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An arbitrary-order discrete de Rham complex on polyhedral meshes. Part I: Exactness and Poincaré inequalities

Daniele A. Di Pietro¹ and Jérôme Droniou²

¹IMAG, Univ Montpellier, CNRS, Montpellier, France, daniele.di-pietro@umontpellier.fr

²School of Mathematics, Monash University, Melbourne, Australia, jerome.droniou@monash.edu

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Abstract

In this series of papers we present a novel arbitrary-order discrete de Rham (DDR) complex on general polyhedral meshes based on the decomposition of polynomial spaces into the ranges of vector calculus operators and complements linked to the spaces in the Koszul complex. The DDR complex is fully discrete, meaning that both the spaces and discrete calculus operators are replaced by discrete counterparts. We prove a complete panel of results required for the analysis of discretisation schemes for partial differential equations based on this complex: exactness properties, uniform Poincaré inequalities, as well as primal and adjoint consistency. We also show how this DDR complex enables the design of a numerical scheme for a magnetostatics problem, and use the aforementioned results to prove stability and optimal error estimates for this scheme.

Key words. Discrete de Rham complex, compatible discretisations, polyhedral methods

MSC2010. 65N30, 65N99

1 Introduction

The design of stable and convergent schemes for the numerical approximation of certain classes of partial differential equations (PDEs) requires to reproduce, at the discrete level, the underlying geometric, topological, and algebraic structures. This leads to the notion of *compatibility*, which can be achieved either in a conforming or non-conforming numerical setting. Relevant examples include PDEs that relate to the de Rham complex. For an open connected polyhedral domain $\Omega \subset \mathbb{R}^3$, this complex reads

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \quad (1.1)$$

where i_Ω denotes the operator that maps a real value to a constant function over Ω , $H^1(\Omega)$ the space of scalar-valued functions over Ω that are square integrable along with their gradient, $\mathbf{H}(\mathbf{curl}; \Omega)$ (resp. $\mathbf{H}(\mathbf{div}; \Omega)$) the space of vector-valued functions over Ω that are square integrable along with their curl (resp. divergence). In order to serve as a basis for the numerical approximation of PDEs, discrete counterparts of this sequence of spaces and operators should enjoy the following key properties:

(P1) Complex and exactness properties. For the sequence to form a complex, the image of each discrete vector calculus operator should be contained in the kernel of the next one. Moreover, the following exactness properties should be reproduced at the discrete level: $\text{Im } i_\Omega = \text{Ker } \mathbf{grad}$ (since Ω is connected); $\text{Im } \mathbf{grad} = \text{Ker } \mathbf{curl}$ if the first Betti number of Ω is zero; $\text{Im } \mathbf{curl} = \text{Ker } \mathbf{div}$ if the second Betti number of Ω is zero; $\text{Im } \mathbf{div} = L^2(\Omega)$ (since we are in dimension three).

- (P2) *Uniform Poincaré inequalities.* Whenever a function from a space in the sequence lies in some orthogonal complement of the kernel of the vector calculus operator defined on this space, its (discrete) L^2 -norm should be controlled by the (discrete) L^2 -norm of the operator up to a multiplicative constant independent of the mesh size.
- (P3) *Primal and adjoint consistency.* The discrete vector calculus operators should satisfy appropriate commutation properties with the interpolators and their continuous counterparts. Additionally, these operators along with the corresponding (scalar or vector) potentials should approximate smooth fields with sufficient accuracy. Finally, whenever a formal integration by parts is used in the weak formulation of the problem at hand, the vector calculus operators should also enjoy suitable adjoint consistency properties. The notion of adjoint consistency accounts for the failure, in non-conforming settings, to exactly verify global integration by parts formulas.

In the context of Finite Element (FE) approximations, discrete counterparts of the de Rham complex are obtained replacing each space in the sequence with a finite-dimensional subspace. These subspaces are built upon a conforming mesh of the domain, whose elements are restricted to a small number of shapes and, in practice, are most often tetrahedra; see [2] for a complete and extremely general exposition including an exhaustive bibliography, and also [14] on the link between Raviart–Thomas–Nédélec differential forms and FE systems. The restriction to conforming meshes made of standard elements can be a major shortcoming in complex applications, limiting, for example, the capacity for local refinement or mesh agglomeration; see, e.g., the preface of [21]. The extension of the FE approach to more general meshes including, e.g., polyhedral elements and non-matching interfaces, is not straightforward. Recent efforts in this direction have been made in [13, 29] (see also references therein), focusing mainly on the lowest-order case and with some limitations on the element shapes in three dimensions. The extension to specific element shapes has also been considered in [18, 27]. A recent generalisation of FE methods is provided by the Isogeometric Analysis, which is designed to facilitate exchanges with Computer Assisted Design software. In this framework, spline spaces and projection operators that verify a de Rham diagram have been developed in [11]; see also [12].

General polytopal meshes can be handled by several lowest-order methods grounded, to a different extent, in the seminal work of Whitney on geometric integration [33]. These methods share the common feature that discrete de Rham complexes are obtained by replacing both the spaces and operators with discrete counterparts. Specifically, the spaces consist of vectors of real numbers attached to mesh entities of dimension equal to the index of the space in the sequence (vertices for $H^1(\Omega)$, edges for $\mathbf{H}(\mathbf{curl}; \Omega)$, faces for $\mathbf{H}(\mathbf{div}; \Omega)$, and elements for $L^2(\Omega)$). In Mimetic Finite Differences, discrete vector calculus operators and L^2 -products are obtained by mimicking the Stokes theorem; see [6] for a complete exposition. Their extension to polytopal meshes has first been carried out in [30, 31], then analysed in [9, 10]; see also [26] for a link with the Mixed Hybrid Finite Volume methods of [25, 28] and [24, Section 2.5] along with [23, Section 3.5] and [1] for links with Hybrid High-Order methods. In the Discrete Geometric Approach, originally introduced in [17] and extended to polyhedral meshes in [15, 16], as well as in Compatible Discrete Operators [7, 8], the key notions are topological vector calculus operators (expressed in terms of incidence matrices) along with the Hodge operator. The role of the latter is to establish a link, through the introduction of physical parameters, between quantities defined on primal and dual mesh entities. All of the above schemes are limited to the lowest-order, and their analysis often relies on an interplay of functional and topological arguments that is not required in our approach.

Discretisation methods that provide arbitrary-order approximations on general polyhedral meshes have only recently appeared in the literature. A first example is provided by Virtual Element Methods, which can be described as FE methods where explicit expressions for the basis functions are not available at each point. A de Rham complex of virtual spaces on polyhedra has been recently proposed in [5],

with polynomial degrees decreasing by one at each application of the exterior derivative; see also the related works [3, 4] concerning applications to magnetostatics. In order to derive an actual discretisation scheme starting from the sequence of virtual spaces, a variational crime involving projections is required, so that the exactness properties of the virtual de Rham complex cannot be directly used in the analysis. A different approach is pursued in [20, 22], where a discrete de Rham (DDR) complex is presented, based on decompositions of full polynomial spaces into the range of vector calculus operators and their L^2 -orthogonal complements. This sequence involves discrete spaces and operators that appear, through discrete L^2 -products, in the formulation of discretisation methods. The analysis in [20, 22] focuses on a subset of properties **(P1)**–**(P2)** involved in the stability analysis of numerical schemes: local exactness ([22, Theorems 4.1 and 5.1]), global complex property, discrete counterparts of $\text{Im grad} = \text{Ker curl}$ for domains that do not enclose voids and $\text{Im div} = L^2(\Omega)$ ([20, Theorem 3]), and Poincaré inequalities for the divergence and the curl ([20, Theorems 18 and 20, respectively]).

In this series of papers, we present a new DDR sequence based on explicit complements of the ranges of vector calculus operators; these complements are easier to implement, and enable a complete proof of the full set of properties **(P1)**–**(P3)**. To the best of our knowledge, this is the first time that such a complete panel of results is available for an arbitrary-order polyhedral method compatible with the de Rham complex. The complements considered here are linked to the spaces appearing in the Koszul complex (see, e.g., [2, Chapter 7]) and enjoy two key properties on general polyhedral meshes: they are hierarchical (see Remark 1 below) and their traces on polyhedral faces or edges lie in appropriate polynomial spaces (cf. Proposition 30). These properties make it possible to prove discrete integration by parts formulas for the discrete potentials (see Remarks 9 below and also [19, Remarks 3, 4, and 5]) that play a key role in the proof of (adjoint) consistency. We note that these discrete integration by parts formula, and the consistency they entail, do not seem available when building the sequence on the orthogonal complements as in [22]. In this first paper we define the (DDR) sequence associated with a polyhedral mesh of a domain and prove properties **(P1)**–**(P2)** along with the commutation property in **(P3)**. The focus of the second paper [19] is on the proof of the remaining consistency results in property **(P3)**, and on the application of the theory developed in both papers to a model problem in magnetostatics.

The rest of the paper is organised as follows. In Section 2 we establish the general setting. Section 3 contains the definition of the DDR sequence along with key intermediate results for the discrete vector calculus operators (including the commutation property in **(P3)**) and the proof of **(P1)**. Discrete Poincaré inequalities corresponding to **(P2)** are proved in Section 4. Finally, Appendix A contains results on local polynomial spaces including those on the traces of the trimmed spaces constructed from the Koszul complements.

2 Setting

2.1 Domain and mesh

For any (measurable) set $Y \subset \mathbb{R}^3$, we denote by $h_Y := \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in Y\}$ its diameter and by $|Y|$ its Hausdorff measure. We consider meshes $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$, where: \mathcal{T}_h is a finite collection of open disjoint polyhedral elements such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$ and $h = \max_{T \in \mathcal{T}_h} h_T > 0$; \mathcal{F}_h is a finite collection of open planar faces; \mathcal{E}_h is the set collecting the open polygonal edges (line segments) of the faces; \mathcal{V}_h is the set collecting the edge endpoints. It is assumed, in what follows, that $(\mathcal{T}_h, \mathcal{F}_h)$ matches the conditions in [21, Definition 1.4]. We additionally assume that the polytopes in $\mathcal{T}_h \cup \mathcal{F}_h$ are simply connected and have connected Lipschitz-continuous boundaries. This notion of mesh is related to that of cellular (or CW) complex from algebraic topology; see, e.g., [32, Chapter 7].

The set collecting the mesh faces that lie on the boundary of a mesh element $T \in \mathcal{T}_h$ is denoted by \mathcal{F}_T . For any mesh element or face $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, we denote, respectively, by \mathcal{E}_Y and \mathcal{V}_Y the set of edges and vertices of Y .

2.2 Orientation of mesh entities and vector calculus operators on faces

For any face $F \in \mathcal{F}_h$, an orientation is set by prescribing a unit normal vector \mathbf{n}_F , and, for any mesh element $T \in \mathcal{T}_h$ sharing F , we denote by $\omega_{TF} \in \{-1, 1\}$ the orientation of F relative to T , that is, $\omega_{TF} = 1$ if \mathbf{n}_F points out of T , -1 otherwise. With this choice, $\omega_{TF}\mathbf{n}_F$ is the unit vector normal to F that points out of T . For any edge $E \in \mathcal{E}_h$, an orientation is set by prescribing the unit tangent vector \mathbf{t}_E . Denoting by $F \in \mathcal{F}_h$ a face such that $E \in \mathcal{E}_F$, its boundary ∂F is oriented counter-clockwise with respect to \mathbf{n}_F , and we denote by $\omega_{FE} \in \{-1, 1\}$ the (opposite of the) orientation of E relative to that ∂F : $\omega_{FE} = 1$ if \mathbf{t}_E points on E in the opposite orientation to ∂F , $\omega_{FE} = -1$ otherwise. We also denote by \mathbf{n}_{FE} the unit vector normal to E lying in the plane of F such that $(\mathbf{t}_E, \mathbf{n}_{FE})$ form a system of right-handed coordinates in the plane of F , so that the system of coordinates $(\mathbf{t}_E, \mathbf{n}_{FE}, \mathbf{n}_F)$ is right-handed in \mathbb{R}^3 . It can be checked that $\omega_{FE}\mathbf{n}_{FE}$ is the normal to E , in the plane where F lies, pointing out of F .

For any mesh face $F \in \mathcal{F}_h$, we denote by \mathbf{grad}_F and div_F the tangent gradient and divergence operators acting on smooth enough functions. Moreover, for any $r : F \rightarrow \mathbb{R}$ and $\mathbf{z} : F \rightarrow \mathbb{R}^2$ smooth enough, we define the two-dimensional vector and scalar curl operators such that

$$\mathbf{rot}_F r := \varrho_{-\pi/2}(\mathbf{grad}_F r) \quad \text{and} \quad \mathbf{rot}_F \mathbf{z} = \text{div}_F(\varrho_{-\pi/2}\mathbf{z}), \quad (2.1)$$

where $\varrho_{-\pi/2}$ is the rotation of angle $-\frac{\pi}{2}$ in the oriented tangent space to F .

2.3 Lebesgue and Sobolev spaces

For Y measured subset of \mathbb{R}^3 , we denote by $L^2(Y)$ the Lebesgue space spanned by functions that are square-integrable over Y . When Y is a subset of an n -dimensional variety, we will use the boldface notation $\mathbf{L}^2(Y) := L^2(Y)^n$ for the space of vector-valued fields over Y with square-integrable components. Given an integer l and $Y \in \{\Omega\} \cup \mathcal{T}_h \cup \mathcal{F}_h$, $H^l(Y)$ will denote the Sobolev space spanned by square-integrable functions whose partial derivatives of order up to l are also square-integrable. Denoting again by n the dimension of Y , we let $\mathbf{H}^l(Y) := H^l(Y)^n$ and $\mathbf{C}^l(Y) := C^l(Y)^n$. For all $F \in \mathcal{F}_h$, we let $\mathbf{H}(\mathbf{rot}; F) := \{\mathbf{v} \in \mathbf{L}^2(F) : \mathbf{rot}_F \mathbf{v} \in \mathbf{L}^2(F)\}$. Similarly, for all $Y \in \{\Omega\} \cup \mathcal{T}_h$, $\mathbf{H}(\mathbf{curl}; Y) := \{\mathbf{v} \in \mathbf{L}^2(Y) : \mathbf{curl} \mathbf{v} \in \mathbf{L}^2(Y)\}$ and $\mathbf{H}(\mathbf{div}; Y) := \{\mathbf{w} \in \mathbf{L}^2(Y) : \text{div} \mathbf{w} \in L^2(Y)\}$.

2.4 Polynomial spaces and decompositions

For a given integer $\ell \geq 0$, \mathbb{P}_n^ℓ denotes the space of n -variate polynomials of total degree $\leq \ell$, with the convention that $\mathbb{P}_0^\ell = \mathbb{R}$ for any ℓ and that $\mathbb{P}_n^{-1} := \{0\}$ for any n . For any $Y \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, we denote by $\mathcal{P}^\ell(Y)$ the space spanned by the restriction to Y of the functions in \mathbb{P}_3^ℓ . Denoting by $1 \leq n \leq 3$ the dimension of Y , $\mathcal{P}^\ell(Y)$ is isomorphic to \mathbb{P}_n^ℓ (see [21, Proposition 1.23]). In what follows, with a little abuse of notation, both spaces are denoted by $\mathcal{P}^\ell(Y)$. We additionally denote by $\pi_{\mathcal{P}, Y}^\ell$ the corresponding L^2 -orthogonal projector and let $\mathcal{P}^{0, \ell}(Y)$ denote the subspace of $\mathcal{P}^\ell(Y)$ made of polynomials with zero average over Y . For the sake of brevity, we also introduce the boldface notations $\mathcal{P}^\ell(T) := \mathcal{P}^\ell(T)^3$ for all $T \in \mathcal{T}_h$ and $\mathcal{P}^\ell(F) := \mathcal{P}^\ell(F)^2$ for all $F \in \mathcal{F}_h$.

Let again an integer $\ell \geq 1$ be given, and denote by $\mathfrak{E} \subset \mathcal{E}_h$ a collection of edges such that $S_{\mathfrak{E}} := \bigcup_{E \in \mathfrak{E}} \overline{E}$ forms a connected set. We denote by $\mathcal{P}_c^\ell(\mathfrak{E}) := \{q_{\mathfrak{E}} \in C^0(S_{\mathfrak{E}}) : (q_{\mathfrak{E}})|_E \in \mathcal{P}^\ell(E) \text{ for all } E \in \mathfrak{E}\}$ the space of functions over $S_{\mathfrak{E}}$ whose restriction to each edge $E \in \mathfrak{E}$ is a polynomial of total degree $\leq \ell$ and that are continuous at the edges endpoints; these endpoints are collected in the set $\mathcal{V}_{\mathfrak{E}} \subset \mathcal{V}_h$. Setting \mathbf{x}_V the coordinates vector of a vertex $V \in \mathcal{V}_h$, it can be easily checked that the following mapping is an isomorphism:

$$\mathcal{P}_c^\ell(\mathfrak{E}) \ni q_{\mathfrak{E}} \mapsto ((\pi_{\mathcal{P}, E}^{\ell-2}(q_{\mathfrak{E}})|_E)_{E \in \mathfrak{E}}, (q_{\mathfrak{E}}(\mathbf{x}_V))_{V \in \mathcal{V}_{\mathfrak{E}}}) \in \left(\bigotimes_{E \in \mathfrak{E}} \mathcal{P}^{\ell-2}(E) \right) \times \mathbb{R}^{\mathcal{V}_{\mathfrak{E}}}. \quad (2.2)$$

For all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, denote by \mathbf{x}_Y a point inside Y such that Y contains a ball centered at \mathbf{x}_Y of radius ρh_Y , where ρ is the mesh regularity parameter in [21, Definition 1.9]. For any mesh face $F \in \mathcal{F}_h$ and any integer $\ell \geq 0$, we define the following relevant subspaces of $\mathcal{P}^\ell(F)$:

$$\mathcal{G}^\ell(F) := \mathbf{grad}_F \mathcal{P}^{\ell+1}(F), \quad \mathcal{G}^{c,\ell}(F) := (\mathbf{x} - \mathbf{x}_F)^\perp \mathcal{P}^{\ell-1}(F), \quad (2.3a)$$

$$\mathcal{R}^\ell(F) := \mathbf{rot}_F \mathcal{P}^{\ell+1}(F), \quad \mathcal{R}^{c,\ell}(F) := (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{\ell-1}(F), \quad (2.3b)$$

(where \mathbf{y}^\perp is a shorthand for the rotated vector $\varrho_{-\pi/2}\mathbf{y}$) so that

$$\mathcal{P}^\ell(F) = \mathcal{G}^\ell(F) \oplus \mathcal{G}^{c,\ell}(F) = \mathcal{R}^\ell(F) \oplus \mathcal{R}^{c,\ell}(F). \quad (2.4)$$

Notice that the direct sums in the above expression are not L^2 -orthogonal in general. The L^2 -orthogonal projectors on the spaces (2.3) are, with obvious notation, $\pi_{\mathcal{G},F}^\ell$, $\pi_{\mathcal{G},F}^{c,\ell}$, $\pi_{\mathcal{R},F}^\ell$, and $\pi_{\mathcal{R},F}^{c,\ell}$. Similarly, for any mesh element $T \in \mathcal{T}_h$ and any integer $\ell \geq 0$ we introduce the following subspaces of $\mathcal{P}^\ell(T)$:

$$\mathcal{G}^\ell(T) := \mathbf{grad} \mathcal{P}^{\ell+1}(T), \quad \mathcal{G}^{c,\ell}(T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{\ell-1}(T), \quad (2.5a)$$

$$\mathcal{R}^\ell(T) := \mathbf{curl} \mathcal{P}^{\ell+1}(T), \quad \mathcal{R}^{c,\ell}(T) := (\mathbf{x} - \mathbf{x}_T) \mathcal{P}^{\ell-1}(T), \quad (2.5b)$$

so that

$$\mathcal{P}^\ell(T) = \mathcal{G}^\ell(T) \oplus \mathcal{G}^{c,\ell}(T) = \mathcal{R}^\ell(T) \oplus \mathcal{R}^{c,\ell}(T). \quad (2.6)$$

Also in this case, the direct sums above are not L^2 -orthogonal in general. The L^2 -orthogonal projectors on the spaces (2.5) are $\pi_{\mathcal{G},T}^\ell$, $\pi_{\mathcal{G},T}^{c,\ell}$, $\pi_{\mathcal{R},T}^\ell$, and $\pi_{\mathcal{R},T}^{c,\ell}$.

