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# Efficient and Accurate Algorithm for the Full Modal Green's Kernel of the Scalar Wave Equation in Helioseismology

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1           **EFFICIENT AND ACCURATE ALGORITHM FOR THE FULL**  
2           **MODAL GREEN’S KERNEL OF THE SCALAR WAVE EQUATION**  
3           **IN HELIOSEISMOLOGY\***

4           HÉLÈNE BARUCQ<sup>†</sup>, FLORIAN FAUCHER<sup>‡</sup>, DAMIEN FOURNIER<sup>§</sup>, LAURENT GIZON<sup>§</sup>,  
5           AND HA PHAM<sup>†</sup>

6           **Abstract.**

7           In this work, we provide an algorithm to compute efficiently and accurately the full outgoing  
8           modal Green’s kernel for the scalar wave equation in local helioseismology under spherical symmetry.  
9           Due to the high computational cost of a full Green’s function, current helioseismic studies only use  
10           its values at a single depth. However, a more realistic modelisation of the helioseismic products  
11           (cross-covariance and power spectrum) requires the full Green’s kernel. In the classical approach,  
12           the Dirac source is discretized and one simulation gives the Green’s function on a line. Here, we  
13           propose a two-step algorithm which, with two simulations, provides the full kernel on the domain.  
14           Moreover, our method is more accurate as the singularity of the solution due to the Dirac source is  
15           described exactly. In addition, it is coupled with the exact Dirichlet-to-Neumann boundary condition,  
16           providing optimal accuracy in approximating the outgoing Green’s kernel, which we demonstrate in  
17           our experiments. In addition, we show that high-frequency approximations of the nonlocal radiation  
18           boundary conditions can represent accurately the helioseismic products.

19           **Key words.** modal Green’s kernel, helioseismology, radiation boundary conditions, helioseismic  
20           observables, Whittaker’s functions, hybridizable discontinuous Galerkin

21           **AMS subject classifications.** 34B27, 33C15, 65N80, 85-04, 85-08, 35J10, 35L05, 35A08

22           **1. Introduction.** In this work, we propose an efficient algorithm to compute  
23           accurately the full outgoing modal Green’s kernel for the scalar wave equation in local  
24           helioseismology under spherical symmetry. The outgoing Green’s kernel, denoted by  
25            $\mathcal{G}(\mathbf{x}, \mathbf{y})$ , is the key ingredient in obtaining simulated observables which are then used in  
26           data analysis in time-distance helioseismology, cf. [section 5](#). At the angular frequency  
27            $\omega$ ,  $\mathcal{G}$  is a solution to, [\[5, 8\]](#),

28           (1.1)           
$$-\nabla \cdot \left( \frac{1}{\rho(\mathbf{x})} \nabla \mathcal{G}(\mathbf{x}, \mathbf{y}) \right) - \frac{\sigma^2(\mathbf{x}, \omega)}{\rho(\mathbf{x}) c^2(\mathbf{x})} \mathcal{G}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}),$$

29           where the background of the Sun is characterized by the density  $\rho$  and the sound speed  
30            $c$ . Here,  $\delta$  is the Dirac function and  $\sigma$  denotes the *complex frequency* that contains a  
31           model of attenuation. In other applications, [\(1.1\)](#) is also called the Bergmann’s equa-  
32           tion ([\[9\]](#)) for inhomogeneous acoustic media, cf. [\[24, Section II.A\]](#). In helioseismology,  
33           it is obtained from the equation for the wave displacement  $\boldsymbol{\xi}$  without background flow  
34           and rotation, cf., e.g., [\[23\]](#),

35           (1.2)           
$$-\frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot \boldsymbol{\xi}) - \sigma^2 \boldsymbol{\xi} + \text{gravity terms} = \mathbf{s}.$$

36           By taking the divergence on both side of [\(1.2\)](#), considering the scalar quantity  $\rho c^2 \nabla \cdot \boldsymbol{\xi}$   
37           as unknown, and if we neglect gravity and the gradient of  $\sigma$ , we obtain the scalar

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operator in (1.1), cf. [17, Section 2]. Due to the omission of the gravity (necessary for the reduction to a scalar problem), equation (1.1) does not take into account buoyancy, and cannot be used to model internal and surface gravity waves. However, it still allows for a good description of acoustic modes and simulates correctly the observed solar data, cf. [17].

Similarly to [8, 7, 1], we work with the *conjugated* equation obtained by a Liouville change of unknowns, since the latter offers a natural setting to define the unique *physical* (also called “outgoing” or “radiation”) kernel  $G$ , solution to, [8],

$$(1.3) \quad \left( -\Delta - \frac{\sigma^2(\mathbf{x}, \omega)}{c^2(\mathbf{x})} + \mathbf{q}(\mathbf{x}) \right) G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad \text{with } \mathbf{q} := \rho^{1/2} \Delta \rho^{-1/2}.$$

The Liouville transform is also discussed in [25, Eq. (2.9)]. In [24, 25], the difference is that the density and the sound speed are considered constant outside of the domain, so this is equivalent to a compact perturbation of the Helmholtz operator. In our case, we have a long-range perturbation and the density is assumed to be exponentially decaying outside of domain, following the solar `Atmo` model, [5]. From  $G$ , the physical Green’s kernel for the original problem (1.1) is defined by

$$(1.4) \quad \mathcal{G}(\mathbf{x}, \mathbf{y}) := \rho^{1/2}(\mathbf{x}) \rho^{1/2}(\mathbf{y}) G(\mathbf{x}, \mathbf{y}).$$

Assuming that the physical parameters only depend on the Sun’s radius, one can decompose  $G$  into the spherical harmonic basis, and compute instead the modal Green’s kernel  $G_\ell(r, s)$  for each spherical mode  $\ell$ , with  $r = |\mathbf{x}|$  the (scaled) radius and  $s$  the source position. It is a fundamental solution of, [8],

$$(1.5) \quad \left( -\frac{d^2}{dr^2} - \frac{\sigma^2(r)}{c^2(r)} + \mathbf{q}(r) + \frac{\ell(\ell+1)}{r^2} \right) G_\ell = \delta(r - s).$$

Current results in local helioseismology assume that the Sun has a “surface” that is well defined, and work with numerical simulations which are obtained from the solution of the wave equation at the surface of the Sun only, i.e.,  $G_\ell(r, s = 1)$  using scaled variables. However, the Sun is a plasma and the observed oscillations represent an average over all depths weighted by the level of transparency (opacity). Therefore, the accurate computations of helioseismic products require the full Green’s function. In addition, it is also required to interpret the multi-height HMI (Helioseismic and Magnetic Imager) data [26] and to image the solar interior using helioseismic holography, [29].

Numerically, the current approach to evaluate the Green’s function is to directly discretize (1.5) using, e.g., a finite element or a finite difference method, cf. [17, 5, 11]. Since each resolution only gives the value for a fixed source, it is expensive to obtain the “complete” Green’s kernel which requires an arbitrarily high number of sources. Additionally, unlike in applications such as seismology or inverse scattering in which one is mostly interested in the far-field or the backscattered data obtained away from the source, in helioseismology, values at the same height of the source are particularly important. The presence of the Dirac however reduces the numerical accuracy of the response in the region around the source. To overcome this problem, one usually refines the mesh around the source, leading to additional computational cost, cf. [11, 16, 5]. With solar background, it is particularly expensive, since this extra-refinement comes in addition to the one needed to capture the profile of the model parameters. In our work, our algorithm provides, with only two simulations, the

81 full Green's function, for each spherical mode  $\ell$  and frequency  $\omega$ , without any extra-  
 82 refinement.

83 The procedure we propose is based on a formula for Green's kernel in ODE the-  
 84 ory and comprises of two steps. One first obtains solutions of two boundary value  
 85 problems: one regular solution in a neighborhood of the origin and an outgoing one in  
 86 an exterior neighborhood. They are then 'assembled' by a Heaviside function to give  
 87 the global modal Green's kernel. The importance here is that while these solutions  
 88 are regular, the singularity of the Green's kernel is captured *exactly* by the Heaviside  
 89 function. In addition, working with regular solutions removes the need to refine the  
 90 mesh around the source position. Namely, with only two problems, one instantly  
 91 obtains the value of the modal Green's kernel on a rectangular region away from the  
 92 origin, and thus of the 3D Green's kernel on a product of two punctured spheres.

93 Other novelties of our work are in the numerical implementation and computation  
 94 of the outgoing Green's kernel, with the following three aspects.

- 95 1. The algorithm is implemented with Hybridizable Discontinuous Galerkin (HDG)  
 96 discretization, [13, 20, 6], which, with unknowns being the numerical trace (bound-  
 97 ary values), reduces the computational effort compared to other scheme.
- 98 2. We compute synthetic helioseismic products (power spectrum) using two atmo-  
 99 spheric models: **S+Atmo** ([12, 5]) and **S+Val-C** ([28]), and compare several forma-  
 100 tion heights. Here, our work is the first to use the exact Dirichlet-to-Neumann  
 101 (DtN) coefficient for the radiation condition.
- 102 3. We investigate the efficiency of approximate radiation boundary conditions (RBC)  
 103 on synthetic helioseismic products using the complete solar background. Until now,  
 104 these comparisons were only carried out in terms of solutions in the atmosphere  
 105 (e.g., [8]). With the exact DtN coefficient as reference, our investigation provides an  
 106 efficient choice of approximate RBC ( $\mathbf{Z}_{\text{S-HF-1a}}$ , see section 5), which is independent  
 107 of the mode  $\ell$ . This is particularly useful for discretization schemes in 3D since,  
 108 being independent of  $\ell$ , it does not contain tangential differential operators.

109 The article is organized as follows. In section 2, we formulate the problem with  
 110 the appropriate boundary conditions, and state, in Proposition 2.4, the main result  
 111 to construct the Green's function. We provide in section 3 the computational steps  
 112 and validate our implementation comparing with analytical solutions to illustrate the  
 113 efficiency of our approach. The helioseismic products are defined in section 4 and  
 114 computed in section 5, illustrating the necessity of using the full Green's kernel.

115 **2. Formulation of the solar Green's functions.** In this section, we provide  
 116 the dimensionless modal equation with the appropriate boundary conditions, and  
 117 give the two-step strategy for the efficient computation of the Green's kernel via  
 118 Proposition 2.4.

119 **2.1. Problem with dimensionless coefficients.** We consider the 3D coordi-  
 120 nate system with the center of the Sun placed at the origin, and denote by  $\check{\mathbf{x}}$  the  
 121 position in this system. Our first task is to derive the adimensional version of the  
 122 problem: with  $R_{\odot}$  denoting the radius of the Sun, ( $R_{\odot} \simeq 695.5 \times 10^6$  m), we introduce  
 123 the scaled coordinates,

$$124 \quad (2.1) \quad \mathbf{x} = \frac{\check{\mathbf{x}}}{R_{\odot}}, \quad \text{such that} \quad \nabla_{\mathbf{x}} = R_{\odot} \nabla_{\check{\mathbf{x}}}.$$

125 We introduce the scaled radius  $r = R/R_{\odot}$ , which is dimensionless, where  $R = |\check{\mathbf{x}}|$ .

126 The associated density function  $\rho$ , the dimensionless inverse scale height  $\alpha$  and

127 the scaled velocity  $\mathbf{c}$  are defined by

$$128 \quad (2.2) \quad r \mapsto \rho(r) = \rho(R_\odot r), \quad \alpha(r) := -\frac{\partial_r \rho(r)}{\rho(r)}, \quad \mathbf{c} := \frac{c}{R_\odot}.$$

129 Since  $\partial_r = R_\odot \partial_R$ , we have that  $\alpha = \alpha/R_\odot$ . Here,  $\mathbf{c}$  is expressed in  $\text{s}^{-1}$ , so that the  
130 quantity  $\sigma/\mathbf{c}$  is dimensionless.

