{k-1}_{\mathcal{R},T}\boldsymbol{v}_{|T}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T}\boldsymbol{v}_{|T})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}^{k-1}_{\mathcal{R},F}\boldsymbol{v}_{\mathrm{t},F}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},F}\boldsymbol{v}_{\mathrm{t},F})_{F\in\mathcal{F}_h}, (\pi^k_{\mathcal{P},E}(\boldsymbol{v}_{|E}\cdot\boldsymbol{t}_E))_{E\in\mathcal{E}_h}\big),$$

where, for all $F\in\mathcal{F}_h$, $\boldsymbol{v}_{\mathrm{t},F} \coloneqq \boldsymbol{n}_F\times(\boldsymbol{v}_{|F}\times\boldsymbol{n}_F)$ denotes the orthogonal projection of $\boldsymbol{v}$ on the plane spanned by F. The role of $\mathbf{H}(\mathrm{div};\Omega)$ is played, at the discrete level, by

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \coloneqq \Big\{ \underline{\boldsymbol{v}}_T = \big((\boldsymbol{v}_{\mathcal{G},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{G},T})_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}\big) \,:\, &(\boldsymbol{v}_{\mathcal{G},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{G},T}) \in \boldsymbol{\mathcal{G}}^{k-1}(T)\times\boldsymbol{\mathcal{G}}^{\mathrm{c},k}(T) \text{ for all } T\in\mathcal{T}_h \text{ and } v_F\in\mathcal{P}^k(F) \text{ for all } F\in\mathcal{F}_h\Big\},\end{aligned}$$

with interpolator $\underline{\boldsymbol{I}}^k_{\mathrm{div},h} : \mathbf{H}^1(\Omega) \to \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ such that, for all $\boldsymbol{v}\in\mathbf{H}^1(\Omega)$,

$$\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{v} \coloneqq \big((\boldsymbol{\pi}^{k-1}_{\mathcal{G},T}\boldsymbol{v}_{|T}, \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{G},T}\boldsymbol{v}_{|T})_{T\in\mathcal{T}_h}, (\pi^k_{\mathcal{P},F}(\boldsymbol{v}_{|F}\cdot\boldsymbol{n}_F))_{F\in\mathcal{F}_T}\big). \tag{2.9}$$

Finally, the discrete counterpart of $\mathrm{L}^2(\Omega)$ in the DDR sequence is

$$\mathcal{P}^k(\mathcal{T}_h) \coloneqq \big\{ q_h\in\mathrm{L}^2(\Omega) \,:\, (q_h)_{|T}\in\mathcal{P}^k(T) \text{ for all } T\in\mathcal{T}_h\big\},$$

equipped with the global L^2-orthogonal projector $\pi^k_{\mathcal{P},h} : \mathrm{L}^2(\Omega)\to\mathcal{P}^k(\mathcal{T}_h)$ such that, for all $q\in\mathrm{L}^2(\Omega)$, $(\pi^k_{\mathcal{P},h}q)_{|T} \coloneqq \pi^k_{\mathcal{P},T}q_{|T}$ for all $T\in\mathcal{T}_h$.

2.4 Local discrete vector calculus operators

Given $\bullet\in\{\mathbf{grad},\mathbf{curl},\mathrm{div}\}$ and a mesh entity Y appearing in the definition of $\underline{X}^k_{\bullet,h}$, we denote by $\underline{X}^k_{\bullet,Y}$ the restriction of this space to Y, gathering the polynomial components on Y and on the geometrical entities on ∂Y. The corresponding local interpolator is denoted by $\underline{I}^k_{\bullet,Y}$.

2.4.1 Gradients

Throughout the rest of the paper, for $E \in \mathcal{E}_h$ and $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$ we set $q_E \coloneqq (q_{\mathcal{E}_h})_{|E} \in \underline{X}^k_{\mathbf{grad},E} = \mathcal{P}^{k+1}(E)$. For any $E \in \mathcal{E}_h$, the edge gradient $G^k_E : \underline{X}^k_{\mathbf{grad},E} \to \mathcal{P}^k(E)$ is such that, for all $q_E \in \underline{X}^k_{\mathbf{grad},E}$,

$$G^k_E q_E \coloneqq q'_E,$$

where the derivative is taken along E according to the orientation of $\boldsymbol{t}_E$. For all $F \in \mathcal{F}_h$, the face gradient $\mathbf{G}^k_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^k(F)$ is such that, for all $\underline{q}_F = (q_F, q_{\mathcal{E}_F}) \in \underline{X}^k_{\mathbf{grad},F}$,

$$\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{w}_F = -\int_F q_F \operatorname{div}_F \boldsymbol{w}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F} (\boldsymbol{w}_F \cdot \boldsymbol{n}_{FE}) \qquad \forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^k(F).$$

The scalar trace $\gamma^{k+1}_F : \underline{X}^k_{\mathbf{grad},F} \to \mathcal{P}^{k+1}(F)$ is such that, for all $\underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$,

$$\int_F \gamma^{k+1}_F \underline{q}_F \operatorname{div}_F \boldsymbol{v}_F = -\int_F \mathbf{G}^k_F \underline{q}_F \cdot \boldsymbol{v}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F} (\boldsymbol{v}_F \cdot \boldsymbol{n}_{FE}) \qquad \forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k+2}(F). \tag{2.10}$$

Remark 1 (Validity of (2.10)). Relation (2.10) also holds for all $\boldsymbol{v}_F \in \boldsymbol{\mathcal{RT}}^{k+1}(F)$; see [10, Remark 9].

Finally, for all $T \in \mathcal{T}_h$, the element gradient $\mathbf{G}^k_T : \underline{X}^k_{\mathbf{grad},T} \to \boldsymbol{\mathcal{P}}^k(T)$ is such that, for all $\underline{q}_T = (q_T, (q_F)_{F\in\mathcal{F}_T}, q_{\mathcal{E}_T}) \in \underline{X}^k_{\mathbf{grad},T}$,

$$\int_T \mathbf{G}^k_T \underline{q}_T \cdot \boldsymbol{w}_T = -\int_T q_T \operatorname{div} \boldsymbol{w}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^{k+1}_F \underline{q}_F (\boldsymbol{w}_T \cdot \boldsymbol{n}_F) \qquad \forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{P}}^k(T). \tag{2.11}$$

2.4.2 Curls

For all $F \in \mathcal{F}_h$, the face curl $C^k_F : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ is such that, for all $\underline{\boldsymbol{v}}_F = \big(\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}, (v_E)_{E\in\mathcal{E}_F}\big) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$,

$$\int_F C^k_F \underline{\boldsymbol{v}}_F \, r_F = \int_F \boldsymbol{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F \, r_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r_F \qquad \forall r_F \in \mathcal{P}^k(F). \tag{2.12}$$

The tangential trace $\boldsymbol{\gamma}^k_{\mathrm{t},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \boldsymbol{\mathcal{P}}^k(F)$ is such that, for all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$, recalling the definition (2.4) of the recovery operator with $(\boldsymbol{\mathcal{X}}, Y) = (\boldsymbol{\mathcal{R}}, F)$,

$$\boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \coloneqq \boldsymbol{\mathfrak{R}}^k_{\mathcal{R},F}(\boldsymbol{\gamma}^k_{\mathrm{t},\mathcal{R},F} \underline{\boldsymbol{v}}_F, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}),$$

where $\boldsymbol{\gamma}^k_{\mathrm{t},\mathcal{R},F} \underline{\boldsymbol{v}}_F \in \boldsymbol{\mathcal{R}}^k(F)$ is defined by

$$\int_F \boldsymbol{\gamma}^k_{\mathrm{t},\mathcal{R},F} \underline{\boldsymbol{v}}_F \cdot \mathbf{rot}_F \, r_F = \int_F C^k_F \underline{\boldsymbol{v}}_F \, r_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E v_E r_F \qquad \forall r_F \in \mathcal{P}^{0,k+1}(F). \tag{2.13}$$

Remark 2 (Validity of (2.13)). We note that this relation actually holds for all $r_F \in \mathcal{P}^{k+1}(F)$ and also with $\boldsymbol{\gamma}^k_{\mathrm{t},F}$ instead of $\boldsymbol{\gamma}^k_{\mathrm{t},\mathcal{R},F}$; see [10, Remark 14].

Finally, for all $T \in \mathcal{T}_h$, the element curl $\mathbf{C}^k_T : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \boldsymbol{\mathcal{P}}^k(T)$ is such that, for all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_{\mathcal{R},T}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T}, (\boldsymbol{v}_{\mathcal{R},F}, \boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F})_{F\in\mathcal{F}_T}, (v_E)_{E\in\mathcal{E}_T}\big) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$,

$$\int_T \mathbf{C}^k_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{w}_T = \int_T \boldsymbol{v}_{\mathcal{R},T} \cdot \mathbf{curl} \, \boldsymbol{w}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F) \qquad \forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{P}}^k(T). \tag{2.14}$$

2.4.3 Divergence

For all $T \in \mathcal{T}_h$, the element divergence $D_T^k : \underline{\boldsymbol{X}}_{\mathrm{div},T}^k \to \mathcal{P}^k(T)$ is defined by: For all $\underline{\boldsymbol{v}}_T = \big(\boldsymbol{v}_{\mathcal{G},T}, \boldsymbol{v}_{\mathcal{G},T}^{\mathrm{c}}, (v_F)_{F\in\mathcal{F}_T}\big) \in \underline{\boldsymbol{X}}_{\mathrm{div},T}^k$,

$$\int_T D_T^k \underline{\boldsymbol{v}}_T \, r_T = -\int_T \boldsymbol{v}_{\mathcal{G},T} \cdot \mathbf{grad}\, r_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F v_F r_T \qquad \forall r_T \in \mathcal{P}^k(T). \tag{2.15}$$

2.5 Global sequence

The global discrete gradient $\underline{\boldsymbol{G}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{grad},h}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$, curl $\underline{\boldsymbol{C}}_h^k : \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \to \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$, and divergence $D_h^k : \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \to \mathcal{P}^k(\mathcal{T}_h)$ are obtained by projecting the local operators onto the corresponding spaces: For all $(\underline{q}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{grad},h}^k \times \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$,

$$\begin{aligned}
\underline{\boldsymbol{G}}_h^k \underline{q}_h &\coloneqq \big((\boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T), \boldsymbol{\pi}_{\mathcal{R},T}^{\mathrm{c},k}(\mathbf{G}_T^k \underline{q}_T))_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathbf{G}_F^k \underline{q}_F))_{F\in\mathcal{F}_h}, (G_E^k q_E)_{E\in\mathcal{E}_h}\big),\\
\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h &\coloneqq \big((\boldsymbol{\pi}_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\boldsymbol{v}}_T), \boldsymbol{\pi}_{\mathcal{G},T}^{\mathrm{c},k}(\mathbf{C}_T^k \underline{\boldsymbol{v}}_T))_{T\in\mathcal{T}_h}, (C_F^k \underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_h}\big),\\
(D_h^k \underline{\boldsymbol{w}}_h)_{|T} &\coloneqq D_T^k \underline{\boldsymbol{w}}_T \qquad \forall T \in \mathcal{T}_h.
\end{aligned}$$

Following our previous notation for local spaces and interpolator, we will use the following notations for the restrictions of these discrete gradients and curl operators to mesh elements and faces:

$$\begin{aligned}
\underline{\boldsymbol{G}}_F^k \underline{q}_F &= \big(\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathbf{G}_F^k \underline{q}_F), (G_E^k q_E)_{E\in\mathcal{F}_E}\big),\\
\underline{\boldsymbol{G}}_T^k \underline{q}_T &= \big(\boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T), \boldsymbol{\pi}_{\mathcal{R},T}^{\mathrm{c},k}(\mathbf{G}_T^k \underline{q}_T), (\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\mathcal{R},F}^{\mathrm{c},k}(\mathbf{G}_F^k \underline{q}_F))_{F\in\mathcal{F}_T}, (G_E^k q_E)_{E\in\mathcal{E}_T}\big),\\
\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T &= \big(\boldsymbol{\pi}_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\boldsymbol{v}}_T), \boldsymbol{\pi}_{\mathcal{G},T}^{\mathrm{c},k}(\mathbf{C}_T^k \underline{\boldsymbol{v}}_T), (C_F^k \underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_T}\big).
\end{aligned}$$

The global sequence reads:

$$\mathbb{R} \xrightarrow{\underline{I}_{\mathbf{grad},h}^k} \underline{\boldsymbol{X}}_{\mathbf{grad},h}^k \xrightarrow{\underline{\boldsymbol{G}}_h^k} \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \xrightarrow{\underline{\boldsymbol{C}}_h^k} \underline{\boldsymbol{X}}_{\mathrm{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}. \tag{2.16}$$

It is proved in [10] that this sequence has exactness properties (the specific nature of which depends on the topology of Ω, as for the continuous de Rham sequence), and that the discrete operators satisfy Poincaré inequalities.

3 Consistency results

3.1 Potential reconstructions and L^2-products on discrete spaces

Let $T \in \mathcal{T}_h$. In this section, we define polynomial potential reconstructions on the discrete spaces $X_{\bullet,T}$ with $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$. These potentials have polynomial consistency properties, and enable the construction of discrete L^2-inner products on DDR spaces that are also polynomially consistent.

3.1.1 Scalar potential on $\underline{X}_{\mathbf{grad},T}^k$

The scalar potential reconstruction $P_{\mathbf{grad},T}^{k+1} : \underline{X}_{\mathbf{grad},T}^k \to \mathcal{P}^{k+1}(T)$ is such that, for all $\underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$,

$$\int_T P_{\mathbf{grad},T}^{k+1} \underline{q}_T \,\mathrm{div}\, \boldsymbol{v}_T = -\int_T \mathbf{G}_T^k \underline{q}_T \cdot \boldsymbol{v}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\boldsymbol{v}_T \cdot \boldsymbol{n}_F) \qquad \forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{R}}^{\mathrm{c},k+2}(T), \tag{3.1}$$

with γ_F^{k+1} defined by (2.10). This relation defines $P_{\mathbf{grad},T}^{k+1} \underline{q}_T$ uniquely since $\mathrm{div} : \boldsymbol{\mathcal{R}}^{\mathrm{c},k+2}(T) \to \mathcal{P}^{k+1}(T)$ is an isomorphism by [1, Corollary 7.3].

Remark 3 (Validity of (3.1)). The definition (2.11) of G_T^k and the identity $\operatorname{div}\mathbf{curl} = 0$ show that both sides of (3.1) vanish when $\boldsymbol{v}_T \in \mathcal{R}^k(T)$. Hence, (3.1) actually holds for any $\boldsymbol{v}_T \in \mathcal{R}^k(T) \oplus \mathcal{R}^{c,k+2}(T) = \mathcal{P}^k(T) + \mathcal{R}^{c,k+2}(T)$, the second equality following from $\mathcal{R}^{c,k}(T) \subset \mathcal{R}^{c,k+2}(T)$ and the decomposition (2.2).

Using the polynomial consistency properties $\mathsf{G}_T^k\big(\underline{I}^k_{\mathbf{grad},T} q\big) = \mathbf{grad}\, q$ and $\gamma_F^{k+1}\big(\underline{I}^k_{\mathbf{grad},F} q_{|F}\big) = q_{|F}$, valid for all $q \in \mathcal{P}^{k+1}(T)$ (see [10, Eqs. (3.13) and (3.11)]), the following polynomial consistency of $P^{k+1}_{\mathbf{grad},T}$ is obtained:

$$P^{k+1}_{\mathbf{grad},T}\big(\underline{I}^k_{\mathbf{grad},T} q\big) = q \qquad \forall q \in \mathcal{P}^{k+1}(T). \tag{3.2}$$

Moreover, applying (3.1) to $\boldsymbol{v}_T \in \mathcal{R}^{c,k}(T)$ (which is possible since $\mathcal{R}^{c,k}(T) \subset \mathcal{R}^{c,k+2}(T)$), using the definition (2.11) of G_T^k with $\boldsymbol{w}_T = \boldsymbol{v}_T$, and recalling that $\operatorname{div} : \mathcal{R}^{c,k}(T) \to \mathcal{P}^{k-1}(T)$ is onto, we obtain

$$\pi^{k-1}_{\mathcal{P},T}\big(P^{k+1}_{\mathbf{grad},T}\underline{q}_T\big) = q_T \qquad \forall \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}. \tag{3.3}$$

3.1.2 Vector potential on $\underline{X}^k_{\mathbf{curl},T}$

The vector potential reconstruction $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{X}^k_{\mathbf{curl},T} \to \boldsymbol{\mathcal{P}}^k(T)$ is such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$,

$$\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \coloneqq \mathfrak{R}^k_{\mathcal{R},T}(\boldsymbol{P}^k_{\mathbf{curl},\mathcal{R},T}\underline{\boldsymbol{v}}_T, \boldsymbol{v}^c_{\mathcal{R},T}), \tag{3.4}$$

where $\boldsymbol{P}^k_{\mathbf{curl},\mathcal{R},T}\underline{\boldsymbol{v}}_T \in \boldsymbol{\mathcal{R}}^k(T)$ is defined, using the isomorphism $\mathbf{curl} : \boldsymbol{\mathcal{G}}^{c,k+1}(T) \to \boldsymbol{\mathcal{R}}^k(T)$ (see [10, Eq. (2.10)]), by

$$\int_T \boldsymbol{P}^k_{\mathbf{curl},\mathcal{R},T}\underline{\boldsymbol{v}}_T \cdot \mathbf{curl}\, \boldsymbol{w}_T = \int_T \mathsf{C}^k_T\underline{\boldsymbol{v}}_T \cdot \boldsymbol{w}_T - \sum_{F\in\mathcal{F}_T} \omega_{TF}\int_F \boldsymbol{\gamma}^k_{t,F}\underline{\boldsymbol{v}}_F \cdot (\boldsymbol{w}_T \times \boldsymbol{n}_F) \qquad \forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{G}}^{c,k+1}(T). \tag{3.5}$$

Remark 4 (Discrete integration by parts formula for $\boldsymbol{P}^k_{\mathbf{curl},T}$). Formula (3.5) can be extended to test functions in the Nédélec space $\boldsymbol{\mathcal{N}}^{k+1}(T)$ defined by (2.3). To check it, simply notice that both sides vanish whenever $\boldsymbol{w}_T \in \boldsymbol{\mathcal{G}}^k(T)$ (use $\mathbf{curl}\,\mathbf{grad} = 0$ and the definition (2.14) of C^k_T). Since $\boldsymbol{\pi}^k_{\mathcal{R},T}\big(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\big) = \boldsymbol{P}^k_{\mathbf{curl},\mathcal{R},T}\underline{\boldsymbol{v}}_T$ (see (3.4) and (2.5)), we infer that

$$\int_T \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \cdot \mathbf{curl}\, \boldsymbol{z}_T = \int_T \mathsf{C}^k_T\underline{\boldsymbol{v}}_T \cdot \boldsymbol{z}_T - \sum_{F\in\mathcal{F}_T} \omega_{TF}\int_F \boldsymbol{\gamma}^k_{t,F}\underline{\boldsymbol{v}}_F \cdot (\boldsymbol{z}_T \times \boldsymbol{n}_F) \qquad \forall \boldsymbol{z}_T \in \boldsymbol{\mathcal{N}}^{k+1}(T). \tag{3.6}$$