Remark 1 (Hierarchical complements). Unlike the L^2 -orthogonal complements considered in [22], the Koszul complements in (2.4) and (2.6) satisfy, for all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$ and all $\ell \geq 1$,

$$\mathcal{G}^{c,\ell-1}(Y) \subset \mathcal{G}^{c,\ell}(Y) \quad \text{and} \quad \mathcal{R}^{c,\ell-1}(Y) \subset \mathcal{R}^{c,\ell}(Y). \quad (2.7)$$

Remark 2 (Vector calculus isomorphisms on local polynomial spaces). For any polygon F , polyhedron T , and polynomial degree $\ell \geq 0$, a consequence of the polynomial exactness [2, Corollary 7.3] is that the following mappings are isomorphisms:

$$\mathbf{rot}_F : \mathcal{P}^{0,\ell}(F) \xrightarrow{\cong} \mathcal{R}^{\ell-1}(F) \quad (2.8)$$

$$\mathbf{div}_F : \mathcal{R}^{c,\ell}(F) \xrightarrow{\cong} \mathcal{P}^{\ell-1}(F), \quad \mathbf{div} : \mathcal{R}^{c,\ell}(T) \xrightarrow{\cong} \mathcal{P}^{\ell-1}(T), \quad (2.9)$$

$$\mathbf{curl} : \mathcal{G}^{c,\ell}(T) \xrightarrow{\cong} \mathcal{R}^{\ell-1}(T). \quad (2.10)$$

An estimate of the norms of the inverses of these differential isomorphisms is provided in Lemma 31 below.

Remark 3 (Composition of L^2 -orthogonal projectors). Let $\mathcal{X} \in \{\mathcal{G}, \mathcal{R}\}$, $\ell \geq -1$, and $Y \in \mathcal{T}_h \cup \mathcal{F}_h$. Using the definition of the L^2 -orthogonal projectors, and denoting by $\pi_{\mathcal{P},Y}^\ell$ the L^2 -orthogonal projector on $\mathcal{P}^\ell(Y)$, it holds

$$\pi_{\mathcal{X},Y}^\ell = \pi_{\mathcal{X},Y}^\ell \circ \pi_{\mathcal{P},Y}^\ell \quad \text{and} \quad \pi_{\mathcal{X},Y}^{c,\ell} = \pi_{\mathcal{X},Y}^{c,\ell} \circ \pi_{\mathcal{P},Y}^\ell. \quad (2.11)$$

In what follows, we will need the local Nédélec and Raviart–Thomas spaces: For $Y \in \mathcal{T}_h \cup \mathcal{F}_h$,

$$\mathcal{N}^\ell(Y) := \mathcal{G}^{\ell-1}(Y) \oplus \mathcal{G}^{c,\ell}(Y), \quad \mathcal{RT}^\ell(Y) := \mathcal{R}^{\ell-1}(Y) \oplus \mathcal{R}^{c,\ell}(Y). \quad (2.12)$$

These spaces sit between $\mathcal{P}^{\ell-1}(Y)$ and $\mathcal{P}^\ell(Y)$ and are therefore referred to as *trimmed* in the finite element literature. Notice that we have selected the index in (2.12) so as to reflect the maximum polynomial degrees of functions in each space and, as a result, it is shifted by +1 with respect to [20, 22].

2.5 Recovery operator

As mentioned above, the direct sums in (2.4) and (2.6) are not L^2 -orthogonal. The following lemma however shows that, for any of these decompositions, a given polynomial can be recovered from its orthogonal projections on each space in the sum.

Lemma 4 (Recovery operator). *Let E be a Euclidean space, S be a subspace of E , and S^c be a complement (not necessarily orthogonal) of S in E . Let π_S and π_S^c be, respectively, the orthogonal projections on S and S^c . Then, the mappings $\text{Id} - \pi_S \pi_S^c : E \rightarrow E$ and $\text{Id} - \pi_S^c \pi_S : E \rightarrow E$ are isomorphisms.*

We can therefore define the recovery operator $\mathfrak{R}_{S,S^c}(\cdot, \cdot) : S \times S^c \rightarrow E$ such that

$$\mathfrak{R}_{S,S^c}(\mathbf{b}, \mathbf{c}) := (\text{Id} - \pi_S \pi_S^c)^{-1}(\mathbf{b} - \pi_S \mathbf{c}) + (\text{Id} - \pi_S^c \pi_S)^{-1}(\mathbf{c} - \pi_S^c \mathbf{b}) \quad \forall (\mathbf{b}, \mathbf{c}) \in S \times S^c, \quad (2.13)$$

and this operator satisfies the following properties:

$$\pi_S(\mathfrak{R}_{S,S^c}(\mathbf{b}, \mathbf{c})) = \mathbf{b} \quad \text{and} \quad \pi_S^c(\mathfrak{R}_{S,S^c}(\mathbf{b}, \mathbf{c})) = \mathbf{c} \quad \forall (\mathbf{b}, \mathbf{c}) \in S \times S^c, \quad (2.14)$$

$$\mathbf{a} = \mathfrak{R}_{S,S^c}(\pi_S \mathbf{a}, \pi_S^c \mathbf{a}) \quad \forall \mathbf{a} \in E. \quad (2.15)$$

Proof. Let us denote by $\|\cdot\|$ the norm in E . To prove that $\text{Id} - \pi_S \pi_S^c$ is invertible, we show that the mapping $\pi_S \pi_S^c$ has a norm < 1 , which implies

$$(\text{Id} - \pi_S \pi_S^c)^{-1} = \sum_{n \geq 0} (\pi_S \pi_S^c)^n. \quad (2.16)$$

The space E being finite dimensional, it suffices to see that, for any $\mathbf{x} \in E$ with $\|\mathbf{x}\| = 1$, we have $\|\pi_S(\pi_S^c \mathbf{x})\| < 1$. Since π_S is an orthogonal projector, by Pythagoras' theorem we have $\|\pi_S(\pi_S^c \mathbf{x})\| \leq \|\pi_S^c \mathbf{x}\|$, with equality only if $\pi_S^c \mathbf{x} \in S$, that is, only if $\pi_S^c \mathbf{x} = \mathbf{0}$ since $\pi_S^c \mathbf{x} \in S^c$. In this case, $\|\pi_S(\pi_S^c \mathbf{x})\| = 0 < 1$. Otherwise, $\|\pi_S(\pi_S^c \mathbf{x})\| < \|\pi_S^c \mathbf{x}\| \leq \|\mathbf{x}\| = 1$, where the second inequality is a consequence of the fact that π_S^c is an orthogonal projection. This concludes the proof that $\text{Id} - \pi_S \pi_S^c$ is an isomorphism. The invertibility of $\text{Id} - \pi_S^c \pi_S$ is obtained similarly, exchanging the roles of S and S^c .

Let us prove the first relation in (2.14). The second follows using the same arguments. We expand $(\text{Id} - \pi_S \pi_S^c)^{-1}$ in (2.13) using the series (2.16) (and similarly for $(\text{Id} - \pi_S^c \pi_S)^{-1}$) to write

$$\begin{aligned} \pi_S(\mathfrak{R}_{S,S^c}(\mathbf{b}, \mathbf{c})) &= \pi_S \sum_{n \geq 0} (\pi_S \pi_S^c)^n (\mathbf{b} - \pi_S \mathbf{c}) + \pi_S \sum_{n \geq 0} (\pi_S^c \pi_S)^n (\mathbf{c} - \pi_S^c \mathbf{b}) \\ &= \left[\pi_S \sum_{n \geq 0} (\pi_S \pi_S^c)^n - \pi_S \sum_{n \geq 0} (\pi_S^c \pi_S)^n \pi_S^c \right] \mathbf{b} + \left[\pi_S \sum_{n \geq 0} (\pi_S^c \pi_S)^n - \pi_S \sum_{n \geq 0} (\pi_S \pi_S^c)^n \pi_S \right] \mathbf{c}. \end{aligned}$$

We have $\pi_S \sum_{n \geq 0} (\pi_S^c \pi_S)^n \pi_S^c = \pi_S \sum_{n \geq 1} (\pi_S \pi_S^c)^n$ (we have used $\pi_S \pi_S = \pi_S$ to introduce the pre-factor π_S) and the operator acting on \mathbf{b} above therefore reduces to π_S , and returns \mathbf{b} since $\mathbf{b} \in S$. As for the operator acting on \mathbf{c} , using again $\pi_S \pi_S = \pi_S$ shows that it is equal to 0. This concludes the proof of the first relation in (2.14).

Fix $\mathbf{a} \in E$ and set $\mathbf{z} := \mathbf{a} - \mathfrak{R}_{S,S^c}(\pi_S \mathbf{a}, \pi_S^c \mathbf{a})$. Applying (2.14) to $\mathbf{b} = \pi_S \mathbf{a}$ and $\mathbf{c} = \pi_S^c \mathbf{a}$ shows that $\pi_S \mathbf{z} = \pi_S^c \mathbf{z} = \mathbf{0}$. Since $E = S \oplus S^c$, we can write $\mathbf{z} = \mathbf{z}_S + \mathbf{z}_S^c$ with $\mathbf{z}_S \in S$ and $\mathbf{z}_S^c \in S^c$, and the definition of the orthogonal projectors on S and S^c therefore yield, with $(\cdot, \cdot)_E$ the scalar product on E ,

$$\|\mathbf{z}\|^2 = (\mathbf{z}, \mathbf{z})_E = (\mathbf{z}, \mathbf{z}_S)_E + (\mathbf{z}, \mathbf{z}_S^c)_E = (\pi_S \mathbf{z}, \mathbf{z}_S)_E + (\pi_S^c \mathbf{z}, \mathbf{z}_S^c)_E = 0.$$

Hence, $\mathbf{z} = \mathbf{0}$ and (2.15) is established. \square

The following lemma shows that the norm of the recovery operator for the decompositions (2.4) and (2.6) is equivalent to the sum of the norms of its arguments, uniformly in h . In other words, it states that the decompositions are not just algebraic but also topological (uniformly in h). Since the recovery operator will mostly be of interest to us for these pairs of spaces, to alleviate the notations from here on we will write

$$\mathfrak{R}_{\mathcal{X},Y}^\ell(\cdot, \cdot) := \mathfrak{R}_{\mathcal{X}^\ell(Y), \mathcal{X}^{c,\ell}(Y)}(\cdot, \cdot) \quad \forall \mathcal{X} \in \{\mathcal{R}, \mathcal{G}\}, \forall Y \in \mathcal{T}_h \cup \mathcal{F}_h. \quad (2.17)$$

Lemma 5 (Estimate on the norm of the recovery operator). *For all $\ell \geq 0$, there exists $\alpha < 1$ depending only on the mesh regularity parameter in [21, Definition 1.9] such that, for all $\mathcal{X} \in \{\mathcal{R}, \mathcal{G}\}$ and all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$,*

$$\|\pi_{\mathcal{X},Y}^\ell \pi_{\mathcal{X},Y}^{c,\ell}\|_Y \leq \alpha \quad \text{and} \quad \|\pi_{\mathcal{X},Y}^{c,\ell} \pi_{\mathcal{X},Y}^\ell\|_Y \leq \alpha, \quad (2.18)$$

where $\|\cdot\|_Y$ denotes the norm induced by $\|\cdot\|_{\mathbf{L}^2(Y)}$ on the space of endomorphisms of $\mathcal{P}^\ell(Y)$. As a result,

$$\|\mathfrak{R}_{\mathcal{X},Y}^\ell(\mathbf{v}, \mathbf{w})\|_{\mathbf{L}^2(Y)} \simeq \|\mathbf{v}\|_{\mathbf{L}^2(Y)} + \|\mathbf{w}\|_{\mathbf{L}^2(Y)} \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathcal{X}^\ell(Y) \times \mathcal{X}^{c,\ell}(Y), \quad (2.19)$$

where $a \simeq b$ means $C^{-1}a \leq b \leq Ca$ with $C > 0$ depending only on the mesh regularity parameter.

Remark 6 (Recovery operator and \mathbf{L}^2 -orthogonal complements). When working with \mathbf{L}^2 -orthogonal complements to $\mathcal{G}^\ell(Y)$ and $\mathcal{R}^\ell(Y)$, instead of the Koszul complements in (2.3) and (2.5), the recovery operator is trivial since it consists in the sum of its two arguments (its topological property (2.19) is also obvious). As mentioned in the introduction, however, the Koszul complements enable proofs of commutation and consistency properties that do not seem straightforward with orthogonal complements; the trade-off lies in having to deal with a less trivial recovery operator (although it purely remains a theoretical tool, see Remark 9), whose topological properties are more complex to establish.

Proof. 1. Proof of (2.18). We estimate $\|\pi_{\mathcal{G},T}^\ell \pi_{\mathcal{G},T}^{c,\ell}\|_T$ for an element $T \in \mathcal{T}_h$, the other cases being identical. The linear mapping $\mathbb{R}^3 \ni \mathbf{x} \mapsto h_T^{-1}(\mathbf{x} - \mathbf{x}_T) \in \mathbb{R}^3$ maps T onto a polyhedron \widehat{T} of diameter 1, transports the spaces $\mathcal{P}^\ell(T)$, $\mathcal{G}^\ell(T)$ and $\mathcal{G}^{c,\ell}(T)$ only their equivalent over \widehat{T} , and simply scales the \mathbf{L}^2 -norm of functions. As a consequence, $\|\pi_{\mathcal{G},T}^\ell \pi_{\mathcal{G},T}^{c,\ell}\|_T = \|\pi_{\mathcal{G},\widehat{T}}^\ell \pi_{\mathcal{G},\widehat{T}}^{c,\ell}\|_{\widehat{T}}$, and we only have to estimate the latter quantity.

Assume that we establish the existence of $\alpha < 1$, depending only on the mesh regularity parameter, such that

$$\int_{\widehat{T}} \mathbf{v} \cdot \pi_{\mathcal{G},\widehat{T}}^\ell \mathbf{v} \leq \alpha^2 \|\mathbf{v}\|_{\mathbf{L}^2(\widehat{T})} \|\pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{v}\|_{\mathbf{L}^2(\widehat{T})} \quad \forall \mathbf{v} \in \mathcal{G}^{c,\ell}(\widehat{T}) = \mathbf{x} \times \mathcal{P}^{\ell-1}(\widehat{T}). \quad (2.20)$$

Notice that, with the selected mapping, \mathbf{x}_T is mapped onto $\mathbf{0} \in \widehat{T}$. Then, for all $\mathbf{w} \in \mathcal{P}^\ell(T)$,

$$\begin{aligned} \int_{\widehat{T}} \pi_{\mathcal{G},\widehat{T}}^\ell(\pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}) \cdot \pi_{\mathcal{G},\widehat{T}}^\ell(\pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}) &= \int_{\widehat{T}} \pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w} \cdot \pi_{\mathcal{G},\widehat{T}}^\ell(\pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}) \\ &\leq \alpha^2 \|\pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}\|_{\mathbf{L}^2(\widehat{T})} \|\pi_{\mathcal{G},\widehat{T}}^\ell \pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}\|_{\mathbf{L}^2(\widehat{T})} \leq \alpha^2 \|\mathbf{w}\|_{\mathbf{L}^2(\widehat{T})}^2, \end{aligned} \quad (2.21)$$

where the first equality comes from the definition of $\pi_{\mathcal{G},\widehat{T}}^\ell$, the first inequality is obtained applying (2.20) to $\mathbf{v} = \pi_{\mathcal{G},\widehat{T}}^{c,\ell} \mathbf{w}$, and the conclusion is obtained using the fact that $\pi_{\mathcal{G},\widehat{T}}^\ell$ and $\pi_{\mathcal{G},\widehat{T}}^{c,\ell}$ are both $\mathbf{L}^2(\widehat{T})$ -orthogonal projectors and have thus norm 1. The bound (2.21) shows that $\|\pi_{\mathcal{G},\widehat{T}}^\ell \pi_{\mathcal{G},\widehat{T}}^{c,\ell}\|_{\widehat{T}} \leq \alpha$ and concludes the proof.

We therefore only have to establish (2.20). Note that, in the rest of the proof, polynomials are indifferently considered over \mathbb{R}^3 or some of its open subsets. We also remark that, by choice of \mathbf{x}_T in

T and of the mapping $T \mapsto \widehat{T}$, we have $B(\rho) \subset \widehat{T} \subset B(1)$, where $B(r)$ is the ball in \mathbb{R}^d centered at $\mathbf{0}$ and of radius r . The proof of (2.20) is done by contradiction: if this relation does not hold, there exists a sequence $(\widehat{T}_n)_{n \in \mathbb{N}}$ of open sets between $B(\rho)$ and $B(1)$, a sequence $(\alpha_n)_{n \in \mathbb{N}}$ converging to 1, and a sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$ in $\mathbf{x} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3)$ such that

$$\int_{\widehat{T}_n} \mathbf{v}_n \cdot \boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n > \alpha_n^2 \|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)} \|\boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)}. \quad (2.22)$$

Upon replacing \mathbf{v}_n by $\mathbf{v}_n / \|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)}$, we can assume that $\|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)} = 1$. Since $B(\rho) \subset \widehat{T}_n$, we infer that $\|\mathbf{v}_n\|_{\mathbf{L}^2(B(\rho))} \leq \|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)} = 1$; hence, $(\mathbf{v}_n)_{n \in \mathbb{N}}$ is bounded for the $\mathbf{L}^2(B(\rho))$ -norm in the finite-dimensional space $\mathbf{x} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3)$, and converges up to a subsequence to some $\mathbf{v} \in \mathbf{x} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3)$. Likewise, we can assume that $\boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n \rightarrow \mathbf{w}$ in $\mathcal{G}^\ell(\mathbb{R}^3)$. The characteristic function $\mathbf{1}_{\widehat{T}_n}$ satisfies $\mathbf{1}_{B(\rho)} \leq \mathbf{1}_{\widehat{T}_n} \leq \mathbf{1}_{B(1)}$ and converges therefore, up to a subsequence, in $L^\infty(B(1))$ weak- \star towards some function θ satisfying $\mathbf{1}_{B(\rho)} \leq \theta \leq \mathbf{1}_{B(1)}$. Noting that

$$\begin{aligned} \int_{\widehat{T}_n} \mathbf{v}_n \cdot \boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n &= \int_{B(1)} \mathbf{1}_{\widehat{T}_n} \mathbf{v}_n \cdot \boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n, \\ \|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)}^2 &= \int_{B(1)} \mathbf{1}_{\widehat{T}_n} |\mathbf{v}_n|^2, \quad \text{and} \quad \|\boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)}^2 = \int_{B(1)} \mathbf{1}_{\widehat{T}_n} |\boldsymbol{\pi}_{\mathcal{G}, \widehat{T}_n}^\ell \mathbf{v}_n|^2, \end{aligned}$$

the aforementioned convergences enable us to take the limit $n \rightarrow \infty$ of (2.22) and find

$$\int_{B(1)} \theta \mathbf{v} \cdot \mathbf{w} \geq \|\sqrt{\theta} \mathbf{v}\|_{\mathbf{L}^2(B(1))} \|\sqrt{\theta} \mathbf{w}\|_{\mathbf{L}^2(B(1))}. \quad (2.23)$$

The Cauchy–Schwarz inequality, on the other hand, gives

$$\int_{B(1)} \theta \mathbf{v} \cdot \mathbf{w} = \int_{B(1)} \sqrt{\theta} \mathbf{v} \cdot \sqrt{\theta} \mathbf{w} \leq \|\sqrt{\theta} \mathbf{v}\|_{\mathbf{L}^2(B(1))} \|\sqrt{\theta} \mathbf{w}\|_{\mathbf{L}^2(B(1))},$$

which, combined with (2.23), shows that,

$$\int_{B(1)} \theta \mathbf{v} \cdot \mathbf{w} = \|\sqrt{\theta} \mathbf{v}\|_{\mathbf{L}^2(B(1))} \|\sqrt{\theta} \mathbf{w}\|_{\mathbf{L}^2(B(1))}.$$

Hence, $\sqrt{\theta} \mathbf{v}$ and $\sqrt{\theta} \mathbf{w}$ are co-linear. Restricted to $B(\rho)$, over which $\theta = 1$, this proves that \mathbf{v} and \mathbf{w} are co-linear. Since $\mathbf{v} \in \mathcal{G}^{c,\ell}(B(\rho))$ and $\mathbf{w} \in \mathcal{G}^\ell(B(\rho))$, we infer that $\mathbf{v} = \mathbf{w} = \mathbf{0}$ on $B(\rho)$, and thus on \mathbb{R}^3 . This leads to $0 = \|\sqrt{\theta} \mathbf{v}\|_{\mathbf{L}^2(B(1))} = \lim_{n \rightarrow \infty} \|\mathbf{v}_n\|_{\mathbf{L}^2(\widehat{T}_n)} = 1$, which yields the sought contradiction.