131 The main equation for the *original* problem is given by (1.1) and we have, for a  
132 generic right-hand side  $\check{\mathbf{f}}$ ,

$$133 \quad (2.3) \quad -\nabla \cdot \left( \frac{1}{\rho} \nabla \check{\mathbf{u}} \right) - \frac{\sigma^2}{\rho \mathbf{c}^2} \check{\mathbf{u}} = \check{\mathbf{f}}, \quad \text{with } \sigma^2(\mathbf{x}, \omega) = \omega^2 + 2i\omega\gamma(\mathbf{x}, \omega).$$

134 We recall that  $\omega > 0$  is the angular frequency, and  $\gamma$  the attenuation. We first write  
135 the field  $\check{\mathbf{u}}$  and the right-hand side  $\check{\mathbf{f}}$  in terms of the scaled radius:

$$136 \quad (2.4) \quad \mathbf{u}(\mathbf{x}) := \check{\mathbf{u}}(R_\odot \mathbf{x}), \quad \mathbf{f}(\mathbf{x}) := \check{\mathbf{f}}(R_\odot \mathbf{x}).$$

137 They satisfy the equation, using (2.1) and multiplying by  $R_\odot^2$ ,

$$138 \quad (2.5) \quad -\nabla \cdot \left( \frac{1}{\rho} \nabla \mathbf{u} \right) - \frac{\sigma^2}{\rho \mathbf{c}^2} \mathbf{u} = R_\odot^2 \mathbf{f}.$$

139 The *conjugated* problem introduces the change of unknown, [8, 1],

$$140 \quad (2.6) \quad u(\mathbf{x}) = \rho(\mathbf{x})^{-1/2} \mathbf{u}(\mathbf{x}).$$

141 The dimensionless potential  $\mathbf{q}$  for the conjugated problem is

$$142 \quad (2.7) \quad \mathbf{q}(\mathbf{x}) := \rho(\mathbf{x})^{1/2} \Delta_{\mathbf{x}} \rho(\mathbf{x})^{-1/2}.$$

143 The conjugated equation with dimensionless coefficients writes as

$$144 \quad (2.8) \quad -\Delta_{\mathbf{x}} u - \frac{\sigma^2}{\mathbf{c}^2} u + \mathbf{q} u = g, \quad \text{with } g = \rho^{1/2} R_\odot^2 \mathbf{f}.$$

145 **2.2. Solar modal Green's function.** Similarly to [8], we work with the con-  
146 jugated problem, given by (2.8) in three dimensions after adimensionalization. Then,  
147 using spherical symmetry, we decompose the solution into one-dimensional *modal* ones  
148 defined on  $[0, r_{\max}]$ , with  $r_{\max} > 1$ . Note that in the adimensionalized version, the  
149 position  $r = 1$  corresponds with the Sun's "surface" (at  $R_\odot$ ). In terms of bound-  
150 ary conditions, a Neumann-like one holds at 0, cf. subsection 2.3, while a radiation  
151 boundary condition is imposed at  $r_{\max}$ , [5, 8].

152 We have denoted by  $\mathcal{G}$  the Green's function of the original problem (cf. (1.1)),  
153 and that of the conjugated problem by  $G$ , which satisfies,

$$154 \quad (2.9) \quad \left( -\Delta_{\mathbf{x}} - \frac{\sigma^2(r)}{\mathbf{c}^2(r)} + \mathbf{q}(r) \right) G(\mathbf{x}, \mathbf{s}) = \delta(\mathbf{x} - \mathbf{s}),$$

155 with

$$156 \quad (2.10) \quad \mathbf{q}(r) = \frac{\alpha^2(r)}{4} + \frac{\alpha'(r)}{2} + \frac{\alpha(r)}{r}.$$

157 *Remark 2.1.* In helioseismology, one usually writes the zero-th order term as

$$158 \quad (2.11) \quad \frac{\sigma^2}{\mathfrak{c}^2} - \mathfrak{q} = \frac{\sigma^2 - \omega_c^2}{\mathfrak{c}^2},$$

159 where  $\omega_c$  is called the local *cut-off frequency*:

$$160 \quad (2.12) \quad \omega_c^2(r) := \mathfrak{c}^2(r)\mathfrak{q}(r) = \mathfrak{c}^2(r) \left( \frac{\alpha^2(r)}{4} + \frac{\alpha'(r)}{2} + \frac{\alpha(r)}{r} \right).$$

161 This is the same expression as in [16, Eq. (11) and Figure 3], where it is shown  
 162 that, despite the simplification of the wave equation (ignoring buoyancy), the cut-off  
 163 frequency is consistent with solar applications, in particular its value at the beginning  
 164 of the atmosphere is around 5.2 mHz. The sign of  $\omega^2 - \omega_c^2$  determines the local  
 165 behaviour of the solution and waves with frequencies  $\omega > \omega_c$  propagate into the  
 166 atmosphere, cf. [16].  $\triangle$

167 As mentioned in the introduction, the Green's functions are related by

$$168 \quad (2.13) \quad \mathcal{G}(\mathbf{x}, \mathbf{y}) = \rho^{1/2}(\mathbf{x}) \rho^{1/2}(\mathbf{y}) G(\mathbf{x}, \mathbf{y}).$$

169 For more details on the transformation, we refer to [8, 1] and [7, Appendix A].

170 **ASSUMPTION 2.2.** *The sound speed  $\mathfrak{c}$  and attenuation coefficient  $\gamma$  are bounded*  
 171 *functions, which are constant outside of a compact set with  $\mathfrak{c}_\infty$  and  $\gamma_\infty$  denoting their*  
 172 *respective values on the exterior domain. In another word, we have,*

$$173 \quad (2.14) \quad \text{Supp}(\mathfrak{c} - \mathfrak{c}_\infty) \text{ is compact} \quad \text{and} \quad \text{Supp}(\gamma - \gamma_\infty) \text{ is compact.}$$

174 **ASSUMPTION 2.3.** *We assume that the background density  $\rho$  is such that  $\alpha =$*   
 175  *$-\rho'/\rho$  satisfies, for constants  $\alpha_\infty > 0$  and  $\epsilon > 0$ ,*

$$176 \quad (2.15a) \quad \alpha(r) \in \mathcal{C}^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \quad \lim_{r \rightarrow \infty} \alpha = \alpha_\infty,$$

$$177 \quad (2.15b) \quad \text{and} \quad \alpha'(r) = \mathcal{O}(1 + r)^{-(1+\epsilon)}, \quad r \rightarrow \infty.$$

179 Let us note that these assumptions are not particularly restrictive and, in par-  
 180 ticular, the solar velocity and density models **S+Atmo** ([12]) or **S+Val-C** ([28]) satisfy  
 181 them. Under **Assumption 2.3**, the potential  $\mathfrak{q}$  has a finite limiting value denoted by  
 182  $\mathfrak{q}_\infty$ , such that

$$183 \quad (2.16) \quad \lim_{r \rightarrow \infty} \mathfrak{q} = \mathfrak{q}_\infty = \frac{\alpha_\infty^2}{4},$$

184 and, as  $r$  tends to 0,

$$185 \quad (2.17) \quad \lim_{r \rightarrow 0} r^2 \left( -\frac{\sigma^2(r)}{\mathfrak{c}^2(r)} + \mathfrak{q}(r) \right) = 0.$$

186 Under **Assumptions 2.2** and **2.3**, we can apply the result of [2, Theorem 6.2]  
 187 to obtain the physical Green's kernel. To state the proposition, we introduce the  
 188 potential  $Q_\ell$  such that,

$$189 \quad (2.18) \quad \frac{Q_\ell(r)}{r^2} = -\frac{\sigma^2(r)}{\mathfrak{c}^2(r)} + \mathfrak{q}(r) + \frac{\ell(\ell+1)}{r^2} = -\frac{\sigma^2(r)}{\mathfrak{c}^2(r)} + \frac{\alpha^2(r)}{4} + \frac{\alpha'(r)}{2} + \frac{\alpha(r)}{r} + \frac{\ell(\ell+1)}{r^2},$$

190 and the operator  $L_\ell$ ,

$$191 \quad (2.19) \quad L_\ell := -\frac{d^2}{dr^2} + \frac{Q_\ell(r)}{r^2} = -\frac{d^2}{dr^2} - \frac{\sigma^2(r)}{c^2(r)} + q(r) + \frac{\ell(\ell+1)}{r^2}.$$

192 Furthermore, we denote by  $Y_\ell^m$  the  $m$ -th spherical harmonic of order  $\ell$ , by  $P_\ell$  the  
193 Legendre polynomial of degree  $\ell$ ,  $H$  is the Heaviside function and the Wronskian is  
194 denoted  $\mathcal{W}(s) := \mathcal{W}\{\psi(s), \tilde{\psi}(s)\}$ . Following [8, 7], with  $\sqrt{\cdot}$  using the argument branch  
195  $[0, 2\pi)$ , we also introduce the conjugate wavenumber  $k$  defined by

$$196 \quad (2.20) \quad k = \sqrt{\frac{\sigma^2}{c_\infty^2} - \frac{\alpha_\infty^2}{4}}.$$

197 **PROPOSITION 2.4** (Green's kernel expansion). *Under Assumptions 2.2 and 2.3,*  
198 *the outgoing Green's function can be written as an expansion in spherical harmonic*  
199 *basis with, using  $\bar{\cdot}$  for the complex conjugation,*

$$200 \quad (2.21) \quad \begin{aligned} G(\mathbf{x}, \mathbf{y}) &= \frac{1}{|\mathbf{x}||\mathbf{y}|} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} G_\ell^m(|\mathbf{x}|, |\mathbf{y}|) Y_\ell^m(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{x}})}, \quad \mathbf{x}, \mathbf{y} \text{ not on the } z\text{-axis} \\ &= \frac{1}{4\pi |\mathbf{x}||\mathbf{y}|} \sum_{\ell=0}^{\infty} (2\ell+1) G_\ell(|\mathbf{x}|, |\mathbf{y}|) P_\ell(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}). \end{aligned}$$

201 Here,  $G_\ell^m(r, s) = G_\ell$  is independent of  $m$  and is the unique distributional solution to

$$202 \quad (2.22) \quad L_\ell G_\ell = \delta(r-s),$$

203 satisfying the boundary condition,

$$204 \quad (2.23) \quad \lim_{r \rightarrow 0} r^{-(\ell+1)} G_\ell(r) = 1,$$

205 at  $r=0$ , and the asymptotic relation at infinity,

$$206 \quad (2.24) \quad G_\ell = e^{i\varphi(r,k)} (1 + o(1)) \quad \text{as } r \rightarrow \infty.$$

207 In (2.24), the phase function is defined for some  $r_0 > 0$  as,

$$208 \quad (2.25) \quad \varphi(r) := \int_{r_0}^r \sqrt{k^2 - \frac{\alpha}{s}} ds = kr - \frac{\alpha}{2k} \log r + k^{-2} o(1).$$