Applying (3.6) to $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}$ with $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(T)$, using the consistency properties $\boldsymbol{\gamma}^k_{t,F}\big(\underline{\boldsymbol{I}}^k_{\mathbf{curl},F}\boldsymbol{v}\big) = \boldsymbol{\pi}^k_{\mathcal{P},F}\boldsymbol{v}_{t,F} = \boldsymbol{v}_{t,F}$ and $\mathsf{C}^k_T\big(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}\big) = \mathbf{curl}\,\boldsymbol{v}$ (see [10, Eqs. (3.22) and (3.26)]), and integrating by parts, and since $\mathbf{curl} : \boldsymbol{\mathcal{N}}^{k+1}(T) \to \boldsymbol{\mathcal{R}}^k(T)$ is onto (due to the isomorphism property [10, Eq. (2.10)]), we see that $\boldsymbol{\pi}^k_{\mathcal{R},T}\big[\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}\big)\big] = \boldsymbol{\pi}^k_{\mathcal{R},T}\boldsymbol{v}$. The definition (3.4) and the property (2.5) of the recovery operator also yield $\boldsymbol{\pi}^{c,k}_{\mathcal{R},T}\big[\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}\big)\big] = \boldsymbol{\pi}^{c,k}_{\mathcal{R},T}\boldsymbol{v}$. As a consequence,

$$\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}\big) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(T). \tag{3.7}$$

Following similar arguments as in the proof of [10, Proposition 15], we also have

$$\boldsymbol{\pi}^{k-1}_{\mathcal{R},T}\big(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\big) = \boldsymbol{v}_{\mathcal{R},T} \quad \text{and} \quad \boldsymbol{\pi}^{c,k}_{\mathcal{R},T}\big(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\big) = \boldsymbol{v}^c_{\mathcal{R},T} \qquad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}. \tag{3.8}$$

3.1.3 Vector potential on $\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$

The vector potential reconstruction $\boldsymbol{P}^k_{\mathrm{div},T} : \underline{\boldsymbol{X}}^k_{\mathrm{div},T} \to \boldsymbol{\mathcal{P}}^k(T)$ is such that, for all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\mathrm{div},T}$,

$$\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T = \mathfrak{R}^k_{\mathcal{G},T}(\boldsymbol{P}^k_{\mathrm{div},\mathcal{G},T}\underline{\boldsymbol{w}}_T, \boldsymbol{w}^{\mathrm{c}}_{\mathcal{G},T}),$$

where $\boldsymbol{P}^k_{\mathrm{div},\mathcal{G},T}\underline{\boldsymbol{w}}_T \in \boldsymbol{\mathcal{G}}^k(T)$ is defined by

$$\int_T \boldsymbol{P}^k_{\mathrm{div},\mathcal{G},T}\underline{\boldsymbol{w}}_T \cdot \mathbf{grad}\, r_T = -\int_T D^k_T\underline{\boldsymbol{w}}_T\, r_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F w_F\, r_T \qquad \forall r_T \in \mathcal{P}^{0,k+1}(T). \tag{3.9}$$

Remark 5 (Discrete integration by parts formula for $\boldsymbol{P}^k_{\mathrm{div},T}$). Observing that $\boldsymbol{P}^k_{\mathrm{div},\mathcal{G},T} = \boldsymbol{\pi}^k_{\mathcal{G},T}\boldsymbol{P}^k_{\mathrm{div},T}$ (use (2.5)) and that (3.9) holds for any $r_T \in \mathcal{P}^{k+1}(T)$ (as can be proved taking r_T constant in T and observing that both sides of this equation vanish due to the definition (2.15) of D^k_T), we infer

$$\int_T \boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T \cdot \mathbf{grad}\, r_T = -\int_T D^k_T\underline{\boldsymbol{w}}_T\, r_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F w_F\, r_T \qquad \forall r_T \in \mathcal{P}^{k+1}(T). \tag{3.10}$$

Writing (3.10) for $\underline{\boldsymbol{w}}_T = \underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}$ with $\boldsymbol{w} \in \boldsymbol{\mathcal{RT}}^{k+1}(T)$, observing that $D^k_T\big(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}\big) = \pi^k_{\mathcal{P},T}(\operatorname{div}\boldsymbol{w}) = \operatorname{div}\boldsymbol{w}$ by [10, Eq. (3.36)] and $\pi^k_{\mathcal{P},F}(\boldsymbol{w}_{|F}\cdot\boldsymbol{n}_F) = \boldsymbol{w}_{|F}\cdot\boldsymbol{n}_F$ for all $F \in \mathcal{F}_T$ by [10, Eq. (A.4)], and integrating by parts the right-hand side of the resulting expression, we infer $\boldsymbol{\pi}^k_{\mathcal{G},T}\big[\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}\big)\big] = \boldsymbol{\pi}^k_{\mathcal{G},T}\boldsymbol{w}$; since $\boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{G},T}\big[\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}\big)\big] = \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{G},T}\boldsymbol{w}$ by definition of $\boldsymbol{P}^k_{\mathrm{div},T}$, $\underline{\boldsymbol{I}}^k_{\mathrm{div},T}$ and (2.5), we deduce that

$$\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}\big) = \boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{w} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{RT}}^{k+1}(T). \tag{3.11}$$

Moreover, following similar arguments as in [10, Proposition 15] we get

$$\boldsymbol{\pi}^{k-1}_{\mathcal{G},T}\big(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T\big) = \boldsymbol{w}_{\mathcal{G},T} \quad \text{and} \quad \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{G},T}\big(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T\big) = \boldsymbol{w}^{\mathrm{c}}_{\mathcal{G},T} \qquad \forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\mathrm{div},T}. \tag{3.12}$$

3.1.4 Discrete L^2-products

We now define discrete L^2-inner products on the DDR spaces. These products are all constructed in a similar way: by assembling local contributions composed of a consistent term based on the potential reconstructions and a stabilisation term that provides a control of the polynomial components on the lower dimensional geometrical objects. Specifically, each L^2-product $(\cdot,\cdot)_{\mathbf{grad},h} : \underline{X}^k_{\mathbf{grad},h}\times\underline{X}^k_{\mathbf{grad},h} \to \mathbb{R}$, $(\cdot,\cdot)_{\mathbf{curl},h} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}\times\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$, and $(\cdot,\cdot)_{\mathrm{div},h} : \underline{\boldsymbol{X}}^k_{\mathrm{div},h}\times\underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathbb{R}$ is the sum over $T \in \mathcal{T}_h$ of its local counterpart defined by:

$$(\underline{r}_T,\underline{q}_T)_{\mathbf{grad},T} \coloneqq \int_T P^{k+1}_{\mathbf{grad},T}\underline{r}_T\; P^{k+1}_{\mathbf{grad},T}\underline{q}_T + \mathrm{s}_{\mathbf{grad},T}(\underline{r}_T,\underline{q}_T) \qquad \forall(\underline{r}_T,\underline{q}_T)\in\underline{X}^k_{\mathbf{grad},T}\times\underline{X}^k_{\mathbf{grad},T}, \tag{3.13a}$$

$$(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T)_{\mathbf{curl},T} \coloneqq \int_T \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T) \quad \forall(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T)\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}\times\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}, \tag{3.13b}$$

$$(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T)_{\mathrm{div},T} \coloneqq \int_T \boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{v}}_T + \mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T) \qquad \forall(\underline{\boldsymbol{w}}_T,\underline{\boldsymbol{v}}_T)\in\underline{\boldsymbol{X}}^k_{\mathrm{div},T}\times\underline{\boldsymbol{X}}^k_{\mathrm{div},T}, \tag{3.13c}$$

where the symmetric, positive semidefinite stabilisation bilinear forms $\mathrm{s}_{\bullet,T}$, $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, are defined as follows:

$$\mathrm{s}_{\mathbf{grad},T}(\underline{r}_T, \underline{q}_T) \coloneqq \sum_{F\in\mathcal{F}_T} h_F \int_F \left(P^{k+1}_{\mathbf{grad},T}\underline{r}_T - \gamma^{k+1}_F \underline{r}_F\right)\left(P^{k+1}_{\mathbf{grad},T}\underline{q}_T - \gamma^{k+1}_F \underline{q}_F\right) + \sum_{E\in\mathcal{E}_T} h_E^2 \int_E \left(P^{k+1}_{\mathbf{grad},T}\underline{r}_F - r_E\right)\left(P^{k+1}_{\mathbf{grad},T}\underline{q}_F - q_E\right), \tag{3.14}$$

$$\mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \sum_{F\in\mathcal{F}_T} h_F \int_F \left((\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T)_{\mathrm{t},F} - \boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{w}}_F\right)\cdot\left((\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T)_{\mathrm{t},F} - \boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F\right) + \sum_{E\in\mathcal{E}_T} h_E^2 \int_E \left(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_F\cdot\boldsymbol{t}_E - w_E\right)\left(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_F\cdot\boldsymbol{t}_E - v_E\right), \tag{3.15}$$

where we recall that the index t, F denotes the tangential trace on F, and

$$\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T) \coloneqq \sum_{F\in\mathcal{F}_T} h_F \int_F \left(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{n}_F - w_F\right)\left(\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{v}}_T\cdot\boldsymbol{n}_F - v_F\right). \tag{3.16}$$

These local stabilisation bilinear forms $\mathrm{s}_{\bullet,T}$ are polynomialy consistent, i.e., they vanish whenever one of their arguments is the interpolate of a polynomial of total degree $\le k+1$ for $\bullet = \mathbf{grad}$, or $\le k$ for $\bullet \in \{\mathbf{curl}, \mathrm{div}\}$. Further consistency properties for interpolates of smooth functions are stated in Theorem 8

For $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, we denote by $\|\cdot\|_{\bullet,T}$ the norm on $\underline{X}^k_{\bullet,T}$ induced by the corresponding local discrete L^2-product $(\cdot,\cdot)_{\bullet,T}$, and by $\|\cdot\|_{\bullet,h}$ the norm on $\underline{X}^k_{\bullet,h}$ corresponding to the global discrete L^2-product $(\cdot,\cdot)_{\bullet,h}$.

3.2 Primal consistency

In this section we state consistency results for the discrete potentials, vector calculus operators, and stabilisation bilinear forms. Because of the nature of the interpolator on $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ (which requires higher regularity of functions), we introduce the following notation: For $T \in \mathcal{T}_h$ and $\boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T)$,

$$|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)} \coloneqq \begin{cases} |\boldsymbol{v}|_{\mathbf{H}^1(T)} + h_T|\boldsymbol{v}|_{\mathbf{H}^2(T)} & \text{if } k=0,\\ |\boldsymbol{v}|_{\mathbf{H}^{k+1}(T)} & \text{if } k\ge 1. \end{cases} \tag{3.17}$$

The corresponding global broken seminorm $|\cdot|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)}$ is such that, for all $\boldsymbol{v} \in \mathbf{H}^{(k+1,2)}(\mathcal{T}_h)$, $|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} \coloneqq \left(\sum_{T\in\mathcal{T}_h} |\boldsymbol{v}|^2_{\mathbf{H}^{(k+1,2)}(T)}\right)^{1/2}$. The proofs of the following theorems are postponed to Section 4.3.

Theorem 6 (Consistency of the potential reconstructions)**.** *It holds, for all $T \in \mathcal{T}_h$,*

$$\|P^{k+1}_{\mathbf{grad},T}(\underline{I}^k_{\mathbf{grad},T}q) - q\|_{\mathrm{L}^2(T)} \lesssim h_T^{k+2}|q|_{\mathrm{H}^{k+2}(T)} \qquad \forall q \in \mathrm{H}^{k+2}(T), \tag{3.18}$$

$$\|\boldsymbol{P}^k_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}) - \boldsymbol{v}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)} \qquad \forall \boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T), \tag{3.19}$$

$$\|\boldsymbol{P}^k_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}) - \boldsymbol{w}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} \qquad \forall \boldsymbol{w} \in \mathbf{H}^{k+1}(T). \tag{3.20}$$

Theorem 7 (Primal consistency of the discrete vector calculus operators)**.** *It holds, for all $T \in \mathcal{T}_h$,*

$$\|\mathsf{G}^k_T(\underline{I}^k_{\mathbf{grad},T}q) - \mathbf{grad}\,q\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1}|q|_{\mathrm{H}^{k+2}(T)}, \qquad \forall q \in \mathrm{H}^{k+2}(T), \tag{3.21}$$

$$\|\mathsf{C}^k_T(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}) - \mathbf{curl}\,\boldsymbol{v}\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1}|\mathbf{curl}\,\boldsymbol{v}|_{\mathbf{H}^{k+1}(T)}, \quad \forall \boldsymbol{v} \in \mathbf{H}^2(T)\ \textit{s.t.}\ \mathbf{curl}\,\boldsymbol{v} \in \mathbf{H}^{k+1}(T), \tag{3.22}$$

$$\|D^k_T(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}) - \mathrm{div}\,\boldsymbol{w}\|_{\mathrm{L}^2(T)} \lesssim h_T^{k+1}|\mathrm{div}\,\boldsymbol{w}|_{\mathrm{H}^{k+1}(T)}, \quad \forall \boldsymbol{w} \in \mathbf{H}^1(T)\ \textit{s.t.}\ \mathrm{div}\,\boldsymbol{w} \in \mathrm{H}^{k+1}(T). \tag{3.23}$$

Theorem 8 (Consistency of stabilisation forms)**.** *For all $T \in \mathcal{T}_h$, the stabilisation forms defined by* (3.14)–(3.16) *satisfy the following consistency properties:*

$$\mathrm{s}_{\mathbf{grad},T}(\underline{I}^k_{\mathbf{grad},T} q, \underline{I}^k_{\mathbf{grad},T} q)^{\frac{1}{2}} \lesssim h_T^{k+2} |q|_{\mathrm{H}^{k+2}(T)} \qquad \forall q \in \mathrm{H}^{k+2}(T), \tag{3.24}$$

$$\mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}, \underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v})^{\frac{1}{2}} \lesssim h_T^{k+1} |\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)} \qquad \forall \boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T), \tag{3.25}$$

$$\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w}, \underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w})^{\frac{1}{2}} \lesssim h_T^{k+1} |\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} \qquad \forall \boldsymbol{w} \in \mathbf{H}^{k+1}(T). \tag{3.26}$$

3.3 Adjoint consistency

Whenever a (formal) integration by parts is used to write the weak formulation of a PDE problem underpinning its discretisation, a form of adjoint consistency is required in the convergence analysis. We state here the adjoint consistency of the operators in the DDR sequence (2.16). Since this sequence does not incorporate boundary conditions, the corresponding adjoint consistency will be based on essential boundary conditions. The regularity requirements will be expressed in terms of the broken Sobolev spaces and norms such that, for any $\ell \geq 1$,

$$\mathrm{H}^\ell(\mathcal{T}_h) \coloneqq \left\{ g \in \mathrm{L}^2(\Omega) \,:\, g_{|T} \in \mathrm{H}^\ell(T) \text{ for all } T \in \mathcal{T}_h \right\} \quad \text{and} \quad |g|_{\mathrm{H}^\ell(\mathcal{T}_h)} \coloneqq \left(\sum_{T \in \mathcal{T}_h} |g_{|T}|^2_{\mathrm{H}^\ell(T)} \right)^{\frac{1}{2}}.$$

The corresponding seminorms for vector-valued functions is denoted, as usual, using boldface letters. We denote in what follows by $\mathrm{H}^1_0(\Omega)$, $\mathbf{H}_0(\mathrm{div};\Omega)$, and $\mathbf{H}_0(\mathbf{curl};\Omega)$ the subspaces of $\mathrm{H}^1(\Omega)$, $\mathbf{H}(\mathrm{div};\Omega)$, and $\mathbf{H}(\mathbf{curl};\Omega)$ spanned by functions whose trace, normal trace, and tangential trace vanish on $\partial\Omega$, respectively.

Theorem 9 (Adjoint consistency for the gradient)**.** *Let $\tilde{\mathcal{E}}_{\mathrm{div},h} : \left(\mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathrm{div};\Omega)\right) \times \underline{X}^k_{\mathbf{grad},h} \to \mathbb{R}$ be such that, for all $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,*

$$\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v}, \underline{q}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \left[(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}_{|T}, \underline{\boldsymbol{G}}^k_T \underline{q}_T)_{\mathbf{curl},T} + \int_T \mathrm{div}\, \boldsymbol{v}\; P^{k+1}_{\mathbf{grad},T} \underline{q}_T \right].$$

Then, it holds, for all $\boldsymbol{v} \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathrm{div};\Omega)$ such that $\boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(\mathcal{T}_h)$ and all $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$,

$$|\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v}, \underline{q}_h)| \lesssim h^{k+1} |\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} \|\underline{\boldsymbol{G}}^k_h \underline{q}_h\|_{\mathbf{curl},h}, \tag{3.27}$$

Proof. See Section 4.4.1. □

Theorem 10 (Adjoint consistency for the curl)**.** *Let $\tilde{\mathcal{E}}_{\mathbf{curl},h} : \left(\mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathbf{curl};\Omega)\right) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$ be such that, for all $(\boldsymbol{w}, \underline{\boldsymbol{v}}_h) \in \left(\mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathbf{curl};\Omega)\right) \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,*

$$\tilde{\mathcal{E}}_{\mathbf{curl},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \left[(\underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w}_{|T}, \underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T)_{\mathrm{div},T} - \int_T \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T \right]. \tag{3.28}$$

Then, for all $\boldsymbol{w} \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}_0(\mathbf{curl};\Omega)$ such that $\boldsymbol{w} \in \mathbf{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$|\tilde{\mathcal{E}}_{\mathbf{curl},h}(\boldsymbol{w}, \underline{\boldsymbol{v}}_h)| \lesssim h^{k+1} \left(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)} \right) \left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} \right). \tag{3.29}$$

Proof. See Section 4.4.2. □

Theorem 11 (Adjoint consistency for the divergence)**.** *Let* $\tilde{\mathcal{E}}_{\mathbf{grad},h} : \big(\mathrm{C}^0(\overline{\Omega}) \cap H_0^1(\Omega)\big) \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathbb{R}$ *be such that, for all* $(q, \underline{\boldsymbol{v}}_h) \in \big(\mathrm{C}^0(\overline{\Omega}) \cap H_0^1(\Omega)\big) \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$*,*

$$\tilde{\mathcal{E}}_{\mathbf{grad},h}(q, \underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega \pi^k_{\mathcal{P},h} q \; D^k_h \underline{\boldsymbol{v}}_h + \sum_{T \in \mathcal{T}_h} \int_\Omega \mathbf{grad}\, q \cdot \boldsymbol{P}^k_{\mathrm{div},T} \underline{\boldsymbol{v}}_T. \tag{3.30}$$

Then, for all $q \in \mathrm{C}^0(\overline{\Omega}) \cap H_0^1(\Omega)$ *such that* $q \in \mathrm{H}^{k+2}(\mathcal{T}_h)$ *and all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$*,*

$$|\tilde{\mathcal{E}}_{\mathbf{grad},h}(q, \underline{\boldsymbol{v}}_h)| \lesssim h^{k+1} |q|_{\mathrm{H}^{k+2}(\mathcal{T}_h)} \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}. \tag{3.31}$$

Proof. See Section 4.4.3. □

4 Proofs of the consistency results

In this section, after establishing some preliminary results, we prove the primal and adjoint consistency results stated in Section 3.