2. Proof of (2.19). By (2.13), recalling the abridged notation (2.17), we have

$$\begin{aligned} \|\mathfrak{R}_{\mathcal{X}, Y}^\ell(\mathbf{v}, \mathbf{w})\|_{\mathbf{L}^2(Y)} &\leq \|(\text{Id} - \boldsymbol{\pi}_{\mathcal{X}, Y}^\ell \boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell})^{-1}\|_Y \left(\|\mathbf{v}\|_{\mathbf{L}^2(Y)} + \|\boldsymbol{\pi}_{\mathcal{X}, Y}^\ell \mathbf{w}\|_{\mathbf{L}^2(Y)} \right) \\ &\quad + \|(\text{Id} - \boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell} \boldsymbol{\pi}_{\mathcal{X}, Y}^\ell)^{-1}\|_Y \left(\|\boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell} \mathbf{v}\|_{\mathbf{L}^2(Y)} + \|\mathbf{w}\|_{\mathbf{L}^2(Y)} \right). \end{aligned}$$

The expansion (2.16) and the estimates (2.18) show that

$$\|(\text{Id} - \boldsymbol{\pi}_{\mathcal{X}, Y}^\ell \boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell})^{-1}\|_Y \leq \sum_{n \geq 0} \|\boldsymbol{\pi}_{\mathcal{X}, Y}^\ell \boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell}\|_Y^n \leq \sum_{n \geq 0} \alpha^n = \frac{1}{1 - \alpha}$$

and, similarly, $\|(\text{Id} - \boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell} \boldsymbol{\pi}_{\mathcal{X}, Y}^\ell)^{-1}\|_Y \leq \frac{1}{1 - \alpha}$. Since $\|\boldsymbol{\pi}_{\mathcal{X}, Y}^\ell \mathbf{w}\|_{\mathbf{L}^2(Y)} \leq \|\mathbf{w}\|_{\mathbf{L}^2(Y)}$ and $\|\boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell} \mathbf{v}\|_{\mathbf{L}^2(Y)} \leq \|\mathbf{v}\|_{\mathbf{L}^2(Y)}$ as both $\boldsymbol{\pi}_{\mathcal{X}, Y}^\ell$ and $\boldsymbol{\pi}_{\mathcal{X}, Y}^{c,\ell}$ are \mathbf{L}^2 -orthogonal projectors, we conclude that

$$\|\mathfrak{R}_{\mathcal{X}, Y}^\ell(\mathbf{v}, \mathbf{w})\|_{\mathbf{L}^2(Y)} \leq \frac{2}{1 - \alpha} \left(\|\mathbf{v}\|_{\mathbf{L}^2(Y)} + \|\mathbf{w}\|_{\mathbf{L}^2(Y)} \right).$$

To prove the converse inequality, we use (2.14) along with the L^2 -boundedness of $\pi_{\mathcal{X},Y}^\ell$ and $\pi_{\mathcal{X},Y}^{c,\ell}$ to write

$$\|\mathbf{v}\|_{L^2(Y)} + \|\mathbf{w}\|_{L^2(Y)} = \|\pi_{\mathcal{X},Y}^\ell \mathfrak{R}_{\mathcal{X},Y}^\ell(\mathbf{v}, \mathbf{w})\|_{L^2(Y)} + \|\pi_{\mathcal{X},Y}^{c,\ell} \mathfrak{R}_{\mathcal{X},Y}^\ell(\mathbf{v}, \mathbf{w})\|_{L^2(Y)} \leq 2\|\mathfrak{R}_{\mathcal{X},Y}^\ell(\mathbf{v}, \mathbf{w})\|_{L^2(Y)}.$$

This concludes the proof of the norm equivalence (2.19). \square

3 Discrete de Rham sequence

We define in this section a discrete counterpart of the de Rham sequence (1.1). Throughout the rest of this section, we fix an integer $k \geq 0$ corresponding to the polynomial degree of the discrete sequence.

3.1 Discrete spaces

The DDR spaces are spanned by vectors of polynomials whose components, each attached to a mesh entity, are selected in order to:

- 1) enable the reconstruction of consistent local discrete vector calculus operators and (scalar or vector) potentials in full polynomial spaces of total degree $\leq k$ (or $\leq k + 1$ for the potentials associated with the gradient);
- 2) give rise to exact local sequences on mesh elements and faces.

Specifically, the discrete counterparts of the spaces $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl}; \Omega)$, $\mathbf{H}(\mathbf{div}; \Omega)$, and $L^2(\Omega)$ are respectively defined as follows:

$$\underline{X}_{\mathbf{grad},h}^k := \left\{ \underline{q}_h = ((q_T)_{T \in \mathcal{T}_h}, (q_F)_{F \in \mathcal{F}_h}, q_{\mathcal{E}_h}) : \begin{aligned} & q_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h, \\ & q_F \in \mathcal{P}^{k-1}(F) \text{ for all } F \in \mathcal{F}_h, \\ & \text{and } q_{\mathcal{E}_h} \in \mathcal{P}_c^{k+1}(\mathcal{E}_h) \end{aligned} \right\},$$

$$\underline{X}_{\mathbf{curl},h}^k := \left\{ \underline{v}_h = ((\mathbf{v}_{\mathcal{R},T}, \mathbf{v}_{\mathcal{R},T}^c)_{T \in \mathcal{T}_h}, (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c)_{F \in \mathcal{F}_h}, (v_E)_{E \in \mathcal{E}_h}) : \begin{aligned} & \mathbf{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \mathbf{v}_{\mathcal{R},T}^c \in \mathcal{R}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h, \\ & \mathbf{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \mathbf{v}_{\mathcal{R},F}^c \in \mathcal{R}^{c,k}(F) \text{ for all } F \in \mathcal{F}_h, \\ & \text{and } v_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h \end{aligned} \right\},$$

$$\underline{X}_{\mathbf{div},h}^k := \left\{ \underline{w}_h = ((\mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c)_{T \in \mathcal{T}_h}, (w_F)_{F \in \mathcal{F}_h}) : \begin{aligned} & \mathbf{w}_{\mathcal{G},T} \in \mathcal{G}^{k-1}(T) \text{ and } \mathbf{w}_{\mathcal{G},T}^c \in \mathcal{G}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h, \\ & \text{and } w_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \end{aligned} \right\},$$

and

$$\mathcal{P}^k(\mathcal{T}_h) := \{q_h \in L^2(\Omega) : (q_h)|_T \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Remark 7 (Component of $\underline{X}_{\mathbf{grad},h}^k$ on the mesh edge skeleton). By the isomorphism (2.2) with $\mathfrak{E} = \mathcal{E}_h$, we can replace the space $\mathcal{P}_c^{k+1}(\mathcal{E}_h)$ in the definition of $\underline{X}_{\mathbf{grad},h}^k$ by the Cartesian product space $(\times_{E \in \mathcal{E}_h} \mathcal{P}^{k-1}(E)) \times \mathbb{R}^{\mathcal{V}_h}$. This product space is in particular easier to manipulate in implementations of the DDR complex.

Remark 8 (Components of $\underline{X}_{\mathbf{curl},h}^k$ and $\underline{X}_{\mathbf{div},h}^k$). For each mesh element or face $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, the pair of components $(\mathbf{v}_{\mathcal{R},Y}, \mathbf{v}_{\mathcal{R},Y}^c)$ of a vector in $\underline{X}_{\mathbf{curl},h}^k$ defines an element in $\mathcal{RT}^k(Y)$. Similarly, for any $T \in \mathcal{T}_h$, each pair of element components $(\mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c)$ of a vector in $\underline{X}_{\mathbf{div},h}^k$ defines an element in $\mathcal{N}^k(T)$.

Index	Space	V	E	F	T
0	$\underline{X}_{\text{grad},h}^k$	$\mathbb{R} = \mathcal{P}^k(V)$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
1	$\underline{X}_{\text{curl},h}^k$		$\mathcal{P}^k(E)$	$\mathcal{R}^{k-1}(F) \times \mathcal{R}^{c,k}(F)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{c,k}(T)$
2	$\underline{X}_{\text{div},h}^k$			$\mathcal{P}^k(F)$	$\mathcal{G}^{k-1}(T) \times \mathcal{G}^{c,k}(T)$
3	$\mathcal{P}^k(\mathcal{T}_h)$				$\mathcal{P}^k(T)$

Table 1: Polynomial components attached to each mesh vertex $V \in \mathcal{V}_h$, edge $E \in \mathcal{E}_h$, face $F \in \mathcal{F}_h$, and element $T \in \mathcal{T}_h$ for each of the DDR spaces.

The polynomial components attached to mesh vertices, edges, faces, and elements for each of the DDR spaces are summarised in Table 1 (notice that we have accounted for Remark 7 for $\underline{X}_{\text{grad},h}^k$). An inspection of Table 1 reveals that its diagonal contains full polynomial spaces on the mesh entities of dimension corresponding to the index of the space in the sequence (with the convention that $\mathcal{P}^k(V) := \mathbb{R}$ for any vertex $V \in \mathcal{V}_h$). The components collected in the upper triangular portion of the table are non-zero only for $k \geq 1$, and encode additional information required for the reconstruction of high-order discrete vector calculus operators and potentials. In particular, the complements $\mathcal{R}^{c,k}(F)$, $\mathcal{R}^{c,k}(T)$, and $\mathcal{G}^{c,k}(T)$ complete the information contained, respectively, in the face curl, element curl and tangential trace, and element divergence to construct the corresponding face or element vector potentials; see [19, Sections 3.1.2 and 3.1.3].

In what follows, given $\bullet \in \{\text{grad}, \text{curl}, \text{div}\}$ and a mesh entity Y of dimension greater than or equal to the index of $\underline{X}_{\bullet,h}^k$, we denote by $\underline{X}_{\bullet,Y}^k$ the restriction of this space to Y , i.e., $\underline{X}_{\bullet,Y}^k$ contains the polynomial components attached to Y and to all the mesh entities that lie on its boundary.

3.2 Interpolators

In the following, for all $q_h \in \underline{X}_{\text{grad},h}^k$, we set

$$q_E := (q_{\mathcal{E}_h})|_E \in \mathcal{P}^{k+1}(E). \quad (3.1)$$

The interpolators on the DDR spaces are defined collecting component-wise L^2 -projections. Specifically $\underline{I}_{\text{grad},h}^k : C^0(\bar{\Omega}) \rightarrow \underline{X}_{\text{grad},h}^k$ is such that, for all $q \in C^0(\bar{\Omega})$,

$$\underline{I}_{\text{grad},h}^k q := ((\pi_{\mathcal{P},T}^{k-1} q|_T)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P},F}^{k-1} q|_F)_{F \in \mathcal{F}_h}, q_{\mathcal{E}_h}) \in \underline{X}_{\text{grad},h}^k$$

where $\pi_{\mathcal{P},E}^{k-1}(q_{\mathcal{E}_h})|_E = \pi_{\mathcal{P},E}^{k-1} q|_E$ for all $E \in \mathcal{E}_h$, and $q_{\mathcal{E}_h}(\mathbf{x}_V) = q(\mathbf{x}_V)$ for all $V \in \mathcal{V}_h$. (3.2)

$\underline{I}_{\text{curl},h}^k : C^0(\bar{\Omega}) \rightarrow \underline{X}_{\text{curl},h}^k$ is defined setting, for all $\mathbf{v} \in C^0(\bar{\Omega})$,

$$\underline{I}_{\text{curl},h}^k \mathbf{v} := ((\pi_{\mathcal{R},T}^{k-1} \mathbf{v}|_T, \pi_{\mathcal{R},T}^{c,k} \mathbf{v}|_T)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{R},F}^{k-1} \mathbf{v}_{t,F}, \pi_{\mathcal{R},F}^{c,k} \mathbf{v}_{t,F})_{F \in \mathcal{F}_h}, (\pi_{\mathcal{P},E}^k(\mathbf{v}|_E \cdot \mathbf{t}_E))_{E \in \mathcal{E}_h}),$$

where $\mathbf{v}_{t,F} := \mathbf{n}_F \times (\mathbf{v}|_F \times \mathbf{n}_F)$ denotes the tangent trace of \mathbf{v} over F . Finally, $\underline{I}_{\text{div},h}^k : \mathbf{H}^1(\Omega) \rightarrow \underline{X}_{\text{div},h}^k$ is such that, for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$,

$$\underline{I}_{\text{div},h}^k \mathbf{w} := ((\pi_{\mathcal{G},T}^{k-1} \mathbf{w}|_T, \pi_{\mathcal{G},T}^{c,k} \mathbf{w}|_T)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P},F}^k(\mathbf{w}|_F \cdot \mathbf{n}_F))_{F \in \mathcal{F}_h}).$$

The restriction of the above interpolators to a mesh entity Y of dimension larger than or equal to the index of the corresponding space in the sequence (see Table 1) is denoted replacing the subscript h by Y . Finally, we let $\pi_{\mathcal{P},h}^k : L^2(\Omega) \rightarrow \mathcal{P}^k(\mathcal{T}_h)$ denote the global L^2 -orthogonal projector such that, for all $q \in L^2(\Omega)$, $(\pi_{\mathcal{P},h}^k q)|_T = \pi_{\mathcal{P},T}^k q|_T$ for all $T \in \mathcal{T}_h$.

3.3 Discrete vector calculus operators

We define in this section the discrete vector calculus operators that appear in the DDR sequence, obtained collecting the L^2 -orthogonal projections of local discrete operators mapping on full polynomial spaces. In what follows, the operators that only appear in the discrete sequence (3.33) through projections are denoted in sans serif font, while those appearing verbatim (without projection) in the sequence are in italic font.

3.3.1 Gradient

The discrete counterpart of the gradient operator in the DDR sequence maps on $\underline{X}_{\text{curl},h}^k$, and therefore requires to define local gradients on mesh edges, faces, and elements.

For any $E \in \mathcal{E}_h$, the *edge gradient* $G_E^k : \underline{X}_{\text{grad},E}^k \rightarrow \mathcal{P}^k(E)$ is defined as: For all $q_E \in \underline{X}_{\text{grad},E}^k = \mathcal{P}^{k+1}(E)$,

$$G_E^k q_E := q_E', \quad (3.3)$$

where the derivative is taken along E according to the orientation of t_E .

For any $F \in \mathcal{F}_h$, the *face gradient* $\mathbf{G}_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(F)$ is such that, for all $\underline{q}_F = (q_F, q_{\mathcal{E}_F}) \in \underline{X}_{\text{grad},F}^k$ and all $\mathbf{w}_F \in \mathcal{P}^k(F)$,

$$\int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{w}_F = - \int_F q_F \operatorname{div}_F \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F} (\mathbf{w}_F \cdot \mathbf{n}_{FE}) \quad (3.4)$$

$$= \int_F \mathbf{grad}_F q_F \cdot \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (q_{\mathcal{E}_F} - q_F) (\mathbf{w}_F \cdot \mathbf{n}_{FE}). \quad (3.5)$$

The existence and uniqueness of $\mathbf{G}_F^k \underline{q}_F$ in $\mathcal{P}^k(F)$ follow from the Riesz representation theorem applied to this space equipped with the usual L^2 -product. Similar considerations hold for the other discrete vector calculus operators defined below, and will not be repeated.

The *scalar trace* $\gamma_F^{k+1} : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F)$ is such that, for all $\underline{q}_F \in \underline{X}_{\text{grad},F}^k$,

$$\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \mathbf{v}_F = - \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F} (\mathbf{v}_F \cdot \mathbf{n}_{FE}) \quad \forall \mathbf{v}_F \in \mathcal{R}^{c,k+2}(F). \quad (3.6)$$

This relation defines $\gamma_F^{k+1} \underline{q}_F$ uniquely in $\mathcal{P}^{k+1}(F)$ owing to the isomorphism property (2.9) with $\ell = k+2$.

Remark 9 (Validity of (3.6)). The relation (3.6) holds, in fact, for any $\mathbf{v}_F \in \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k+2}(F)$. To check it, take $\mathbf{v}_F \in \mathcal{R}^k(F)$ and notice that the left-hand side vanishes owing to $\operatorname{div}_F \mathbf{v}_F = 0$, while the right-hand side vanishes owing to the definition (3.4) of $\mathbf{G}_F^k \underline{q}_F$ and again $\operatorname{div}_F \mathbf{v}_F = 0$. This means, in particular, that (3.6) holds for any $\mathbf{v}_F \in \mathcal{RT}^{k+1}(F) \subset \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k+2}(F)$ (see Remark 1).

For all $T \in \mathcal{T}_h$, the *element gradient* $\mathbf{G}_T^k : \underline{X}_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)$ is defined such that, for all $\underline{q}_T = (q_T, (q_F)_{F \in \mathcal{F}_T}, q_{\mathcal{E}_T}) \in \underline{X}_{\text{grad},T}^k$ and all $\mathbf{w}_T \in \mathcal{P}^k(T)$,

$$\int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{w}_T = - \int_T q_T \operatorname{div} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\mathbf{w}_T \cdot \mathbf{n}_F) \quad (3.7)$$

$$= \int_T \mathbf{grad} q_T \cdot \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (\gamma_F^{k+1} \underline{q}_F - q_T) (\mathbf{w}_T \cdot \mathbf{n}_F). \quad (3.8)$$

Lemma 10 (Consistency properties). *The edge, face, and element gradients, and scalar trace satisfy the following consistency properties:*

$$\forall E \in \mathcal{E}_h \quad G_E^k(\underline{I}_{\text{grad},E}^k q) = \pi_{\mathcal{P},E}^k(q') \quad \forall q \in H^1(E), \quad (3.9)$$

$$\forall F \in \mathcal{F}_h \quad \mathbf{G}_F^k(\underline{I}_{\text{grad},F}^k q) = \mathbf{grad}_F q \quad \forall q \in \mathcal{P}^{k+1}(F), \quad (3.10)$$

$$\forall F \in \mathcal{F}_h \quad \gamma_F^{k+1}(\underline{I}_{\text{grad},F}^k q) = q \quad \forall q \in \mathcal{P}^{k+1}(F), \quad (3.11)$$

$$\forall F \in \mathcal{F}_h \quad \pi_{\mathcal{P},F}^{k-1}(\gamma_F^{k+1} \underline{q}_F) = q_F \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k, \quad (3.12)$$

$$\forall T \in \mathcal{T}_h \quad \mathbf{G}_T^k(\underline{I}_{\text{grad},T}^k q) = \mathbf{grad} q \quad \forall q \in \mathcal{P}^{k+1}(T). \quad (3.13)$$

Proof. Let us prove (3.9). Take $q \in H^1(E)$. For all $r_E \in \mathcal{P}^k(E)$, denoting by \mathbf{x}_{V_1} and \mathbf{x}_{V_2} the coordinates of the vertices V_1 and V_2 of E , oriented so that \mathbf{t}_E points from V_1 to V_2 , we have

$$\begin{aligned} \int_E (\underline{I}_{\text{grad},E}^k q)' r_E &= (\underline{I}_{\text{grad},E}^k q r_E)(\mathbf{x}_{V_2}) - (\underline{I}_{\text{grad},E}^k q r_E)(\mathbf{x}_{V_1}) - \int_E (\underline{I}_{\text{grad},E}^k q) r'_E \\ &= (q r_E)(\mathbf{x}_{V_2}) - (q r_E)(\mathbf{x}_{V_1}) - \int_E q r'_E = \int_E q' r_E, \end{aligned}$$

where we have used an integration by parts in the first line, obtained the second equality applying the definition of $\underline{I}_{\text{grad},E}^k q \in \mathcal{P}^{k+1}(E)$ (which satisfies $(\underline{I}_{\text{grad},E}^k q)(\mathbf{x}_V) = q(\mathbf{x}_V)$ for all $V \in \mathcal{V}_E$ and $\pi_{\mathcal{P},E}^{k-1}(\underline{I}_{\text{grad},E}^k q) = \pi_{\mathcal{P},E}^{k-1} q$) together with $r'_E \in \mathcal{P}^{k-1}(E)$, and used another integration by parts to conclude. This proves that $(\underline{I}_{\text{grad},E}^k q)' = \pi_{\mathcal{P},E}^k(q')$.