209 Furthermore, if  $\tilde{\psi}_\ell$  and  $\psi_\ell$  are two homogeneous solutions to  $L_\ell w = 0$  on  $(0, s)$  and  
210  $(s, \infty)$  respectively, with  $\tilde{\psi}_\ell$  satisfying the boundary condition (2.23) at  $r=0$ , and  $\psi_\ell$   
211 the condition (2.24) as  $r \rightarrow \infty$ , we have

$$212 \quad (2.26) \quad G_\ell(r, s) = \frac{-H(s-r) \psi(r) \tilde{\psi}(s) - H(r-s) \tilde{\psi}(r) \psi(s)}{\mathcal{W}(s)}.$$

213 **Remark 2.5.** The kernel for the original equation is

$$214 \quad (2.27) \quad \mathcal{G}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi |\mathbf{x}||\mathbf{y}|} \sum_{\ell=0}^{\infty} (2\ell+1) \mathcal{G}_\ell(|\mathbf{x}|, |\mathbf{y}|) P_\ell(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}),$$

$$215 \quad (2.28) \quad \text{where } \mathcal{G}_\ell(|\mathbf{x}|, |\mathbf{y}|) = \rho^{1/2}(|\mathbf{x}|) \rho^{1/2}(|\mathbf{y}|) G_\ell(|\mathbf{x}|, |\mathbf{y}|).$$

217 **2.3. Boundary conditions.**

218 **2.3.1. Boundary condition at zero.** Using the Frobenius theory, e.g. [14,  
219 Theorem 4 p.165], the indicial equation at  $r = 0$  for (2.19) is,

$$220 \quad (2.29) \quad -\lambda^2 + \lambda + \lim_{r \rightarrow 0} r^2 \left( -\frac{\sigma^2(r)}{\mathfrak{c}^2(r)} + \mathfrak{q}(r) \right) + \ell(\ell + 1) = 0.$$

221 Under (2.17), this simplifies to

$$222 \quad (2.30) \quad \lambda^2 - \lambda - \ell(\ell + 1) = 0 \quad \Rightarrow \quad \lambda = -\ell \quad \text{or} \quad \lambda = \ell + 1.$$

223 The regular solution at  $r = 0$  is given by the exponent  $\lambda = \ell + 1$ , which explains  
224 the boundary condition (2.23). Under (2.17), for operator (2.19) (or generally other  
225 regular singular ODE with indicial roots of opposite signs), (2.23) can be replaced by  
226 the boundary condition

$$227 \quad (2.31) \quad \lim_{r \rightarrow 0} r \frac{d}{dr} \left( \frac{G_\ell(r)}{r} \right) = 0,$$

228 which also selects the regular solution at  $r = 0$ . This can be seen as follows.

229 A generic solution in the neighborhood of zero is a linear combination of a function  
230 that decays in  $r^{\ell+1}$  and one that blows up in  $r^{-\ell}$ ; in particular, cf. [14, Theorem 4  
231 p.165],

$$232 \quad (2.32) \quad w(r) = \mathfrak{a} (r^{-\ell} h(r) + \mathfrak{c}_\ell (\log r) r^{\ell+1} g(r)) + \mathfrak{b} r^{\ell+1} g(r) \quad , \quad r > 0,$$

233 for some linear combination of constants  $\mathfrak{a}, \mathfrak{b}$ . Here,  $\mathfrak{c}_\ell$  is a fixed constant (depending  
234 on  $\ell$ ) and functions  $g(r)$  and  $h(r)$  are  $C^1$  up to  $r = 0$  with

$$235 \quad (2.33) \quad g(0) \neq 0, \quad h(0) \neq 0.$$

236 The presence of the log term is due to the integral difference of the two indicial  
237 exponents, i.e.  $(\ell + 1) - (-\ell) \in \mathbb{N}$ . We have

$$238 \quad (2.34) \quad r \left( \frac{w}{r} \right)' = \mathfrak{a} ((-\ell - 1)r^{-\ell-1} h(r) + \mathfrak{c} r^\ell g(r) + \ell \mathfrak{c} (\log r) r^\ell g(r)) + \mathfrak{b} \ell r^\ell g(r) \\ + \mathfrak{a} (r^{-\ell} h'(r) + \mathfrak{c} (\log r) r^{\ell+1} g'(r)) + \mathfrak{b} r^{\ell+1} g'(r).$$

239

$$240 \quad (2.35) \quad \Rightarrow \quad \lim_{r \rightarrow 0} r \left( \frac{w(r)}{r} \right)' = \lim_{r \rightarrow 0} \mathfrak{a} ((-\ell - 1)r^{-\ell-1} h(r) + r^{-\ell} h'(r)), \quad \ell > 0; \\ \lim_{r \rightarrow 0} r \left( \frac{w(r)}{r} \right)' = \mathfrak{a} \left( \mathfrak{c} g(0) + h'(0) - \lim_{r \rightarrow 0} \frac{h(r)}{r} \right), \quad \ell = 0.$$

241 It follows immediately that, for  $w$  of the form (2.32) and for all  $\ell \in \mathbb{N}$ ,

$$242 \quad (2.36) \quad \lim_{r \rightarrow 0} r \left( \frac{w(r)}{r} \right)' < \infty \quad \Rightarrow \quad \mathfrak{a} = 0; \\ \text{and} \quad \mathfrak{a} = 0 \quad \Leftrightarrow \quad \lim_{r \rightarrow 0} r \left( \frac{w(r)}{r} \right)' = 0.$$

243 *Remark 2.6.* One can also consider a condition such that  $\lim_{r \rightarrow 0} r^2(w/r)' = 0$ ,  
 244 which also selects the regular family, i.e. forces  $\mathbf{a} = 0$ . The reasoning for  $\ell > 0$  is the  
 245 same as above, due to the extra factor in  $r$ . For  $\ell = 0$ , we have

$$246 \quad (2.37) \quad r^2 \left( \frac{w}{r} \right)' = \mathbf{a} (-h(r) + \mathbf{c} r g(r)) + \mathbf{a} (r h'(r) + \mathbf{c} (\log r) r^2 g'(r)) + \mathbf{b} r^2 g'(r);$$

$$\implies \lim_{r \rightarrow 0} r^2 \left( \frac{w}{r} \right)' = -\mathbf{a} h(0).$$

247 In using  $h(0) \neq 0$ , cf. (2.33), we also obtain, for  $\ell = 0$ ,

$$248 \quad \lim_{r \rightarrow 0} r^2 \left( \frac{w}{r} \right)' = 0 \implies \mathbf{a} = 0.$$

249 In addition to being  $\ell$ -independent, the condition (2.31) is more natural and is  
 250 simpler to implement in a discretization scheme, in the sense that it will lead to a  
 251 Dirichlet-type boundary condition.

252 **2.3.2. Outgoing boundary condition.** For the numerical discretization on a  
 253 finite domain, we need to replace the outgoing condition (2.24) by one on an ar-  
 254 tificial boundary. In particular in the case of the solar atmosphere with  $\rho$  and  $\mathbf{c}$   
 255 extended according to the `Atmo` model on  $(r_{\max}, \infty)$  (i.e.,  $\mathbf{c}$  becomes constant and  
 256  $\rho$  exponentially decaying in the atmosphere), we have an explicit expression for the  
 257 Dirichlet-to-Neumann (DtN) coefficient, cf. [8, Section 4.1],

$$258 \quad (2.38) \quad \mathbf{Z}_{\text{DtN}}^\ell(r) := -2ik \frac{W'_{-\frac{i\alpha}{2k}, \ell+1/2}(-2ikr)}{W_{-\frac{i\alpha}{2k}, \ell+\frac{1}{2}}(-2ikr)},$$

259 where  $W$  is the Whittaker's special function, cf. [8, 7], and  $k$  is defined by (2.20) (Whit-  
 260 taker's functions are also discussed in [24] when considering exponential variation in  
 261 density). The outgoing condition (2.24) is then replaced by

$$262 \quad (2.39) \quad \partial_n G_\ell = \mathbf{Z}_{\text{DtN}}^\ell G_\ell.$$

263 *Remark 2.7.* Under more general assumptions of extensions, one does not have an  
 264 explicit description of the DtN. However, if the extension still maintains the structure  
 265 of the potential, as described in Assumptions 2.2 and 2.3, and since we work mode by  
 266 mode, we can use the *nonlocal* radiation boundary condition and its high-frequency  
 267 approximations, cf. subsection 5.3.  $\triangle$

268 **3. Numerical calculation of the Green's function.** For the computation  
 269 of the modal Green's function that solves (2.22), we consider the generic problem  
 270 associated with a right-hand side  $g$ . By using the unknown  $u = G/r$  and omitting the  
 271 index  $\ell$  for clarity, we thus consider, on the interval  $[0, r_{\max}]$ , the numerical calculation  
 272 of the solution to

$$273 \quad (3.1) \quad \begin{cases} -(r^2 u'(r))' + Q(r) u(r) = r g(r), & r \in (0, r_{\max}); \\ \lim_{r \rightarrow 0} (r u'(r)) = 0; & u'(r_{\max}) = \left( -\frac{1}{r_{\max}} + \mathbf{Z}_\bullet(r_{\max}) \right) u(r_{\max}). \end{cases}$$

274 We recall, from (2.18),

$$275 \quad (3.2) \quad \frac{Q(r)}{r^2} = -\frac{\sigma^2(r)}{\mathbf{c}^2(r)} + \frac{\alpha^2(r)}{4} + \frac{\alpha'(r)}{2} + \frac{\alpha(r)}{r} + \frac{\ell(\ell+1)}{r^2}.$$

276 If the coefficient  $\mathbf{Z}_{\text{DtN}}^\ell$  (2.38) is used in the impedance condition at  $r_{\max}$ , then we have  
 277 the exact outgoing solution (modulo the numerical discretization error). We will also  
 278 investigate the effectiveness of approximate RBC coefficients, listed in (5.3) and (5.4).

279 **3.1. First-order formulation for HDG discretization.** In Proposition 2.4,  
 280 the Green's function is constructed using the formula (2.26), which relies on the Wron-  
 281 skian, hence on the derivative of the solution to (3.1). This motivates the numerical  
 282 implementation of the first-order formulation for the problem (3.1), where we intro-  
 283 duce the new variables  $w$  and  $v$  such that,

$$284 \quad (3.3) \quad w := r u \quad , \quad v := r u' \quad , \quad \text{first-order variables.}$$

285 With this choice of variables, the problem (3.1) is written as

$$286 \quad (3.4) \quad \begin{cases} -r(rv(r))' + Q(r)w(r) = r^2g(r), & r \in (0, r_{\max}); \\ rw'(r) - w(r) = rv(r), & r \in (0, r_{\max}); \\ \lim_{r \rightarrow 0} v(r) = 0, & v(r_{\max}) = \left(-\frac{1}{r_{\max}} + \mathbf{Z}_\bullet(r_{\max})\right)w(r_{\max}). \end{cases}$$

287 Here, the radiation condition in terms of  $(w, v)$  is obtained by multiplying both sides  
 288 of  $u' = \left(-\frac{1}{r_{\max}} + \mathbf{Z}_\bullet\right)u$  by  $r_{\max}$ , and replacing  $u$  and  $u'$  by  $w$  and  $v$ . Note that  
 289 by solving the first-order problem, both  $w$  and  $v$  (i.e.,  $u$  and  $u'$ ) are obtained with  
 290 the same accuracy while if we were to implement the second-order problem (3.1), the  
 291 derivative of  $u$  (needed for the Wronskian in (2.26)) would be retrieved with one order  
 292 less accuracy than  $u$ .