4.1 Component norms and bounds on potentials

We recall the definition of the component L^2-norm on $\underline{\boldsymbol{X}}^k_{\mathbf{grad},T}$, $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ and $\underline{\boldsymbol{X}}^k_{\mathrm{div},T}$ introduced in [10, Section 4.1], and which correspond to the L^2-norms of the components of the vectors of polynomials, with scaling appropriate to the dimensions of the geometrical objects on which these components are defined:

$$\begin{aligned}
|||\underline{q}_T|||_{\mathbf{grad},T} &\coloneqq \Big(\|q_T\|^2_{\mathrm{L}^2(T)} + \sum_{F \in \mathcal{F}_T} h_F |||\underline{q}_F|||^2_{\mathbf{grad},F}\Big)^{1/2} \quad \text{for all } \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T},\\
\text{where } |||\underline{q}_F|||_{\mathbf{grad},F} &\coloneqq \Big(\|q_F\|^2_{\mathrm{L}^2(F)} + \sum_{E \in \mathcal{E}_F} h_E \|q_E\|^2_{\mathrm{L}^2(E)}\Big)^{1/2} \text{ for all } F \in \mathcal{F}_T,\\
|||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T} &\coloneqq \Big(\|\boldsymbol{v}_{\mathcal{R},T}\|^2_{\mathbf{L}^2(T)} + \|\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},T}\|^2_{\mathbf{L}^2(T)} + \sum_{F \in \mathcal{F}_T} h_F |||\underline{\boldsymbol{v}}_F|||^2_{\mathbf{curl},F}\Big)^{1/2} \quad \text{for all } \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T},\\
\text{where } |||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} &\coloneqq \Big(\|\boldsymbol{v}_{\mathcal{R},F}\|^2_{\mathbf{L}^2(F)} + \|\boldsymbol{v}^{\mathrm{c}}_{\mathcal{R},F}\|^2_{\mathbf{L}^2(F)} + \sum_{E \in \mathcal{E}_F} h_E \|v_E\|^2_{\mathrm{L}^2(E)}\Big)^{1/2} \text{ for all } F \in \mathcal{F}_T,
\end{aligned} \tag{4.1}$$

and

$$|||\underline{\boldsymbol{w}}_T|||_{\mathrm{div},T} \coloneqq \Big(\|\boldsymbol{w}_{\mathcal{G},T}\|^2_{\mathbf{L}^2(T)} + \|\boldsymbol{w}^{\mathrm{c}}_{\mathcal{G},T}\|^2_{\mathbf{L}^2(T)} + \sum_{F \in \mathcal{F}_T} h_F \|w_F\|^2_{\mathrm{L}^2(F)}\Big)^{1/2} \quad \text{for all } \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\mathrm{div},T}.$$

The next proposition follows from (2.7) and [10, Lemma 31], in a similar way as in the proof of [11, Proposition 13].

Proposition 12 (Boundedness of local potentials)**.** *It holds, for all* $T \in \mathcal{T}_h$ *and all* $F \in \mathcal{F}_T$*,*

$$\|\gamma^{k+1}_F \underline{q}_F\|_{\mathrm{L}^2(F)} \lesssim |||\underline{q}_F|||_{\mathbf{grad},F} \;\; \textit{and} \;\; \|P^{k+1}_{\mathbf{grad},T} \underline{q}_T\|_{\mathrm{L}^2(T)} \lesssim |||\underline{q}_T|||_{\mathbf{grad},T} \qquad \forall \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}, \tag{4.2}$$

$$\|\boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F\|_{\mathbf{L}^2(F)} \lesssim |||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} \;\; \textit{and} \;\; \|\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} \lesssim |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T} \qquad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}, \tag{4.3}$$

$$\|\boldsymbol{P}^k_{\mathrm{div},T} \underline{\boldsymbol{w}}_T\|_{\mathbf{L}^2(T)} \lesssim |||\underline{\boldsymbol{w}}_T|||_{\mathrm{div},T} \qquad \forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\mathrm{div},T}. \tag{4.4}$$

For $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathrm{div}\}$, using triangle inequalities as in [11, Proposition 14], invoking the bounds of Proposition 12, the projection properties (3.3), (3.8) (and similar for $\boldsymbol{\gamma}^k_{\mathrm{t},F}$, see [10, Proposition 15]) or (3.12), and recalling (2.7), we have the norm equivalence: For all $T \in \mathcal{T}_h$

$$|||\underline{z}_T|||_{\bullet,T} \simeq \|\underline{z}_T\|_{\bullet,T} \qquad \forall \underline{z}_T \in \underline{X}^k_{\bullet,T}. \tag{4.5}$$

Lemma 13 (Boundedness of local interpolators)**.** *It holds, for all $T \in \mathcal{T}_h$,*

$$|||\underline{I}^k_{\mathbf{grad},T} q|||_{\mathbf{grad},T} \lesssim \|q\|_{\mathrm{L}^2(T)} + h_T |q|_{\mathrm{H}^1(T)} + h_T^2 |q|_{\mathrm{H}^2(T)} \qquad \forall q \in \mathrm{H}^2(T), \tag{4.6}$$

$$|||\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}|||_{\mathbf{curl},T} \lesssim \|\boldsymbol{v}\|_{\mathbf{L}^2(T)} + h_T |\boldsymbol{v}|_{\mathbf{H}^1(T)} + h_T^2 |\boldsymbol{v}|_{\mathbf{H}^2(T)} \qquad \forall \boldsymbol{v} \in \mathbf{H}^2(T), \tag{4.7}$$

$$|||\underline{\boldsymbol{I}}^k_{\mathrm{div},T} \boldsymbol{w}|||_{\mathrm{div},T} \lesssim \|\boldsymbol{w}\|_{\mathbf{L}^2(T)} + h_T |\boldsymbol{w}|_{\mathbf{H}^1(T)} \qquad \forall \boldsymbol{w} \in \mathbf{H}^1(T). \tag{4.8}$$

Proof. The definition of $\underline{I}^k_{\mathbf{grad},T}$ (see (2.8)) clearly shows that $|||\underline{I}^k_{\mathbf{grad},T} q|||_{\mathbf{grad},T} \lesssim |T|^{1/2} \max_T |q|$. By [12, Eq. (5.110)], it holds

$$\max_T |q| \lesssim |T|^{-\frac{1}{2}} \sum_{r=0}^{2} h_T^r |q|_{\mathrm{H}^r(T)},$$

which concludes the proof of (4.6). The estimate (4.7) is obtained the same way. As for (4.8), by the continuous trace inequality of [12, Lemma 1.31], we have

$$\|\pi^k_{\mathcal{P},F}(\boldsymbol{w} \cdot \boldsymbol{n}_F)\|_{\mathrm{L}^2(F)} \le \|\boldsymbol{w}\|_{\mathbf{L}^2(F)} \lesssim h_F^{-\frac{1}{2}} \|\boldsymbol{w}\|_{\mathbf{L}^2(T)} + h_F^{\frac{1}{2}} |\boldsymbol{w}|_{\mathbf{H}^1(T)}.$$

Using this bound in the definition (2.9) of $\underline{\boldsymbol{I}}^k_{\mathrm{div},T}$ yields (4.8). □

4.2 Links between discrete vector potentials and vector calculus operators

Proposition 14 (Link between discrete vector potentials and vector calculus operators)**.** *For all $T \in \mathcal{T}_h$, it holds*

$$\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{G}}^k_T \underline{q}_T\big) = \mathsf{G}^k_T \underline{q}_T \qquad \forall \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}, \tag{4.9}$$

$$\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T\big) = \mathsf{C}^k_T \underline{\boldsymbol{v}}_T \qquad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}. \tag{4.10}$$

Proof. 1. *Proof of* (4.9). By the second projection property in (3.8), we have $\boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T}\big[\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{G}}^k_T \underline{q}_T\big)\big] = \boldsymbol{\pi}^{\mathrm{c},k}_{\mathcal{R},T}\big(\mathsf{G}^k_T \underline{q}_T\big)$. To infer the conclusion, it then suffices to prove that

$$\boldsymbol{\pi}^k_{\mathcal{R},T}\big[\boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{G}}^k_T \underline{q}_T\big)\big] = \boldsymbol{\pi}^k_{\mathcal{R},T}\big(\mathsf{G}^k_T \underline{q}_T\big) \tag{4.11}$$

and invoke (2.6). To prove (4.11), we take $\boldsymbol{z}_T \in \boldsymbol{\mathcal{N}}^{k+1}(T)$ and apply (3.6) with $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{G}}^k_T \underline{q}_T$. Using the inclusion $\operatorname{Im} \underline{\boldsymbol{G}}^k_T \subset \operatorname{Ker} \mathsf{C}^k_T$ (see [10, Remark 21]) and the relation $\boldsymbol{\gamma}^k_{\mathrm{t},F}\big(\underline{\boldsymbol{G}}^k_F \underline{q}_F\big) = \mathsf{G}^k_F \underline{q}_F$ valid for all $F \in \mathcal{F}_T$ (see [10, Proposition 15]), we obtain

$$\int_T \boldsymbol{P}^k_{\mathbf{curl},T}\big(\underline{\boldsymbol{G}}^k_T \underline{q}_T\big) \cdot \mathbf{curl}\, \boldsymbol{z}_T = -\sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \mathsf{G}^k_F \underline{q}_F \cdot (\boldsymbol{z}_T \times \boldsymbol{n}_F) = \int_T \mathsf{G}^k_T \underline{q}_T \cdot \mathbf{curl}\, \boldsymbol{z}_T,$$

the conclusion following from the link between element and face gradient, see [10, Proposition 11]. By the isomorphism $\mathbf{curl} : \boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(T) \to \boldsymbol{\mathcal{R}}^k(T)$ of [10, Eq. (2.10)] and since $\boldsymbol{\mathcal{G}}^{\mathrm{c},k+1}(T) \subset \boldsymbol{\mathcal{N}}^{k+1}(T)$, this establishes (4.11) and concludes the proof of (4.9).

2. *Proof of* (4.10). The second projection property in (3.12) ensures that $\boldsymbol{\pi}^{\mathrm{c},k}_{\boldsymbol{\mathcal{G}},T}\big[\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T\big)\big] = \boldsymbol{\pi}^{\mathrm{c},k}_{\boldsymbol{\mathcal{G}},T}\big(\mathsf{C}^k_T \underline{\boldsymbol{v}}_T\big)$. As before, it therefore remains to analyse the projections on $\boldsymbol{\mathcal{G}}^k(T)$. Apply (3.10) to $\underline{\boldsymbol{w}}_T = \underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T$ and a generic $r_T \in \mathcal{P}^{k+1}(T)$, and use the inclusion $\operatorname{Im} \underline{\boldsymbol{C}}^k_T \subset \operatorname{Ker} D^k_T$ (see [10, Proposition 17]) to get

$$\int_T \boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T\big) \cdot \mathbf{grad}\, r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C^k_F \underline{\boldsymbol{v}}_F\, r_T = \int_T \mathsf{C}^k_T \underline{\boldsymbol{v}}_T \cdot \mathbf{grad}\, r_T,$$

where the conclusion is obtained applying the link between element and face curls of [10, Proposition 16]. This establishes that $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},T}\big[\boldsymbol{P}^k_{\mathrm{div},T}\big(\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T\big)\big] = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},T}\big(\mathsf{C}^k_T \underline{\boldsymbol{v}}_T\big)$, proving (4.10). □

Corollary 15 (Bound on discrete gradients and curl)**.** *For all $F \in \mathcal{F}_h$, it holds*

$$\|\mathsf{G}_F^k \underline{q}_F\|_{\mathbf{L}^2(F)}^2 + \sum_{E\in\mathcal{E}_F} h_E \|G_E^k q_E\|_{\mathrm{L}^2(E)}^2 \lesssim |||\underline{\boldsymbol{G}}_F^k \underline{q}_F|||_{\mathbf{curl},F} \qquad \forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k. \tag{4.12}$$

For all $T \in \mathcal{T}_h$, it holds

$$\|\mathsf{G}_T^k \underline{q}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F\in\mathcal{F}_T} h_F \|\mathsf{G}_F^k \underline{q}_F\|_{\mathbf{L}^2(F)}^2 + \sum_{E\in\mathcal{E}_T} h_E^2 \|G_E^k q_E\|_{\mathrm{L}^2(E)}^2 \lesssim |||\underline{\boldsymbol{G}}_T^k \underline{q}_T|||_{\mathbf{curl},T}$$
$$\forall \underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k, \tag{4.13}$$

$$\|\mathsf{C}_T^k \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F\in\mathcal{F}_T} h_F \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathbf{L}^2(F)}^2 \lesssim |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T} \qquad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k. \tag{4.14}$$

Proof. The definitions of $|||\cdot|||_{\mathbf{curl},F}$, $|||\cdot|||_{\mathbf{curl},T}$, $\underline{\boldsymbol{G}}_F^k$ and $\underline{\boldsymbol{G}}_T^k$ show that the edge gradient contributions in the left-hand sides of (4.12) and (4.13) are bounded by the corresponding right-hand sides. To bound the face and element gradient contributions in the left-hand sides of (4.12) and (4.13), simply apply (4.3) to $\underline{\boldsymbol{v}}_T = \underline{\boldsymbol{G}}_T^k \underline{q}_T$, use $\boldsymbol{\gamma}_{\mathrm{t},F}^k \circ \underline{\boldsymbol{G}}_F^k = \mathsf{G}_F^k$ (see [10, Proposition 15]) and (4.9). The estimate (4.14) is established in a similar way. □

4.3 Primal consistency

Proof of Theorem 6. Let us start with (3.18). Since $\mathrm{H}^2(T) \subset \mathrm{C}^0(\overline{T})$, the mapping $P_{\mathbf{grad},T}^{k+1} \circ \underline{I}_{\mathbf{grad},T}^k : \mathrm{H}^2(T) \to \mathcal{P}^{k+1}(T)$ is well-defined and, owing to (3.2), it is a projector. Moreover, combining (4.6) and (4.2), it satisfies the $\mathrm{L}^2(T)$-boundedness

$$\|P_{\mathbf{grad},T}^{k+1}(\underline{I}_{\mathbf{grad},T}^k q)\|_{\mathrm{L}^2(T)} \lesssim \|q\|_{\mathrm{L}^2(T)} + h_T |q|_{\mathrm{H}^1(T)} + h_T^2 |q|_{\mathrm{H}^2(T)} \qquad \forall q \in \mathrm{H}^2(T).$$

The approximation property (3.18) is thus a direct consequence of [12, Lemma 1.43]. The proofs of (3.19) (for $k \geq 1$) and (3.20) are similar, using the fact that the considered operators are projectors onto $\mathcal{P}^k(T)$ (see (3.7) and (3.11)) and invoking Proposition 12 and Lemma 13 to establish their L^2-boundedness. In the case $k = 0$, since $\boldsymbol{P}_{\mathbf{curl},T}^0 \circ \underline{\boldsymbol{I}}_{\mathbf{curl},T}^0$ requires the $\mathbf{H}^2$-regularity of its argument, with $2 > k + 1$, (3.19) cannot be deduced directly from [12, Lemma 1.43]. However, using the bounds (4.3) and (4.7) a direct proof can be done by introducing $\boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v} = \boldsymbol{P}_{\mathbf{curl},T}^0 (\underline{\boldsymbol{I}}_{\mathbf{curl},T}^0 \boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v})$:

$$\begin{aligned}\|\boldsymbol{P}_{\mathbf{curl},T}^0 (\underline{\boldsymbol{I}}_{\mathbf{curl},T}^0 \boldsymbol{v}) - \boldsymbol{v}\|_{\mathbf{L}^2(T)} &\leq \|\boldsymbol{P}_{\mathbf{curl},T}^0 [\underline{\boldsymbol{I}}_{\mathbf{curl},T}^0 (\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v})]\|_{\mathbf{L}^2(T)} + \|\boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v} - \boldsymbol{v}\|_{\mathbf{L}^2(T)} \\ &\lesssim \|\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v}\|_{\mathbf{L}^2(T)} + h_T |\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v}|_{\mathbf{H}^1(T)} + h_T^2 |\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v}|_{\mathbf{H}^2(T)},\end{aligned}$$

and (3.19) follows using the approximation properties of $\boldsymbol{\pi}_{\mathcal{P},T}^0$, the fact that the $\mathbf{H}^1(T)$- and $\mathbf{H}^2(T)$-seminorms of $\boldsymbol{\pi}_{\mathcal{P},T}^0 \boldsymbol{v}$ vanish, and the definition (3.17) of $|\cdot|_{\mathbf{H}^{(k+1,2)}(T)}$. □

Proof of Theorem 7. Let us prove (3.21). For any $\underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$, taking $\boldsymbol{w}_T = \mathsf{G}_T^k \underline{q}_T$ in (2.11) and using Cauchy–Schwarz inequalities along with discrete inverse and trace inequalities, it is inferred, after simplification,

$$\|\mathsf{G}_T^k \underline{q}_T\|_{\mathbf{L}^2(T)} \lesssim h_T^{-1} \|q_T\|_{\mathrm{L}^2(T)} + \sum_{F\in\mathcal{F}_T} h_F^{-1/2} \|\gamma_F^{k+1} \underline{q}_F\|_{\mathrm{L}^2(F)} \lesssim h_T^{-1} |||\underline{q}|||_{\mathbf{grad},T},$$

where the conclusion follows from the estimate on $\gamma_F^{k+1} \underline{q}_F$ in (4.2) and from the definition of $|||\cdot|||_{\mathbf{grad},T}$. As a result, for any $r \in \mathrm{H}^2(T)$, making $\underline{q}_T = \underline{I}_{\mathbf{grad},T}^k r$ and invoking (4.6), we infer

$$\|\mathsf{G}_T^k (\underline{I}_{\mathbf{grad},T}^k r)\|_{\mathbf{L}^2(T)} \lesssim h_T^{-1} \|r\|_{\mathrm{L}^2(T)} + |r|_{\mathrm{H}^1(T)} + h_T |r|_{\mathrm{H}^2(T)}. \tag{4.15}$$

Letting now $q \in \mathrm{H}^{k+2}(T)$, we use the polynomial consistency [10, Eq. (3.13)] of G_T^k followed by a triangle inequality to write

$$\|\mathsf{G}_T^k\left(\underline{I}_{\mathbf{grad},T}^k q\right) - \mathbf{grad}\, q\|_{\mathbf{L}^2(T)} \le \|\mathsf{G}_T^k\left[\underline{I}_{\mathbf{grad},T}^k\left(q - \pi_{\mathcal{P},T}^{k+1} q\right)\right]\|_{\mathbf{L}^2(T)} + \|\,\mathbf{grad}\left(\pi_{\mathcal{P},T}^{k+1} q - q\right)\|_{\mathbf{L}^2(T)}$$

and conclude using (4.15) with $r = q - \pi_{\mathcal{P},T}^{k+1} q$ for the first term in the right-hand side followed by the approximation properties of $\pi_{\mathcal{P},T}^{k+1}$ (see [12, Theorem 1.45]).