Relation (3.10) can be deduced as in [22, Proposition 4.1]. To prove (3.11), we write (3.6) for $\underline{q}_F = \underline{I}_{\text{grad},F}^k q$ with $q \in \mathcal{P}^{k+1}(T)$, use (3.10) and notice that $q_{\mathcal{E}_F} = q|_{\partial F}$ (since $q|_{\partial F} \in \mathcal{P}_c^{k+1}(\mathcal{E}_F)$) to get, for all $\mathbf{v}_F \in \mathcal{R}^{c,k+2}(F)$,

$$\int_F \gamma_F^{k+1}(\underline{I}_{\text{grad},F}^k q) \operatorname{div}_F \mathbf{v}_F = - \int_F \mathbf{grad}_F q \cdot \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q|_{\partial F} (\mathbf{v}_F \cdot \mathbf{n}_{FE}) = \int_F q \operatorname{div}_F \mathbf{v}_F.$$

The isomorphism property (2.9) with $\ell = k + 2$ then concludes the proof that $\gamma_F^{k+1}(\underline{I}_{\text{grad},F}^k q) = q$.

The equality (3.12) follows from (3.6) written for $\mathbf{v}_F \in \mathcal{R}^{c,k}(F)$ (this choice is made possible by (2.7)) after replacing the full face gradient \mathbf{G}_F^k by its definition (3.4), simplifying the boundary terms, and invoking again the isomorphism property (2.9), this time with $\ell = k$.

Finally, (3.13) can be established from (3.11) following the ideas in [22, Lemma 5.1]. \square

The following proposition contains a stronger version of [22, Eq. (5.16)], with test function taken in the Nédélec space $\mathcal{N}^{k+1}(T)$ instead of $\mathcal{P}^k(T)$.

Proposition 11 (Link between element and face gradients). *For all $T \in \mathcal{T}_h$ and all $(\underline{q}_T, \mathbf{z}_T) \in \underline{X}_{\text{grad},T}^k \times \mathcal{N}^{k+1}(T)$,*

$$\int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{curl} \mathbf{z}_T = - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \mathbf{G}_F^k \underline{q}_F \cdot (\mathbf{z}_T \times \mathbf{n}_F). \quad (3.14)$$

Proof. Writing (3.7) with $\mathbf{w}_T = \mathbf{curl} \mathbf{z}_T \in \mathcal{P}^k(T)$ and recalling that $\operatorname{div} \mathbf{curl} \mathbf{z}_T = 0$, we have

$$\int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{curl} \mathbf{z}_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\mathbf{curl} \mathbf{z}_T \cdot \mathbf{n}_F) = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F (\mathbf{z}_T \times \mathbf{n}_F),$$

the last equality being a consequence of [22, Eq. (3.7)]. To conclude, we invoke (3.6) with $\mathbf{v}_F = (\mathbf{z}_T)|_F \times \mathbf{n}_F \in \mathcal{R}\mathcal{T}^{k+1}(F)$ (cf. (A.5) and Remark 9) and cancel the edge terms using [22, Eqs. (5.13) and (5.14)]. \square

The *global gradient* $\underline{\mathbf{G}}_h^k : \underline{X}_{\text{grad},h}^k \rightarrow \underline{X}_{\text{curl},h}^k$ is obtained collecting the projections of each local gradient on the space attached to the corresponding mesh entity: For all $\underline{q}_h \in \underline{X}_{\text{grad},h}^k$,

$$\underline{\mathbf{G}}_h^k \underline{q}_h := ((\pi_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T), \pi_{\mathcal{R},T}^{c,k}(\mathbf{G}_T^k \underline{q}_T))_{T \in \mathcal{T}_h}, (\pi_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F))_{F \in \mathcal{F}_h}, (G_E^k q_E)_{E \in \mathcal{E}_h}). \quad (3.15)$$

3.3.2 Curl

We next consider the DDR counterpart of the curl operator, which maps on $\underline{X}_{\text{div},h}^k$ and therefore has components at mesh faces and inside mesh elements. For all $F \in \mathcal{F}_h$, the *face curl* $C_F^k : \underline{X}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)$ is such that, for all $\underline{v}_F = (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c, (v_E)_{E \in \mathcal{E}_F}) \in \underline{X}_{\text{curl},F}^k$,

$$\int_F C_F^k \underline{v}_F r_F = \int_F \mathbf{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \quad \forall r_F \in \mathcal{P}^k(F). \quad (3.16)$$

Reasoning as in [22, Proposition 4.3], we get

$$C_F^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}) = \pi_{\mathcal{P},F}^k(\mathbf{rot}_F \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(F). \quad (3.17)$$

Proposition 12 (Local complex property). *Let $F \in \mathcal{F}_h$ and denote by $\underline{\mathbf{G}}_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \underline{X}_{\text{curl},F}^k$ the restriction to F of the global gradient $\underline{\mathbf{G}}_h^k$ defined by (3.15). Then, it holds*

$$\text{Im } \underline{\mathbf{G}}_F^k \subset \text{Ker } C_F^k \quad \forall F \in \mathcal{F}_h. \quad (3.18)$$

Remark 13 (Two-dimensional complex). The relations (3.10) and (3.18) show that the following two-dimensional sequence forms a complex:

$$\mathbb{R} \xrightarrow{\underline{\mathbf{I}}_{\text{grad},F}^k} \underline{X}_{\text{grad},F}^k \xrightarrow{\underline{\mathbf{G}}_F^k} \underline{X}_{\text{curl},F}^k \xrightarrow{C_F^k} \mathcal{P}^k(F) \xrightarrow{0} \{0\}.$$

Having assumed F simply connected, adapting the arguments of [22, Theorem 4.1], one can additionally prove that this complex is exact, that is, $\text{Ker } \underline{\mathbf{G}}_F^k = \underline{\mathbf{I}}_{\text{grad},F}^k \mathbb{R}$, $\text{Im } \underline{\mathbf{G}}_F^k = \text{Ker } C_F^k$, and $\text{Im } C_F^k = \mathcal{P}^k(F)$.

Proof of Proposition 12. Let $\underline{q}_F \in \underline{X}_{\text{grad},F}^k$. Using the definition (3.16) of C_F^k and (3.15) of $\underline{\mathbf{G}}_h^k$ we have, for all $r_F \in \mathcal{P}^k(F)$,

$$\begin{aligned} \int_F C_F^k(\underline{\mathbf{G}}_F^k \underline{q}_F) r_F &= \int_F \pi_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G_E^k \underline{q}_F r_F \\ &= \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G_E^k \underline{q}_F r_F \\ &= \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E [q_{\mathcal{E}_F}(\mathbf{rot}_F r_F \cdot \mathbf{n}_{FE}) - q'_E r_F] = 0, \end{aligned}$$

where the suppression of $\pi_{\mathcal{R},F}^{k-1}$ in the second line is possible since $\mathbf{rot}_F r_F \in \mathcal{R}^{k-1}(F)$, the third line is obtained using the definitions (3.4) of \mathbf{G}_F^k with $\mathbf{w}_F = \mathbf{rot}_F r_F$ (additionally noticing that $\text{div}_F(\mathbf{rot}_F r_F) = 0$) and (3.3) of G_E^k , while the conclusion is obtained reasoning as in [22, Point 2. of Proposition 4.4] (see in particular Eq. (4.19) therein). \square

The tangential trace $\gamma_{\mathfrak{t},F}^k : \underline{X}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)$ is such that, for all $\underline{v}_F \in \underline{X}_{\text{curl},F}^k$, recalling the notation (2.17),

$$\gamma_{\mathfrak{t},F}^k \underline{v}_F := \mathfrak{R}_{\mathcal{R},\mathcal{F}}^k(\gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F, \mathbf{v}_{\mathcal{R},F}^c), \quad (3.19)$$

where $\gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F \in \mathcal{R}^k(F)$ is defined, using the isomorphism property (2.8) with $\ell = k + 1$, by

$$\int_F \gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F \cdot \mathbf{rot}_F r_F = \int_F C_F^k \underline{v}_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \quad \forall r_F \in \mathcal{P}^{0,k+1}(F). \quad (3.20)$$

Remark 14 (Validity of (3.20)). Observing that both sides of (3.20) vanish when $r_F \in \mathcal{P}^0(F)$, it is inferred that this relation holds in fact for any $r_F \in \mathcal{P}^{k+1}(F)$. We also notice that, since $\pi_{\mathcal{R},F}^k(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F$ (by virtue of (3.19) and (2.14)), $\gamma_{\mathfrak{t},\mathcal{R},F}^k$ can be replaced by $\gamma_{\mathfrak{t},F}^k$ in the left-hand side of (3.20).

The actual computation of $\gamma_{\mathfrak{t},F}^k$ does not require the implementation of the recovery operator in the right-hand side of (3.19), but rather hinges on the solution of the following equation: For all $(r_F, \mathbf{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F)$,

$$\int_F \gamma_{\mathfrak{t},F}^k \underline{v}_F \cdot (\mathbf{rot}_F r_F + \mathbf{w}_F) = \int_F C_F^k \underline{v}_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F + \int_F \mathbf{v}_{\mathcal{R},F}^c \cdot \mathbf{w}_F.$$

Indeed, the test functions of the form $(r_F, \mathbf{0})$ with r_F spanning $\mathcal{P}^{0,k+1}(F)$ enforce that $\pi_{\mathcal{R},F}^k(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F$ satisfies (3.20), while the test functions of the form $(0, \mathbf{w}_F)$ with \mathbf{w}_F spanning $\mathcal{R}^{c,k}(F)$ enforce that $\pi_{\mathcal{R},F}^{c,k}(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \mathbf{v}_{\mathcal{R},F}^c$. These two conditions combined yield (3.19). Similar considerations hold for the three-dimensional potential reconstructions defined in [19, Sections 3.1.2 and 3.1.3].

Proposition 15 (Properties of the tangential trace). *It holds*

$$\pi_{\mathcal{R},F}^{k-1}(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \mathbf{v}_{\mathcal{R},F} \quad \text{and} \quad \pi_{\mathcal{R},F}^{c,k}(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \mathbf{v}_{\mathcal{R},F}^c \quad \forall \underline{v}_F \in \underline{X}_{\text{curl},F}^k, \quad (3.21)$$

$$\gamma_{\mathfrak{t},F}^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}) = \pi_{\mathcal{P},F}^k \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{N}^{k+1}(F), \quad (3.22)$$

$$\gamma_{\mathfrak{t},F}^k(\underline{\mathbf{G}}_F^k \underline{q}_F) = \mathbf{G}_F^k \underline{q}_F \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k. \quad (3.23)$$

Proof. 1. Proof of (3.21). Since $\mathcal{R}^{k-1}(F) \subset \mathcal{R}^k(F)$, we have $\pi_{\mathcal{R},F}^{k-1} = \pi_{\mathcal{R},F}^{k-1} \pi_{\mathcal{R},F}^k$ and thus, using (2.14) and Remark 14, we obtain $\pi_{\mathcal{R},F}^{k-1}(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \pi_{\mathcal{R},F}^{k-1}(\pi_{\mathcal{R},F}^k \gamma_{\mathfrak{t},F}^k \underline{v}_F) = \pi_{\mathcal{R},F}^{k-1}(\gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F)$. Applying the definitions (3.20) of $\gamma_{\mathfrak{t},\mathcal{R},F}^k$ and (3.16) of C_F^k with a generic $r_F \in \mathcal{P}^{0,k}(F)$ leads to $\int_F \gamma_{\mathfrak{t},\mathcal{R},F}^k \underline{v}_F \cdot \mathbf{rot}_F r_F = \int_F \mathbf{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F$, hence $\pi_{\mathcal{R},F}^{k-1}(\gamma_{\mathfrak{t},F}^k \underline{v}_F) = \mathbf{v}_{\mathcal{R},F}$. This proves the first relation in (3.21). The second relation is a straightforward consequence of (3.19) and (2.14).

2. Proof of (3.22). Let $\mathbf{v} \in \mathcal{N}^{k+1}(F)$. Writing (3.20) for $\underline{v}_F = \underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}$, observing that $C_F^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}) = \mathbf{rot}_F \mathbf{v} \in \mathcal{P}^k(F)$ by (3.17) and that $v_E = \mathbf{v}|_E \cdot \mathbf{t}_E$ for all $E \in \mathcal{E}_F$ by (A.1) with $\ell = k + 1$, and integrating by parts the right-hand side, it is inferred that $\gamma_{\mathfrak{t},\mathcal{R},F}^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}) = \pi_{\mathcal{R},F}^k \mathbf{v}$. Thus, by (3.19), $\gamma_{\mathfrak{t},F}^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}) = \mathfrak{R}_{\mathcal{R},F}^k(\pi_{\mathcal{R},F}^k \mathbf{v}, \pi_{\mathcal{R},F}^{c,k} \mathbf{v}) = \pi_{\mathcal{P},F}^k \mathbf{v}$, where the conclusion results from (2.11) with $(\mathcal{X}, \mathcal{Y}, \ell) = (\mathcal{R}, F, k)$ followed by (2.15).

3. Proof of (3.23). Let $\underline{q}_F \in \underline{X}_{\text{grad},F}^k$. For all $r_F \in \mathcal{P}^{k+1}(F)$, it holds

$$\sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G_E^k \underline{q}_E r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \underline{q}_E (\mathbf{rot}_F r_F \cdot \mathbf{n}_{FE}) = \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{rot}_F r_F, \quad (3.24)$$

where the first equality follows recalling that $G_E^k \underline{q}_E = \underline{q}'_E$ on E , integrating by parts on each edge, noting that $(r_F)'|_E = -\mathbf{rot}_F r_F \cdot \mathbf{n}_{FE}$ (see [22, Eq. (4.20)]), and cancelling out the vertex values that

appear twice with opposite sign, while the conclusion is obtained recalling the definition (3.4) of \mathbf{G}_F^k and observing that $\operatorname{div}_F(\mathbf{rot}_F r_F) = 0$. Writing (3.20) for $\underline{v}_F = \underline{\mathbf{G}}_F^k q_F$, we obtain

$$\int_F \gamma_{\mathcal{R},F}^k(\underline{\mathbf{G}}_F^k q_F) \cdot \mathbf{rot}_F r_F = \int_F \cancel{C_F^k(\underline{\mathbf{G}}_F^k q_F)} r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G_E^k q_E r_F = \int_F \mathbf{G}_F^k q_F \cdot \mathbf{rot}_F r_F,$$

where we have used the inclusion (3.18) in the cancellation, while the conclusion follows from (3.24). This implies $\gamma_{\mathcal{R},F}^k(\underline{\mathbf{G}}_F^k q_F) = \pi_{\mathcal{R},F}^k(\mathbf{G}_F^k q_F)$. By definition, the component of $\underline{\mathbf{G}}_F^k q_F$ on $\mathcal{R}^{c,k}(F)$ is $\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k q_F)$. Plugging the above results into (3.19) with $\underline{v}_F = \underline{\mathbf{G}}_F^k q_F$ and using the recovery formula (2.15) with $(S, S^c) = (\mathcal{R}^k(F), \mathcal{R}^{c,k}(F))$ and $\mathbf{a} = \mathbf{G}_F^k q_F$ concludes the proof. \square

For all $T \in \mathcal{T}_h$, the *element curl* $\mathbf{C}_T^k : \underline{\mathbf{X}}_{\operatorname{curl},T}^k \rightarrow \mathcal{P}^k(T)$ is defined such that, for all $\underline{v}_T = (\mathbf{v}_{\mathcal{R},T}, \mathbf{v}_{\mathcal{R},T}^c, (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c)_{F \in \mathcal{F}_T}, (\mathbf{v}_E)_{E \in \mathcal{E}_T}) \in \underline{\mathbf{X}}_{\operatorname{curl},T}^k$,

$$\int_T \mathbf{C}_T^k \underline{v}_T \cdot \mathbf{w}_T = \int_T \mathbf{v}_{\mathcal{R},T} \cdot \mathbf{curl} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\mathcal{R},F}^k \underline{v}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \quad \forall \mathbf{w}_T \in \mathcal{P}^k(T). \quad (3.25)$$

The following polynomial consistency property is proved as in [22, Lemma 5.2] (recall the shift of exponent in the notation of the Nédélec space with respect to this reference):

$$\forall T \in \mathcal{T}_h \quad \mathbf{C}_T^k(\underline{\mathbf{I}}_{\operatorname{curl},T}^k \mathbf{v}) = \mathbf{curl} \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{N}^{k+1}(T). \quad (3.26)$$

Proposition 16 (Link between element and face curls). *For all $(\underline{v}_T, r_T) \in \underline{\mathbf{X}}_{\operatorname{curl},T}^k \times \mathcal{P}^{k+1}(T)$, it holds*

$$\int_T \mathbf{C}_T^k \underline{v}_T \cdot \mathbf{grad} r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^k \underline{v}_F r_T. \quad (3.27)$$

Proof. For any $r_T \in \mathcal{P}^{k+1}(T)$, writing (3.25) for $\mathbf{w}_T = \mathbf{grad} r_T \in \mathcal{P}^k(T)$ and using the fact that $\mathbf{curl}(\mathbf{grad} r_T) = \mathbf{0}$ and that $(\mathbf{grad} r_T)|_F \times \mathbf{n}_F = \mathbf{rot}(r_T|_F)$ for all $F \in \mathcal{F}_T$ (see [22, Eq. (3.6)]), we infer that $\int_T \mathbf{C}_T^k \underline{v}_T \cdot \mathbf{grad} r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\mathcal{R},F}^k \underline{v}_F \cdot \mathbf{rot}(r_T|_F)$. Using Remark 14, we arrive at

$$\int_T \mathbf{C}_T^k \underline{v}_T \cdot \mathbf{grad} r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \left[\int_F C_F^k \underline{v}_F r_T + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_T \right].$$

By [22, Eq. (5.13)], the edge terms in the above expression can be cancelled, thereby proving (3.27). \square

The *global curl* $\underline{\mathbf{C}}_h^k : \underline{\mathbf{X}}_{\operatorname{curl},h}^k \rightarrow \underline{\mathbf{X}}_{\operatorname{div},h}^k$ is such that, for all $\underline{v}_h \in \underline{\mathbf{X}}_{\operatorname{curl},h}^k$,

$$\underline{\mathbf{C}}_h^k \underline{v}_h := ((\pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{v}_T), \pi_{\mathcal{G},T}^{c,k}(\mathbf{C}_T^k \underline{v}_T))_{T \in \mathcal{T}_h}, (C_F^k \underline{v}_F)_{F \in \mathcal{F}_h}). \quad (3.28)$$

3.3.3 Divergence

For all $T \in \mathcal{T}_h$, the *element divergence* $D_T^k : \underline{\mathbf{X}}_{\operatorname{div},T}^k \rightarrow \mathcal{P}^k(T)$ is defined by: For all $\underline{w}_T = (\mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c, (\mathbf{w}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{X}}_{\operatorname{div},T}^k$,

$$\int_T D_T^k \underline{w}_T q_T = - \int_T \mathbf{w}_{\mathcal{G},T} \cdot \mathbf{grad} q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F q_T \quad \forall q_T \in \mathcal{P}^k(T). \quad (3.29)$$

The *global divergence* $D_h^k : \underline{\mathbf{X}}_{\operatorname{div},h}^k \rightarrow \mathcal{P}^k(\mathcal{T}_h)$ is obtained setting, for all $\underline{w}_h \in \underline{\mathbf{X}}_{\operatorname{div},h}^k$,

$$(D_h^k \underline{w}_h)|_T := D_T^k \underline{w}_T \quad \forall T \in \mathcal{T}_h, \quad (3.30)$$

Proposition 17 (Local exactness property). *It holds, for all $T \in \mathcal{T}_h$,*

$$\text{Im } \underline{\mathbf{C}}_T^k = \text{Ker } D_T^k, \quad (3.31)$$

where $\underline{\mathbf{C}}_T^k$ denotes the restriction to T of the global curl $\underline{\mathbf{C}}_h^k$ defined by (3.28)

Proof. Let us start by proving that $D_T^k(\underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T) = 0$ for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$, that is, $\text{Im } \underline{\mathbf{C}}_T^k \subset \text{Ker}(D_T^k)$. By Proposition 16, for all $q_T \in \mathcal{P}^k(T)$,

$$\int_T \pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) \cdot \mathbf{grad } q_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^k \underline{\mathbf{v}}_F q_T, \quad (3.32)$$

where we have used $\mathbf{grad } q_T \in \mathcal{G}^{k-1}(T)$ to introduce the projector $\pi_{\mathcal{G},T}^{k-1}$. Hence, using the definition (3.29) of D_T^k , we have, for all $q_T \in \mathcal{P}^k(T)$,

$$\int_T D_T^k(\underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T) q_T = - \int_T \pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) \cdot \mathbf{grad } q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^k \underline{\mathbf{v}}_F q_T = 0.$$

Since q_T is arbitrary in $\mathcal{P}^k(T)$, this shows that $D_T^k(\underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T) = 0$.