293 For the implementation, we use the Hybridizable Discontinuous Galerkin (HDG,  
 294 [4, 13, 20]) discretization method which, we believe, is the most appropriate. In other  
 295 discretization methods such as finite elements or finite differences, the discretization  
 296 leads to a linear system whose size is the number of degrees of freedom for all unknowns  
 297 (here,  $w$  and  $v$ ). On the other hand, with the HDG method, the global linear system  
 298 is only composed of the degrees of freedom of the numerical trace for one variable  
 299 ( $w$ ), that is, those that are on the faces of the cells that constitute the discretized  
 300 domain. Then, the volume solutions (for  $w$  and  $v$ ) are constructed locally via small  
 301 (hence numerically cheap) linear systems.

302 The HDG method is specifically designed for first-order problems as it allows  
 303 to maintain a small linear system compared to other discretizations, as observed by,  
 304 e.g., [13, 22, 10, 15]. The implementation follows two levels with first the global  
 305 system for the numerical trace and then local systems for the volume solution. For  
 306 our problem (3.4), these are detailed in our extended report, [6, Section 6].

307 **3.2. Computation of the Green's function: Approach 1.** The first, and  
 308 most natural, approach for the computation of the Green's function  $G_\ell$  is to solve  
 309 numerically (2.22), that is, with a Dirac distribution as a right-hand side. Using the  
 310 HDG first-order system, it amounts to Algorithm 3.1.

---

**Algorithm 3.1** Computational steps for the evaluation of the Green's function using  
 Approach 1 that discretizes directly the problem with a Dirac right-hand side.

---

**for** each source position  $s$  **do**

    Generate a mesh which is refined around the source position.

    Find  $(v, w)$  that solves (3.4) using  $g(r) := \delta(r - s)$ .

    Save the Green's function at the position  $s$ :  $G_\ell(r, s) := w(r)$ .

**end for**

---

311 Here, for each resolution of the problem (3.4), i.e., for a fixed value of  $s_0 \in$   
 312  $(0, r_{\max})$ , one obtains the value of the Green's kernel  $G_\ell(r, s_0)$  only on the vertical  
 313 segment  $[0, r_{\max}] \times s_0$ , and its symmetric reflection across the diagonal  $r = s_0$ , see  
 314 Figure 1(a). This means that to precisely evaluate the Green's function (i.e., using  
 315 several sources), one needs to solve many problems. One can take advantage of the  
 316 multiple right-hand sides features of direct solver such as MUMPS ([3]) to calculate at  
 317 the same time several solutions and obtain the value of  $G_\ell$  on several vertical line  
 318 segments (and hence their symmetric reflexion). This, however, requires the use of  
 319 the same domain discretization for all of the right-hand sides, which is not appropriate  
 320 with the discretization of a Dirac function, as we illustrate in subsection 3.4.

321 **3.3. Computation of the Green's function: Approach 2.** In the second  
 322 approach, we make use of the formula (2.26) for the evaluation of the Green's function.  
 323 Consider  $0 < r_a < r_b \leq r_{\max}$ , we construct the two solutions,  $\psi$  on  $[0, r_b]$ , and  $\tilde{\psi}$  on  
 324  $[r_a, r_{\max}]$ , with each solving a boundary value problem, and we assemble the Green's  
 325 function using (2.26). We detail the computational steps in Algorithm 3.2.

---

**Algorithm 3.2** Computational steps for the evaluation of the Green's function using  
 Approach 2 that uses Proposition 2.4. The complete Green's function is obtained  
 from the solution of two boundary value problems.

---

**Step 1a.** Find  $(v, w)$  that solves

$$(3.5) \quad \begin{cases} -r(rv)' + Qw = 0, & \text{on } (0, r_b); \\ rw' - w = rv, & \text{on } (0, r_b); \\ v(0) = 0, & w(r_b) = 1. \end{cases}$$

Set  $\psi := w$  and  $\psi' := v + w/r$ .

**Step 1b.** Find  $(v, w)$  that solves

$$(3.6) \quad \begin{cases} -r(rv)' + Qw = 0, & \text{on } (r_a, r_{\max}); \\ rw' - w = rv, & \text{on } (r_a, r_{\max}); \\ w(r_a) = 1, & v(r_{\max}) = \left(-\frac{1}{r_{\max}} + \mathbf{Z}\bullet\right)w(r_{\max}). \end{cases}$$

Set  $\tilde{\psi} := w$  and  $\tilde{\psi}' := v + w/r$ .

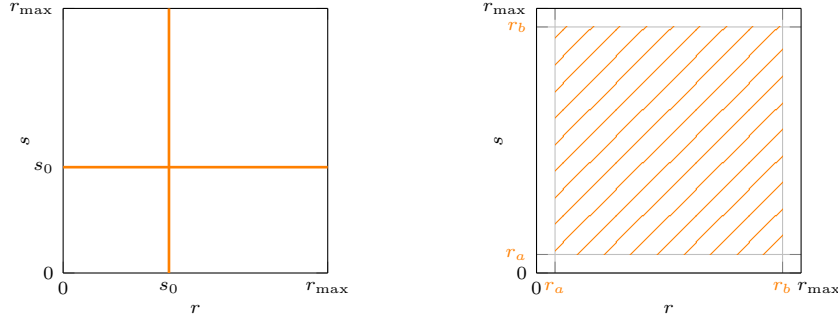
**Step 2.** Using  $\mathcal{W}(s) = \psi(s)\tilde{\psi}'(s) - \psi'(s)\tilde{\psi}(s)$ , assemble the Green's function,

$$(3.7) \quad G_\ell(r, s) = \frac{-\mathbf{H}(s-r)\psi(r)\tilde{\psi}(s) - \mathbf{H}(r-s)\tilde{\psi}(r)\psi(s)}{\mathcal{W}(s)}.$$


---

326 Using this approach, from the solution of two boundary value problems, the  
 327 Green's function  $G_\ell(r, s)$  is obtained on the domain  $[r_a, r_b] \times [r_a, r_b]$ , as we illustrate  
 328 in Figure 1. Approach 2 offers the following advantages compared to Approach 1.

- 329 – The Green's function is obtained for *all* pairs  $(r, s)$  with  $r, s \in [r_a, r_b]$  from the  
 330 solutions of two problems, while with Approach 1, one problem only gives the  
 331 Green's function at a fixed  $s$ .
- 332 – The functions  $\psi$  and  $\tilde{\psi}$  are regular on their corresponding domain of computation.  
 333 They correspond to boundary value problems and they do not have a singular



(a) Approach 1: the solution of one problem for a Dirac in  $s_0$  only gives  $G_\ell(r, s = s_0)$  and  $G_\ell(r = s_0, s)$ .

(b) Approach 2: from the solutions of two boundary value problems,  $G_\ell(r, s)$  is obtained for any position between  $r_a$  and  $r_b$ .

FIG. 1. Domain of definition (orange) of the modal Green's kernel for Approach 1 and 2, respectively given by Algorithms 3.1 and 3.2.

334 source. Because of this, one does not need to refine the mesh around the singularities  
 335 of the source as in Approach 1, cf. subsection 3.4 and Remark 5.1.

336 *Remark 3.1.* The outgoing solution  $\tilde{\psi}$  is a continuous solution on  $[\epsilon, \infty)$  for ar-  
 337 bitrarily small  $\epsilon > 0$ , however it contains a singularity term that behaves like a  
 338 constant  $\times r^{-\ell}$  as  $r \rightarrow 0$  for  $\ell > 0$ . Therefore, the Dirichlet condition (3.6) cannot be  
 339 extended to  $r_a = 0$  for  $\ell > 0$ . This means  $r_a$  is positive but can be arbitrarily small,  
 340 cf. (3.13) for its numerical value in the experiment.  $\triangle$

341 **3.4. Numerical validation with the analytical solutions.** To validate our  
 342 numerical code based upon the HDG discretization (see [6, Section 6]), we compare  
 343 the numerical and the analytical solutions, obtained by considering the first-order  
 344 problem (3.4) with,

$$345 \quad (3.8) \quad \begin{aligned} & \text{constant } \mathbf{c} \text{ and } \alpha \text{ for } Q \text{ in (3.2) (i.e., } \alpha' = 0), \\ & \text{a right-hand side } g = \delta(r - s), \text{ and the exact DtoN condition } \mathbf{Z}_{\text{DtN}}^\ell. \end{aligned}$$

346 **3.4.1. Analytical solution.** Under (3.8), the exact solution  $w_{\text{ref}}$  to (3.4) is  
 347 given by (the details of which are given in our extended report [6, Section 7])

$$348 \quad (3.9) \quad w_{\text{ref}}(r, s) = \frac{-\text{H}(s - r) \psi(r) \tilde{\psi}(s) - \text{H}(r - s) \tilde{\psi}(r) \psi(s)}{\mathcal{W}\{\psi(s), \tilde{\psi}(s)\}},$$

349 where  $\psi$  and  $\tilde{\psi}$  are given in terms of the Whittaker's function  $W$  and of the regular  
 350 Whittaker's (or Buchholtz) function  $M$ ,

$$351 \quad (3.10) \quad \psi = M_{\frac{i\alpha}{2k}, \ell+1/2}(2ikr), \quad \tilde{\psi} = W_{-\frac{i\alpha}{2k}, \ell+1/2}(-2ikr).$$

352 We note that  $\psi$  and  $\tilde{\psi}$  can be chosen as *constant multiples* of the solutions of the  
 353 boundary value problems. The Wronskian of  $\psi$  and  $\tilde{\psi}$  defined in (3.10) is given  
 354 explicitly by ([6, Section 7]),

$$355 \quad (3.11) \quad \mathcal{W}\{\psi(s), \tilde{\psi}(s)\} = 2i k \frac{\Gamma(2\ell + 2)}{\Gamma(1 + \ell + \frac{i\alpha}{2k})} (-1)^{\ell+1},$$

356 where  $\Gamma$  is the (complex-valued) Gamma function.

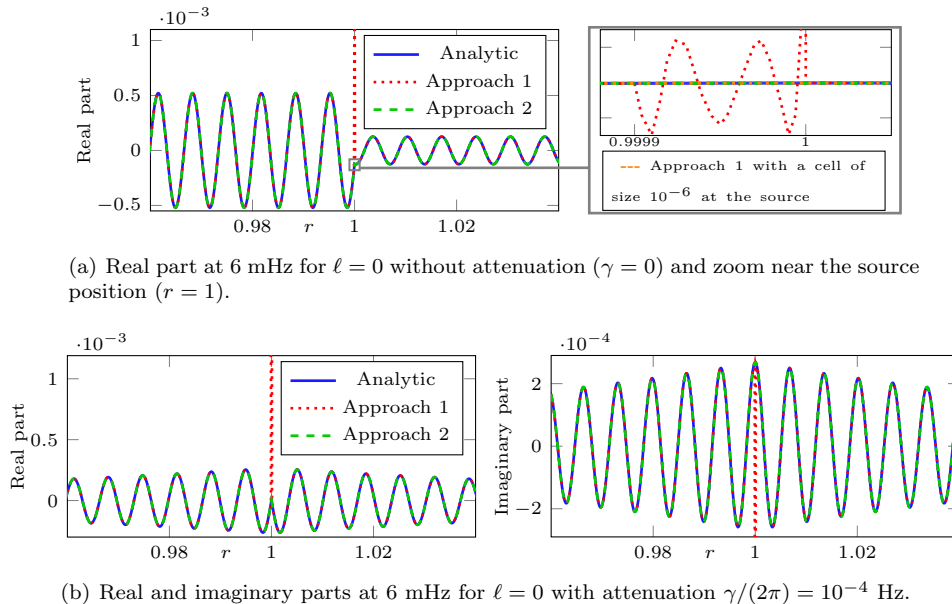
357 **3.4.2. Numerical experiments.** We choose constant parameters that are rep-  
 358 resentative of the solar atmosphere. They are extracted from the `Atmo` model and are  
 359 scaled such that,

$$360 \quad (3.12) \quad \mathfrak{c} = 6.86 \times 10^9 / R_\odot = 9.87 \times 10^{-6} \text{ s}^{-1}; \quad \alpha = 6663.62.$$

361 The (scaled) radius varies from 0 to 1.05 and we consider a fixed source at  $s = 1$ .  
 362 The comparison between the two approaches and the analytical solution is pictured  
 363 in [Figure 2](#), for a frequency of 6 mHz, with and without attenuation. Due to the fast  
 364 oscillations of the solutions, we zoom in an interval around the source location. In  
 365 Approach 2, we take the parameters

$$366 \quad (3.13) \quad r_a = 10^{-25}, \quad r_b = r_{\max} = 1.05,$$

367 which give the interval of computation for  $\psi$  and  $\tilde{\psi}$ , see (3.5) and (3.6).