To prove (3.22), we notice that $\mathsf{C}_T^k\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}\right) = \boldsymbol{P}_{\mathrm{div},T}^k\left[\underline{\boldsymbol{C}}_T^k\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}\right)\right] = \boldsymbol{P}_{\mathrm{div},T}^k\left[\underline{\boldsymbol{I}}_{\mathrm{div},T}^k\left(\mathbf{curl}\,\boldsymbol{v}\right)\right]$ owing to (4.10) along with the commutation property [10, Eq. (3.35)], and conclude using the approximation properties (3.20) with $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{v}$.

Finally, (3.23) is a straightforward consequence of the commutation property $D_T^k\left(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{w}\right) = \pi_{\mathcal{P},T}^k(\mathrm{div}\,\boldsymbol{w})$ stated in [10, Eq. (3.36)] together with [12, Theorem 1.45]. □

Remark 16 (Alternative proof of (3.21)). When $q \in \mathrm{C}^1(\overline{T})$ is such that $\mathbf{grad}\, q \in \mathbf{H}^{\max(k+1,2)}(T)$, the proof of (3.21) can be done following similar arguments as for (3.22), i.e., we write $\mathsf{G}_T^k\left(\underline{I}_{\mathbf{grad},T}^k q\right) = \boldsymbol{P}_{\mathbf{curl},T}^k\left[\underline{\boldsymbol{G}}_T^k\left(\underline{I}_{\mathbf{grad},T}^k q\right)\right] = \boldsymbol{P}_{\mathbf{curl},T}^k\left[\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\left(\mathbf{grad}\, q\right)\right]$ using (4.9) followed by [10, Eq. (3.34)], and conclude using the approximation properties (3.19) with $\boldsymbol{v} = \mathbf{grad}\, q$. This argument, however, requires additional regularity on q with respect to the one used above.

Proof of Theorem 8. We only prove (3.25), the other consistency properties being established in a similar way. Let $\boldsymbol{v} \in \mathbf{H}^{\max(k+1,2)}(T)$. By the polynomial consistency [10, Eq. (3.22)] of $\boldsymbol{\gamma}_{\mathrm{t},F}^k$ and (3.7) of $\boldsymbol{P}_{\mathbf{curl},T}^k$, it is easily checked that, for all $z_T \in \mathcal{P}^k(T)$ and all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$, it holds $\mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k z_T, \underline{\boldsymbol{w}}_T\right) = 0$. Applying this with $z_T = \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}$ we infer

$$\mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}, \underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}\right) = \mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\left(\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}\right), \underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\left(\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}\right)\right) \lesssim |||\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\left(\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}\right)|||_{\mathbf{curl},T}^2,$$

the conclusion following from the definition of $\|\cdot\|_{\mathbf{curl},T}$ and the norm equivalence (4.5). Invoking then (4.7) we infer

$$\mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}, \underline{\boldsymbol{I}}_{\mathbf{curl},T}^k \boldsymbol{v}\right)^{\frac{1}{2}} \lesssim \|\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}\|_{\mathbf{L}^2(T)} + h_T |\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}|_{\mathbf{H}^1(T)} + h_T^2 |\boldsymbol{v} - \boldsymbol{\pi}_{\mathcal{P},T}^k \boldsymbol{v}|_{\mathbf{H}^2(T)}$$

and the estimate (3.25) follows from the approximation properties of $\boldsymbol{\pi}_{\mathcal{P},T}^k$, see [12, Theorem 1.45], and the definition (3.17) of $|\cdot|_{\mathbf{H}^{(k+1,2)}(T)}$, using in the case $k = 0$ the same arguments as in the proof of Theorem 6. □

4.4 Adjoint consistency

4.4.1 Adjoint consistency for the gradient

Lemma 17 (Estimates on local H^1-seminorms of potentials)**.** *For all $F \in \mathcal{F}_h$ and all $\underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k$, it holds*

$$\|\,\mathbf{grad}\,\gamma_F^{k+1}\underline{q}_F\|_{\mathbf{L}^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E^{-1}\|\gamma_F^{k+1}\underline{q}_F - q_E\|_{\mathrm{L}^2(E)}^2 \lesssim |||\underline{\boldsymbol{G}}_F^k \underline{q}_F|||_{\mathbf{curl},F}^2. \tag{4.16}$$

For all $T \in \mathcal{T}_h$ and all $\underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$, it holds

$$\|\,\mathbf{grad}\, P_{\mathbf{grad},T}^{k+1}\underline{q}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1}\|P_{\mathbf{grad},T}^{k+1}\underline{q}_T - \gamma_F^{k+1}\underline{q}_F\|_{\mathrm{L}^2(F)}^2 \lesssim |||\underline{\boldsymbol{G}}_T^k \underline{q}_T|||_{\mathbf{curl},T}^2. \tag{4.17}$$

Proof. The proof follows arguments similar to [10, Lemma 29].

1. *Proof of* (4.16). Let $\underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$ and define $A_{q,\partial F} \in \mathbb{R}$ as the average of $q_{\mathcal{E}_F}$ over ∂F. Introducing $A_{q,\partial F} = \gamma^{k+1}_F\big(\underline{I}^k_{\mathbf{grad},F} A_{q,\partial F}\big)$ (see [10, Eq. (3.11)]), using $h_E \simeq h_F$ and $\mathrm{card}(\mathcal{E}_F) \lesssim 1$, and invoking a discrete trace inequality on $\gamma^{k+1}_F\big(\underline{q}_F - A_{q,\partial F}\big)$, we have

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|\gamma^{k+1}_F\underline{q}_F - q_E\|^2_{\mathrm{L}^2(E)} \lesssim \sum_{E\in\mathcal{E}_F} h_E^{-1}\|q_E - A_{q,\partial F}\|^2_{\mathrm{L}^2(E)} + h_F^{-2}\|\gamma^{k+1}_F\big(\underline{q}_F - A_{q,\partial F}\big)\|^2_{\mathrm{L}^2(F)}. \quad (4.18)$$

Since $q_{\mathcal{E}_F}$ is continuous, recalling that $q_E = (q_{\mathcal{E}_h})_{|E}$ for all $E \in \mathcal{E}_F$ and using a Poincaré–Wirtinger inequality along ∂F followed by the definition (4.1) of $|||\cdot|||_{\mathbf{curl},F}$ yields

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|q_E - A_{q,\partial F}\|^2_{\mathrm{L}^2(E)} \lesssim h_F \sum_{E\in\mathcal{E}_F} \|G^k_E q_E\|^2_{\mathrm{L}^2(E)} \lesssim |||\underline{\boldsymbol{G}}^k_F\underline{q}_F|||^2_{\mathbf{curl},F}. \quad (4.19)$$

We now turn to the second term in (4.18). Select $\boldsymbol{v}_F \in \mathcal{R}^{\mathrm{c},k+2}(F)$ such that $\operatorname{div}\boldsymbol{v}_F = \gamma^{k+1}_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F} A_{q,\partial F}\big)$. By the L^2-estimate on $\boldsymbol{v}_F$ coming from [10, Lemma 31], the discrete trace inequality of [12, Lemma 1.32], and the consistency property [10, Eq. (3.10)] of $\mathbf{G}^k_F$, we have

$$\|\boldsymbol{v}_F\|_{\mathrm{L}^2(F)} + \left(\sum_{E\in\mathcal{E}_F} h_E\|\boldsymbol{v}_F\|^2_{\mathrm{L}^2(E)}\right)^{\frac12} \lesssim h_F\|\gamma^{k+1}_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F}\big)\|_{\mathrm{L}^2(F)},$$
$$\mathbf{G}^k_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F}\big) = \mathbf{G}^k_F\underline{q}_F.$$

Hence, applying the definition (2.10) of γ^{k+1}_F to $\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F} \in \underline{X}^k_{\mathbf{grad},F}$, taking $\boldsymbol{v}_F$ above as a test function, and using Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}\|\gamma^{k+1}_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F}\big)\|^2_{\mathrm{L}^2(F)} &\lesssim h_F\|\mathbf{G}^k_F\underline{q}_F\|_{\mathbf{L}^2(F)}\|\gamma^{k+1}_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F}\big)\|_{\mathrm{L}^2(F)}\\ &\quad + \left(\sum_{E\in\mathcal{E}_F} h_E^{-1}\|q_E - A_{q,\partial F}\|^2_{\mathrm{L}^2(E)}\right)^{\frac12} h_F\|\gamma^{k+1}_F\big(\underline{q}_F - \underline{I}^k_{\mathbf{grad},F}A_{q,\partial F}\big)\|_{\mathrm{L}^2(F)}.\end{aligned}$$

Simplifying and recalling (4.12) and (4.19), we infer $\|\gamma^{k+1}_F\big(\underline{q}_F - A_{q,\partial F}\big)\|_{\mathrm{L}^2(F)} \lesssim h_F|||\underline{\boldsymbol{G}}^k_F\underline{q}_F|||_{\mathbf{curl},F}$ which, plugged together with (4.19) into (4.18), gives the following estimate on the second term in the left-hand side of (4.16):

$$\sum_{E\in\mathcal{E}_F} h_E^{-1}\|\gamma^{k+1}_F\underline{q}_F - q_E\|^2_{\mathrm{L}^2(E)} \lesssim |||\underline{\boldsymbol{G}}^k_F\underline{q}_F|||^2_{\mathbf{curl},F}. \quad (4.20)$$

Integrating by parts the definition (2.10) of γ^{k+1}_F applied to $\boldsymbol{v}_F \in \boldsymbol{\mathcal{P}}^k(F)$ (see Remark 1), we have

$$\int_F \mathbf{grad}_F\,\gamma^{k+1}_F\underline{q}_F\cdot\boldsymbol{v}_F = \int_F \mathbf{G}^k_F\underline{q}_F\cdot\boldsymbol{v}_F + \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E(\gamma^{k+1}_F\underline{q}_F - q_{\mathcal{E}_F})(\boldsymbol{v}_F\cdot\boldsymbol{n}_{FE}).$$

Making $\boldsymbol{v}_F = \mathbf{grad}_F\,\gamma^{k+1}_F\underline{q}_F$, using Cauchy–Schwarz inequalities, (4.12), a discrete trace inequality, and (4.20) then yields the bound on the first term in the left-hand side of (4.16).

2. *Proof of* (4.17). The ideas are similar to those used to prove (4.16), provided we can establish a Poincaré–Writinger inequality for face potentials (which is not straightforward given their discontinuity). Let

$$A_{q,\partial T} \coloneqq \frac{1}{|\partial T|}\sum_{F\in\mathcal{F}_T}|F|A_{q,F} \quad\text{with}\quad A_{q,F} \coloneqq \frac{1}{|F|}\int_F \gamma^{k+1}_F\underline{q}_F$$

denote the average over ∂T of the piecewise polynomial function defined by $(\gamma_F^{k+1}\underline{q}_F)_{F\in\mathcal{F}_T}$. We write, using triangle and Cauchy–Schwarz inequalities,

$$\sum_{F\in\mathcal{F}_T} h_F^{-1}\|P^{k+1}_{\mathbf{grad},T}\underline{q}_T - \gamma_F^{k+1}\underline{q}_F\|^2_{\mathrm{L}^2(F)} \lesssim \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\gamma_F^{k+1}\underline{q}_F - A_{q,F}\|^2_{\mathrm{L}^2(F)} + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|A_{q,F} - A_{q,\partial T}\|^2_{\mathrm{L}^2(F)} + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|P^{k+1}_{\mathbf{grad},T}\underline{q}_T - A_{q,\partial T}\|^2_{\mathrm{L}^2(F)} =: \mathfrak{T}_1+\mathfrak{T}_2+\mathfrak{T}_3. \quad (4.21)$$

The first term is estimated using a Poincaré–Wirtinger inequality on $\gamma_F^{k+1}\underline{q}_F$ and invoking (4.16) together with the definition (4.1) of $|||\cdot|||_{\mathbf{curl},T}$ to get

$$\mathfrak{T}_1 \lesssim \sum_{F\in\mathcal{F}_T} h_F^{-1}\left(h_F\|\mathbf{grad}_F\,\gamma_F^{k+1}\underline{q}_F\|_{\mathbf{L}^2(F)}\right)^2 \lesssim \sum_{F\in\mathcal{F}_T} h_F\,|||\underline{\boldsymbol{G}}_F^k\underline{q}_F|||^2_{\mathbf{curl},F} \lesssim |||\underline{\boldsymbol{G}}_T^k\underline{q}_T|||^2_{\mathbf{curl},T}. \quad (4.22)$$

For the second term in (4.21), we follow the same steps as in [10, Lemma 29], working from face to face through common edges and using (4.16) to get $\mathfrak{T}_2 \lesssim |||\underline{\boldsymbol{G}}_T^k\underline{q}_T|||^2_{\mathbf{curl},T}$. Finally, for $\mathfrak{T}_3$, we apply the definition (3.1) of $P^{k+1}_{\mathbf{grad},T}\left(\underline{q}_T - \underline{I}^k_{\mathbf{grad},T}A_{q,\partial T}\right)$ with $\boldsymbol{v}_T \in \mathcal{R}^{\mathrm{c},k+2}(T)$ such that $\operatorname{div}\boldsymbol{v}_T = P^{k+1}_{\mathbf{grad},T}(\underline{q}_T - \underline{I}^k_{\mathbf{grad},T}A_{q,\partial T})$ and $\|\boldsymbol{v}_T\|_{\mathbf{L}^2(T)} \lesssim h_T\|P^{k+1}_{\mathbf{grad},T}(\underline{q}_T - \underline{I}^k_{\mathbf{grad},T}A_{q,\partial T})\|_{\mathrm{L}^2(T)}$, see [10, Lemma 31]. Using the consistency properties (3.2) of $P^{k+1}_{\mathbf{grad},T}$, [10, Eq. (3.13)] of G_T^k and [10, Eq. (3.11)] of γ_F^{k+1}, and a discrete trace inequality, this gives

$$\begin{aligned}\|P^{k+1}_{\mathbf{grad},T}\underline{q}_T - A_{q,\partial T}\|_{\mathrm{L}^2(T)} &\lesssim h_T\|\mathsf{G}_T^k\underline{q}_T\|_{\mathbf{L}^2(T)} + h_T\sum_{F\in\mathcal{F}_T} h_F^{-\frac12}\|\gamma_F^{k+1}\underline{q}_F - A_{q,\partial T}\|_{\mathrm{L}^2(F)}\\ &\lesssim h_T\,|||\underline{\boldsymbol{G}}_T^k\underline{q}_T|||_{\mathbf{curl},T} + h_T\left(\mathfrak{T}_1^{\frac12}+\mathfrak{T}_2^{\frac12}\right), \end{aligned} \quad (4.23)$$

where the second line follows from (4.13), and a triangle inequality to write

$$\sum_{F\in\mathcal{F}_T} h_F^{-\frac12}\|\gamma_F^{k+1}\underline{q}_F - A_{q,\partial T}\|_{\mathrm{L}^2(F)} \le \sum_{F\in\mathcal{F}_T} h_F^{-\frac12}\|\gamma_F^{k+1}\underline{q}_F - A_{q,F}\|_{\mathrm{L}^2(F)} + \sum_{F\in\mathcal{F}_T} h_F^{-\frac12}\|A_{q,F} - A_{q,\partial T}\|_{\mathrm{L}^2(F)}.$$

Using discrete trace inequalities and the previous estimates on $\mathfrak{T}_1$ and $\mathfrak{T}_2$, (4.23) leads to

$$\mathfrak{T}_3 \lesssim h_T^{-2}\|P^{k+1}_{\mathbf{grad},T}\underline{q}_T - A_{q,\partial T}\|^2_{\mathrm{L}^2(T)} \lesssim |||\underline{\boldsymbol{G}}_T^k\underline{q}_T|||^2_{\mathbf{curl},T}$$

Plugging this bound together with the estimates on $\mathfrak{T}_1$ and $\mathfrak{T}_2$ into (4.21) concludes the proof of the bound on the second term in the right-hand side of (4.17) To bound the first term in the left-hand side of (4.17), we proceed as for $\mathbf{grad}_F\,\gamma_F^{k+1}\underline{q}_F$ in Step 1, using an integration by parts in the definition (3.1) of $P^{k+1}_{\mathbf{grad},T}\underline{q}_T$ and selecting the test function $\boldsymbol{v}_T = \mathbf{grad}\,P^{k+1}_{\mathbf{grad},T}\underline{q}_T$ (see Remark 3). □

Proof of Theorem 9. It holds, by definition (3.13b) of the local discrete L^2-product in $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and (4.9),

$$\begin{aligned}&\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v},\underline{q}_h)\\ &\quad= \sum_{T\in\mathcal{T}_h}\left[\int_T \boldsymbol{P}^k_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}\right)\cdot\mathsf{G}_T^k\underline{q}_T + \mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}_{|T},\underline{\boldsymbol{G}}_T^k\underline{q}_T\right) + \int_T \operatorname{div}\boldsymbol{v}\;P^{k+1}_{\mathbf{grad},T}\underline{q}_T\right]. \end{aligned} \quad (4.24)$$

Using Remark 3, we have, for all $\boldsymbol{w}_T \in \boldsymbol{\mathcal{P}}^k(T)$,

$$\int_T P^{k+1}_{\mathbf{grad},T}\underline{q}_T\;\operatorname{div}\boldsymbol{w}_T + \int_T \mathsf{G}_T^k\underline{q}_T\cdot\boldsymbol{w}_T - \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \gamma_F^{k+1}\underline{q}_F(\boldsymbol{w}_T\cdot\boldsymbol{n}_F) = 0.$$

Subtracting this quantity from (4.24), we obtain

$$\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v},\underline{q}_h) = \sum_{T\in\mathcal{T}_h}\left[\int_T\left(\boldsymbol{P}^k_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}) - \boldsymbol{w}_T\right)\cdot\mathbf{G}^k_T\underline{q}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}_{|T},\underline{\boldsymbol{G}}^k_T\underline{q}_T)\right]$$
$$+\sum_{T\in\mathcal{T}_h}\left[\int_T \mathrm{div}(\boldsymbol{v}-\boldsymbol{w}_T)P^{k+1}_{\mathbf{grad},T}\underline{q}_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F(\boldsymbol{w}_T-\boldsymbol{v})\cdot\boldsymbol{n}_F\gamma^{k+1}_F\underline{q}_F\right],$$

where $\boldsymbol{v}$ is introduced in the boundary term by single-valuedness of the discrete trace, and using $\boldsymbol{v}_{|F}\cdot\boldsymbol{n}_F = 0$ whenever $F\in\mathcal{F}_h^{\mathrm{b}}$. Integrating by parts the third term in the right-hand side of the above expression, we obtain

$$\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v},\underline{q}_h) = \sum_{T\in\mathcal{T}_h}\left[\int_T\left(\boldsymbol{P}^k_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}) - \boldsymbol{w}_T\right)\cdot\mathbf{G}^k_T\underline{q}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}_{|T},\underline{\boldsymbol{G}}^k_T\underline{q}_T)\right]$$
$$+\sum_{T\in\mathcal{T}_h}\left[-\int_T(\boldsymbol{v}-\boldsymbol{w}_T)\cdot\mathbf{grad}\,P^{k+1}_{\mathbf{grad},T}\underline{q}_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F(\boldsymbol{w}_T-\boldsymbol{v})\cdot\boldsymbol{n}_F(\gamma^{k+1}_F\underline{q}_F - P^{k+1}_{\mathbf{grad},T}\underline{q}_T)\right]. \tag{4.25}$$