Let us now prove the inclusion $\text{Ker}(D_T^k) \subset \text{Im } \underline{\mathbf{C}}_T^k$. We fix an element $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\text{div},T}^k$ such that $D_T^k \underline{\mathbf{w}}_T = 0$ and prove the existence of $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$ such that $\underline{\mathbf{w}}_T = \underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T$. Enforcing $D_T^k \underline{\mathbf{w}}_T = 0$ in (3.29) with $q_T = 1$, we infer that $\sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \mathbf{w}_F = 0$. Thus, [22, Lemma 5.3], which remains valid in the present context, provides $(\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c)_{F \in \mathcal{F}_T}$ and $(\mathbf{v}_E)_{E \in \mathcal{E}_T}$ such that, for all $F \in \mathcal{F}_T$, letting $\underline{\mathbf{v}}_F := (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c, (\mathbf{v}_E)_{E \in \mathcal{E}_F})$, it holds $\mathbf{w}_F = C_F^k \underline{\mathbf{v}}_F$ for all $F \in \mathcal{F}_T$. Enforcing again $D_T^k \underline{\mathbf{w}}_T = 0$ in (3.29), this time for a generic test function $q_T \in \mathcal{P}^k(T)$, and accounting for the previous result, we can write, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$ with boundary values as above,

$$\int_T \mathbf{w}_{\mathcal{G},T} \cdot \mathbf{grad } q_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^k \underline{\mathbf{v}}_F q_T = \int_T \pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) \cdot \mathbf{grad } q_T,$$

where the conclusion follows from the relation (3.32) linking volume and face curls. Since $\mathbf{grad } q_T$ spans $\mathcal{G}^{k-1}(T)$ as q_T spans $\mathcal{P}^k(T)$, this proves that $\pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) = \mathbf{w}_{\mathcal{G},T}$. Finally, we select $\mathbf{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T)$ in such a way as to have $\pi_{\mathcal{G},T}^{c,k}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) = \mathbf{w}_{\mathcal{G},T}^c$, that is, recalling (3.25),

$$\int_T \mathbf{v}_{\mathcal{R},T} \cdot \mathbf{curl } \mathbf{z}_T = \int_T \mathbf{w}_{\mathcal{G},T}^c \cdot \mathbf{z}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{t,F}^k \underline{\mathbf{v}}_F \cdot (\mathbf{z}_T \times \mathbf{n}_F) \quad \forall \mathbf{z}_T \in \mathcal{G}^{c,k}(T).$$

By the isomorphism (2.10), this condition defines $\mathbf{v}_{\mathcal{R},T}$ uniquely. \square

3.4 Discrete sequence

Recalling the definitions (3.2), (3.15), (3.28), and (3.30) of the global discrete operators, the DDR sequence reads:

$$\mathbb{R} \xrightarrow{I_{\text{grad},h}^k} \underline{\mathbf{X}}_{\text{grad},h}^k \xrightarrow{\underline{\mathbf{G}}_h^k} \underline{\mathbf{X}}_{\text{curl},h}^k \xrightarrow{\underline{\mathbf{C}}_h^k} \underline{\mathbf{X}}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}. \quad (3.33)$$

3.5 Commutation properties

Lemma 18 (Local commutation properties). *It holds, for all $T \in \mathcal{T}_h$,*

$$\underline{\mathbf{G}}_T^k(\underline{\mathbf{I}}_{\text{grad},T}^k q) = \underline{\mathbf{I}}_{\text{curl},T}^k(\mathbf{grad} q) \quad \forall q \in C^1(\bar{T}), \quad (3.34)$$

$$\underline{\mathbf{C}}_T^k(\underline{\mathbf{I}}_{\text{curl},T}^k \mathbf{v}) = \underline{\mathbf{I}}_{\text{div},T}^k(\mathbf{curl} \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^2(T), \quad (3.35)$$

$$D_T^k(\underline{\mathbf{I}}_{\text{div},T}^k \mathbf{w}) = \pi_{\mathcal{P},T}^k(\text{div} \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}^1(T). \quad (3.36)$$

Proof. We start by noticing that all the interpolates defined in (3.34)–(3.36) are well-defined under the assumed regularities.

1. *Proof of (3.34).* By (3.9) it holds, for all $E \in \mathcal{E}_T$, $G_E^k(\underline{\mathbf{I}}_{\text{grad},E}^k q|_E) = \pi_{\mathcal{P},E}^k(q'|_E) = \pi_{\mathcal{P},E}^k((\mathbf{grad} q)|_E \cdot \mathbf{t}_E)$. Let now $F \in \mathcal{F}_T$. Writing the definition (3.4) of \mathbf{G}_F^k with $q|_F = \underline{\mathbf{I}}_{\text{grad},F}^k q|_F$ and $\mathbf{w}_F \in \mathcal{RT}^k(F)$, and recalling (A.2) to replace $q_{\mathcal{E}_F}$ with $\pi_{\mathcal{P},E}^{k-1}(q_{\mathcal{E}_F})|_E = \pi_{\mathcal{P},E}^{k-1} q|_E$ (see (3.2)) in each edge integral, we infer

$$\begin{aligned} \int_F \mathbf{G}_F^k(\underline{\mathbf{I}}_{\text{grad},F}^k q|_F) \cdot \mathbf{w}_F &= - \int_F \pi_{\mathcal{P},F}^{k-1} q|_F \text{div}_F \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^{k-1} q|_E (\mathbf{w}_F \cdot \mathbf{n}_{FE}) \\ &= - \int_F q \text{div}_F \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q (\mathbf{w}_F \cdot \mathbf{n}_{FE}) = \int_F \mathbf{grad}_F q|_F \cdot \mathbf{w}_F, \end{aligned}$$

where we have removed the projectors using their definition to pass to the second line and we have integrated by parts to conclude. Recalling the definition (2.12) of $\mathcal{RT}^k(F)$, we can first let \mathbf{w}_F span $\mathcal{R}^{k-1}(F)$ to infer $\pi_{\mathcal{R},F}^{k-1}[\mathbf{G}_F^k(\underline{\mathbf{I}}_{\text{grad},F}^k q|_F)] = \pi_{\mathcal{R},F}^{k-1}(\mathbf{grad}_F q|_F)$, and then $\mathcal{R}^{c,k}(F)$ to infer $\pi_{\mathcal{R},F}^{c,k}[\mathbf{G}_F^k(\underline{\mathbf{I}}_{\text{grad},F}^k q|_F)] = \pi_{\mathcal{R},F}^{c,k}(\mathbf{grad}_F q|_F)$. The proof that $\pi_{\mathcal{R},T}^{k-1}[\mathbf{G}_T^k(\underline{\mathbf{I}}_{\text{grad},T}^k q)] = \pi_{\mathcal{R},T}^{k-1}(\mathbf{grad} q)$ and $\pi_{\mathcal{R},T}^{c,k}[\mathbf{G}_T^k(\underline{\mathbf{I}}_{\text{grad},T}^k q)] = \pi_{\mathcal{R},T}^{c,k}(\mathbf{grad} q)$ is similar: we write the definition (3.7) of \mathbf{G}_T^k for $q|_T = \underline{\mathbf{I}}_{\text{grad},T}^k q$ and $\mathbf{w}_T \in \mathcal{RT}^k(T)$, use property (A.4) along with (3.12) to replace $\gamma_F^{k+1}(\underline{\mathbf{I}}_{\text{grad},F}^k q|_F)$ with $\pi_{\mathcal{P},F}^{k-1}[\gamma_F^{k+1}(\underline{\mathbf{I}}_{\text{grad},F}^k q|_F)] = \pi_{\mathcal{P},F}^{k-1} q|_F$ in each face integral, remove the projectors using their definitions, and integrate by parts. This concludes the proof of (3.34).

2. *Proof of (3.35).* For all $F \in \mathcal{F}_T$, by (3.17) it holds $C_F^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}|_F) = \pi_{\mathcal{P},F}^k((\mathbf{curl} \mathbf{v})|_F \cdot \mathbf{n}_F)$, where we have used $\text{rot}_F \mathbf{v}_{t,F} = (\mathbf{curl} \mathbf{v})|_F \cdot \mathbf{n}_F$, see [22, Eq. (3.7)]. Writing the definition (3.25) for $\mathbf{w}_T \in \mathcal{N}^k(T)$, we have

$$\int_T \mathbf{C}_T^k(\underline{\mathbf{I}}_{\text{curl},T}^k \mathbf{v}) \cdot \mathbf{w}_T = \int_T \cancel{\pi_{\mathcal{R},T}^{k-1}} \mathbf{v} \cdot \mathbf{curl} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \cancel{\pi_{\mathcal{R},T,F}^k} [\gamma_{t,F}^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}_{t,F})] \cdot (\mathbf{w}_T \times \mathbf{n}_F), \quad (3.37)$$

where we have removed $\pi_{\mathcal{R},T}^{k-1}$ using its definition and, recalling (A.5), we have introduced the L^2 -orthogonal projector $\pi_{\mathcal{R},T,F}^k$ on $\mathcal{RT}^k(F)$ in the boundary integral. By (2.12) together with (2.15) written for $(E, S, S^c) = (\mathcal{RT}^k(F), \mathcal{R}^{k-1}(F), \mathcal{R}^{c,k}(F))$ and (3.21),

$$\pi_{\mathcal{R},T,F}^k[\gamma_{t,F}^k(\underline{\mathbf{I}}_{\text{curl},F}^k \mathbf{v}_{t,F})] = \mathfrak{R}_{\mathcal{R}^{k-1}(F), \mathcal{R}^{c,k}(F)}(\pi_{\mathcal{R},F}^{k-1} \mathbf{v}_{t,F}, \pi_{\mathcal{R},F}^{c,k} \mathbf{v}_{t,F}) = \pi_{\mathcal{R},T,F}^k \mathbf{v}_{t,F}.$$

Plugging this relation into (3.37), we infer

$$\int_T \mathbf{C}_T^k(\underline{\mathbf{I}}_{\text{curl},T}^k \mathbf{v}) \cdot \mathbf{w}_T = \int_T \mathbf{v} \cdot \mathbf{curl} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \cancel{\pi_{\mathcal{R},T,F}^k} \mathbf{v}_{t,F} \cdot (\mathbf{w}_T \times \mathbf{n}_F) = \int_T \mathbf{curl} \mathbf{v} \cdot \mathbf{w}_T,$$

where we have used again (A.5) to remove the projector in the boundary term and we have integrated by parts to conclude. Letting \mathbf{w}_T span $\mathcal{G}^{k-1}(T)$ (resp. $\mathcal{G}^{c,k}(T)$), this yields $\pi_{\mathcal{G},T}^{k-1}[\mathbf{C}_T^k(\underline{\mathbf{I}}_{\text{curl},T}^k \mathbf{v})] =$

$\pi_{\underline{\mathcal{G}},T}^{k-1}(\mathbf{curl} \nu)$ (resp. $\pi_{\underline{\mathcal{G}},T}^{c,k}[\mathbf{C}_T^k(\underline{I}_{\mathbf{curl},T}^k \nu)] = \pi_{\underline{\mathcal{G}},T}^{c,k}(\mathbf{curl} \nu)$), thus concluding the proof of (3.35).

3. *Proof of (3.36).* The proof is done as in [22, Lemma 5.4], noticing that the cancellation of the component in the complement of $\underline{\mathcal{G}}^k(T)$, obtained therein by orthogonality of this complement, is not required here since this component is absent from the definition (3.29) of D_T^k . \square

Remark 19 (Global commutation properties). Global commutation properties can be readily inferred from the local ones stated in Lemma 18 when interpolating functions that have sufficient global regularity.

3.6 Complex and exactness properties

The properties collected in the following theorem show that the sequence (3.33) forms a (cochain) complex.

Theorem 20 (Complex property). *It holds*

$$\underline{I}_{\mathbf{grad},h}^k \mathbb{R} = \text{Ker } \underline{\mathbf{G}}_h^k, \quad (3.38)$$

$$\text{Im } \underline{\mathbf{G}}_h^k \subset \text{Ker } \underline{\mathbf{C}}_h^k, \quad (3.39)$$

$$\text{Im } \underline{\mathbf{C}}_h^k \subset \text{Ker } D_h^k, \quad (3.40)$$

$$\text{Im } D_h^k = \mathcal{P}^k(\mathcal{T}_h). \quad (3.41)$$

Proof. 1. *Proof of (3.38).* From the consistency properties (3.9), (3.10) and (3.13) of the full gradients and the definition (3.15) of $\underline{\mathbf{G}}_h^k$, it is readily inferred that $\underline{\mathbf{G}}_h^k(\underline{I}_{\mathbf{grad},h}^k C) = \mathbf{0}$ for all $C \in \mathbb{R}$, hence $\underline{I}_{\mathbf{grad},h}^k \mathbb{R} \subset \text{Ker } \underline{\mathbf{G}}_h^k$.

To prove converse inclusion $\text{Ker } \underline{\mathbf{G}}_h^k \subset \underline{I}_{\mathbf{grad},h}^k \mathbb{R}$, let $\underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k$ be such that $\underline{\mathbf{G}}_h^k \underline{q}_h = \mathbf{0}$. By the definitions (3.15) of $\underline{\mathbf{G}}_h^k$ and (3.3) of G_E^k , this means that $q'_E = 0$ for all $E \in \mathcal{E}_h$, that is, $(q_{\mathcal{E}_h})|_E$ is constant over E . Since Ω has only one connected component, accounting for the single-valuedness of $q_{\mathcal{E}_h}$ at vertices, we thus infer the existence of $C \in \mathbb{R}$ such that $q_{\mathcal{E}_h} = C$. Let now $F \in \mathcal{F}_h$ and $\mathbf{w}_F \in \mathcal{R}^{c,k}(F)$. We have $\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F) = \mathbf{0}$, and thus

$$0 = \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{w}_F = - \int_F q_F \text{div}_F \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F} (\mathbf{w}_F \cdot \mathbf{n}_{FE}) = \int_F (C - q_F) \text{div}_F \mathbf{w}_F,$$

where the second equality comes from the definition (3.4) of $\mathbf{G}_F^k \underline{q}_F$, and the conclusion is obtained accounting for the fact that $q_{\mathcal{E}_F} = C$ and integrating by parts. Since \mathbf{w}_F is generic in $\mathcal{R}^{c,k}(F)$, recalling the isomorphism (2.9) this implies $\pi_{\mathcal{P},F}^{k-1}(q_F - C) = 0$, and thus $q_F = \pi_{\mathcal{P},F}^{k-1} C$. As, for all $F \in \mathcal{F}_h$, the previous results give $\underline{q}_F = (q_F, q_{\mathcal{E}_F}) = \underline{I}_{\mathbf{grad},F}^k C$, we also have $\gamma_F^{k+1} \underline{q}_F = C$ by (3.11). Similarly, let $T \in \mathcal{T}_h$ and $\mathbf{w}_T \in \mathcal{R}^{c,k}(T)$. Writing the definition (3.7) of $\mathbf{G}_T^k \underline{q}_T$ for $\mathbf{w}_T \in \mathcal{R}^{c,k}(T)$, and accounting for $\pi_{\mathcal{R},T}^{c,k}(\mathbf{G}_T^k \underline{q}_T) = \mathbf{0}$ and $\gamma_F^{k+1} \underline{q}_F = C$, it is inferred

$$0 = \int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{w}_T = - \int_T q_T \text{div } \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C (\mathbf{w}_T \cdot \mathbf{n}_F) = \int_T (C - q_T) \text{div } \mathbf{w}_T,$$

which implies, invoking the isomorphism (2.9), $\pi_{\mathcal{P},T}^{k-1}(q_T - C) = 0$ since \mathbf{w}_T is generic in $\mathcal{R}^{c,k}(T)$. Hence $q_T = \pi_{\mathcal{P},T}^{k-1} C$ for all $T \in \mathcal{T}_h$, which concludes the proof that $\underline{q}_h = \underline{I}_{\mathbf{grad},h}^k C$.

2. *Proof of (3.39).* The inclusion (3.39) follows from the local property:

$$\text{Im } \underline{\mathbf{G}}_T^k \subset \text{Ker } \underline{\mathbf{C}}_T^k \quad \forall T \in \mathcal{T}_h, \quad (3.42)$$

i.e., $\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T) = \mathbf{0}$ for all $\underline{q}_T \in \underline{X}_{\text{grad},T}^k$. Let $T \in \mathcal{T}_h$. The relation (3.18) implies $C_F^k(\underline{\mathbf{G}}_F^k \underline{q}_F) = 0$ for all $F \in \mathcal{F}_T$. The fact that $\pi_{\mathcal{G},T}^{k-1}[\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T)] = \mathbf{0}$ then follows from (3.32). We next notice that it holds, for all $\mathbf{w}_T \in \mathcal{G}^{c,k}(T)$,

$$\int_T \underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T) \cdot \mathbf{w}_T = \int_T \pi_{\mathcal{R},T}^{k-1}(\underline{\mathbf{G}}_T^k \underline{q}_T) \cdot \text{curl } \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \underline{\mathbf{G}}_F^k \underline{q}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) = 0,$$

where we have used the definition (3.25) of $\underline{\mathbf{C}}_T^k$ and the property (3.23) of the tangential trace reconstruction in the first equality, the fact that $\text{curl } \mathbf{w}_T \in \mathcal{R}^{k-1}(T)$ to cancel the projector, and the link (3.14) between volume and face gradients to conclude. This shows that $\pi_{\mathcal{G},T}^{c,k}[\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T)] = \mathbf{0}$ and concludes the proof of (3.42).