(a) Real part at 6 mHz for  $\ell = 0$  without attenuation ( $\gamma = 0$ ) and zoom near the source position ( $r = 1$ ).

(b) Real and imaginary parts at 6 mHz for  $\ell = 0$  with attenuation  $\gamma/(2\pi) = 10^{-4}$  Hz.

FIG. 2. Comparison of the solutions  $w$  to the first-order conjugated problem (3.4) under (3.8) and (3.12) at 6 mHz for mode  $\ell = 0$  without and with attenuation for a source located at  $s = 1$ . The discretization uses a homogeneous mesh with elements of size  $10^{-4}$  and polynomials of order 6.

368 We observe that, away from the source location, the two approaches cannot be  
 369 distinguished visually from the analytical solution. However, in the Approach 1,  
 370 which discretizes the Dirac source function on the right-hand side, the singularity  
 371 remains, leading to drastic inaccuracy, not only at the exact source location, but in  
 372 the whole cell where it is contained, as highlighted in [Figure 2\(a\)](#), and here we need to  
 373 divide the size of the cell by 100 to obtain an accurate solution. While the singularity  
 374 is only in the real part when there is no attenuation ( $\gamma = 0$ ), the inaccuracy also  
 375 appears in the imaginary part in the general case  $\gamma \neq 0$ . This observation highlights  
 376 a major difficulty of the naive discretization approach: the discretization of the source  
 377 needs to be carefully addressed. This is even more important as, in helioseismology,  
 378 one is mostly interested by the value of the solution at the source location (namely,  
 379  $G_\ell(1,1)$ ). In order to overcome the issue, one needs to refine around the source

380 position, as advocated in [11, 16, 5]. This can however be cumbersome when several  
 381 sources must be taken into account, as the mesh should be updated for every source,  
 382 or be over-refined everywhere. On the other hand, Approach 2 provides the accurate  
 383 solution at the source position, without any need for refinement. Note also that, while  
 384  $r_a$  must be different of 0 in Approach 2, here, an arbitrarily small value ( $10^{-25}$ ) works  
 385 well.

386 In Table 1, we give the relative errors between the approaches for different fre-  
 387 quencies and modes

$$388 \quad (3.14) \quad \mathcal{E}_w = \frac{\|w_{\text{ref}} - w_{\bullet}\|_{L^2}}{\|w_{\text{ref}}\|_{L^2}},$$

389 where  $w_{\bullet}$  is the computed solution using Approach 1 or 2. The error is computed  
 390 on the whole interval (from 0 to  $r_{\text{max}}$ ). It confirms the accuracy of the numerical  
 391 solutions, with a relative error smaller than 0.1%. However, Approach 1 suffers from  
 392 the inaccuracy near the source position, that leads to high error if the cell is not  
 393 sufficiently refined.

TABLE 1

Relative error (3.14) between the analytical solution and the computations using Approach 1 or 2. The original mesh uses cells of size  $10^{-4}$ . The refined mesh is obtained by reducing the size of the cell (of the original mesh) that contains the source by 100. The error is multiplied by 100 to give the result in percent.

| 3 mHz  |                                | Approach 1<br>(original mesh) | Approach 1<br>(refined mesh) | Approach 2<br>(original mesh) |
|--------|--------------------------------|-------------------------------|------------------------------|-------------------------------|
|        | $\ell = 0, \gamma = 0$         | $3.2 \times 10^1\%$           | $2.4 \times 10^{-2}\%$       | $6.9 \times 10^{-3}\%$        |
|        | $\ell = 100, \gamma = 0$       | $3.1 \times 10^1\%$           | $1.8 \times 10^{-2}\%$       | $6.6 \times 10^{-3}\%$        |
|        | $\ell = 0, \gamma = 10^{-4}$   | $2.1 \times 10^2\%$           | $1.8 \times 10^{-4}\%$       | $5.5 \times 10^{-5}\%$        |
|        | $\ell = 100, \gamma = 10^{-4}$ | $2.1 \times 10^2\%$           | $1.8 \times 10^{-4}\%$       | $5.5 \times 10^{-5}\%$        |
| 10 mHz |                                | Approach 1<br>(original mesh) | Approach 1<br>(refined mesh) | Approach 2<br>(original mesh) |
|        | $\ell = 0, \gamma = 0$         | $1.1 \times 10^1\%$           | $2.1 \times 10^{-2}\%$       | $9.9 \times 10^{-3}\%$        |
|        | $\ell = 100, \gamma = 0$       | $1.1 \times 10^1\%$           | $9.4 \times 10^{-3}\%$       | $9.4 \times 10^{-3}\%$        |
|        | $\ell = 0, \gamma = 10^{-4}$   | $5.8 \times 10^1\%$           | $6.3 \times 10^{-3}\%$       | $8.3 \times 10^{-3}\%$        |
|        | $\ell = 100, \gamma = 10^{-4}$ | $5.9 \times 10^1\%$           | $6.3 \times 10^{-3}\%$       | $8.3 \times 10^{-3}\%$        |

#### 394 4. Observables and data analysis in local helioseismology.

395 **4.1. Measured Observations and helioseismic products.** In helioseismol-  
 396 ogy, the measured data are time series representing line-of-sight Doppler velocities  
 397  $\Phi_{\text{obs}}(\hat{\mathbf{r}}, t_j)$  at spatial points  $\hat{\mathbf{r}}$  on the CCD (charge-coupled device) camera and at  
 398 times  $t_j$ . Here, a spatial point is represented either by a two-dimensional vector  $\hat{\mathbf{r}}$ , or  
 399 by its polar and azimuthal spherical angles  $(\theta, \phi)$ . These observables can come from  
 400 interferometers on-board of satellites such as the Michelson Doppler Imager (MDI) or  
 401 the Helioseismic and Magnetic Imager (HMI), or from ground-based telescopes such as  
 402 the Global Oscillation Network Group (GONG). Upon performing a discrete Fourier  
 403 transform, the time-series data can be turned into frequency observations  $\Phi_{\text{obs}}(\hat{\mathbf{r}}, \omega)$ .  
 404 In practice, the signals can be filtered depending on the purpose of investigation, but  
 405 we do not consider filtering in this paper.

406 Since the solar oscillations are driven by turbulent convection, the signals are  
 407 realizations of random processes and analyzed by statistical methods. As the sources

408 have zero mean, the expectation value of the observable  $\mathbb{E}[\Phi(\hat{\mathbf{r}}, \omega)] = 0$ , which means  
 409 that the data cannot be directly interpreted on the observed signal. Instead, time-  
 410 distance helioseismology is based on the *temporal cross-covariance*  $\mathcal{C}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, t)$ , between  
 411 any two points, for a given  $\Phi$ ,

$$412 \quad (4.1) \quad C(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, t) = \frac{1}{T} \int_{-T/2}^{T/2} \Phi(\hat{\mathbf{r}}_1, s) \Phi(\hat{\mathbf{r}}_2, t + s) ds,$$

413 where  $T$  is the observation time. In the frequency domain, the definition of the *cross-*  
 414 *covariance* is given by a multiplication,

$$415 \quad (4.2) \quad C(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega) = \overline{\Phi(\hat{\mathbf{r}}_1, \omega)} \Phi(\hat{\mathbf{r}}_2, \omega).$$

416 The main quantities of interest in helioseismology are given below.

417 1. The *power spectrum*  $\mathcal{P}$  represents the strength of the signal as a function of har-  
 418 monic degrees and frequency. It is obtained from the harmonic spherical decom-  
 419 position of the signal

$$420 \quad (4.3) \quad \mathcal{P}^{\ell, m}(\omega) = \mathbb{E}[|\Phi_\ell^m(\omega)|^2], \quad \text{with} \quad \Phi_\ell^m(\omega) = \int_0^{2\pi} \int_0^\pi \Phi(\hat{\mathbf{r}}, \omega) \overline{Y_\ell^m(\hat{\mathbf{r}})} d\hat{\mathbf{r}}.$$

421 2. The expectation value of the cross-covariance is used to represent the *time-distance*  
 422 *diagram*, which visualizes how waves propagating through the solar interior are  
 423 observed at the surface as a function of time and distance. It is given by

$$424 \quad (4.4) \quad \mathcal{C}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, t) := \mathbb{E}[C(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, t)], \quad \mathcal{C}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega) := \mathbb{E}[C(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega)].$$

425 **4.2. Synthetic signals.** The synthetic signals are created from solutions  $\phi$  of  
 426 the scalar wave equation (2.5) with a stochastic source  $\mathfrak{s}$  on the right-hand side, i.e.

$$427 \quad (4.5) \quad \phi(\mathbf{r}, \omega) = \int \mathcal{G}(\mathbf{r}, \mathbf{s}, \omega) \mathfrak{s}(\mathbf{s}, \omega) d\mathbf{s}.$$

428 As the height (relative to the solar surface) variable plays a special role, we will often  
 429 separate the vertical and horizontal variables and write  $\mathbf{r} = (r, \hat{\mathbf{r}})$ . We also denote by  
 430  $\mathfrak{M}(\mathbf{s}, \mathbf{s}', \omega)$  the source covariance matrix

$$431 \quad (4.6) \quad \mathfrak{M}(\mathbf{s}, \mathbf{s}', \omega) = \mathbb{E}[\overline{\mathfrak{s}(\mathbf{s}, \omega)} \mathfrak{s}(\mathbf{s}', \omega)].$$

432 This is a distribution in  $\mathbb{R}^3 \times \mathbb{R}^3$  depending on the parameter  $\omega$ . Below, we write  $\langle, \rangle$   
 433 to denote the distribution pairing.