We set $\boldsymbol{w}_T = \boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{v}$ and use (3.19) and the approximation properties of $\boldsymbol{\pi}^k_{\mathcal{P},T}$ stated in [12, Theorem 1.45] to see that

$$\|\boldsymbol{P}^k_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v}) - \boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{v}\|_{\mathbf{L}^2(T)} + \|\boldsymbol{v}-\boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{v}\|_{\mathbf{L}^2(T)} + \sum_{F\in\mathcal{F}_T}h_F^{\frac12}\|\boldsymbol{v}-\boldsymbol{\pi}^k_{\mathcal{P},T}\boldsymbol{v}\|_{\mathbf{L}^2(F)} \lesssim h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)}.$$

Using Cauchy–Schwarz inequalities on the integrals and on the stabilisation bilinear form in (4.25), the bound (4.13) together with the norm equivalence (4.5), and the consistency property (3.25) of the stabilisation term, we arrive at

$$\left|\tilde{\mathcal{E}}_{\mathrm{div},h}(\boldsymbol{v},\underline{q}_h)\right| \le \sum_{T\in\mathcal{T}_h}h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)}\|\underline{\boldsymbol{G}}^k_T\underline{q}_T\|_{\mathbf{curl},T} + \sum_{T\in\mathcal{T}_h}h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)}\|\mathbf{grad}\,P^{k+1}_{\mathbf{grad},T}\underline{q}_T\|_{\mathbf{L}^2(T)}$$
$$+\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}h_T^{k+1}|\boldsymbol{v}|_{\mathbf{H}^{(k+1,2)}(T)}\,h_F^{-\frac12}\|\gamma^{k+1}_F\underline{q}_F - P^{k+1}_{\mathbf{grad},T}\underline{q}_T\|_{\mathrm{L}^2(F)}.$$

The conclusion follows from the estimate (4.17), and Cauchy–Schwarz inequalities on the sums. □

4.4.2 Adjoint consistency for the curl

The proof of the adjoint consistency for the curl hinges on liftings defined as solutions of local problems. For any $F\in\mathcal{F}_h$, the *face lifting* $\boldsymbol{R}_{\mathbf{curl},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}\to\mathbf{H}(\mathrm{rot};F)\cap\mathbf{H}(\mathrm{div};F)$ is such that, for all $\underline{\boldsymbol{v}}_F\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$, $\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F = \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F} + \mathbf{grad}_F\,\psi_{\underline{\boldsymbol{v}}_F}$ with $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}\in\mathbf{H}(\mathrm{rot};F)\cap\mathbf{H}(\mathrm{div};F)$ such that

$$\mathrm{rot}_F\,\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F} = C^k_F\underline{\boldsymbol{v}}_F \qquad \text{in } F, \tag{4.26a}$$
$$\mathrm{div}_F\,\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F} = 0 \qquad \text{in } F, \tag{4.26b}$$
$$\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}\cdot\boldsymbol{t}_E = v_E \qquad \text{on all } E\in\mathcal{E}_F, \tag{4.26c}$$

while $\psi_{\underline{\boldsymbol{v}}_F}\in C^\infty_{\mathrm{c}}(F)$ is such that

$$-\int_F\psi_{\underline{\boldsymbol{v}}_F}\,\mathrm{div}_F\,z_F = \int_F(\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F - \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F})\cdot z_F \qquad \forall z_F\in\mathcal{R}^{\mathrm{c},k+1}(F). \tag{4.27}$$

Let now $T \in \mathcal{T}_h$. The *curl correction* $\boldsymbol{\delta}_T : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathbf{H}(\mathbf{curl};T) \cap \mathbf{H}(\mathrm{div};T)$ is such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$,

$$\begin{aligned}
\mathrm{div}\, \boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T &= -\mathrm{div}\, \mathbf{C}^k_T \underline{\boldsymbol{v}}_T && \text{in } T, && (4.28a)\\
\mathbf{curl}\, \boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T &= 0 && \text{in } T, && (4.28b)\\
\boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_F &= C^k_F \underline{\boldsymbol{v}}_F - \mathbf{C}^k_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_F && \text{on all } F \in \mathcal{F}_T. && (4.28c)
\end{aligned}$$

The curl correction lifts the difference between the face curl $C^k_F \underline{\boldsymbol{v}}_F$ and the normal component of the element curl $\mathbf{C}^k_T \underline{\boldsymbol{v}}_T$ as a function defined over T. Its role is to ensure the well-posedness of the problem defining the *element lifting* $\boldsymbol{R}_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \mathbf{H}(\mathbf{curl};T) \cap \mathbf{H}(\mathrm{div};T)$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$,

$$\begin{aligned}
\mathbf{curl}\, \boldsymbol{R}_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T &= \mathbf{C}^k_T \underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T && \text{in } T, && (4.29a)\\
\mathrm{div}\, \boldsymbol{R}_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T &= 0 && \text{in } T, && (4.29b)\\
(\boldsymbol{R}_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T)_{\mathrm{t},F} &= \boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F && \text{on all } F \in \mathcal{F}_T. && (4.29c)
\end{aligned}$$

In Appendix A we prove that these lifting operators are well-defined, and that they satisfy the following two key properties:

- *Orthogonality of the face lifting:* For all $F \in \mathcal{F}_h$,

$$\int_F (\boldsymbol{\gamma}^k_{\mathrm{t},F} \underline{\boldsymbol{v}}_F - \boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F) \cdot \boldsymbol{z}_F = 0 \qquad \forall (\underline{\boldsymbol{v}}_F, \boldsymbol{z}_F) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \times \boldsymbol{\mathcal{RT}}^{k+1}(F); \tag{4.30}$$

- *Boundedness of the element lifting:* For all $T \in \mathcal{T}_h$,

$$\|\boldsymbol{R}_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} + \|\mathbf{curl}\, \boldsymbol{R}_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} \lesssim \|\underline{\boldsymbol{v}}_T\|_{\mathbf{curl},T} + \|\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T\|_{\mathrm{div},T} \quad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}. \tag{4.31}$$

Lemma 18 (Approximation properties of $\boldsymbol{\mathcal{N}}^{k+1}(T)$ on polyhedral elements)**.** *For all $T \in \mathcal{T}_h$ and all $\boldsymbol{w} \in \mathbf{H}^{k+2}(T)$, there exists $\boldsymbol{z}_T \in \boldsymbol{\mathcal{N}}^{k+1}(T)$ such that*

$$\begin{aligned}
\|\boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(T)} &\lesssim h_T^{k+1} \big(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}\big), && (4.32)\\
\|\mathbf{curl}\, \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{z}_T\|_{\mathbf{L}^2(T)} &\lesssim h_T^{k+1} |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}. && (4.33)
\end{aligned}$$

Proof. By the mesh regularity assumption, there is a simplex $S \subset T$ whose inradius is $\gtrsim h_T$. Following the arguments in the proof of [12, Lemma 1.25], we infer the norm equivalence

$$\|q\|_{\mathrm{L}^2(S)} \simeq \|q\|_{\mathrm{L}^2(T)} \qquad \forall q \in \mathcal{P}^{k+1}(T). \tag{4.34}$$

Let us take $\boldsymbol{z}_T$ as the Nédélec interpolant in $\boldsymbol{\mathcal{N}}^{k+1}(S)$ of $\boldsymbol{w}$; $\boldsymbol{z}_T$ can be uniquely extended as an element of $\boldsymbol{\mathcal{N}}^{k+1}(T)$. By the arguments in the proof of [16, Theorem 3.14 and Corollary 3.17], and since $S \subset T$, it holds

$$\begin{aligned}
\|\boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(S)} &\lesssim h_T^{k+1} \big(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}\big),\\
\|\mathbf{curl}\, \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{z}_T\|_{\mathbf{L}^2(S)} &\lesssim h_T^{k+1} |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}.
\end{aligned} \tag{4.35}$$

We then write, introducing $\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T} \boldsymbol{w}$ and using triangle inequalities,

$$\begin{aligned}
\|\boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(T)} &\lesssim \|\boldsymbol{w} - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T} \boldsymbol{w}\|_{\mathbf{L}^2(T)} + \|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T} \boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(T)}\\
&\lesssim h_T^{k+1} |\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + \|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T} \boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(S)}\\
&\lesssim h_T^{k+1} (|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}),
\end{aligned}$$

where we have used the approximation property of $\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}$ together with the norm equivalence (4.34) in the second equality, and concluded by introducing $\boldsymbol{w}$ and invoking (4.35) to write

$$\begin{aligned}\|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(S)} &\le \|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w} - \boldsymbol{w}\|_{\mathbf{L}^2(S)} + \|\boldsymbol{w} - \boldsymbol{z}_T\|_{\mathbf{L}^2(S)}\\ &\lesssim h_T^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + h_T^{k+1}(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(T)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}).\end{aligned}$$

This concludes the proof of (4.32). The proof of (4.33) is done in a similar way, introducing $\mathbf{curl}(\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w})$ and using the approximation property $\|\mathbf{curl}\,\boldsymbol{w} - \mathbf{curl}(\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w})\|_{\mathbf{L}^2(T)} \lesssim h_T^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+2}(T)}$. □

Proof of Theorem 10. For all $T \in \mathcal{T}_h$, select $\boldsymbol{z}_T \in \boldsymbol{\mathcal{N}}^{k+1}(T)$ given by Lemma 18. Using (3.13c) to expand $(\cdot,\cdot)_{\mathrm{div},h}$ together with (4.10), and recalling (3.6), we see that it holds, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathbf{curl},h}(\boldsymbol{w},\underline{\boldsymbol{v}}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \big(\boldsymbol{P}^k_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_{|T}) - \boldsymbol{z}_T\big)\cdot\mathbf{C}^k_T\underline{\boldsymbol{v}}_T + \sum_{T\in\mathcal{T}_h}\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_{|T},\underline{\boldsymbol{C}}^k_T\underline{\boldsymbol{v}}_T)\\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}(\boldsymbol{z}_T - \boldsymbol{w})\cdot\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F(\boldsymbol{z}_T\times\boldsymbol{n}_F)\cdot\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_F \qquad (4.36)\\ &=: \mathfrak{T}_1+\mathfrak{T}_2+\mathfrak{T}_3+\mathfrak{T}_4.\end{aligned}$$

Using Cauchy–Schwarz and triangle inequalities, it is readily inferred for the first term

$$\begin{aligned}|\mathfrak{T}_1| &\lesssim \left[\sum_{T\in\mathcal{T}_h}\Big(\|\boldsymbol{P}^k_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}) - \boldsymbol{w}\|^2_{\mathbf{L}^2(T)} + \|\boldsymbol{w}-\boldsymbol{z}_T\|^2_{\mathbf{L}^2(T)}\Big)\right]^{\frac12}\left(\sum_{T\in\mathcal{T}_h}\|\mathbf{C}^k_T\underline{\boldsymbol{v}}_T\|^2_{\mathbf{L}^2(T)}\right)^{\frac12}\\ &\lesssim h^{k+1}\Big(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\Big)\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}, \qquad (4.37)\end{aligned}$$

where the conclusion follows using the approximation properties (3.20) and (4.32) to bound the first factor, and (4.14) along with the norm equivalence (4.5) to bound the second.

For $\mathfrak{T}_2$, combining the consistency property (3.26) of $\mathrm{s}_{\mathrm{div},T}$ with discrete Cauchy–Schwarz inequalities and the definition of the $\|\cdot\|_{\mathrm{div},h}$-norm readily gives

$$|\mathfrak{T}_2| \lesssim h^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)}\|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}. \qquad (4.38)$$

For $\mathfrak{T}_3$, Cauchy–Schwarz inequalities, the approximation property (4.35), and the definition of the norm $\|\cdot\|_{\mathbf{curl},h}$ yield

$$|\mathfrak{T}_3| \le \left(\sum_{T\in\mathcal{T}_h}\|\mathbf{curl}(\boldsymbol{z}_T-\boldsymbol{w})\|^2_{\mathbf{L}^2(T)}\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\|^2_{\mathbf{L}^2(T)}\right)^{\frac12} \lesssim h^{k+1}|\boldsymbol{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h}. \qquad (4.39)$$

Let us now consider the last term in the right-hand side of (4.36). Since $(\boldsymbol{z}_T)_{|F}\times\boldsymbol{n}_F \in \boldsymbol{\mathcal{RT}}^{k+1}(F)$ (see [10, Proposition 30]), by (4.30) we can replace $\boldsymbol{\gamma}^k_{\mathrm{t},F}\underline{\boldsymbol{v}}_T$ by $\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F$ in the boundary integral. Using the fact that both $\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F$ and the (rotated) tangential component of $\boldsymbol{w}$ are continuous across interfaces, along with the fact that $\omega_{T_1F}+\omega_{T_2F}=0$ for all $F\in\mathcal{F}_h$ between two elements T_1,T_2, and $\boldsymbol{w}_{|F}\times\boldsymbol{n}_F = \mathbf{0}$ for all $F\in\mathcal{F}_h^{\mathrm{b}}$, we then have

$$\begin{aligned}\mathfrak{T}_4 &= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F(\boldsymbol{z}_T-\boldsymbol{w})\times\boldsymbol{n}_F\cdot\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F\\ &= \sum_{T\in\mathcal{T}_h}\left(\int_T(\boldsymbol{z}_T-\boldsymbol{w})\cdot\mathbf{curl}\,\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T - \int_T\mathbf{curl}(\boldsymbol{z}_T-\boldsymbol{w})\cdot\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\right),\end{aligned}$$

where the conclusion follows recalling that, by definition (4.29), $\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F = (\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T)_{\mathrm{t},F}$ for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, and by integrating by parts. Using Cauchy–Schwarz inequalities, it is inferred

$$|\mathfrak{T}_4| \le \left[\sum_{T\in\mathcal{T}_h}\left(\|\boldsymbol{z}_T - \boldsymbol{w}\|_{\mathbf{L}^2(T)}^2 + \|\mathbf{curl}(\boldsymbol{z}_T - \boldsymbol{w})\|_{\mathbf{L}^2(T)}^2\right)\right]^{\frac12} \times \left[\sum_{T\in\mathcal{T}_h}\left(\|\mathbf{curl}\,\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)}^2 + \|\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)}^2\right)\right]^{\frac12}.$$

The approximation properties (4.35) of $\boldsymbol{z}_T$ along with the boundedness (4.31) of $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T$ yield

$$|\mathfrak{T}_4| \lesssim h^{k+1}\left(|\boldsymbol{w}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{w}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right)\left(\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\right). \tag{4.40}$$

Plugging (4.37)–(4.40) into (4.36), (3.29) follows. □

4.4.3 Adjoint consistency for the divergence

Proof of Theorem 11. Combining the definition (3.30) of the adjoint consistency error for the divergence with (3.10) summed over $T \in \mathcal{T}_h$, we infer that it holds, for all $(q, \underline{\boldsymbol{v}}_h)$ as in the theorem and all $q_h \in \mathcal{P}^{k+1}(\mathcal{T}_h)$ with $q_T \coloneqq (q_h)_{|T}$ for all $T \in \mathcal{T}_h$,

$$\tilde{\mathcal{E}}_{\mathbf{grad},h}(q,\underline{\boldsymbol{v}}_h) = \sum_{T\in\mathcal{T}_h}\left[\int_T (\cancel{\pi_{\mathcal{P},T}^k} q - q_T) D_T^k\underline{\boldsymbol{v}}_T + \int_T \mathbf{grad}(q - q_T)\cdot\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F (q_T - q)v_F\right],$$

where the cancellation of $\pi_{\mathcal{P},T}^k$ is justified by its definition along with $D_T^k\underline{\boldsymbol{v}}_T \in \mathcal{P}^k(T)$, while the insertion of q into the boundary integral is possible thanks to its single-valuedness at interfaces along with the fact that it vanishes on $\partial\Omega$. Taking absolute values and using Cauchy–Schwarz inequalities in the right-hand side along with $h_F \simeq h_T$ for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, we infer

$$\begin{aligned}\left|\tilde{\mathcal{E}}_{\mathbf{grad},h}(q,\underline{\boldsymbol{v}}_h)\right| \lesssim{}& \left[\sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|q - q_T\|_{\mathrm{L}^2(T)}^2 + \|\mathbf{grad}(q - q_T)\|_{\mathbf{L}^2(T)}^2 + h_T^{-1}\|q_T - q\|_{\partial T}^2\right)\right]^{\frac12}\\ &\times\left[\sum_{T\in\mathcal{T}_h}\left(h_T^2\|D_T^k\underline{\boldsymbol{v}}_T\|_{\mathrm{L}^2(T)}^2 + \|\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F\in\mathcal{F}_T}h_F\|v_F\|_{\mathrm{L}^2(F)}^2\right)\right]^{\frac12}.\end{aligned} \tag{4.41}$$

Taking q_h such that $q_T = \pi_{\mathcal{P},T}^{k+1}q_{|T}$ for all $T \in \mathcal{T}_h$ and using the approximation properties of the L^2-orthogonal projector [12, Theorem 1.45], it is inferred that the first factor in the right-hand side of (4.41) is $\lesssim h^{k+1}|q|_{\mathrm{H}^{k+2}(\mathcal{T}_h)}$. Moving to the second factor, we use, for all $T \in \mathcal{T}_h$, [14, Lemma 8] followed by the local seminorm equivalence (4.5) to write $h_T\|D_T^k\underline{\boldsymbol{v}}_T\|_{\mathrm{L}^2(T)} \lesssim |||\underline{\boldsymbol{v}}_T|||_{\mathrm{div},T} \lesssim \|\underline{\boldsymbol{v}}_T\|_{\mathrm{div},T}$. The same norm equivalence and the definition of the $\|\cdot\|_{\mathrm{div},T}$-norm also yields $\|\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F\in\mathcal{F}_T}h_F\|v_F\|_{\mathrm{L}^2(F)}^2 \lesssim \|\underline{\boldsymbol{v}}_T\|_{\mathrm{div},T}$. The second factor in the right-hand side of (4.41) is therefore $\lesssim \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}$, and the proof is complete. □

5 Convergence analysis for a DDR discretisation of magnetostatics

We analyse in this section the DDR approximation of the following magnetostatics model, in which the unknowns are the magnetic field $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl};\Omega)$ and the vector potential $\boldsymbol{A} \in \mathbf{H}(\mathrm{div};\Omega)$:

$$\begin{aligned}&\mu\boldsymbol{H} - \mathbf{curl}\,\boldsymbol{A} = \boldsymbol{0}, \quad \mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}, \quad \mathrm{div}\,\boldsymbol{A} = 0 && \text{in } \Omega,\\ &\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{0} && \text{on } \partial\Omega.\end{aligned} \tag{5.1}$$

The free current $\boldsymbol{J}$ belongs to $\mathbf{curl}\,\mathbf{H}(\mathbf{curl};\Omega)$ and we assume, for the sake of simplicity, that the magnetic permeability μ is piecewise-constant on the considered meshes, with $\mu \in [\mu_-, \mu_+]$ for some constant numbers $0 < \mu_- \le \mu_+$.