3. Proof of (3.40). Immediate consequence of (3.31) after observing that $\underline{\mathbf{C}}_T^k$ and D_T^k are the restrictions of $\underline{\mathbf{C}}_h^k$ and D_h^k to T , respectively.

4. Proof of (3.41). The inclusion $\text{Im } D_h^k \subset \mathcal{P}^k(\mathcal{T}_h)$ is an obvious consequence of the definition (3.30) of the global divergence. To prove the converse inclusion, let $q_h \in \mathcal{P}^k(\mathcal{T}_h)$. Since the continuous divergence operator $\text{div} : \mathbf{H}^1(\Omega) \rightarrow L^2(\Omega)$ is onto (see, e.g., [21, Lemma 8.3]), there exists $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\text{div } \mathbf{v} = q_h$. Setting $\underline{\mathbf{v}}_h := \underline{\mathbf{I}}_{\text{div},h}^k \mathbf{v} \in \underline{X}_{\text{div},h}^k$, the commutation property (3.36) and the definition (3.30) of the global divergence yield $D_h^k \underline{\mathbf{v}}_h = \pi_{\mathcal{P},h}^k(\text{div } \mathbf{v}) = \pi_{\mathcal{P},h}^k q_h = q_h$. This shows that $\mathcal{P}^k(\mathcal{T}_h) \subset \text{Im } D_h^k$, thereby concluding the proof of (3.41). \square

Remark 21 (Kernel of the full curl operator). Combining the inclusion (3.18) with the relation (3.27) linking element and face curls, it is inferred that, for all $\underline{q}_T \in \underline{X}_{\text{grad},T}^k$, $\pi_{\mathcal{G},T}^k[\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T)] = \mathbf{0}$. On the other hand, (3.42) implies $\pi_{\mathcal{G},T}^{c,k}[\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T)] = \mathbf{0}$. Hence $\underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T) = \mathbf{0}$ by (2.15) with $\mathbf{a} = \underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T)$ and $(S, S^c) = (\mathcal{G}^k(T), \mathcal{G}^{c,k}(T))$. This shows that $\text{Im } \underline{\mathbf{G}}_T^k \subset \text{Ker } \underline{\mathbf{C}}_T^k$.

The exactness properties of the DDR sequence, depending on the topology of the domain, are collected in the following theorem.

Theorem 22 (Exactness). *Denoting by (b_0, b_1, b_2, b_3) the Betti numbers of Ω (with $b_0 = 1$ since Ω is connected and $b_3 = 0$ since $\Omega \subset \mathbb{R}^3$), we have*

$$b_1 = 0 \implies \text{Im } \underline{\mathbf{G}}_h^k = \text{Ker } \underline{\mathbf{C}}_h^k, \quad (3.43)$$

$$b_2 = 0 \implies \text{Im } \underline{\mathbf{C}}_h^k = \text{Ker } D_h^k. \quad (3.44)$$

Remark 23 (Meaning of vanishing Betti numbers). In broad terms, the condition $b_1 = 0$ means that Ω does not have any tunnel, while $b_2 = 0$ means that Ω does not enclose any void. A typical example of Ω that has $b_1 \neq 0$ is (the interior of) a torus, and an example of Ω with $b_2 \neq 0$ is a domain enclosed between two concentric spheres.

Proof. 1. Proof of (3.43). Recalling (3.39), we only have to show the inclusion

$$\text{Ker } \underline{\mathbf{C}}_h^k \subset \text{Im } \underline{\mathbf{G}}_h^k, \quad (3.45)$$

that is, for all $\underline{\mathbf{v}}_h \in \underline{X}_{\text{curl},h}^k$ such that $\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h = \mathbf{0}$, there exists $\underline{q}_h \in \underline{X}_{\text{grad},h}^k$ such that $\underline{\mathbf{v}}_h = \underline{\mathbf{G}}_h^k \underline{q}_h$. In what follows, we show how to construct such a \underline{q}_h .

We start by constructing a function $q_{\mathcal{E}_h} \in \mathcal{P}_c^{k+1}(\mathcal{E}_h)$ such that $v_E = (q_{\mathcal{E}_h})'$ for all $E \in \mathcal{E}_h$. Let $V_0, V \in \mathcal{V}_h$ be two distinct mesh vertices of coordinates \mathbf{x}_{V_0} and \mathbf{x}_V , respectively, and denote by $\mathcal{E}_P \subset \mathcal{E}_h$ a set of edges that form a connected path P from V_0 to V (such a path always exists since Ω is

connected). By the fundamental theorem of calculus, there is a unique function $q_{\mathcal{E}_P} \in \mathcal{P}_c^{k+1}(\mathcal{E}_P)$ such that $q_{\mathcal{E}_P}(\mathbf{x}_{V_0}) = 0$ and $(q_{\mathcal{E}_P})'|_E = v_E$ for all $E \in \mathcal{E}_P$, the derivative being taken in the direction of E ($q_{\mathcal{E}_P}$ is obtained integrating, in the direction defined on each edge E by \mathbf{t}_E , the functions $(v_E)_{E \in \mathcal{E}_P}$). We want to show that the value $q_{\mathcal{E}_P}(\mathbf{x}_V)$ taken at V is independent of the choice of the path P . To this end, denote by \tilde{P} another path from V_0 to V formed by the edges in $\mathcal{E}_{\tilde{P}}$, and denote by $-\tilde{P}$ the same path but with reversed orientation. We assume, for the moment, that \mathcal{E}_P and $\mathcal{E}_{\tilde{P}}$ are disjoint. By similar considerations as before, there exists a unique $q_{\mathcal{E}_{\tilde{P}}} \in \mathcal{P}_c^{k+1}(\mathcal{E}_{\tilde{P}})$ such that $q_{\mathcal{E}_{\tilde{P}}}(\mathbf{x}_{V_0}) = 0$ and $(q_{\mathcal{E}_{\tilde{P}}})'|_E = v_E$ for all $E \in \mathcal{E}_{\tilde{P}}$. Since $b_1 = 0$ (i.e., there is no “tunnel” crossing Ω), the path $B := P - \tilde{P}$ formed by the edges in $\mathcal{E}_B := \mathcal{E}_P \cup \mathcal{E}_{\tilde{P}}$ is a 1-boundary, i.e., there is a set of faces $\mathcal{F}_B \subset \mathcal{F}_h$ giving rise to a connected surface $S_B := \bigcup_{F \in \mathcal{F}_B} \bar{F}$ such that $B = \partial S_B$. We fix an orientation for S_B and, for all $F \in \mathcal{F}_B$, we denote by $\omega_{BF} \in \{-1, 1\}$ the orientation of F relative to S_B . For all $E \in \mathcal{E}_B$, there is a unique face $F \in \mathcal{F}_B$ such that $E \in \mathcal{E}_F$, and we let $\omega_{BE} := \omega_{BF} \omega_{FE}$ denote the orientation of E relative to S_B . Since $C_F^k \underline{\mathbf{v}}_F = 0$ for all $F \in \mathcal{F}_B$, it holds

$$0 = \sum_{F \in \mathcal{F}_B} \omega_{BF} \int_F C_F^k \underline{\mathbf{v}}_F = - \sum_{F \in \mathcal{F}_B} \omega_{BF} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E = - \sum_{E \in \mathcal{E}_B} \omega_{BE} \int_E v_E,$$

where the second equality is obtained from (3.16) with r_F identically equal to 1, while the conclusion follows observing that all the edges that are interior to S_B appear exactly twice in the sum, with opposite signs. Thus, reasoning as in [22, Proposition 4.2], there exists $q_{\mathcal{E}_B} \in \mathcal{P}_c^{k+1}(\mathcal{E}_B)$ such that $(q_{\mathcal{E}_B})' = v_E$ for all $E \in \mathcal{E}_B$, which we can be uniquely identified by additionally prescribing that $q_{\mathcal{E}_B}(\mathbf{x}_{V_0}) = 0$. Under this condition, by uniqueness we infer $(q_{\mathcal{E}_B})|_E = (q_{\mathcal{E}_P})|_E$ for all $E \in \mathcal{E}_P$ and $(q_{\mathcal{E}_B})|_E = (q_{\mathcal{E}_{\tilde{P}}})|_E$ for all $E \in \mathcal{E}_{\tilde{P}}$. Since $q_{\mathcal{E}_B}$ is continuous at the vertices of B , this shows that $q_{\mathcal{E}_P}(\mathbf{x}_V) = q_{\mathcal{E}_{\tilde{P}}}(\mathbf{x}_V)$. This reasoning can be extended to paths P and \tilde{P} such that $\mathcal{E}_P \cap \mathcal{E}_{\tilde{P}} \neq \emptyset$, the only difference being that one should reason, in this case, on each connected component of the manifold S_B (corresponding to a “loop” inside the path $B = P - \tilde{P}$). The details are left to the reader.

Repeating this reasoning for each vertex $V \in \mathcal{V}_h$ and all possible paths connecting V_0 and V , we conclude that there exists a unique $q_{\mathcal{E}_h} \in \mathcal{P}_c^{k+1}(\mathcal{E}_h)$ such that $q_{\mathcal{E}_h}(\mathbf{x}_{V_0}) = 0$ and, recalling the notation (3.1),

$$q'_E = G_E^k q_E = v_E \quad \forall E \in \mathcal{E}_h. \quad (3.46)$$

Let now $F \in \mathcal{F}_h$. We look for a $q_F \in \mathcal{P}^{k-1}(F)$ such that $\underline{q}_F := (q_F, q_{\mathcal{E}_F}) \in \underline{X}_{\text{grad}, F}^k$ satisfies $\underline{\mathbf{v}}_F = \mathbf{G}_F^k \underline{q}_F$. Plugging $C_F^k \underline{\mathbf{v}}_F = 0$ into (3.16), we infer, for all $r_F \in \mathcal{P}^k(F)$,

$$\int_F \mathbf{v}_{\mathcal{R}, F} \cdot \mathbf{rot}_F r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E G_E^k q_E r_F = \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{rot}_F r_F,$$

where the second equality is a consequence of (3.46), while the conclusion follows from (3.24). This shows that $\mathbf{v}_{\mathcal{R}, F} = \pi_{\mathcal{R}, F}^{k-1}(\mathbf{G}_F^k \underline{q}_F)$ for all $q_F \in \mathcal{P}^{k-1}(F)$. Let us now enforce $\mathbf{v}_{\mathcal{R}, F}^c = \pi_{\mathcal{R}, F}^{c, k}(\mathbf{G}_F^k \underline{q}_F)$, that is, for all $\mathbf{w}_F \in \mathcal{R}^{c, k}(F)$,

$$\int_F \mathbf{v}_{\mathcal{R}, F} \cdot \mathbf{w}_F = \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{w}_F = - \int_F q_F \operatorname{div}_F \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{\mathcal{E}_F}(\mathbf{w}_F \cdot \mathbf{n}_{FE}),$$

where we have used the definition (3.4) of \mathbf{G}_F^k in the second equality. Recalling the isomorphism (2.9), the above condition defines the sought $q_F \in \mathcal{P}^{k-1}(F)$ uniquely.

Writing the definition (3.25) of $\mathbf{C}_T^k \underline{\mathbf{v}}_T$ with $\mathbf{w}_T \in \mathcal{G}^{c, k}(T)$, using $\pi_{\mathcal{G}, T}^{c, k}(\mathbf{C}_T^k \underline{\mathbf{v}}_T) = \mathbf{0}$, invoking (3.23) to write $\gamma_{\mathbf{t}, F}^k \underline{\mathbf{v}}_F = \mathbf{G}_F^k \underline{q}_F$ and using the link (3.14) between element and face gradients with $\mathbf{z}_T = \mathbf{w}_T$, we

see that, for any $T \in \mathcal{T}_h$, $\mathbf{v}_{\mathcal{R},T} = \boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T)$ with $\underline{q}_T := (q_T, (q_F)_{F \in \mathcal{F}_T}, q_{\mathcal{E}_T})$ for all $q_T \in \mathcal{P}^{k-1}(T)$. Proceeding then as for q_F above, we can select $q_T \in \mathcal{P}^{k-1}(T)$ to additionally have $\mathbf{v}_{\mathcal{R},T}^c = \boldsymbol{\pi}_{\mathcal{R},T}^{c,k}(\mathbf{G}_T^k \underline{q}_T)$. This concludes the proof of (3.45).

2. *Proof of (3.44).* The proof can be obtained reasoning as in [20, Point 2b) of Theorem 3]. As a matter of fact, this argument is based on a local exactness property analogous to (3.31) together with a topological assembly of the mesh valid for domains that do not enclose voids ($b_2 = 0$), and it therefore does not depend on the specific choice of the complements in (2.5) and (2.3). \square

4 Poincaré inequalities

In this section we state and prove Poincaré-type inequalities for the operators in the DDR sequence.

4.1 Component L^2 -norms

We equip the DDR spaces with the L^2 -like norms naturally associated with the choices of polynomial components. Specifically we set, for all $\underline{q}_h \in \underline{X}_{\text{grad},h}^k$,

$$\begin{aligned} \|\underline{q}_h\|_{\text{grad},h} &:= \left(\sum_{T \in \mathcal{T}_h} \|\underline{q}_T\|_{\text{grad},T}^2 \right)^{\frac{1}{2}} \text{ with} \\ \forall T \in \mathcal{T}_h, \quad \|\underline{q}_T\|_{\text{grad},T} &:= \left(\|q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|\underline{q}_F\|_{\text{grad},F}^2 \right)^{\frac{1}{2}} \quad \forall \underline{q}_T \in \underline{X}_{\text{grad},T}^k, \\ \forall F \in \mathcal{F}_h, \quad \|\underline{q}_F\|_{\text{grad},F} &:= \left(\|q_F\|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E \|q_E\|_{L^2(E)}^2 \right)^{\frac{1}{2}} \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k. \end{aligned} \quad (4.1)$$

Similarly, for all $\underline{v}_h \in \underline{X}_{\text{curl},h}^k$,

$$\begin{aligned} \|\underline{v}_h\|_{\text{curl},h} &:= \left(\sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{\text{curl},T}^2 \right)^{\frac{1}{2}} \text{ with} \\ \forall T \in \mathcal{T}_h, \quad \|\underline{v}_T\|_{\text{curl},T} &:= \left(\|\mathbf{v}_{\mathcal{R},T}\|_{L^2(T)}^2 + \|\mathbf{v}_{\mathcal{R},T}^c\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|\underline{v}_F\|_{\text{curl},F}^2 \right)^{\frac{1}{2}} \quad \forall \underline{v}_T \in \underline{X}_{\text{curl},T}^k, \\ \forall F \in \mathcal{F}_h, \quad \|\underline{v}_F\|_{\text{curl},F} &:= \left(\|\mathbf{v}_{\mathcal{R},F}\|_{L^2(F)}^2 + \|\mathbf{v}_{\mathcal{R},F}^c\|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E \|\mathbf{v}_E\|_{L^2(E)}^2 \right)^{\frac{1}{2}} \quad \forall \underline{v}_F \in \underline{X}_{\text{curl},F}^k. \end{aligned} \quad (4.2)$$

Finally, for all $\underline{w}_h \in \underline{X}_{\text{div},h}^k$,

$$\begin{aligned} \|\underline{w}_h\|_{\text{div},h} &:= \left(\sum_{T \in \mathcal{T}_h} \|\underline{w}_T\|_{\text{div},T}^2 \right)^{\frac{1}{2}} \text{ with} \\ \forall T \in \mathcal{T}_h, \quad \|\underline{w}_T\|_{\text{div},T} &:= \left(\|\mathbf{w}_{\mathcal{G},T}\|_{L^2(T)}^2 + \|\mathbf{w}_{\mathcal{G},T}^c\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{w}_F\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \quad \forall \underline{w}_T \in \underline{X}_{\text{div},T}^k. \end{aligned}$$

4.2 Discrete Poincaré inequalities

In order to state the Poincaré inequality for the gradient in the case $k = 0$, we need to reconstruct face and element values from edge values. To do so, we select, for all $F \in \mathcal{F}_h$, $(\lambda_{FE})_{E \in \mathcal{E}_F} \in [0, 1]^{\mathcal{E}_F}$ such that $\sum_{E \in \mathcal{E}_F} \lambda_{FE} = 1$ and, for all $T \in \mathcal{T}_h$, $(\lambda_{TF})_{F \in \mathcal{F}_T} \in [0, 1]^{\mathcal{F}_T}$ such that $\sum_{F \in \mathcal{F}_T} \lambda_{TF} = 1$. We then define face and element averages as follows:

$$\begin{aligned} \forall F \in \mathcal{F}_h \quad \bar{q}_F &:= \sum_{E \in \mathcal{E}_F} \lambda_{FE} \pi_{\varphi, E}^0 q_E & \forall \underline{q}_F \in \underline{X}_{\text{grad}, F}^0, \\ \forall T \in \mathcal{T}_h \quad \bar{q}_T &:= \sum_{F \in \mathcal{F}_T} \lambda_{TF} \bar{q}_F & \forall \underline{q}_T \in \underline{X}_{\text{grad}, T}^0. \end{aligned} \quad (4.3)$$

We recall that $q_E = (q_{\mathcal{E}_F})|_E$ (see Remark 7). If $\underline{q}_h \in \underline{X}_{\text{grad}, h}^0$, we still denote by \bar{q}_F and \bar{q}_T the averages generated by the restrictions $\underline{q}_F \in \underline{X}_{\text{grad}, F}^0$ and $\underline{q}_T \in \underline{X}_{\text{grad}, T}^0$ of \underline{q}_h .

Theorem 24 (Poincaré inequality for the gradient). *Let $\underline{q}_h \in \underline{X}_{\text{grad}, h}^k$ be such that*

$$\begin{cases} \sum_{T \in \mathcal{T}_h} |T| \bar{q}_T = 0 & \text{if } k = 0, \\ \sum_{T \in \mathcal{T}_h} \int_T q_T = 0 & \text{if } k = 1. \end{cases} \quad (4.4)$$

$$\begin{cases} \sum_{T \in \mathcal{T}_h} \int_T q_T = 0 & \text{if } k = 1. \end{cases} \quad (4.5)$$

Then, there exists a real number $C > 0$ independent of h and \underline{q}_h , and depending only on Ω , k , and the mesh regularity parameter, such that

$$\|\underline{q}_h\|_{\text{grad}, h} \leq C \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\text{curl}, h}. \quad (4.6)$$

Proof. See Section 4.3. □

Remark 25 (Choice of weights in (4.3)). Assume that, for all $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, $\mathbf{x}_Y \in Y$ is a point such that Y is star-shaped with respect to \mathbf{x}_Y . The weights in (4.3) can be selected as follows: for all $F \in \mathcal{F}_h$, $\lambda_{FE} = \frac{d_{FE}|E|}{2|F|}$ for all $E \in \mathcal{E}_F$, with d_{FE} denoting the orthogonal distance between \mathbf{x}_F and E ; for all $T \in \mathcal{T}_h$, $\lambda_{TF} = \frac{d_{TF}|F|}{3|T|}$ for all $F \in \mathcal{F}_T$, with d_{TF} denoting the orthogonal distance between \mathbf{x}_T and F . With this choice, the condition (4.4) becomes

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \sum_{E \in \mathcal{E}_F} |P_{TFE}| \pi_{\varphi, E}^0 q_E = \frac{1}{2} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \sum_{E \in \mathcal{E}_F} \sum_{V \in \mathcal{V}_E} |P_{TFE}| q_{\mathcal{E}_F}(\mathbf{x}_V) = 0,$$

where, for any mesh element $T \in \mathcal{T}_h$, face $F \in \mathcal{F}_T$, and edge $E \in \mathcal{E}_F$ of vertices V_1 and V_2 , P_{TFE} is the tetrahedron of vertices \mathbf{x}_T , \mathbf{x}_F , \mathbf{x}_{V_1} , and \mathbf{x}_{V_2} . When $\mathbf{x}_Y = \frac{1}{|Y|} \int_Y \mathbf{x}$, we recover the construction on the dual barycentric mesh of [8, Section 4.1].