434 **ASSUMPTION 4.1.** *We suppose that the observed signal is coming from the ob-*  
 435 *servaion height  $r_{\text{obs}}$  such that the synthetic signal  $\Phi$  is given by the trace of  $\phi$  at*  
 436  $r = r_{\text{obs}}$ ,

$$437 \quad (4.7) \quad \Phi(\hat{\mathbf{r}}, \omega) = \phi(r_{\text{obs}}, \hat{\mathbf{r}}, \omega).$$

438 Using [Proposition 2.4](#) and [Remark 2.5](#), it can be written in spherical harmonic ex-  
 439 pansion and in terms of the modal Green's function  $\mathcal{G}_\ell$ ,

$$440 \quad (4.8) \quad \Phi(\hat{\mathbf{r}}, \omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{\mathbb{R}^3} \frac{\mathcal{G}_\ell(r_{\text{obs}}, \mathbf{s}; \omega)}{r_{\text{obs}} s} Y_\ell^m(\hat{\mathbf{r}}) \overline{Y_\ell^m(\hat{\mathbf{s}})} \mathfrak{s}(\mathbf{s}, \omega) d\mathbf{s},$$

441 with its spherical harmonic projection,

$$442 \quad (4.9) \quad \Phi_\ell^m(\omega) = \int_{\mathbb{R}^3} \frac{\mathcal{G}_\ell(r_{\text{obs}}, s; \omega)}{r_{\text{obs}} s} \overline{Y_\ell^m(\hat{\mathbf{s}})} \mathfrak{s}(\mathbf{s}, \omega) \, d\mathbf{s}.$$

443 In the above expressions,  $\mathbf{s} = (s, \hat{\mathbf{s}})$  and  $d\mathbf{s} = s^2 \, ds \, d\hat{\mathbf{s}}$ .

444 **4.3. Synthetic helioseismic products.** Using the notation  $\mathfrak{M}$  in (4.6), we  
445 substitute expression (4.9) into the definition of the power spectrum (4.3), to obtain

$$446 \quad (4.10) \quad \mathcal{P}_\ell^m(\omega) = \left\langle \mathfrak{M}(\mathbf{s}, \mathbf{s}', \omega), \frac{\overline{\mathcal{G}_\ell(r_{\text{obs}}, s; \omega)}}{r_{\text{obs}} s} \frac{\mathcal{G}_\ell(r_{\text{obs}}, s'; \omega)}{r_{\text{obs}} s'} Y_\ell^m(\hat{\mathbf{s}}) \overline{Y_\ell^m(\hat{\mathbf{s}'})} \right\rangle.$$

447 To simplify the computation of the helioseismic observables, we consider the fol-  
448 lowing assumption on the source, (cf. the Section 8 of our extended report [6]) for  
449 more details.

450 ASSUMPTION 4.2. *The source  $\mathfrak{s}$  is a random process with zero mean and covari-*  
451 *ance  $\mathfrak{M}$ ,*

$$452 \quad (4.11) \quad \mathfrak{M}(\mathbf{r}, \mathbf{r}', \omega) = \Pi(\omega) \mathcal{M}(r) \delta(\mathbf{r} - \mathbf{r}').$$

453 *This form implies that the sources are spatially uncorrelated. Here,  $\Pi$  is a function*  
454 *that is linked to the time correlation of the sources, and  $\mathcal{M}(s)$  is a distribution in the*  
455 *variable  $s$ .*

456 By the notation (4.11), we mean a distribution whose action is defined as, for a smooth  
457 compactly supported function  $u(\mathbf{s}; \mathbf{s}') = u(s, \hat{\mathbf{s}}; s', \hat{\mathbf{s}}') \in \mathcal{C}_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ ,

$$458 \quad (4.12) \quad \langle \mathfrak{M}(\mathbf{s}, \mathbf{s}'; \omega), u(\mathbf{s}, \mathbf{s}') \rangle := \Pi(\omega) \int_{\mathbb{S}^2} \langle \mathcal{M}(s), s^4 u(s, \hat{\mathbf{s}}; s, \hat{\mathbf{s}}) \rangle \, d\hat{\mathbf{s}}.$$

459

460 *Remark 4.3.* Assumption 4.2 is routinely employed in helioseismology, even if a  
461 more accurate description is possible, for example by replacing the delta function in  
462 (4.11) by a Gaussian with a given correlation length [18]. This is not a limitation for  
463 our approach. Specifically, we could work with the general formula (4.10). See also  
464 further discussion at the end of the subsection.  $\triangle$

465 Under Assumption 4.2, using the orthonormality of the spherical harmonics, the ex-  
466 pression of the power spectrum (4.10) simplifies to

$$467 \quad (4.13) \quad \mathcal{P}_\ell^m(\omega) = \frac{\Pi(\omega)}{r_{\text{obs}}^2} \langle \mathcal{M}(s), s^2 |\mathcal{G}_\ell(r_{\text{obs}}, s; \omega)|^2 \rangle.$$

468 *Convenient source.* In order to simplify this expression, it is possible to find a  
469 convenient form of the function  $\mathcal{M}$  such that the power spectrum directly relates to  
470 the Green's function for a source and receiver located at the observation height.

471 ASSUMPTION 4.4.  *$\mathcal{M}$  is given by the following distribution  $\mathcal{M}_{\text{eq}}$ ,*

$$472 \quad (4.14) \quad \langle \mathcal{M}_{\text{eq}}(s), u(s) \rangle := \int_0^\infty \frac{\gamma(s)}{\rho(s) \epsilon^2(s) s^2} u(s) \, ds, \quad u \in \mathcal{C}_c^\infty((0, \infty)).$$

473 Under [Assumption 4.4](#), the power spectrum has the simplified form,

$$474 \quad (4.15) \quad \mathcal{P}_{\text{eq}}^\ell(\omega) = \frac{\Pi(\omega)}{2\omega r_{\text{obs}}^2} \text{Im}[\mathcal{G}_\ell(r_{\text{obs}}, r_{\text{obs}}, \omega)].$$

475 For derivation, see [Appendix A](#). This relation between the power spectrum and the  
 476 imaginary part of the Green's function is classic in geophysics under the hypothesis of  
 477 energy equipartition (the energy current is identical in all directions, see, e.g., [\[27\]](#)).  
 478 It has been used in helioseismology in [\[17\]](#) and reproduces the main features of the  
 479 observed power spectrum. Moreover, under this hypothesis, travel-time sensitivity  
 480 kernels were computed in order to infer the meridional flow in the Sun, [\[19\]](#).

481 *Single-depth assumptions.* Another assumption used in helioseismology is that  
 482 the sources are coming from a single depth denoted  $r_{\text{src}}$ , [\[18\]](#). In this case, we use the  
 483 following assumption.

484 **ASSUMPTION 4.5.** *The radial dependency  $\mathcal{M}$ , that we denote  $\mathcal{M}_{\text{sd}}$  is given by*

$$485 \quad (4.16) \quad \mathcal{M}_{\text{sd}}(s) = \delta(s - r_{\text{src}}).$$

486 The expression of the power spectrum from [\(4.13\)](#) simplifies to,

$$487 \quad (4.17) \quad \mathcal{P}_{\text{sd}}^\ell(\omega) = \frac{\Pi(\omega) r_{\text{src}}^2}{r_{\text{obs}}^2} |\mathcal{G}_\ell(r_{\text{obs}}, r_{\text{src}}, \omega)|^2.$$

488 As in the case of energy equipartition, the power spectrum is related to the Green's  
 489 function at the observation height but the second point is now at the source location.

490 As the power spectrum is independant of  $m$ , the expectation value of the cross-  
 491 covariance corresponds to the Legendre transform of the power spectrum [\[6, Section 8\]](#)

$$492 \quad (4.18) \quad \mathcal{C}_\bullet(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega) = \sum_{\ell=0}^{L_{\text{max}}} \frac{2\ell+1}{4\pi} \mathcal{P}_\bullet^\ell(\omega) P_\ell(\cos \theta_{\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2}) \quad \text{with } \bullet = \text{eq, sd}.$$

493 The synthetic time-distance diagram is obtained from the inverse Fourier transform  
 494 of the synthetic  $\mathcal{C}_\bullet(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega)$  with

$$495 \quad (4.19) \quad \mathcal{C}_\bullet(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, t_j) = h_\omega \sum_{k=-N_t/2}^{N_t/2-1} \mathcal{C}_\bullet(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \omega) e^{-i\omega t_j}; \quad \text{with } \bullet = \text{eq, sd}.$$

496 In order to compute the power spectrum, one only needs the Green's function for  
 497 a source location at  $r_{\text{obs}}$  and a receiver at  $r_{\text{obs}}$  or  $r_{\text{src}}$  depending on the assumption  
 498 on the source. However, we made two major simplifications to obtain these formulae:  
 499 1. the observation height  $r_{\text{obs}}$  is a single height that is the same everywhere on the  
 500 CCD. A more realistic assumption would require to integrate over depth depending  
 501 on the opacity of the solar surface.  
 502 2. the sources are coming from a single depth  $r_{\text{src}}$  ([Assumption 4.5](#)) or the energy  
 503 is equipartitioned ([Assumption 4.4](#)). These are simplifying assumptions as the  
 504 sources should come from a range of depths and the energy distribution in the Sun  
 505 is complex.

506 Relaxing these two hypotheses, the full Green's kernel is required in order to compute  
 507 the power spectrum or the expectation value of the cross-covariance [6, Section 8].  
 508 In particular, one would need to evaluate a quantity such as

$$509 \quad (4.20) \quad \int \int f(r, s) \overline{\mathcal{G}_\ell(r, r', \omega)} \mathcal{G}_\ell(s, r', \omega) dr ds,$$

510 for a given function  $f$ , e.g., [6, Eq. (8.59)] in order to compute a power spectrum.  
 511 Due to the fast oscillations of the Green's function, evaluating this integral accu-  
 512 rately requires several thousand sources and receivers per frequency and mode, see  
 513 [Remark 5.1](#).

514 Even without relaxing the two previous hypotheses, the observation height is  
 515 dependent on the instrument used for the observations and the source height is not  
 516 known. It is thus interesting to have the full Green's kernel in order to vary the two  
 517 parameters  $r_{\text{obs}}$  and  $r_{\text{src}}$  without any additional computational cost. Computing the  
 518 expectation value of the cross-covariance requires summing over the harmonic degrees  
 519 and thus computing the Green's function for all  $\ell$ . If one wants to compute the time-  
 520 distance diagram, many frequencies should be computed in order to make the inverse  
 521 Fourier transform. An efficient algorithm to compute  $G_\ell(r, s)$  is thus primordial in  
 522 order to interpret helioseismic measurements.

523 **5. Numerical experiments of helioseismic observables.** In this section, we  
 524 perform numerical experiments using inhomogeneous medium parameters that follow  
 525 the solar profiles, using our numerical setup with the HDG method, validated above<sup>1</sup>  
 526 We use the Green's functions computed with Approach 2 to obtain the synthetic helio-  
 527 seismic quantities as defined in [section 4](#). It allows us to compare with observational  
 528 data, to evaluate the performance of the boundary conditions and to illustrate the  
 529 importance of the full Green's functions to generate synthetic data.

530 **5.1. Solar Green's kernels for models S+Atmo and S+Val-C.** The propa-  
 531 gation of the scalar waves is governed by the medium wave speed  $\mathbf{c}$  and the inverse  
 532 density scale height  $\alpha$ , together with its derivative  $\alpha'$ . In the interior of the Sun, these  
 533 are extracted from the model **S** of [12]. In the atmosphere, we consider two models:  
 534 **Atmo**, where the velocity is smoothly extended to a constant and the density follows  
 535 an exponential decay ([16, 17]); and the model **Val-C** of [28]. At the end of the model  
 536 **Val-C**, we can further extend the model following the **Atmo** principles, to define the  
 537 **Val-C+Atmo** model. The representations of the coefficients  $\mathbf{c}$  and  $\alpha$  are given in [Fig-](#)  
 538 [ure 3](#) for two solar profiles, **S+Atmo** and **S+Val-C**. The atmospheric profiles given by  
 539 the model **Val-C** contains a drastic increase of both parameters  $\mathbf{c}$  and  $\alpha$ , while these  
 540 quantities are constant in the **Atmo** model.