5.1 Scheme

As shown in [11], a scheme based on the discrete de Rham tools can be written by replacing, in the weak formulation of (5.1), the continuous L^2-products by discrete ones built on the local products. Denote by μ_T the constant value of μ over $T \in \mathcal{T}_h$ and define the bilinear forms $a_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \to \mathbb{R}$, $b_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathbb{R}$, and $c_h : \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathbb{R}$ as follows: For all $\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{\zeta}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{\nu}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$,

$$\mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{\zeta}}_h) \coloneqq \sum_{T\in\mathcal{T}_h} \mu_T\, (\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{\zeta}}_T)_{\mathbf{curl},T}, \quad \mathrm{b}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h) \coloneqq (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h)_{\mathrm{div},h},$$

$$\mathrm{c}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{\nu}}_h) \coloneqq \int_\Omega D^k_h \underline{\boldsymbol{w}}_h\, D^k_h \underline{\boldsymbol{\nu}}_h.$$

The discrete problem then reads: Find $\underline{\boldsymbol{H}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{A}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ such that

$$\begin{aligned}
&\mathrm{a}_h(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\zeta}}_h) - \mathrm{b}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{A}}_h) = 0 && \forall \underline{\boldsymbol{\zeta}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
&\mathrm{b}_h(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{\nu}}_h) + \mathrm{c}_h(\underline{\boldsymbol{A}}_h, \underline{\boldsymbol{\nu}}_h) = \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{J}\cdot \boldsymbol{P}^k_{\mathrm{div},T} \underline{\boldsymbol{\nu}}_T && \forall \underline{\boldsymbol{\nu}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}.
\end{aligned} \tag{5.2}$$

The equations of this problem can be recast in the standard variational form $\mathcal{A}_h((\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h), (\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h)) = \mathcal{L}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h)$, where $\mathcal{A}_h : (\underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h})^2 \to \mathbb{R}$ and $\mathcal{L}_h : \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \to \mathbb{R}$ are the bilinear and linear forms, respectively, such that

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h), (\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h)) \coloneqq \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{\zeta}}_h) - \mathrm{b}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{w}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{\nu}}_h) + \mathrm{c}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{\nu}}_h),$$

$$\mathcal{L}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\nu}}_h) \coloneqq \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{J}\cdot \boldsymbol{P}^k_{\mathrm{div},T} \underline{\boldsymbol{\nu}}_T.$$

5.2 Error estimate

We establish an error estimate using the stability results of the companion paper [10] and the consistency results presented in Section 3. To measure the error, we introduce the following $\mathbf{H}(\mathbf{curl};\Omega)$- and $\mathbf{H}(\mathrm{div};\Omega)$-like norms on $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$, respectively:

$$\begin{aligned}
\|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h} &\coloneqq \left(\mathrm{a}_h(\underline{\boldsymbol{\zeta}}_h, \underline{\boldsymbol{\zeta}}_h) + \|\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\zeta}}_h\|^2_{\mathrm{div},h}\right)^{\frac12} && \forall \underline{\boldsymbol{\zeta}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},\\
\|\underline{\boldsymbol{\nu}}_h\|_{\mathrm{div},1,h} &\coloneqq \left(\|\underline{\boldsymbol{\nu}}_h\|^2_{\mathrm{div},h} + \|D^k_h \underline{\boldsymbol{\nu}}_h\|^2_{\mathrm{L}^2(\Omega)}\right)^{\frac12} && \forall \underline{\boldsymbol{\nu}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}.
\end{aligned}$$

Theorem 19 (Error estimate for the magnetostatics problem)**.** *Assume that both the first and second Betti numbers of Ω are zero (i.e., Ω is not crossed by any tunnel and does not enclose any void). Then, there exists a unique solution $(\underline{\boldsymbol{H}}_h, \underline{\boldsymbol{A}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ to* (5.2)*. Moreover, letting $(\boldsymbol{H}, \boldsymbol{A}) \in \mathbf{H}(\mathbf{curl};\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ be the weak solution to* (5.1) *and assuming that $\boldsymbol{H} \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{H}^{k+2}(\mathcal{T}_h)$ and $\boldsymbol{A} \in \mathbf{C}^0(\overline{\Omega}) \times \mathbf{H}^{k+2}(\mathcal{T}_h)$, we have*

$$\begin{aligned}
&\|\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{H}\|_{\mu,\mathbf{curl},1,h} + \|\underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{A}\|_{\mathrm{div},1,h}\\
&\qquad \lesssim h^{k+1} \left(|\,\mathbf{curl}\,\boldsymbol{H}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{H}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right),
\end{aligned} \tag{5.3}$$

where the hidden constant in $\lesssim$ only depends on Ω, k, the mesh regularity parameter, and μ_-, μ_+.

Proof. As shown in the proof of [11, Theorem 10], the exactness of the rightmost part of the sequence (2.16), which holds owing to [10, Eqs. (3.44) and (3.41)], and the Poincaré inequalities for $\underline{\boldsymbol{C}}_h^k$ and D_h^k (see [10, Theorems 26 and 27]) enable a reproduction of the arguments of the continuous inf-sup condition (see, e.g., [13, Section 2] or [1, Theorem 4.9]) to see that $\mathcal{A}_h$ satisfies a uniform inf-sup condition with respect to the norm on $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$ induced by $\|\cdot\|_{\mu,\mathbf{curl},1,h}$ and $\|\cdot\|_{\mathrm{div},1,h}$.

Using the Third Strang Lemma [9], we therefore obtain (5.3) provided we can prove that the consistency error

$$\mathcal{E}_h((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) \coloneqq \mathcal{L}_h(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h) - \mathcal{A}_h((\underline{\boldsymbol{I}}_{\mathbf{curl},h}^k\boldsymbol{H},\underline{\boldsymbol{I}}_{\mathrm{div},h}^k\boldsymbol{A}),(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h))$$

satisfies, for all $(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k \times \underline{\boldsymbol{X}}_{\mathrm{div},h}^k$,

$$\begin{aligned}\mathcal{E}_h((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) \lesssim h^{k+1}\Big(|\,\mathbf{curl}\,\boldsymbol{H}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{H}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\Big)\\ \times\Big(\|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h} + \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},1,h}\Big).\end{aligned} \tag{5.4}$$

Expanding according to the respective definitions $\mathcal{A}_h$, $\mathcal{L}_h$, a_h, b_h, and c_h, we have

$$\begin{aligned}\mathcal{E}_h((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) = \mathcal{E}_{h,1}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h))\\ + \mathcal{E}_{h,2}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) + \mathcal{E}_{h,3}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)),\end{aligned} \tag{5.5}$$

with

$$\begin{aligned}\mathcal{E}_{h,1}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) &\coloneqq \sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{J}\cdot\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T - (\underline{\boldsymbol{C}}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H}),\underline{\boldsymbol{v}}_T)_{\mathrm{div},T}\right),\\
\mathcal{E}_{h,2}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) &\coloneqq -\sum_{T\in\mathcal{T}_h}\int_T D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k\boldsymbol{A})D_T^k\underline{\boldsymbol{v}}_T\\
\mathcal{E}_{h,3}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) &\coloneqq -\sum_{T\in\mathcal{T}_h}\left(\mu_T(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H},\underline{\boldsymbol{\zeta}}_T)_{\mathbf{curl},T} - (\underline{\boldsymbol{C}}_T^k\underline{\boldsymbol{\zeta}}_T,\underline{\boldsymbol{I}}_{\mathrm{div},T}^k\boldsymbol{A})_{\mathrm{div},T}\right).\end{aligned}$$

Let us first estimate $\mathcal{E}_{h,1}$. Recalling that $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$ and expanding $(\cdot,\cdot)_{\mathrm{div},T}$ according to its definition (3.13b), we have

$$\begin{aligned}&\mathcal{E}_{h,1}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h))\\
&\quad= \sum_{T\in\mathcal{T}_h}\left[\int_T\Big(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{P}_{\mathrm{div},T}^k\big[\underline{\boldsymbol{C}}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H})\big]\Big)\cdot\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T - \mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{C}}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H}),\underline{\boldsymbol{v}}_T)\right]\\
&\quad= \sum_{T\in\mathcal{T}_h}\int_T\Big(\mathbf{curl}\,\boldsymbol{H} - \mathsf{C}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H})\Big)\cdot\boldsymbol{P}_{\mathrm{div},T}^k\underline{\boldsymbol{v}}_T - \sum_{T\in\mathcal{T}_h}\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k(\mathbf{curl}\,\boldsymbol{H}),\underline{\boldsymbol{v}}_T)\end{aligned}$$

where the second line comes from the relation (4.10) and the commutation formula $\underline{\boldsymbol{C}}_T^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H}) = \underline{\boldsymbol{I}}_{\mathrm{div},T}^k(\mathbf{curl}\,\boldsymbol{H})$, see [10, Eq. (3.35)]. Using Cauchy–Schwarz inequalities on the integrals, on $\mathrm{s}_{\mathrm{div},T}$, and on the sums over $T\in\mathcal{T}_h$, and recalling the consistency properties (3.22) and (3.26), we infer

$$\mathcal{E}_{h,1}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) \le h^{k+1}|\,\mathbf{curl}\,\boldsymbol{H}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)}\|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}. \tag{5.6}$$

To handle $\mathcal{E}_{h,2}$, we invoke the commutation formula [10, Eq. (3.36)] to see that $D_T^k(\underline{\boldsymbol{I}}_{\mathrm{div},T}^k\boldsymbol{A}) = \pi_{\mathcal{P},T}^k(\operatorname{div}\boldsymbol{A}) = 0$, and thus

$$\mathcal{E}_{h,2}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) = 0. \tag{5.7}$$

Finally, we turn to $\mathcal{E}_{h,3}$. Since $\boldsymbol{A} \in \mathbf{H}_0(\mathbf{curl};\Omega)$, the adjoint consistency Theorem 10 enables us to replace, in this consistency error, the term $(\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{\zeta}}_T, \underline{\boldsymbol{I}}_{\mathrm{div},T}^k \boldsymbol{A})_{\mathrm{div},T}$ with $\int_T \mathbf{curl}\,\boldsymbol{A} \cdot \boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{\zeta}}_T = \int_T \mu_T \boldsymbol{H} \cdot \boldsymbol{P}_{\mathbf{curl},T}^k \underline{\boldsymbol{\zeta}}_T$ up to a term that is controlled, i.e.,

$$\begin{aligned}
\mathcal{E}_{h,3}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) \le & -\sum_{T\in\mathcal{T}_h}\left(\mu_T(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H},\underline{\boldsymbol{\zeta}}_T)_{\mathbf{curl},T} - \mu_T\int_T \boldsymbol{H}\cdot\boldsymbol{P}_{\mathbf{curl},T}^k\underline{\boldsymbol{\zeta}}_T\right)\\
& + h^{k+1}\left(|\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right)\|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h}\\
\le & -\sum_{T\in\mathcal{T}_h}\left(\mu_T\int_T\left[\boldsymbol{P}_{\mathbf{curl},T}^k(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H}) - \boldsymbol{H}\right]\cdot\boldsymbol{P}_{\mathbf{curl},T}^k\underline{\boldsymbol{\zeta}}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{I}}_{\mathbf{curl},T}^k\boldsymbol{H},\underline{\boldsymbol{\zeta}}_T)\right)\\
& + h^{k+1}\left(|\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right)\|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h},
\end{aligned}$$

where we have used $\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{\zeta}}_h\|_{\mathrm{div},h} \lesssim \|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h}$ and the second inequality comes from expanding $(\cdot,\cdot)_{\mathbf{curl},T}$ according to its definition. Cauchy–Schwarz inequalities and the consistency properties (3.19) and (3.25) then lead to

$$\mathcal{E}_{h,3}((\boldsymbol{H},\boldsymbol{A});(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{v}}_h)) \lesssim h^{k+1}|\boldsymbol{H}|_{\mathbf{H}^{(k+1,2)}(\mathcal{T}_h)}\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} + h^{k+1}\left(|\boldsymbol{A}|_{\mathbf{H}^{k+1}(\mathcal{T}_h)} + |\boldsymbol{A}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)}\right)\|\underline{\boldsymbol{\zeta}}_h\|_{\mu,\mathbf{curl},1,h}.$$

Plugging this estimate together with (5.6) and (5.7) into (5.5), we infer that (5.4) holds, which concludes the proof. □

5.3 Numerical tests

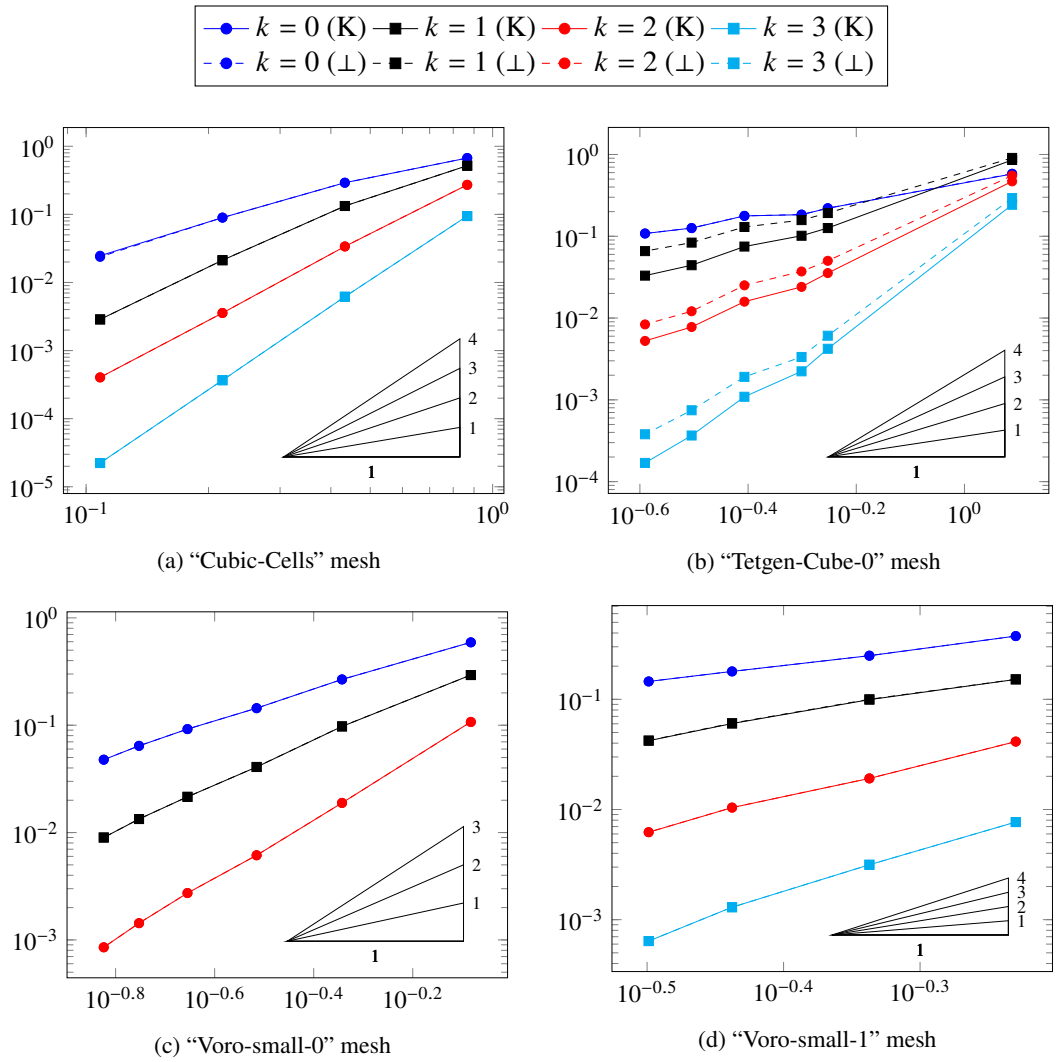
We present here the results of some numerical tests obtained with the DDR scheme (5.2) for the magnetostatics model (5.1), focusing on comparing outputs obtained using either the complements (2.1), hereafter denoted by (K), or the orthogonal complements of [11, 13], denoted by (⊥). Both versions of the DDR complex, and related schemes, have been implemented in the `HArDCore3D` C++ framework (see `https://github.com/jdroniou/HArDCore`), using linear algebra facilities from the `Eigen3` library (see `http://eigen.tuxfamily.org`) and the `Intel MKL PARDISO` library (see `https://software.intel.com/en-us/mkl`) for the resolution of the global sparse linear system. All tests were run on a 16-inch 2019 MacBook Pro equipped with an 8-core Intel Core i9 processor (I9-9980HK) and 32Gb of RAM and running macOS Big Sur version 11.1. We consider a constant permeability $\mu = 1$, and the same exact smooth solution and mesh families as in [11, Section 4.4].

Figure 1 presents the errors, for various values of k, computed in the relative discrete $\mathbf{H}(\mathbf{curl};\Omega)\times\mathbf{H}(\mathrm{div};\Omega)$ norm:

$$\frac{\left(\|\underline{\boldsymbol{H}}_h - \underline{\boldsymbol{I}}_{\mathbf{curl},h}^k\boldsymbol{H}\|_{\mu,\mathbf{curl},1,h}^2 + \|\underline{\boldsymbol{A}}_h - \underline{\boldsymbol{I}}_{\mathrm{div},h}^k\boldsymbol{A}\|_{\mathrm{div},1,h}^2\right)^{1/2}}{\left(\|\underline{\boldsymbol{I}}_{\mathbf{curl},h}^k\boldsymbol{H}\|_{\mu,\mathbf{curl},1,h}^2 + \|\underline{\boldsymbol{I}}_{\mathrm{div},h}^k\boldsymbol{A}\|_{\mathrm{div},1,h}^2\right)^{1/2}}.$$

In the case of the Koszul complements, Theorem 19 states that this error should decrease as $O(h^{k+1})$ with the mesh size. No such estimate is known for the DDR scheme using orthogonal complements and, due to the lack of key properties of these complements (hierarchical inclusions, structure of traces), it is not clear whether the analysis carried out in the rest of this paper could be adapted to such complements. Nonetheless, the graphs in Figure 1 show that both schemes converge with an order $k+1$. The errors between (K) and (⊥) are essentially indistinguishable, except for $k \ge 1$ on tetrahedral meshes, where (⊥) leads to slightly larger errors than (K) – about twice as large on the finest mesh with $k = 3$.

The assembly of the (⊥)-DDR scheme requires, for any $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, to compute bases for the L^2-orthogonal complements in $\boldsymbol{\mathcal{P}}^\ell(Y)$ of $\boldsymbol{\mathcal{G}}^\ell(Y)$ and $\boldsymbol{\mathcal{R}}^\ell(Y)$, which is done by computing the kernels of local matrices through a full pivot LU algorithm [11, Section 5.1]. On the contrary, in the (K) version, explicit



(a) "Cubic-Cells" mesh

(b) "Tetgen-Cube-0" mesh

(c) "Voro-small-0" mesh

(d) "Voro-small-1" mesh

Figure 1: Relative error estimates in discrete $\mathbf{H}(\mathbf{curl};\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ norm vs. h, for the Koszul complements of (2.1) [(K), continuous lines], and the orthogonal complements of [13][(⊥), dashed lines].