For the sake of completeness, we state in what follows Poincaré inequalities for the curl and the divergence that are easy consequences of the results of [20].

Theorem 26 (Poincaré inequality for the curl). *Denote by (b_0, b_1, b_2, b_3) the Betti numbers of Ω (with $b_0 = 1$ and $b_3 = 0$) and assume $b_2 = 0$. Let $(\text{Ker } \underline{\mathbf{C}}_h^k)^\perp$ be the orthogonal of $\text{Ker } \underline{\mathbf{C}}_h^k$ in $\underline{\mathbf{X}}_{\text{curl}, h}^k$ for an inner product whose norm is, uniformly in h , equivalent to $\|\cdot\|_{\text{curl}, h}$. Then, $\underline{\mathbf{C}}_h^k : (\text{Ker } \underline{\mathbf{C}}_h^k)^\perp \rightarrow \text{Ker } D_h^k$ is an isomorphism. Further assuming that $b_1 = 0$, there exists $C > 0$ independent of h , and depending only on Ω , k and the mesh regularity parameter, such that*

$$\|\underline{\mathbf{v}}_h\|_{\text{curl}, h} \leq C \|\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h\|_{\text{div}, h} \quad \forall \underline{\mathbf{v}}_h \in (\text{Ker } \underline{\mathbf{C}}_h^k)^\perp. \quad (4.7)$$

Proof. The isomorphism property is a consequence of (3.44). In order to prove the Poincaré inequality (4.7), combine [20, Theorem 20] with [20, Proposition 16] (which requires the additional assumption $b_1 = 0$) and the norm equivalence of [20, Proposition 14] (see also [19, Eq. (4.5)]). \square

Theorem 27 (Poincaré inequality for the divergence). *Let $(\text{Ker } D_h^k)^\perp$ be the orthogonal of $\text{Ker } D_h^k$ in $\underline{X}_{\text{div},h}^k$ for an inner product whose norm is, uniformly in h , equivalent to $\|\cdot\|_{\text{div},h}$. Then, $D_h^k : (\text{Ker } D_h^k)^\perp \rightarrow \mathcal{P}^k(\mathcal{T}_h)$ is an isomorphism and there exists $C > 0$ independent of h such that*

$$\|\underline{w}_h\|_{\text{div},h} \leq C \|D_h^k \underline{w}_h\|_{L^2(\Omega)} \quad \forall \underline{w}_h \in (\text{Ker } D_h^k)^\perp. \quad (4.8)$$

Proof. The isomorphism property is a consequence of (3.41). The Poincaré inequality (4.8) follows from [20, Theorem 18] accounting for the norm equivalence stated in [20, Proposition 14] or [19, Eq. (4.5)]. \square

4.3 Proof of the discrete Poincaré inequality for the gradient

4.3.1 Preliminary results

We need the preliminary results established in the following lemmas.

Lemma 28 (Estimates of local H^1 -seminorms for $k = 0$). *It holds, recalling (4.3),*

$$\forall F \in \mathcal{F}_h, \quad \sum_{E \in \mathcal{E}_F} h_E^{-1} \|\bar{q}_F - \pi_{\mathcal{P},E}^0 q_E\|_{L^2(E)}^2 \lesssim \|\underline{G}_F^0 \underline{q}_F\|_{\text{curl},F}^2 \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^0, \quad (4.9)$$

$$\forall T \in \mathcal{T}_h, \quad \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\bar{q}_T - \bar{q}_F\|_{L^2(F)}^2 \lesssim \|\underline{G}_T^0 \underline{q}_T\|_{\text{curl},T}^2 \quad \forall \underline{q}_T \in \underline{X}_{\text{grad},T}^0, \quad (4.10)$$

where $a \lesssim b$ means $a \leq Cb$ with $C > 0$ depending only on the mesh regularity parameter.

Proof. Throughout the proof, \lesssim has the same meaning as in the statement. Let $F \in \mathcal{F}_h$ and $\underline{q}_F = (0, q_{\mathcal{E}_F}) \in \underline{X}_{\text{grad},F}^0$. For all $E \in \mathcal{E}_F$, recalling that $q_E = (q_{\mathcal{E}_F})|_E$ it holds

$$|\pi_{\mathcal{P},E}^0 q_E - q_{\mathcal{E}_F}(\mathbf{x}_V)| \leq \frac{h_E}{2} |G_E^0 q_E| \lesssim h_E^{\frac{1}{2}} \|G_E^0 q_E\|_{L^2(E)} \quad \forall V \in \mathcal{V}_E. \quad (4.11)$$

Denote by E_1 and E_2 two distinct edges in \mathcal{E}_F . Constructing a connected path of edges in \mathcal{E}_F that starts in E_1 and ends in E_2 (which is possible since the boundary of F is connected), inserting $\pm q_{\mathcal{E}_h}(\mathbf{x}_V)$ for all $V \in \mathcal{V}_F$ vertex internal to the path, using triangle inequalities, and invoking (4.11) along with $|q_{\mathcal{E}_F}(\mathbf{x}_{V_1}) - q_{\mathcal{E}_F}(\mathbf{x}_{V_2})| \leq h_E^{\frac{1}{2}} \|G_E^0 q_E\|_{L^2(E)}$ for all $E \in \mathcal{E}_F$ of vertices V_1 and V_2 , it is readily inferred that

$$|\pi_{\mathcal{P},E_1}^0 q_{E_1} - \pi_{\mathcal{P},E_2}^0 q_{E_2}| \lesssim \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \|G_E^0 q_E\|_{L^2(E)} \leq \|\underline{G}_F^0 \underline{q}_F\|_{\text{curl},F}, \quad (4.12)$$

where we have invoked a discrete Cauchy–Schwarz inequality, $\text{card}(\mathcal{E}_F) \lesssim 1$, and the definition (4.2) of $\|\cdot\|_{\text{curl},F}$ to conclude. We next notice that, for all $\tilde{E} \in \mathcal{E}_F$, it holds

$$|\bar{q}_F - \pi_{\mathcal{P},\tilde{E}}^0 q_{\tilde{E}}| = \left| \sum_{E \in \mathcal{E}_F} \lambda_{FE} (\pi_{\mathcal{P},E}^0 q_E - \pi_{\mathcal{P},\tilde{E}}^0 q_{\tilde{E}}) \right| \lesssim \|\underline{G}_F^0 \underline{q}_F\|_{\text{curl},F},$$

where we have used the definition (4.3) of \bar{q}_F along with $\sum_{E \in \mathcal{E}_F} \lambda_{FE} = 1$ in the equality, and (4.12) along with $\lambda_{FE} \geq 0$ and again $\sum_{E \in \mathcal{E}_F} \lambda_{FE} = 1$ to conclude. Hence, (4.9) follows by writing

$$\sum_{E \in \mathcal{E}_F} h_E^{-1} \|\bar{q}_F - \pi_{\mathcal{P},E}^0 q_E\|_{L^2(E)}^2 = \sum_{E \in \mathcal{E}_F} |\bar{q}_F - \pi_{\mathcal{P},E}^0 q_E|^2 \lesssim \|\underline{G}_F^0 \underline{q}_F\|_{\text{curl},F}^2.$$

The proof of (4.10) is similar. Letting $T \in \mathcal{T}_h$ and $\underline{q}_T \in \underline{X}_{\text{grad},T}^0$, the first step consists in leveraging (4.9) as to pass from (4.11) to (4.12) in order to prove that, for all F_1 and F_2 distinct faces of \mathcal{F}_T ,

$$|\bar{q}_{F_1} - \bar{q}_{F_2}| \lesssim \sum_{F \in \mathcal{F}_T} \|\underline{\mathbf{G}}_F^0 \underline{q}_F\|_{\text{curl},F}. \quad (4.13)$$

We then write, for all $\tilde{F} \in \mathcal{F}_T$,

$$\begin{aligned} h_{\tilde{F}}^{-1} \|\bar{q}_T - \bar{q}_{\tilde{F}}\|_{L^2(\tilde{F})}^2 &= h_{\tilde{F}}^{-1} |\tilde{F}| |\bar{q}_T - \bar{q}_{\tilde{F}}|^2 \lesssim h_{\tilde{F}} \left| \sum_{F \in \mathcal{F}_T} \lambda_{TF} (\bar{q}_F - \bar{q}_{\tilde{F}}) \right|^2 \\ &\lesssim \sum_{F \in \mathcal{F}_T} h_{\tilde{F}} \|\underline{\mathbf{G}}_F^0 \underline{q}_F\|_{\text{curl},F}^2 \lesssim \|\underline{\mathbf{G}}_T^0 \underline{q}_T\|_{\text{curl},T}^2, \end{aligned}$$

where we have used the definition (4.3) of \bar{q}_T along with $\sum_{F \in \mathcal{F}_T} \lambda_{TF} = 1$ and $|\tilde{F}| \lesssim h_{\tilde{F}}^2$ in the first inequality, a Cauchy–Schwarz inequality, (4.13), $\lambda_{TF} \geq 0$, $\sum_{F \in \mathcal{F}_T} \lambda_{TF} = 1$, and the fact that $\text{card}(\mathcal{F}_T) \lesssim 1$ (owing to mesh regularity) in the second inequality, and invoked again a Cauchy–Schwarz inequality and $\text{card}(\mathcal{F}_T) \lesssim 1$, together with $h_{\tilde{F}} \lesssim h_F$ for all $F \in \mathcal{F}_T$ and the definition (4.2) of $\|\cdot\|_{\text{curl},T}$ to conclude. Summing the previous inequality over the faces \tilde{F} of T yields (4.10). \square

Lemma 29 (Estimates of local H^1 -seminorms for $k \geq 1$). *Assume $k \geq 1$. Then, it holds:*

$$\forall F \in \mathcal{F}_h, \quad \|\mathbf{grad}_F q_F\|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E^{-1} \|q_F - q_E\|_{L^2(E)}^2 \lesssim \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\text{curl},F}^2 \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k, \quad (4.14)$$

$$\forall T \in \mathcal{T}_h, \quad \|\mathbf{grad}_T q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_T - q_F\|_{L^2(F)}^2 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\text{curl},T}^2, \quad \forall \underline{q}_T \in \underline{X}_{\text{grad},T}^k, \quad (4.15)$$

where $a \lesssim b$ means $a \leq Cb$ with $C > 0$ depending only on k and the mesh regularity parameter.

Proof of Lemma 29. Throughout the proof, \lesssim has the same meaning as in the statement, and we also use $a \simeq b$ as a shortcut for “ $a \lesssim b$ and $b \lesssim a$ ”.

1. *Proof of (4.14).* Let $\underline{q}_F \in \underline{X}_{\text{grad},F}^k$ and define $A_{q,\partial F} \in \mathbb{R}$ as the average of $q_{\mathcal{E}_F}$ over ∂F . Using triangle and trace inequalities along with the fact that $\text{card}(\mathcal{E}_F) \lesssim 1$ by mesh regularity, we infer that

$$\sum_{E \in \mathcal{E}_F} h_E^{-1} \|q_F - q_E\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_F} h_E^{-1} \|q_E - A_{q,\partial F}\|_{L^2(E)}^2 + h_F^{-2} \|q_F - A_{q,\partial F}\|_{L^2(F)}^2 =: \mathfrak{I}_1 + \mathfrak{I}_2. \quad (4.16)$$

We proceed to estimate the terms in the right-hand side. Since $q_{\mathcal{E}_F}$ is continuous, a Poincaré–Wirtinger inequality along ∂F together with $h_E \simeq h_F$ readily yields

$$\mathfrak{I}_1 \lesssim \sum_{E \in \mathcal{E}_F} h_E \|G_E^k q_E\|_{L^2(E)}^2 \leq \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\text{curl},F}^2. \quad (4.17)$$

To estimate \mathfrak{I}_2 , we apply the definition (3.4) of \mathbf{G}_F^k to $\underline{q}_F - I_{\text{grad},F}^k A_{q,\partial F} \in \underline{X}_{\text{grad},F}^k$ and take as a test function $\mathbf{v}_F \in \mathcal{R}^{c,k}(F)$ such that $\text{div}_F \mathbf{v}_F = q_F - A_{q,\partial F}$ (this is possible because $k \geq 1$) and $\|\mathbf{v}_F\|_{L^2(F)} \lesssim h_F \|q_F - A_{q,\partial F}\|_{L^2(F)}$ (see Lemma 31 below). Using the definition of $\pi_{\mathcal{R},F}^{c,k}$ and recalling the consistency property (3.10) of \mathbf{G}_F^k , we have

$$\int_F \mathbf{G}_F^k (\underline{q}_F - I_{\text{grad},F}^k A_{q,\partial F}) \cdot \mathbf{v}_F = \int_F \pi_{\mathcal{R},F}^{c,k} (\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{v}_F$$

and thus, using Cauchy–Schwarz and discrete trace inequalities, we infer

$$\begin{aligned}
\|q_F - A_{q,\partial F}\|_{\mathbf{L}^2(F)}^2 &\leq \|\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} \|\mathbf{v}_F\|_{\mathbf{L}^2(F)} + \sum_{E \in \mathcal{E}_F} \|q_E - A_{q,\partial F}\|_{\mathbf{L}^2(E)} \|\mathbf{v}_F \cdot \mathbf{n}_{FE}\|_{\mathbf{L}^2(E)} \\
&\lesssim h_F \|\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} \|q_F - A_{q,\partial F}\|_{\mathbf{L}^2(F)} \\
&\quad + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \|q_E - A_{q,\partial F}\|_{\mathbf{L}^2(E)} \|q_F - A_{q,\partial F}\|_{\mathbf{L}^2(F)}. \tag{4.18}
\end{aligned}$$

Simplifying by $\|q_F - A_{q,\partial F}\|_{\mathbf{L}^2(F)}$, accounting for (4.17) in the last term, using a discrete Cauchy–Schwarz inequality, squaring, and dividing by h_F^2 yields

$$\mathfrak{I}_2 \lesssim \|\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)}^2 + \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2 \lesssim \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2. \tag{4.19}$$

Plugging (4.17) and (4.19) into (4.16) yields

$$\sum_{E \in \mathcal{E}_F} h_E^{-1} \|q_F - q_E\|_{\mathbf{L}^2(E)}^2 \lesssim \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2. \tag{4.20}$$

To estimate the first term in the left-hand side of (4.14), we write the definition (3.5) of $\mathbf{G}_F^k \underline{q}_F$ with $\mathbf{w}_F = \mathbf{grad}_F q_F$ to infer

$$\begin{aligned}
\|\mathbf{grad}_F q_F\|_{\mathbf{L}^2(F)}^2 &= \int_F \pi_{\mathcal{P},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F) \cdot \mathbf{w}_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (q_{\mathcal{E}_F} - q_F)(\mathbf{w}_F \cdot \mathbf{n}_{FE}) \\
&\lesssim \|\pi_{\mathcal{P},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} \|\mathbf{w}_F\|_{\mathbf{L}^2(F)} + \sum_{E \in \mathcal{E}_F} h_E^{-\frac{1}{2}} \|q_{\mathcal{E}_F} - q_F\|_{\mathbf{L}^2(E)} \|\mathbf{w}_F\|_{\mathbf{L}^2(F)}, \tag{4.21}
\end{aligned}$$

where we have used the fact that $\mathbf{w}_F \in \mathcal{P}^{k-1}(F)$ to insert the projector in front of $\mathbf{G}_F^k \underline{q}_F$ in the first line, and we have invoked Cauchy–Schwarz and trace inequalities to pass to the second line. We next notice that $\pi_{\mathcal{P},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F) = \mathfrak{R}_{\mathcal{R},F}^{k-1}(\pi_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \pi_{\mathcal{R},F}^{c,k-1} \pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F))$ since, by (2.11), $\pi_{\mathcal{R},F}^{k-1} \pi_{\mathcal{P},F}^{k-1} = \pi_{\mathcal{R},F}^{k-1}$ and $\pi_{\mathcal{R},F}^{c,k-1} \pi_{\mathcal{P},F}^{k-1} = \pi_{\mathcal{R},F}^{c,k-1} = \pi_{\mathcal{R},F}^{c,k-1} \pi_{\mathcal{R},F}^{c,k}$ given that $\mathcal{R}^{c,k-1}(F) \subset \mathcal{R}^{c,k}(F)$ by (2.7). By virtue of (2.19) along with the $\mathbf{L}^2(F)$ -boundedness of $\pi_{\mathcal{R},F}^{c,k-1}$, this gives $\|\pi_{\mathcal{P},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} \lesssim \|\pi_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} + \|\pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F)\|_{\mathbf{L}^2(F)} \lesssim \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}$. Using this result along with (4.20) in (4.21) and simplifying gives

$$\|\mathbf{grad}_F q_F\|_{\mathbf{L}^2(F)}^2 \lesssim \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2, \tag{4.22}$$

which, summed to (4.20), yields (4.14).

2. Proof of (4.15). The ideas to prove (4.15) are similar to those used for F above, but the Poincaré–Wirtinger inequality leading to (4.17) is not readily available for face potentials, as they are discontinuous from one face to the other, and needs to be separately established. Let

$$A_{q,\partial T} := \frac{1}{|\partial T|} \sum_{F \in \mathcal{F}_T} |F| A_{q,F} \quad \text{with} \quad A_{q,F} := \frac{1}{|F|} \int_F q_F \quad \text{for all } F \in \mathcal{F}_T$$

denote the average over ∂T of the piecewise polynomial function defined by $(q_F)_{F \in \mathcal{F}_T}$. As a starting point to estimate the second term in the left-hand side of (4.15), using triangle and trace inequalities along with the fact that $\text{card}(\mathcal{F}_T) \lesssim 1$ by mesh regularity, we write

$$\begin{aligned}
\sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_T - q_F\|_{\mathbf{L}^2(F)}^2 &\lesssim \sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_F - A_{q,F}\|_{\mathbf{L}^2(F)}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|A_{q,F} - A_{q,\partial T}\|_{\mathbf{L}^2(F)}^2 \\
&\quad + h_T^{-2} \|q_T - A_{q,\partial T}\|_{\mathbf{L}^2(T)}^2 =: \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \tag{4.23}
\end{aligned}$$

The Poincaré–Wirtinger inequality $\|q_F - A_{q,F}\|_{L^2(F)} \lesssim h_F \|\mathbf{grad}_F q_F\|_{L^2(F)}$ combined with (4.22) readily yields for the first term

$$\mathfrak{I}_1 \lesssim \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{grad}_F q_F\|_{L^2(F)}^2 \lesssim \sum_{F \in \mathcal{F}_T} h_F \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2 \leq \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2. \quad (4.24)$$

Since $A_{q,\partial T}$ is a weighted average of all $(A_{q,F})_{F \in \mathcal{F}_T}$, the bound

$$\mathfrak{I}_2 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2 \quad (4.25)$$

follows if we prove that, for all $F, F' \in \mathcal{F}_T$,

$$h_F^{-1} \|A_{q,F} - A_{q,F'}\|_{L^2(F)}^2 = h_F^{-1} |F| |A_{q,F} - A_{q,F'}|^2 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2. \quad (4.26)$$

Creating a sequence $(F = F_0, F_1, \dots, F_m = F')$ of faces in \mathcal{F}_T such that, for all $i = 0, \dots, m-1$, the faces F_i, F_{i+1} share an edge E_i , inserting

$$0 = -\pi_{\mathcal{P},E_0}^0 q_{E_0} + \sum_{i=0}^{m-2} \left[\left(\pi_{\mathcal{P},E_i}^0 q_{E_i} - A_{q,F_{i+1}} \right) - \left(\pi_{\mathcal{P},E_{i+1}}^0 q_{E_{i+1}} - A_{q,F_{i+1}} \right) \right] + \pi_{\mathcal{P},E_{m-1}}^0 q_{E_{m-1}}$$

into $|A_{q,F} - A_{q,F'}|$, using triangle inequalities and the fact that $h_{F_i} \simeq h_{F_{i+1}}$ and $|F_i| \simeq |F_{i+1}|$ for all $i = 0, \dots, m-1$ by mesh regularity, and recalling the definition (4.2) of $\|\cdot\|_{\mathbf{curl},h}$, (4.26) is a consequence of

$$\forall F \in \mathcal{F}_T, \quad h_F^{-1} |F| |A_{q,F} - \pi_{\mathcal{P},E}^0 q_E|^2 \lesssim h_F \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2 \quad \forall E \in \mathcal{E}_F. \quad (4.27)$$

To prove this relation, we write

$$\begin{aligned} \|A_{q,F} - \pi_{\mathcal{P},E}^0 q_E\|_{L^2(E)}^2 &\leq \|A_{q,F} - q_E\|_{L^2(E)}^2 \\ &\lesssim h_F^{-1} \|A_{q,F} - q_F\|_{L^2(F)}^2 + \|q_F - q_E\|_{L^2(E)}^2 \lesssim h_F \|\underline{\mathbf{G}}_F^k \underline{q}_F\|_{\mathbf{curl},F}^2, \end{aligned}$$

where the first inequality comes from the $L^2(E)$ -boundedness of $\pi_{\mathcal{P},E}^0$, the second inequality is obtained introducing q_F and using a triangle inequality together with a discrete trace inequality, while we have used (4.14) and the same arguments that lead to (4.24) in the conclusion. The relation (4.27) follows noticing that $|F| \simeq h_F |E|$, so that $|F| |A_{q,F} - \pi_{\mathcal{P},E}^0 q_E|^2 \simeq h_F \|A_{q,F} - \pi_{\mathcal{P},E}^0 q_E\|_{L^2(E)}^2$. This concludes the proof of (4.26), hence of (4.25).