541 The interval for the computations ends as soon as the wave speed is constant and  
 542 the density exponentially decaying (i.e.,  $\alpha$  is constant), such that

$$543 \quad (5.1) \quad \begin{aligned} r &\in [0, 1.0008], && \text{interval for S+Atmo, up to 550km above the surface.} \\ r &\in [0, 1.00365], && \text{interval for S+Val-C, up to 2.5Mm above the surface.} \end{aligned}$$

544 The attenuation is kept fixed to the value  $\gamma/2\pi = 20 \mu\text{Hz}$  and the scaling function  $\Pi$

<sup>1</sup>Our code, `hawen`, is written in `Fortran90`, and combines `mpi` and `OpenMp` parallelism, it is available at <https://ffaucher.gitlab.io/hawen-website/>. It is linked with the library `Arb`, [21] for the efficient computation of the special functions (i.e., the Whittaker's functions for the DtN coefficient).

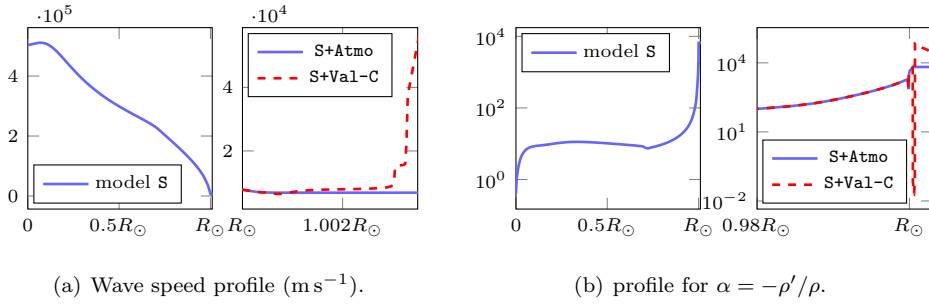


FIG. 3. Profiles of the solar parameters, the model  $S$  is used for the interior and is combined with the atmospheric models  $Atmo$  or  $Val-C$ .

545 is given by [17, Eq. (85)],

$$546 \quad (5.2) \quad \Pi(\omega) = \left( 1 + \left( \frac{|\omega| - \omega_0}{\omega_1} \right)^2 \right)^{-1}, \quad \text{with } \frac{\omega_0}{2\pi} = 3.3 \text{ mHz} \quad \text{and} \quad \frac{\omega_1}{2\pi} = 0.6 \text{ mHz}.$$

547 We compare the modal Green's kernels for the two solar models,  $S+Atmo$  and  $S+Val-C$   
 548 in Figure 4 with  $\mathbf{Z}_{DtN}^\ell$  as boundary conditions (see subsection 5.3). Here, we use  
 549 Approach 2 which gives access to the complete Green's kernel  $G$  (of the conjugated  
 550 problem) from two simulations. Then,  $\mathcal{G}$  (for the original problem) is obtained after  
 551 multiplying by the density, according to (2.13). We see that, for the two choices  
 552 of models, the wavelength reduces when the waves are getting closer to the surface.  
 553 Between the models  $S+Atmo$  and  $S+Val-C$ , there is a difference in amplitude and the  
 554 patterns of the waves are also different. These are even more pronounced when we  
 555 zoom near the surface (right of Figure 4), which is the area that is used to compute  
 556 the following power spectrum.

557 *Remark 5.1* (Computational cost of Approach 1). While Approach 2 gives the  
 558 Green's function from two simulations, Approach 1 requires as many simulations as  
 559 the number of sources, which must be at least a few thousand to later approximate the  
 560 integral in (4.20). The multiple right-hand sides feature of direct solvers, such as MUMPS  
 561 [3], allows to mitigate the computational cost of having thousand of sources, however,  
 562 as highlighted in Figure 2, one must also refine the mesh near the sources to obtain  
 563 accurate solutions, consequently increasing the number of degrees of freedom. This  
 564 need for an extra refinement was also emphasized in [17] in order to obtain accurate  
 565 helioseismic observables. This is in particular crucial for  $\mathcal{P}_{eq}$ , (4.15) that relies on  
 566  $\mathcal{G}_\ell(r, r)$ . We illustrate the two approaches in Figure 5, where we have 4000 sources and  
 567 we need to use cells of size  $10^{-7}$  near the sources to ensure the accuracy of Approach 1.  
 568 In our experiments, we consequently observe an increase in the computational time  
 569 by a factor from 15 (for 4000 sources) to 30 (for 10 000 sources) with Approach 1.  
 570 Moreover, Approach 1 requires a large amount of memory to store the solutions for all  
 571 sources while Approach 2 only requires the two simulations (this is not to be neglected  
 572 when considering the number of frequencies and modes one needs to compute).  $\triangle$

573 **5.2. Computational experiments of power spectrum.** Using the computa-  
 574 tional Approach 2, we obtain the full Green's functions and instantly have access to  
 575 the power spectrum of subsection 4.2, associated with any heights for the source and  
 576 the receiver. In our computations, we use frequencies from 1 to 12 mHz and modes

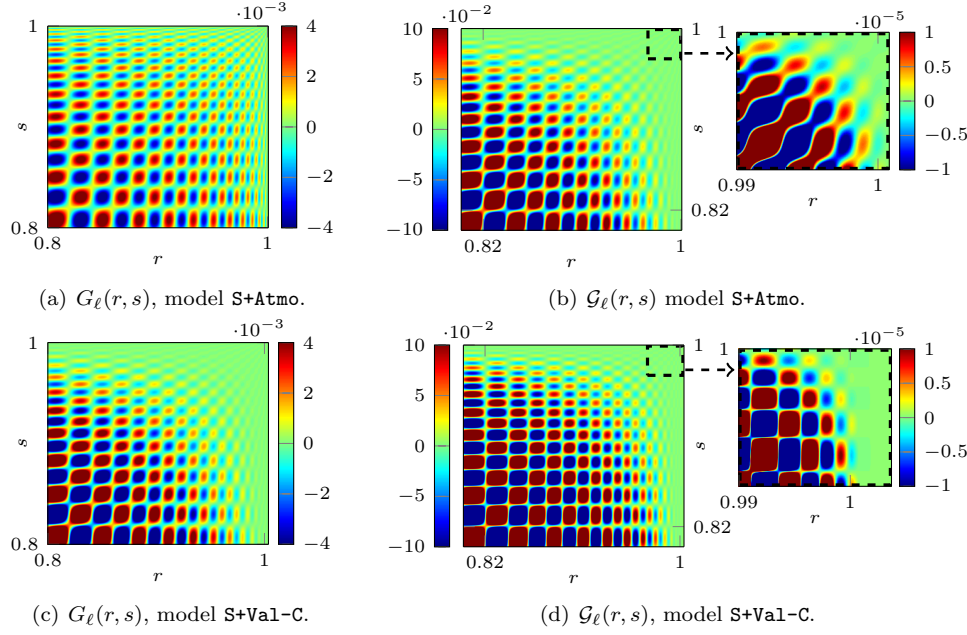


FIG. 4. Imaginary part of the Solar modal Green's functions at 7 mHz for mode  $\ell = 100$  for the models *S+Atmo* (top) and *S+Val-C* (bottom). The solution to the original problem and the conjugated one, respectively  $\mathcal{G}_\ell$  and  $G_\ell$ , are related by (2.13).

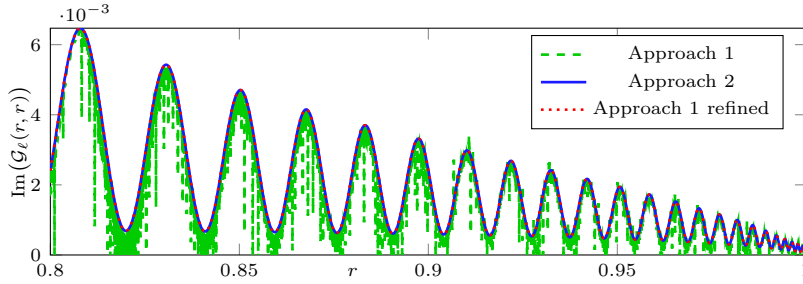


FIG. 5. Comparison of the Green's function for model *S+Atmo* at frequency 7 mHz and mode  $\ell = 100$ . While Approach 2 depends on two simulations, Approach 1 needs one simulation for each of the 4000 sources. In Approach 1 with the refined mesh, the cell at the positions of the sources is of size  $10^{-7}$ .

577 from  $\ell = 0$  to 1000, with a constant attenuation  $\gamma/(2\pi) = 20 \mu\text{Hz}$ . Note that, con-  
 578 trary to the existing literature, our computation uses the exact Dirichlet-to-Neumann  
 579 map condition  $\mathbf{Z}_{\text{DtN}}^\ell$  given in (2.38). In solar applications, the source depth  $r_{\text{src}}$  corre-  
 580 sponds to the location where the waves are excited and is generally considered to be  
 581 a bit below the surface (usually a few hundreds kilometers). The observation height  
 582 depends on the instrument and is usually located slightly above the surface (up to 500  
 583 km). It might appear as a small interval as 1 Mm represents approximately  $1 \times 10^{-3}$   
 584 in the scaled axis, nonetheless, we shall see that it leads to drastic differences.

585 **5.2.1. Influence of the source assumption and its height.** We compare  
 586 the power spectra of [subsection 4.2](#):  $\mathcal{P}_{\text{eq}}$  under the hypotheses of energy equipartition  
 587 and  $\mathcal{P}_{\text{sd}}$  for the single depth source. While the source position,  $r_{\text{src}}$ , is allowed to  
 588 vary for  $\mathcal{P}_{\text{sd}}$ , it is fixed at the observation height for  $\mathcal{P}_{\text{eq}}$ , cf. (4.15) and (4.17). In  
 589 [Figure 6](#), the receiver is fixed at the surface ( $r_{\text{obs}} = 1$ ) and, for  $\mathcal{P}_{\text{sd}}$ , we provide the  
 590 power spectrum for different source positions, indicated relative to the solar surface.  
 591 The computations use the model **S+Atmo**, with the exact DtN for boundary condition.  
 592 For visualization, the spectra are represented on a logarithmic scale.

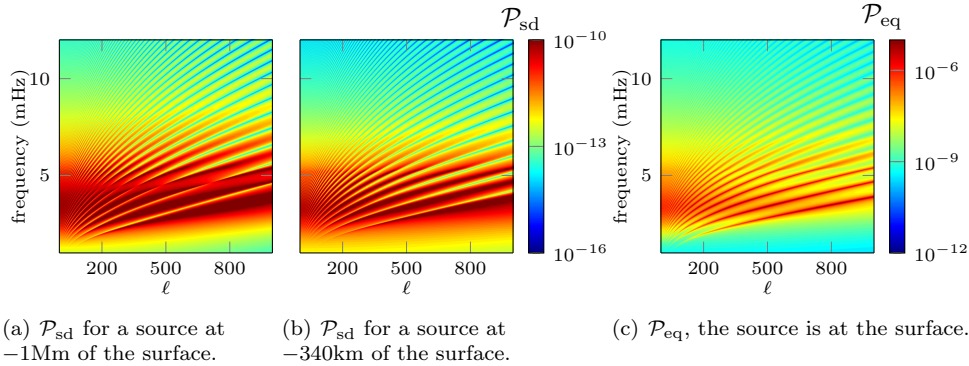


FIG. 6. Power spectra  $\mathcal{P}_{\text{eq}}$  and  $\mathcal{P}_{\text{sd}}$ , (4.15) and (4.17), for a receiver positioned at the surface, using the model **S+Atmo** with  $\mathbf{Z}_{\text{DtN}}^{\ell}$  for RBC. For  $\mathcal{P}_{\text{sd}}$ , the different formation source heights are given relative to the solar surface  $R_{\odot}$ .