Mesh	$\mathrm{card}(\mathcal{T}_h)$	$\mathrm{card}(\mathcal{F}_h)$	$\mathrm{card}(\mathcal{E}_h)$
Cubic_Cells	4 096	13 056	13 872
Tetgen_Cube-0	2 925	6 228	3 965
Voro-small-0	2 197	15 969	27 546
Voro-small-1	356	2 376	4 042

(a) Number of relevant mesh entities

Mesh	$\dim(\underline{\boldsymbol{X}}^0_{\mathbf{curl},h})$	$\dim(\underline{\boldsymbol{X}}^1_{\mathbf{curl},h})$	$\dim(\underline{\boldsymbol{X}}^2_{\mathbf{curl},h})$	$\dim(\underline{\boldsymbol{X}}^3_{\mathbf{curl},h})$
Cubic_Cells	13 872	83 296	207 504	398 784
Tetgen_Cube-0	3 956	38 314	105 594	214 580
Voro-small-0	27 546	111 787	243 345	—
Voro-small-1	4 042	16 636	36 474	64 624

(b) Dimension of the space $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ for $k \in \{0, \ldots, 3\}$

Mesh	$\dim(\underline{\boldsymbol{X}}^0_{\mathrm{div},h})$	$\dim(\underline{\boldsymbol{X}}^1_{\mathrm{div},h})$	$\dim(\underline{\boldsymbol{X}}^2_{\mathrm{div},h})$	$\dim(\underline{\boldsymbol{X}}^3_{\mathrm{div},h})$
Cubic_Cells	13 056	63 744	160 256	314 880
Tetgen_Cube-0	6 228	36 234	95 868	193 905
Voro-small-0	15 969	61 089	139 754	—
Voro-small-1	2 376	9 264	21 376	39 780

(c) Dimension of the space $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ for $k \in \{0, \ldots, 3\}$

Table 1: Dimension of meshes and spaces considered for the evaluation of computational times in the numerical tests of Section 5.3.

bases for $\mathcal{G}^{\mathrm{c},\ell}(Y)$ and $\mathcal{R}^{\mathrm{c},\ell}(Y)$ can be devised; even though these bases are then orthonormalised to ensure a better numerical stability of the scheme (especially on non-isotropic elements, see the discussion in [12, Section B.1.1] on this topic), the computational cost of creating the polynomial bases in (⊥) is expected to be larger than in (K). Figure 2 compares the processor times for the two DDR schemes required for (a) the creation of the bases for local polynomial spaces and (b) the model construction (computation of the discrete operators, potentials, and L^2-products, and global system assembly). We do not compare the linear system resolution times as they are very close for both schemes. In all the cases, the finest mesh of each sequence is considered; see Table 1. In the left column of Figure 2 we report the total CPU time, which constitutes the most reliable measure to assess performance. Since our code relies on multi-threading, we also report, in the right column, wall-clock times, which are more representative of real-life performance on the selected architecture. Wall-clock times are subject to outside influences, such as the impact of other processes, and should therefore be regarded with caution.

As expected, (K) polynomial bases are faster to create than (⊥) polynomial bases, but not by a large factor. There is a more pronounced difference when comparing the time for model construction, which is mostly dedicated to the creation of the discrete vector calculus operators and potentials in $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ and $\underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ (once these are created, assemblying the global linear system itself takes only a small fraction of the total model construction time). One possible explanation for the largest model construction time noticed with (⊥) complements is that the local systems defining the operators and potentials, solved using the `Eigen::LDLT` direct solver, have a worse condition number than those with Koszul complements. Drawing more definitive conclusions is always difficult, as running times highly depend on specific implementation choices, and our implementation is designed for flexibility rather than for efficiency on

one given model. The results presented in this section seems to show, however, that the DDR complex using Koszul complements is not only theoretically better (as it allows for complete consistency analysis and error estimates), but also requires less computational resources. The comparison of CPU times and wall clock times also confirms that the assembly step strongly benefits from parallel implementations.

A Curl lifting

We prove here that the face $\boldsymbol{R}_{\mathbf{curl},F}$ and element $\boldsymbol{R}_{\mathbf{curl},T}$ liftings, detailed in Section 4.4.2, are well defined and satisfy the key properties (4.30) and (4.31).

A.1 Face lifting $\boldsymbol{R}_{\mathbf{curl},F}$

A.1.1 Existence of $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}$

Owing to (4.26b), we look for $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F} = \mathbf{rot}_F\, q_F$ for some $q_F \in \mathrm{H}^1(F)$. Using the property $\mathrm{rot}_F(\mathbf{rot}_F) = -\mathrm{div}_F(\mathbf{grad}_F) = -\Delta_F$ (which stems from [10, Eq. (2.1)]) and that $\mathbf{rot}_F\, q_F$ (resp. $\boldsymbol{t}_E$) is $\mathbf{grad}_F\, q_F$ (resp. $\boldsymbol{n}_{FE}$) rotated by $-\pi/2$ in the plane spanned by F, we see that (4.26) reduces to the following Neumann problem on q_F:

$$\begin{aligned} -\Delta_F q_F &= C_F^k \underline{\boldsymbol{v}}_F \qquad \text{in } F,\\ \mathbf{grad}_F\, q_F \cdot (\omega_{FE}\boldsymbol{n}_{FE}) &= \omega_{FE} v_E \quad \forall E \in \mathcal{E}_F. \end{aligned} \tag{A.1}$$

Recalling that $\omega_{FE}\boldsymbol{n}_{FE}$ is the outer normal, in the plane spanned by F, to F on E, we see that the compatibility condition of this Neumann problem simply amounts to the definition (2.12) of C_F^k with $r_F = 1$. There exists therefore a unique $q_F \in \mathrm{H}^1(F)$ solution of this problem with $\int_F q_F = 0$. Using q_F as a test function in the weak formulation and applying Cauchy–Schwarz inequalities leads to

$$\begin{aligned} \|\mathbf{grad}_F\, q_F\|_{\mathbf{L}^2(F)}^2 &\le \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} \|q_F\|_{\mathrm{L}^2(F)} + \sum_{E\in\mathcal{E}_F} \|v_E\|_{\mathrm{L}^2(E)} \|q_F\|_{\mathrm{L}^2(E)}\\ &\lesssim h_F \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} \|\mathbf{grad}_F\, q_F\|_{\mathbf{L}^2(F)} + \left(\sum_{E\in\mathcal{E}_F} h_E \|v_E\|_{\mathrm{L}^2(E)}^2\right)^{\frac12} \|\mathbf{grad}_F\, q_F\|_{\mathbf{L}^2(F)}, \end{aligned}$$

where the second line follows from the Poincaré–Wirtinger inequality $\|q_F\|_{\mathrm{L}^2(F)} \lesssim h_F \|\mathbf{grad}_F\, q_F\|_{\mathbf{L}^2(F)}$ together with the continuous trace inequality $\|q_F\|_{\mathrm{L}^2(E)} \lesssim h_E^{-1/2}\|q_F\|_{\mathrm{L}^2(F)} + h_E^{1/2}\|\mathbf{grad}_F\, q_F\|_{\mathbf{L}^2(F)}$, see [12, Remark 1.46 and Lemma 1.31]. As a consequence,

$$\|\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}\|_{\mathbf{L}^2(F)} \lesssim \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} + |||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F}. \tag{A.2}$$

A.1.2 Existence of $\psi_{\underline{\boldsymbol{v}}_F}$

Fix $\varpi_F \in \mathrm{C}_c^\infty(F)$ such that $\varpi_F = 1$ on a ball $B_F \subset F$ of radius $\eqsim h_F$ (the existence of such a ball follows from the mesh regularity assumption) and $0 \le \varpi_F \le 1$. We look for $\psi_{\underline{\boldsymbol{v}}_F}$ under the form $\varpi_F r_F$ with $r_F \in \mathcal{P}^k(F)$. Since $\mathrm{div}_F : \mathcal{R}^{\mathrm{c},k+1}(F) \to \mathcal{P}^k(F)$ is an isomorphism, denoting as in [10, Lemma 31] its inverse by $(\mathrm{div}_F)^{-1}$, the relation (4.27) is equivalent to

$$\int_F \varpi_F r_F w_F = \int_F (\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F - \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}) \cdot (\mathrm{div}_F)^{-1} w_F \qquad \forall w_F \in \mathcal{P}^k(F).$$

Since $\varpi_F \ge 0$ is strictly positive on a ball, the mapping $(r_F, w_F) \mapsto \int_F \varpi_F r_F w_F$ is an inner product on $\mathcal{P}^k(F)$ and there exists therefore a unique $r_F \in \mathcal{P}^k(F)$ that satisfies this property. This establishes the existence of $\psi_{\underline{\boldsymbol{v}}_F}$.

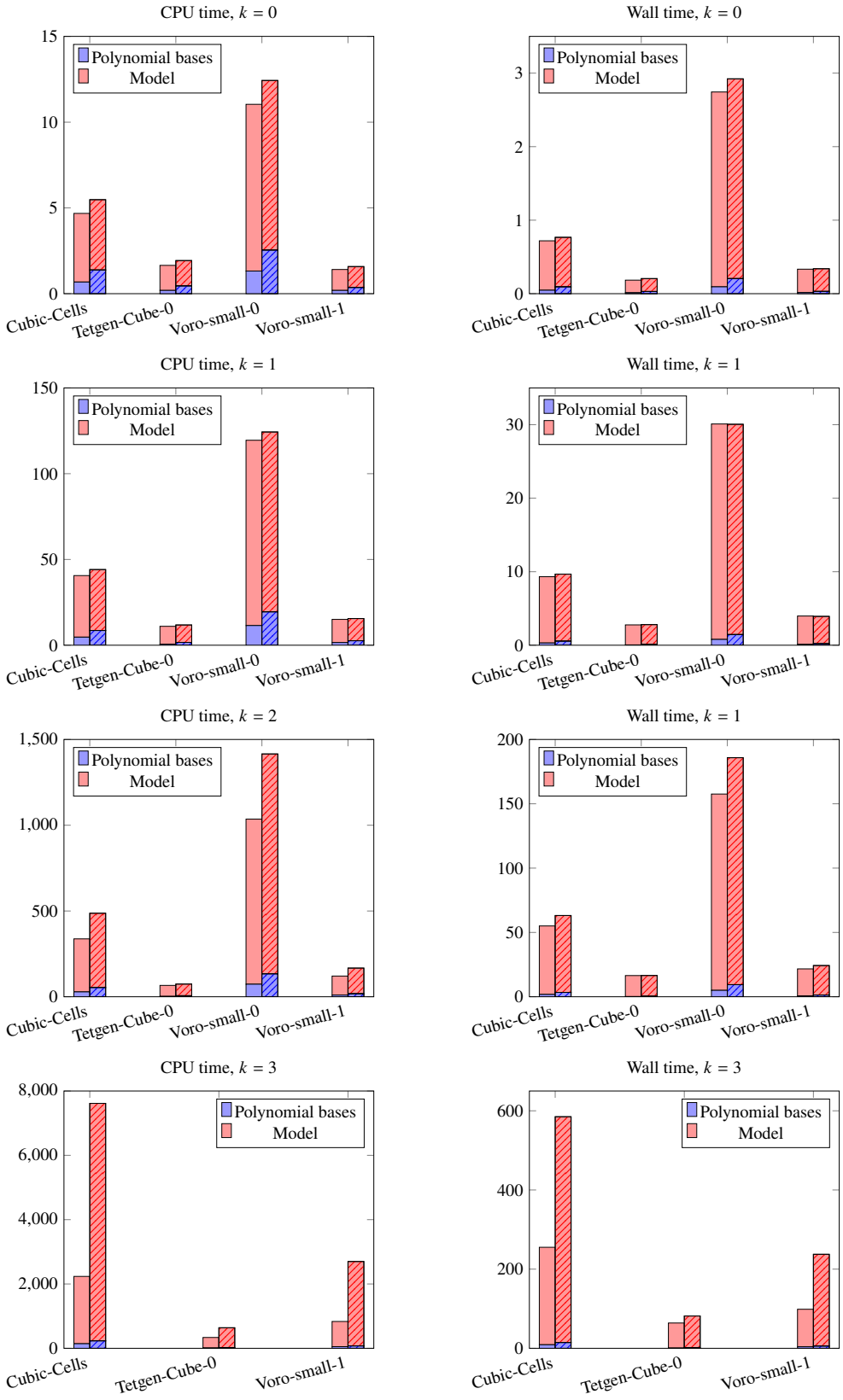


Figure 2: Comparison of CPU (left column) and wall times (right column), both measured in seconds, for the computation of the DDR bases ("Polynomial bases") and of the model construction ("Model") for Koszul (solid fill) and orthogonal (pattern fill) complements on the finest mesh of each sequence; see Table 1.

Moreover, since $\varpi_F = 1$ on B_F and $\|\cdot\|_{\mathrm{L}^2(B_F)}$ and $\|\cdot\|_{\mathrm{L}^2(F)}$ are uniformly equivalent on $\mathcal{P}^k(F)$ (see the proof of [12, Lemma 1.25]), using $w_F = r_F$ above leads to

$$\begin{aligned}\|r_F\|_{\mathrm{L}^2(F)}^2 \lesssim \int_F \varpi_F r_F^2 &\le \|\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F - \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}\|_{\mathbf{L}^2(F)} \|(\mathrm{div}_F)^{-1} r_F\|_{\mathbf{L}^2(F)} \\ &\lesssim \big(|||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} + \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)}\big) h_F \|r_F\|_{\mathrm{L}^2(F)},\end{aligned}$$

where the conclusion follows from a triangle inequality along with the boundedness (4.3) of $\boldsymbol{\gamma}_{\mathrm{t},F}^k$ and the estimate (A.2) for the first factor, and [10, Lemma 31] for the second factor. Simplifying, we obtain

$$\|r_F\|_{\mathrm{L}^2(F)} \lesssim h_F \big(|||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} + \|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)}\big). \tag{A.3}$$

A.1.3 Orthogonality property of $\boldsymbol{R}_{\mathbf{curl},F}$

We prove here the relation (4.30). Notice first that, since $\psi_{\underline{\boldsymbol{v}}_F}$ vanishes on ∂F and $\mathrm{rot}_F \,\mathbf{grad}_F = 0$, by (4.26) it holds

$$\mathrm{rot}_F(\boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F) = C_F^k \underline{\boldsymbol{v}}_F \quad \text{ and } \quad (\boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F) \cdot \boldsymbol{t}_E = v_E \quad \forall E \in \mathcal{E}_F. \tag{A.4}$$

Let $z_F \in \mathcal{R}^k(F)$ and write $z_F = \mathbf{rot}_F\, r_F$ with $r_F \in \mathcal{P}^{0,k+1}(F)$. By (2.13) and Remark 2, we have

$$\begin{aligned}\int_F \boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F \cdot z_F &= \int_F C_F^k \underline{\boldsymbol{v}}_F r_F + \sum_{E\in\mathcal{F}_E} \omega_{FE} \int_E v_E r_F \\ &= \int_F \mathrm{rot}_F(\boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F) r_F + \sum_{E\in\mathcal{F}_E} \omega_{FE} \int_E (\boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F)\cdot \boldsymbol{t}_E\, r_F = \int_F \boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F \cdot z_F,\end{aligned}$$

where the second equality follows from (A.4), and the conclusion has been obtained using an integration by parts. This proves that (4.30) holds for $z_F \in \mathcal{R}^k(F)$.

Let us now take $z_F \in \mathcal{R}^{\mathrm{c},k+1}(F)$. Integrating the left-hand side of (4.27) by parts yields

$$\int_F \mathbf{grad}\, \psi_{\underline{\boldsymbol{v}}_F} \cdot z_F = \int_F (\boldsymbol{\gamma}_{\mathrm{t},F}^k \underline{\boldsymbol{v}}_F - \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}) \cdot z_F.$$

Since $\boldsymbol{R}_{\mathbf{curl},F} \underline{\boldsymbol{v}}_F = \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F} + \mathbf{grad}\, \psi_{\underline{\boldsymbol{v}}_F}$, this establishes that (4.30) also holds for $z_F \in \mathcal{R}^{\mathrm{c},k+1}(F)$, which completes the proof of this orthogonality relation since $\mathcal{RT}^{k+1}(F) = \mathcal{R}^k(F) \oplus \mathcal{R}^{\mathrm{c},k+1}(F)$.

A.2 Element lifting $\boldsymbol{R}_{\mathbf{curl},T}$

A.2.1 Existence of $\boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T$

Owing to (4.28b), we look for $\boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T$ under the form of a potential gradient $\mathbf{grad}\, q_T$ with $q_T \in \mathrm{H}^1(T)$. Equations (4.28a) and (4.28c) then show that q_T must solve the Neumann problem

$$\begin{aligned}\Delta q_T &= -\,\mathrm{div}\, \mathbf{C}_T^k \underline{\boldsymbol{v}}_T && \text{in } T, \\ \mathbf{grad}\, q_T \cdot (\omega_{TF} \boldsymbol{n}_F) &= \omega_{TF}(C_F^k \underline{\boldsymbol{v}}_F - \mathbf{C}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_F) && \forall F \in \mathcal{F}_T,\end{aligned} \tag{A.5}$$

where we recall that $\omega_{TF}\boldsymbol{n}_F$ is the outer normal to T on F. The compatibility condition of this problem is

$$\sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F (C_F^k \underline{\boldsymbol{v}}_F - \mathbf{C}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_F) = -\int_T \mathrm{div}\, \mathbf{C}_T^k \underline{\boldsymbol{v}}_T = -\sum_{T\in\mathcal{F}_T} \omega_{TF} \int_F \mathbf{C}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_F,$$

which holds true owing to [10, Eq. (3.27)] with $r_T = 1$. There exists therefore a unique $q_T \in \mathrm{H}^1(T)$ with $\int_T q_T = 0$ solution to (A.5). Using q_T as a test function in the weak formulation of (A.5) yields

$$\|\mathbf{grad}\, q_T\|_{\mathbf{L}^2(T)}^2 \le \|\mathbf{C}_T^k \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} \|\mathbf{grad}\, q_T\|_{\mathbf{L}^2(T)} + \sum_{F\in\mathcal{F}_T} \|C_F^k \underline{\boldsymbol{v}}_T\|_{\mathrm{L}^2(F)} \|q_T\|_{\mathrm{L}^2(F)}.$$

Using the Poincaré–Wirtinger and continuous trace inequalities as we did to obtain (A.2), and recalling that $\boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T = \mathbf{grad}\, q_T$, we infer

$$\|\boldsymbol{\delta}_T \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} \lesssim \|\mathbf{C}_T^k \underline{\boldsymbol{v}}_T\|_{\mathbf{L}^2(T)} + \left(\sum_{F\in\mathcal{F}_T} h_F \|C_F^k \underline{\boldsymbol{v}}_T\|_{\mathrm{L}^2(F)}^2\right)^{\frac12} \lesssim |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T}, \tag{A.6}$$

where the conclusion follows from (4.14).