Finally, to estimate \mathfrak{I}_3 , we apply the definition (3.7) of the element gradient to $\underline{q}_T - \underline{I}_{\mathbf{grad},T}^k A_{q,\partial T} \in \underline{X}_{\mathbf{grad},T}^k$ and take as a test function $\mathbf{w}_T \in \mathcal{R}^{c,k}(T)$ such that $\operatorname{div} \mathbf{w}_T = q_T - A_{q,\partial T}$. By Lemma 31 below, we have $\|\mathbf{w}_T\|_{L^2(T)} \lesssim h_T \|q_T - A_{q,\partial T}\|_{L^2(T)}$. Noticing that, for all $F \in \mathcal{F}_T$, $(\mathbf{w}_T)|_F \cdot \mathbf{n}_F \in \mathcal{P}^{k-1}(F)$ (see (A.6)), using (3.12) to replace $\gamma_F^{k+1}(\underline{q}_F - \underline{I}_{\mathbf{grad},F}^k A_{q,\partial T})$ by $q_F - A_{q,\partial T}$, proceeding as in (4.18) and noticing that

$$\sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_F - A_{q,\partial T}\|_{L^2(F)}^2 \lesssim \mathfrak{I}_1 + \mathfrak{I}_2,$$

we infer

$$\mathfrak{I}_3 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2. \quad (4.28)$$

Plugging (4.24), (4.25), and (4.28) into (4.23) yields

$$\sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_T - q_F\|_{L^2(F)}^2 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2. \quad (4.29)$$

Proceeding in a similar way as for the proof of (4.22), that is, writing (3.8) with $\mathbf{w}_T = \mathbf{grad} q_T$, and additionally noticing that, owing to (3.12), $\gamma_F^{k+1} \underline{q}_F$ can be replaced by q_F in the boundary term since $(\mathbf{w}_T)|_F \cdot \mathbf{n}_F \in \mathcal{P}^{k-1}(F)$ for all $F \in \mathcal{F}_T$, we infer $\|\mathbf{grad} q_T\|_{L^2(T)}^2 \lesssim \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl},T}^2$. Summing this estimate together with (4.29) yields (4.15). \square

4.3.2 Proof of Theorem 24

We are now ready to prove Theorem 24.

Proof of Theorem 24. Let us first consider the case $k \geq 1$. By [21, Theorem 6.5] with $p = q = 2$, (4.5) implies

$$\sum_{T \in \mathcal{T}_h} \|q_T\|_{L^2(T)}^2 \lesssim \sum_{T \in \mathcal{T}_h} \left(\|\mathbf{grad} q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|q_T - q_F\|_{L^2(F)}^2 \right) \lesssim \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\mathbf{curl}, h}^2, \quad (4.30)$$

where the conclusion is a consequence of (4.15) followed by the definition (4.2) of the $\|\cdot\|_{\mathbf{curl}, h}$ -norm. Let now $T \in \mathcal{T}_h$ and write, using triangle inequalities along with a trace inequality and the bound $\text{card}(\mathcal{F}_T) \lesssim 1$ resulting from mesh regularity,

$$\sum_{F \in \mathcal{F}_T} h_F \|q_F\|_{L^2(F)}^2 \lesssim \|q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \|q_T - q_F\|_{L^2(F)}^2 \lesssim \|q_T\|_{L^2(T)}^2 + h_T^2 \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl}, T}^2, \quad (4.31)$$

where the conclusion follows from $h_F \leq h_T$ for all $F \in \mathcal{F}_T$ along with (4.15). Similarly, triangle and trace inequalities along with the geometric bound $\text{card}(\mathcal{E}_F) \lesssim 1$ yield

$$\begin{aligned} \sum_{F \in \mathcal{F}_T} h_F \sum_{E \in \mathcal{E}_F} h_E \|q_E\|_{L^2(E)}^2 &\lesssim \sum_{F \in \mathcal{F}_T} h_F \|q_F\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_T} h_F \sum_{E \in \mathcal{E}_F} h_E \|q_F - q_E\|_{L^2(E)}^2 \\ &\lesssim \|q_T\|_{L^2(T)}^2 + h_T^2 \|\underline{\mathbf{G}}_T^k \underline{q}_T\|_{\mathbf{curl}, T}^2, \end{aligned} \quad (4.32)$$

where the conclusion follows using (4.31) for the first term and (4.14) along with $h_E \leq h_T$ for the second. Adding (4.31) to (4.32), summing the resulting inequality over $T \in \mathcal{T}_h$, recalling the definition (4.1) of the $\|\cdot\|_{\mathbf{grad}, F}$ -norm, using the fact that $h_T \leq h \leq h_\Omega$ for all $T \in \mathcal{T}_h$, and invoking (4.30), we get

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\underline{q}_F\|_{\mathbf{grad}, F}^2 \lesssim \sum_{T \in \mathcal{T}_h} \|q_T\|_{L^2(T)}^2 + h_\Omega^2 \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\mathbf{curl}, h}^2 \lesssim \|\underline{\mathbf{G}}_h^k \underline{q}_h\|_{\mathbf{curl}, h}^2. \quad (4.33)$$

Summing (4.30) to (4.33) yields (4.6).

The proof for the case $k = 0$ is the same provided that we replace, in the above expressions, (q_T, q_F, q_E) with $(\bar{q}_T, \bar{q}_F, \pi_{\mathcal{P}, E}^0 q_E)$ defined by (4.3) (additionally noticing that $\mathbf{grad} \bar{q}_T = \mathbf{0}$), (4.14) with (4.9), and (4.15) with (4.10), and that we notice that, by a Poincaré–Wirtinger inequality on each edge, $\|q_E - \pi_{\mathcal{P}, E}^0 q_E\|_{L^2(E)} \leq h_E \|G_E^k q_E\|_{L^2(E)}$. \square

A Results on local spaces

This section collects miscellaneous results on the Koszul complements defined in (2.3) and (2.5), as well as on the trimmed spaces (2.12) obtained from the latter.

Proposition 30 (Traces of Nédélec and Raviart–Thomas functions). *It holds, for all $F \in \mathcal{F}_h$,*

$$\forall E \in \mathcal{E}_F \quad (\mathbf{v}_F)|_E \cdot \mathbf{t}_E \in \mathcal{P}^{\ell-1}(E) \quad \forall \mathbf{v}_F \in \mathcal{N}^\ell(F), \quad (\text{A.1})$$

$$\forall E \in \mathcal{E}_F \quad (\mathbf{w}_F)|_E \cdot \mathbf{n}_{FE} \in \mathcal{P}^{\ell-1}(E) \quad \forall \mathbf{w}_F \in \mathcal{RT}^\ell(F) \quad (\text{A.2})$$

and, for all $T \in \mathcal{T}_h$,

$$\forall E \in \mathcal{E}_T \quad (\mathbf{v}_T)|_E \cdot \mathbf{t}_E \in \mathcal{P}^{\ell-1}(E) \quad \forall \mathbf{v}_T \in \mathcal{N}^\ell(T), \quad (\text{A.3})$$

$$\forall F \in \mathcal{F}_T \quad (\mathbf{w}_T)|_F \cdot \mathbf{n}_F \in \mathcal{P}^{\ell-1}(F) \quad \forall \mathbf{w}_T \in \mathcal{RT}^\ell(T), \quad (\text{A.4})$$

$$\forall F \in \mathcal{F}_T \quad (\mathbf{v}_T)|_F \times \mathbf{n}_F \in \mathcal{RT}^\ell(F) \quad \forall \mathbf{v}_T \in \mathcal{N}^\ell(T). \quad (\text{A.5})$$

Proof. 1. Proof of (A.1) and (A.3). The tangent edge traces of functions in $\mathcal{G}^{\ell-1}(F)$ (resp. $\mathcal{G}^{\ell-1}(T)$) are in $\mathcal{P}^{\ell-1}(E)$ for all $E \in \mathcal{E}_F$ (resp. $E \in \mathcal{E}_T$). To prove (A.1), it then suffices to recall the definition (2.3a) and observe that the quantity $(\mathbf{x} - \mathbf{x}_F)^\perp \cdot \mathbf{t}_E = \underbrace{(\mathbf{x} - \mathbf{x}_E)^\perp \cdot \mathbf{t}_E}_{\text{constant}} + (\mathbf{x}_E - \mathbf{x}_F)^\perp \cdot \mathbf{t}_E$ is constant over E , the cancellation coming from the fact that $(\mathbf{x} - \mathbf{x}_E)$ and \mathbf{t}_E are parallel for all $\mathbf{x} \in E$. To prove (A.3), recall the definition (2.5a) of $\mathcal{G}^{\mathbf{c},\ell}(T)$ and observe that, for all $\mathbf{v} \in \mathcal{P}^{\ell-1}(T)$ and all $\mathbf{x} \in E$, $[(\mathbf{x} - \mathbf{x}_T) \times \mathbf{v}] \cdot \mathbf{t}_E = \underbrace{[(\mathbf{x} - \mathbf{x}_E) \times \mathbf{v}] \cdot \mathbf{t}_E}_{\text{constant}} + [(\mathbf{x}_E - \mathbf{x}_T) \times \mathbf{v}] \cdot \mathbf{t}_E \in \mathcal{P}^{\ell-1}(E)$, where the cancellation follows observing, as before, that the vectors $(\mathbf{x} - \mathbf{x}_E)$ and \mathbf{t}_E are parallel.

2. Proof of (A.2) and (A.4). The normal traces of functions in $\mathcal{R}^{\ell-1}(F)$ (resp. $\mathcal{R}^{\ell-1}(T)$) are in $\mathcal{P}^{\ell-1}(E)$ (resp. $\mathcal{P}^{\ell-1}(F)$) for all $E \in \mathcal{E}_F$ (resp. $F \in \mathcal{F}_T$). To conclude, recall the definition (2.3b) (resp. (2.5b)) of $\mathcal{R}^{\mathbf{c},\ell}(F)$ (resp. $\mathcal{R}^{\mathbf{c},\ell}(T)$) and observe, using similar arguments as above, that the quantity $(\mathbf{x} - \mathbf{x}_F) \cdot \mathbf{n}_{FE}$ (resp. $(\mathbf{x} - \mathbf{x}_T) \cdot \mathbf{n}_F$) is constant for any $\mathbf{x} \in E$ (resp. $\mathbf{x} \in F$). This implies, in particular, for all $T \in \mathcal{T}_h$,

$$\forall F \in \mathcal{F}_T, \quad (\mathbf{z}_T)|_F \cdot \mathbf{n}_F \in \mathcal{P}^{\ell-1}(F) \quad \forall \mathbf{z}_T \in \mathcal{R}^{\mathbf{c},\ell}(T). \quad (\text{A.6})$$

3. Proof of (A.5). For all $\mathbf{v}_T \in \mathcal{G}^{\ell-1}(T) \subset \mathcal{P}^{\ell-1}(T)$ we have $(\mathbf{v}_T)|_F \times \mathbf{n}_F \in \mathcal{P}^{\ell-1}(F) \subset \mathcal{RT}^\ell(F)$ since \mathbf{n}_F is constant. It therefore suffices to prove (A.5) for $\mathbf{v}_T \in \mathcal{G}^{\mathbf{c},\ell}(T)$. Recalling (2.5a), there is $\mathbf{z}_T \in \mathcal{P}^{\ell-1}(T)$ such that $\mathbf{v}_T = (\mathbf{x} - \mathbf{x}_T) \times \mathbf{z}_T$. Thus, we can write

$$\begin{aligned} (\mathbf{v}_T)|_F \times \mathbf{n}_F &= ((\mathbf{x} - \mathbf{x}_T) \times \mathbf{z}_T)|_F \times \mathbf{n}_F \\ &= ((\mathbf{x} - \mathbf{x}_T)|_F \cdot \mathbf{n}_F)(\mathbf{z}_T)|_F + ((\mathbf{z}_T)|_F \cdot \mathbf{n}_F)(\mathbf{x}_T - \mathbf{x})|_F \\ &= ((\mathbf{x}_F - \mathbf{x}_T)|_F \cdot \mathbf{n}_F)(\mathbf{z}_T)|_F + ((\mathbf{z}_T)|_F \cdot \mathbf{n}_F)(\mathbf{x}_F - \mathbf{x})|_F - ((\mathbf{z}_T)|_F \cdot \mathbf{n}_F)(\mathbf{x}_F - \mathbf{x}_T) \\ &= \underbrace{((\mathbf{x}_F - \mathbf{x}_T) \times \mathbf{z}_T)|_F \times \mathbf{n}_F}_{\in \mathcal{P}^{\ell-1}(F)} + \underbrace{((\mathbf{z}_T)|_F \cdot \mathbf{n}_F)(\mathbf{x}_F - \mathbf{x})|_F}_{\in \mathcal{R}^{\mathbf{c},\ell}(F)}, \end{aligned}$$

where we have used the vector algebra identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \quad \forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^3 \quad (\text{A.7})$$

with $\mathbf{A} = \mathbf{x} - \mathbf{x}_T$, $\mathbf{B} = \mathbf{z}_T$, and $\mathbf{C} = \mathbf{n}_F$ to pass to the second line; to pass to the third line, we have noticed that $(\mathbf{x} - \mathbf{x}_T) \cdot \mathbf{n}_F$ is constant on F for the first term, and we have added $\pm \mathbf{x}_F$ inside the last parentheses and developed; the last line follows from an application of (A.7) with $\mathbf{A} = \mathbf{x}_F - \mathbf{x}_T$, $\mathbf{B} = \mathbf{z}_T$, and $\mathbf{C} = \mathbf{n}_F$. Since $\mathcal{P}^{\ell-1}(F) = \mathcal{R}^{\ell-1}(F) \oplus \mathcal{R}^{\mathbf{c},\ell-1}(F) \subset \mathcal{R}^{\ell-1}(F) \oplus \mathcal{R}^{\mathbf{c},\ell}(F) = \mathcal{RT}^\ell(F)$, this concludes the proof. \square

Lemma 31 (Norms of the inverses of local differential isomorphisms). *The norms of the inverses of the isomorphisms defined in (2.8)–(2.10) satisfy, for all $F \in \mathcal{F}_h$ or $T \in \mathcal{T}_h$,*

$$\|(\mathbf{rot}_F)^{-1}\| \lesssim h_F, \quad \|(\mathbf{div}_F)^{-1}\| \lesssim h_F, \quad \|(\mathbf{div})^{-1}\| \lesssim h_T, \quad \text{and} \quad \|(\mathbf{curl})^{-1}\| \lesssim h_T$$

where, above, $\|\cdot\|$ denotes the norm of the corresponding isomorphism when its domain and co-domains are endowed with their L^2 -norms, and $a \lesssim b$ means that $a \leq Cb$ with C depending only on the polynomial degree ℓ and on the mesh regularity parameter.

Proof. We only prove the estimate on $\|(\mathbf{curl})^{-1}\|$, since the other ones follow from similar arguments. The idea is to use the transport $T \ni \mathbf{x} \mapsto \widehat{\mathbf{x}} = h_T^{-1}(\mathbf{x} - \mathbf{x}_T) \in \widehat{T}$ as in the proof of Lemma 5. We recall that $B(\rho) \subset \widehat{T} \subset B(1)$, where ρ is the mesh regularity parameter and $B(r) = \{\mathbf{y} \in \mathbb{R}^3 : |\mathbf{y}| < r\}$.

Let $\mathbf{v} \in \mathcal{R}^{\ell-1}(T)$ and set $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) := \mathbf{v}(\mathbf{x})$. Since the transport $\mathbf{x} \mapsto \widehat{\mathbf{x}}$ is linear, $\widehat{\mathbf{v}}$ belongs to $\mathcal{R}^{\ell-1}(\widehat{T})$, and can be considered as a polynomial in $\mathcal{R}^{\ell-1}(\mathbb{R}^3)$. As $\mathbf{curl} : \widehat{\mathbf{x}} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3) \rightarrow \mathcal{R}^{\ell-1}(\mathbb{R}^3)$ is an

isomorphism, it has a continuous inverse for any pair of norms we choose on the domain and co-domain; we endow $\widehat{\mathbf{x}} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3)$ with the $\mathbf{L}^2(B(\rho))$ -norm and $\mathcal{R}^{\ell-1}(\mathbb{R}^3)$ with the $\mathbf{L}^2(B(1))$ -norm. The continuity of the inverse of this **curl** operator gives $\widehat{\mathbf{w}} \in \widehat{\mathbf{x}} \times \mathcal{P}^{\ell-1}(\mathbb{R}^3)$ such that $\mathbf{curl} \widehat{\mathbf{w}} = \widehat{\mathbf{v}}$ on \mathbb{R}^3 and $\|\widehat{\mathbf{w}}\|_{\mathbf{L}^2(B(1))} \lesssim \|\widehat{\mathbf{v}}\|_{\mathbf{L}^2(B(\rho))}$, where the hidden constant depends only on the spaces and their norms, that is, on ℓ and ρ . Since $B(\rho) \subset \widehat{T} \subset B(1)$, this shows that $\mathbf{curl} \widehat{\mathbf{w}} = \widehat{\mathbf{v}}$ on \widehat{T} and $\|\widehat{\mathbf{w}}\|_{\mathbf{L}^2(\widehat{T})} \lesssim \|\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\widehat{T})}$.

For $\mathbf{x} \in T$, define $\mathbf{w}(\mathbf{x}) := h_T \widehat{\mathbf{w}}(\widehat{\mathbf{x}})$. Then, $\mathbf{w} \in (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{\ell-1}(T)$, $\mathbf{curl} \mathbf{w} = \mathbf{v}$ (the scaling by h_T cancels out the factor h_T^{-1} which appears when differentiating $\mathbf{x} \mapsto \widehat{\mathbf{w}}(h_T^{-1}(\mathbf{x} - \mathbf{x}_T))$), and, denoting by \mathbf{J}_T the Jacobian of the transport $\widehat{T} \rightarrow T$, we have

$$\|\mathbf{w}\|_{\mathbf{L}^2(T)} = h_T |\mathbf{J}_T|^{\frac{1}{2}} \|\widehat{\mathbf{w}}\|_{\mathbf{L}^2(\widehat{T})} \lesssim h_T |\mathbf{J}_T|^{\frac{1}{2}} \|\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\widehat{T})} = h_T \|\mathbf{v}\|_{\mathbf{L}^2(T)},$$

which concludes the proof. \square

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