A.2.2 Existence of $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T$

The equation (4.29b) suggests to look for $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T = \mathbf{curl}\, z_T$. Since adding a gradient to z_T does not change its curl, we can look for z_T in the space

$$z_T \in (\mathbf{grad}\,\mathrm{H}^1(T))^\perp := \left\{\boldsymbol{w}\in\mathbf{H}(\mathbf{curl};T)\ :\ \int_T \boldsymbol{w}\cdot\mathbf{grad}\, r = 0 \quad \forall r\in\mathrm{H}^1(T)\right\}. \tag{A.7}$$

The equations (4.29a) and (4.29c) then lead to a curl-curl problem on z_T, whose variational form is: Find $z_T \in (\mathbf{grad}\,\mathrm{H}^1(T))^\perp$ such that

$$\int_T \mathbf{curl}\, z_T\cdot\mathbf{curl}\,\boldsymbol{w} = \int_T (\mathbf{C}_T^k\underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{w} - \langle \omega_{T\partial T}\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T}, \boldsymbol{w}\times\boldsymbol{n}_{\partial T}\rangle_{\mathrm{H}_{/\!/}^{1/2}(\partial T),\mathrm{H}_{/\!/}^{-1/2}(\partial T)} \quad \forall \boldsymbol{w}\in(\mathbf{grad}\,\mathrm{H}^1(T))^\perp, \tag{A.8}$$

where $\omega_{T,\partial T}\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T}$ and $\boldsymbol{w}\times\boldsymbol{n}_{\partial T}$ are the functions defined on ∂T by setting $(\omega_{T,\partial T}\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T})_{|F} := (\omega_{TF}\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F)_{\mathrm{t},F}$ and $(\boldsymbol{w}\times\boldsymbol{n}_{\partial T})_{|F} := \boldsymbol{w}_{|F}\times\boldsymbol{n}_F$ for all $F\in\mathcal{F}_T$, $\mathrm{H}_{/\!/}^{1/2}(\partial T)$ is the set of functions on ∂T whose restriction to each face $F\in\mathcal{F}_T$ belongs to $\mathrm{H}^{1/2}(F)$, and whose tangential traces on the edges are weakly continuous (see [2, Definition 3.1.2] for details), and $\mathrm{H}_{/\!/}^{-1/2}(\partial T)$ is its dual space. Since the solution to (A.1) belongs to $\mathrm{H}^{3/2}(F)$ (see [8, Corollary 23.5]), the edge tangential trace property in (A.4) ensures that $\omega_{T,\partial T}\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T}$ indeed belongs to $\mathrm{H}_{/\!/}^{1/2}(\partial T)$.

Owing to the Poincaré inequality (A.15) and to the fact that $(\mathbf{grad}\,\mathrm{H}^1(T))^\perp$ is a closed subspace of $\mathbf{H}(\mathbf{curl};T)$, there exists a unique solution to (A.8). We now prove that z_T satisfies (A.8) for all $\boldsymbol{w}\in\mathbf{H}(\mathbf{curl};T) = \mathbf{grad}\,\mathrm{H}^1(T)\oplus(\mathbf{grad}\,\mathrm{H}^1(T))^\perp$, which amounts to showing that the right-hand side vanishes whenever $\boldsymbol{w} = \mathbf{grad}\, r$ for some $r\in\mathrm{H}^1(T)$. By density of smooth functions in $\mathrm{H}^1(T)$, we only need to prove this result for $r\in\mathrm{C}^\infty(\overline{T})$. Plugging $\boldsymbol{w} = \mathbf{grad}\, r$ in the right-hand side of (A.8), the duality product can be written as standard integrals (since $\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F\in\mathbf{L}^2(F)$ for all $F\in\mathcal{F}_T$) and, integrating by parts, we obtain

$$\begin{aligned}
&\int_T (\mathbf{C}_T^k\underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T\underline{\boldsymbol{v}}_T)\cdot\mathbf{grad}\, r - \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F\cdot(\mathbf{grad}\, r\times\boldsymbol{n}_F)\\
&\quad = -\int_T \cancel{\mathrm{div}(\mathbf{C}_T^k\underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T\underline{\boldsymbol{v}}_T)}\, r + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F (\mathbf{C}_T^k\underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{n}_F\, r\\
&\qquad - \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F\cdot\mathbf{rot}_F(r_{|F})\\
&\quad = \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F C_F^k\underline{\boldsymbol{v}}_F\, r - \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \mathrm{rot}_F(\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F)\, r_{|F}\\
&\qquad - \sum_{F\in\mathcal{F}_T}\sum_{E\in\mathcal{F}_E}\omega_{TF}\omega_{FE}\int_E (\boldsymbol{R}_{\mathbf{curl},F}\underline{\boldsymbol{v}}_F\cdot\boldsymbol{t}_E)\, r_{|F},
\end{aligned}$$

where we have used (4.28a) to cancel the term in the first equality, and (4.28c) together with integrations by parts on each face in the second equality. Recalling (A.4) and that $\omega_{TF_1}\omega_{F_1E} + \omega_{TF_2}\omega_{F_2E} = 0$ if F_1, F_2 are the two faces of T that share the edge E, the right-hand side above vanish, which shows that (A.8) indeed holds for $\boldsymbol{w} = \mathbf{grad}\, r$, and thus for all $\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}; T)$.

Since $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T = \mathbf{curl}\, z_T$, applying this relation to a generic $\boldsymbol{w} \in \mathbf{C}_c^\infty(T)$ and integrating by parts yields (4.29a); using then a generic $\boldsymbol{w} \in \mathbf{C}^\infty(\overline{T})$ and again integrating by parts, we infer (4.29c).

A.2.3 Bound on $\boldsymbol{R}_{\mathbf{curl},T}$

We prove here the estimate (4.31). The estimate on $\mathbf{curl}\,\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T$ follows from (4.29a), (4.14) and (A.6). It remains to bound the L^2-norm of $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T$. To do so, we use $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T}$ provided by Lemma 20 below and an integration by parts [2, Eq. (2.27)] to re-cast (A.8) as

$$\int_T \mathbf{curl}\, z_T \cdot \mathbf{curl}\, \boldsymbol{w} = \int_T (\mathbf{C}_T^k\underline{\boldsymbol{v}}_T + \boldsymbol{\delta}_T\underline{\boldsymbol{v}}_T)\cdot\boldsymbol{w} + \int_T \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{g}_{\underline{\boldsymbol{v}}_T} - \int_T \boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{g}_{\underline{\boldsymbol{v}}_T}.$$

Making $\boldsymbol{w} = z_T$, we deduce

$$\|\,\mathbf{curl}\, z_T\,\|_{\mathbf{L}^2(T)}^2 \lesssim |||\underline{\boldsymbol{C}}_T^k\underline{\boldsymbol{v}}_T|||_{\mathrm{div},T}\, h_T\,\|\,\mathbf{curl}\, z_T\,\|_{\mathbf{L}^2(T)} + \|\,\mathbf{curl}\, z_T\,\|_{\mathbf{L}^2(T)}\left(|||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T} + |||\underline{\boldsymbol{C}}_T^k\underline{\boldsymbol{v}}_T|||_{\mathrm{div},T}\right),$$

where we have invoked (4.14), (A.6), the Poincaré inequality (A.15), and (A.9) below. Simplifying, using the norm equivalences (4.5), and recalling that $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T = \mathbf{curl}\, z_T$ concludes the proof of the L^2-estimate on $\boldsymbol{R}_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T$ stated in (4.31).

Lemma 20 (Lifting in $\mathbf{H}^1(T)$)**.** *There exists $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T} \in \mathbf{H}^1(T)$ such that the tangential trace of $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T}$ on ∂T is $\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T}$, and*

$$\|\boldsymbol{g}_{\underline{\boldsymbol{v}}_T}\|_{\mathbf{L}^2(T)} + h_T\,\|\,\mathbf{curl}\,\boldsymbol{g}_{\underline{\boldsymbol{v}}_T}\,\|_{\mathbf{L}^2(T)} \lesssim |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T} + |||\underline{\boldsymbol{C}}_T^k\underline{\boldsymbol{v}}_T|||_{\mathrm{div},T}. \tag{A.9}$$

Proof. Recalling that

$$\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T} = \boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}} + \mathbf{grad}_{\partial T}\,\psi_{\underline{\boldsymbol{v}}_{\partial T}}, \tag{A.10}$$

with obvious notations (each of these functions, restricted to a face $F \in \mathcal{F}_T$, corresponds to the function obtained replacing ∂T by F), we construct $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T} = \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}} + \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$, each addend corresponding to the addends in the decomposition (A.10) of $\boldsymbol{R}_{\mathbf{curl},\partial T}\underline{\boldsymbol{v}}_{\partial T}$.

1. *Construction of $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}$.* We assume, for the moment, that $h_T = 1$. By [8, Corollary 23.5] and inverse inequalities on the polynomials $C_F^k\underline{\boldsymbol{v}}_F$ and $(v_E)_{E\in\mathcal{F}_E}$ (recalling that $1 = h_T \simeq h_F \simeq h_E$ for all $F \in \mathcal{F}_T$ and $E \in \mathcal{E}_T$), there exists $\epsilon \in (0, 1/2)$ such that $\mathbf{grad}_F\, q_F \in \mathbf{H}^{1/2+\epsilon}(F)$ and

$$\|\,\mathbf{grad}_F\, q_F\,\|_{\mathbf{H}^{1/2+\epsilon}(F)} \lesssim \|C_F^k\underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} + \sum_{E\in\mathcal{E}_F}\|v_E\|_{\mathrm{L}^2(E)} \lesssim \|C_F^k\underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} + |||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F}.$$

Above, when invoking [8, Corollary 23.5], we have used the fact that, since $\epsilon < 1/2$, the $\mathrm{H}^\epsilon(\partial F)$-norm is equivalent to the sum of the $\mathrm{H}^\epsilon(E)$-norms over $E \in \mathcal{F}_E$. By construction, $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}$ has strongly continuous tangential traces on the edges of T so

$$\begin{aligned}|\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}|_{\mathbf{H}^{1/2}_{/\!/}(\partial T)}^2 &\lesssim \sum_{F\in\mathcal{F}_T}|\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_F}|_{\mathbf{H}^{1/2}(F)}^2 = \sum_{F\in\mathcal{F}_T}|\,\mathbf{grad}_F\, q_F|_{\mathbf{H}^{1/2}(F)}^2\\ &\lesssim \sum_{F\in\mathcal{F}_T}\|\,\mathbf{grad}_F\, q_F\|_{\mathbf{H}^{1/2+\epsilon}(F)}^2 \lesssim \sum_{F\in\mathcal{F}_T}\left(\|C_F^k\underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)} + |||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F}\right)^2.\end{aligned}$$

Combined with (A.2) and recalling that the local length scales are $\simeq 1$, this leads to

$$\|\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}\|_{\mathbf{L}^2(\partial T)} + |\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}|_{\mathbf{H}^{1/2}_{/\!/}(\partial T)} \lesssim |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T} + |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T}.$$

Since $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}$ belongs to $\mathbf{H}^{1/2}_{/\!/}(\partial T)$, by [2, Theorem 3.1.3] there exists $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}} \in \mathbf{H}^1(T)$ such that the tangential trace of $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}$ is $\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}$ and

$$\|\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}\|_{\mathbf{L}^2(T)} + \|\mathbf{curl}\, \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}\|_{\mathbf{L}^2(T)} \lesssim \|\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}\|_{\mathbf{L}^2(\partial T)} + |\boldsymbol{\phi}_{\underline{\boldsymbol{v}}_{\partial T}}|_{\mathbf{H}^{1/2}_{/\!/}(\partial T)} \lesssim |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T} + |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T}.$$

This was done under the assumption that $h_T = 1$. Using a scaling argument, we infer from the estimate above that, for an element T of generic diameter h_T,

$$\|\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}\|_{\mathbf{L}^2(T)} + h_T\|\mathbf{curl}\, \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}}\|_{\mathbf{L}^2(T)} \lesssim |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T} + |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T}. \tag{A.11}$$

2. *Construction of* $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$. By definition, $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$ is the lifting of $\mathbf{grad}_{\partial T}\, \psi_{\underline{\boldsymbol{v}}_{\partial T}}$. Recalling the construction of each $\psi_{\underline{\boldsymbol{v}}_F} = \varpi_F r_F$, for $F \in \mathcal{F}_T$, we can extend r_F into a polynomial $r_{TF} \in \mathcal{P}^k(T)$ (for example, by making r_{TF} independent of the coordinate perpendicular to F). We then have, by (A.3),

$$\|r_{TF}\|_{\mathrm{L}^2(T)} \lesssim h_T^{\frac12}\|r_F\|_{\mathrm{L}^2(F)} \lesssim h_T\left(h_F^{\frac12}|||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} + h_F^{\frac12}\|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)}\right). \tag{A.12}$$

The smooth, compactly supported function ϖ_F can be extended in T into ϖ_{TF} such that $0 \le \varpi_{TF} \le 1$, ϖ_{TF} has a compact support in a ball of radius $\simeq h_T$ that does not touch the faces in $\mathcal{F}_T\backslash\{F\}$, and $|\mathbf{grad}\, \varpi_{TF}| \lesssim h_T^{-1}$. Then, for each $F \in \mathcal{F}_T$, the chain rule yields

$$\begin{aligned}\|\mathbf{grad}(\varpi_{TF} r_{TF})\|_{\mathbf{L}^2(T)} &\lesssim \|\mathbf{grad}\, r_{TF}\|_{\mathbf{L}^2(T)} + h_T^{-1}\|r_{TF}\|_{\mathrm{L}^2(T)}\\ &\lesssim h_F^{\frac12}|||\underline{\boldsymbol{v}}_F|||_{\mathbf{curl},F} + h_F^{\frac12}\|C_F^k \underline{\boldsymbol{v}}_F\|_{\mathrm{L}^2(F)},\end{aligned} \tag{A.13}$$

where the second inequality follows from an inverse inequality and (A.12). We then set

$$\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi} = \sum_{F\in\mathcal{F}_T} \mathbf{grad}(\varpi_{TF} r_{TF}) \in \mathbf{C}^\infty(\overline{T}).$$

By choice of the supports of $(\varpi_{TF})_{F\in\mathcal{F}_T}$, the tangential trace of $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$ on each face $F \in \mathcal{F}_T$ is $\mathbf{grad}_F(\varpi_{TF} r_{TF})_{|F} = \mathbf{grad}_F\, \psi_{\underline{\boldsymbol{v}}_F}$. Moreover, the estimate (A.13) gives

$$\|\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}\|_{\mathbf{L}^2(T)} \lesssim \left[\sum_{F\in\mathcal{F}_T}\left(h_F|||\underline{\boldsymbol{v}}_F|||^2_{\mathbf{curl},F} + h_F\|C_F^k \underline{\boldsymbol{v}}_F\|^2_{\mathrm{L}^2(F)}\right)\right]^{\frac12} \lesssim |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T} + |||\underline{\boldsymbol{C}}_T^k \underline{\boldsymbol{v}}_T|||_{\mathrm{div},T}. \tag{A.14}$$

Since $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$ is a gradient, we also have $\mathbf{curl}\, \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi} = \mathbf{0}$ and thus, combining (A.11) and (A.14) yields the estimate (A.9) on $\boldsymbol{g}_{\underline{\boldsymbol{v}}_T} = \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\boldsymbol{\phi}} + \boldsymbol{g}_{\underline{\boldsymbol{v}}_T,\psi}$. □

Lemma 21 (Local Poincaré inequality for **curl**). *With* $(\mathbf{grad}\, \mathrm{H}^1(T))^\perp$ *defined by* (A.7)*, it holds*

$$\|\boldsymbol{w}\|_{\mathbf{L}^2(T)} \lesssim h_T\|\mathbf{curl}\, \boldsymbol{w}\|_{\mathbf{L}^2(T)} \qquad \forall \boldsymbol{w} \in (\mathbf{grad}\, \mathrm{H}^1(T))^\perp. \tag{A.15}$$

Proof. By [2, Theorem 3.4.1], for all $\boldsymbol{v} \in \mathbf{H}(\mathrm{div};T)$ such that $\mathrm{div}\, \boldsymbol{v} = 0$ and $\langle \boldsymbol{v}\cdot\boldsymbol{n}_T, 1\rangle_{\partial T} = 0$ (where $\langle\cdot,\cdot\rangle_{\partial T}$ is the $\mathrm{H}^{-\frac12}(\partial T)$–$\mathrm{H}^{\frac12}(\partial T)$ duality product and $\boldsymbol{n}_T$ is the outer normal to T), there exists $z \in \mathbf{H}(\mathbf{curl};T)$ such that $\int_T z = 0$ and $\boldsymbol{v} = \mathbf{curl}\, z$. Moreover, $\|z\|_{\mathbf{L}^2(T)} \le C_0\|\boldsymbol{v}\|_{\mathbf{L}^2(T)} = C_0\|\mathbf{curl}\, z\|_{\mathbf{L}^2(T)}$

and an inspection of the proof shows that $C_0 \lesssim h_T$ (this estimate is obtained via a scaling argument, and noticing that, if $h_T = 1$, the constants appearing in the proof of [2, Theorem 3.4.1] do not depend on T under our mesh regularity assumptions).

Take $\boldsymbol{w} \in (\mathbf{grad}\,\mathrm{H}^1(T))^\perp$ and let $(\boldsymbol{w}_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbf{C}^\infty(\overline{T})$ which converges to $\boldsymbol{w}$ in $\mathbf{H}(\mathbf{curl};T)$, see [2, Proposition 2.2.12]. Apply the result above to $\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{w}_m$, which satisfies the requirements since, on each $F \in \mathcal{F}_T$, we have $\mathbf{curl}\,\boldsymbol{w}_m \cdot \boldsymbol{n}_{TF} = \mathrm{rot}_F((\boldsymbol{w}_m)_{t,F})$ (where $\boldsymbol{n}_{TF} = (\boldsymbol{n}_T)_{|F}$ and, as before, $(\boldsymbol{w}_m)_{t,F}$ is the tangential trace of $\boldsymbol{w}_m$ on F, oriented here according to $\boldsymbol{n}_{TF}$), and $\boldsymbol{w}_m$ is continuous on ∂T. This yields $z_m \in \mathbf{H}(\mathbf{curl};T)$ such that $\mathbf{curl}(\boldsymbol{w}_m - z_m) = 0$ and $\|z_m\|_{\mathbf{L}^2(T)} \lesssim h_T \|\mathbf{curl}\,\boldsymbol{w}_m\|_{\mathbf{L}^2(T)}$. In particular, since the second Betti number of T is zero, $\boldsymbol{w}_m - z_m \in \mathbf{grad}\,\mathrm{H}^1(T)$, and thus $\int_T (\boldsymbol{w}_m - z_m)\cdot\boldsymbol{w} = 0$. Hence,

$$\int_T \boldsymbol{w}_m\cdot\boldsymbol{w} = \int_T z_m\cdot\boldsymbol{w} \lesssim \|\boldsymbol{w}\|_{\mathbf{L}^2(T)}\|z_m\|_{\mathbf{L}^2(T)} \lesssim \|\boldsymbol{w}\|_{\mathbf{L}^2(T)} h_T \|\mathbf{curl}\,\boldsymbol{w}_m\|_{\mathbf{L}^2(T)}.$$

The conclusion follows by letting $m \to \infty$ and simplifying by $\|\boldsymbol{w}\|_{\mathbf{L}^2(T)}$. □

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