593 To highlight the differences, we picture a line of the spectra, for a fixed mode  
 594  $\ell = 200$ , in [Figures 7](#) and [8](#). We show the results for different source heights using the  
 595 original or normalized spectra. We see that there is a strong difference in amplitudes  
 596 depending on the formulation, in part due to the different scaling in (4.15) and (4.17).  
 597 Comparing between the different source heights for  $\mathcal{P}_{\text{sd}}$ , we can visually note the  
 598 differences in the shape of the ridges in [Figure 6](#). This is further confirmed by the  
 599 section at mode  $\ell = 200$  in [Figures 7](#) and [8](#). At low frequencies, not all of the  
 600 peaks appear, for instance the ones between 4 and 5 mHz in [Figure 8](#). If the sources  
 601 of excitation are too deep then some modes are not excited leading to the absence  
 602 of ridge in the power spectrum. At high frequencies, the spectra are very different  
 603 depending on the assumption on the source covariance: here, both the amplitude and  
 604 the phase are affected, cf. [Figure 8](#).

605 **5.2.2. Influence of the formation height.** We represent the power spectra  
 606  $\mathcal{P}_{\text{sd}}$  (4.17) for two solar models with a constant source height in [Figure 9](#). We vary  
 607 the position of the receiver,  $r_{\text{obs}}$ , which corresponds to the observation height and can  
 608 be different depending on the instrument. We show the normalized section at mode  
 609  $\ell = 200$  in [Figure 10](#).

610 We observe strong differences in both amplitude and phase depending on the  
 611 positions of the observation height. While the two models, **S+Atmo** and **S+Val-C**  
 612 corresponds well at low frequency, see [Figure 10](#), the high-frequency profiles are totally  
 613 different, with sharp or smooth peaks. In addition, we observe lines of reduced power  
 614 with **S+Val-C** in the complete spectrum of [Figure 9](#).

615 **5.3. Efficiency of the radiation boundary conditions (RBC).** Contrary  
 616 to previous work, we analyze the performance of the RBCs in the context of helioseis-

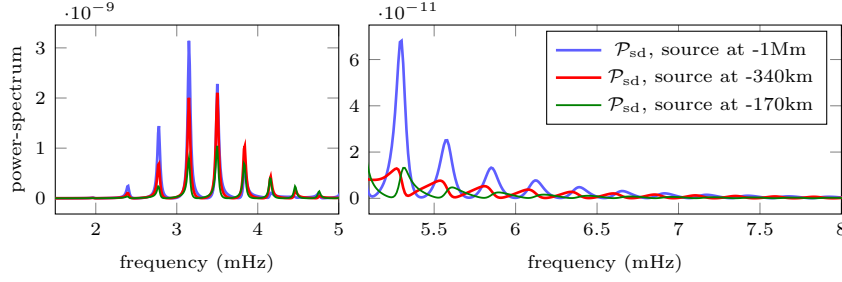


FIG. 7. Comparison of the line of the power spectrum  $\mathcal{P}_{sd}$ , (4.17), at a fixed mode  $\ell = 200$ , for different source heights, given relative to the solar surface. The receiver is positioned at the solar surface and the computations use model  $S+Atmo$  with  $\mathbf{Z}_{DtN}^\ell$  for RBC. We separate the low and high frequencies, which are visualized on different scales.

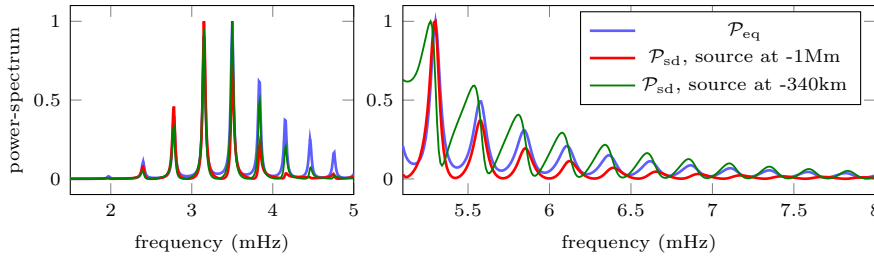


FIG. 8. Comparison of the line of the power spectra at a fixed mode  $\ell = 200$ , for different source heights, given relative to the solar surface. The receiver is positioned at the solar surface and the computations use model  $S+Atmo$  with  $\mathbf{Z}_{DtN}^\ell$  for RBC. The low and high frequencies are independently normalized.

617 mic products, and using the exact DtN as the reference solution. We have in mind  
 618 discretization in dimensions higher than one, where it is necessary to select a RBC  
 619 that does not depend on the mode  $\ell$ , i.e., one that does not contain a tangential  
 620 differential operator. We first recall the nonlocal RBC from [5, 8],

$$621 \quad (5.3) \quad \mathbf{Z}_{\text{nonlocal}}^\ell(r) = i \left( \frac{\sigma^2(r)}{c^2(r)} - \mathbf{q}(r) - \frac{\ell(\ell+1)}{r^2} \right)^{1/2}.$$

622 Different high-frequency approximations of  $\mathbf{Z}_{\text{nonlocal}}^\ell$  have been given in [5, 8] and we  
 623 investigate the performance of (in the notation of [8]):

$$624 \quad (5.4) \quad \begin{aligned} \mathbf{Z}_{\text{SAI-0}} &= ik \left( 1 - \frac{\alpha_\infty}{r_{\max}} \frac{1}{k^2} \right)^{1/2}, & \mathbf{Z}_{\text{S-HF-0}} &= ik, \\ \mathbf{Z}_{\text{S-HF-1a}} &= ik - \frac{i}{2k} \frac{1}{r_{\max}} \alpha_\infty, & \mathbf{Z}_{\text{A-RBC-1}} &:= \frac{1}{r_{\max}} + ik; \\ \mathbf{Z}_{\text{A-HF-1}}^\ell &= i \frac{\sigma}{c_\infty} + \frac{c_\infty}{2i\sigma} \left( \frac{\ell(\ell+1)}{r_{\max}^2} + \frac{\alpha_\infty}{r_{\max}} + \frac{\alpha_\infty^2}{4} \right). \end{aligned}$$

625 The exponent  $\ell$  indicates that the condition depends on the mode. Note that, while  
 626 we provide comparisons for helioseismic quantities below, the performance of the  
 627 conditions is evaluated *analytically* in our extended report, [6, Section 5].

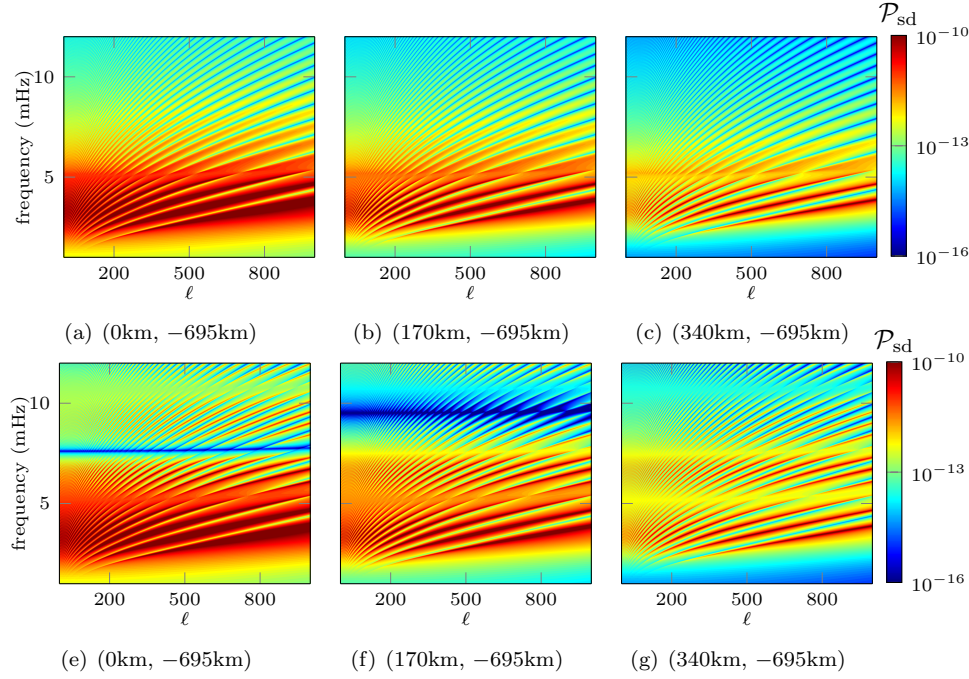


FIG. 9. Power spectrum  $\mathcal{P}_{\text{sd}}$ , (4.17), at different formation heights on a logarithmic scale. It uses the solar model  $\text{S+Atmo}$  on top and  $\text{S+Val-C}$  at the bottom. Each spectrum corresponds to a position of a receiver and a source, which are indicated in parenthesis and given relative to the solar surface  $R_{\odot}$ , i.e., the source is fixed at 695 km below the surface.

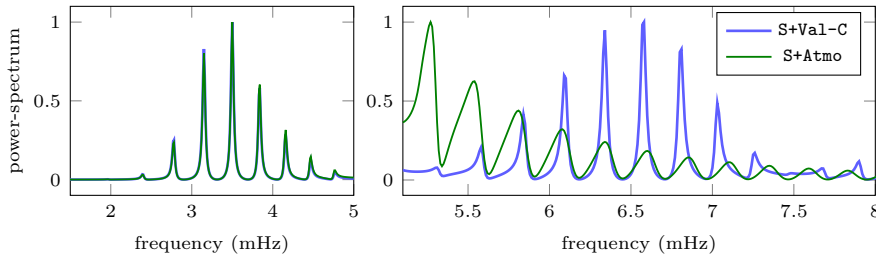


FIG. 10. Comparison of the line of the power spectrum, (4.17), at a fixed mode  $\ell = 200$ , for a receiver and a source at respectively 340 km and  $-695$  km of the solar surface. The low and high frequencies are independently normalized.

628 **5.3.1. Radiation boundary conditions for the power spectrum.** We use  
 629 the solar model  $\text{S+Atmo}$ , where the reference solution is given by the solution using the  
 630 exact Dirichlet-to-Neumann map condition  $\mathbf{Z}_{\text{DtN}}^{\ell}$ . We compute the power spectrum  
 631  $\mathcal{P}_{\text{eq}}$  of (4.15) for  $r_{\text{obs}} = 1$ , with different boundary conditions, and evaluate the  
 632 performance with the relative error

$$633 \quad (5.5) \quad \epsilon_{\mathbf{Z}_{\bullet}}^{\mathcal{P}}(\omega, \ell) = \frac{\left\| \mathcal{P}_{\text{eq}, \mathbf{Z}_{\bullet}}^{\ell}(\omega) - \mathcal{P}_{\text{eq}, \mathbf{Z}_{\text{DtN}}^{\ell}}^{\ell}(\omega) \right\|}{\left\| \mathcal{P}_{\text{eq}, \mathbf{Z}_{\text{DtN}}^{\ell}}^{\ell}(\omega) \right\|},$$

634 where the index  $\mathbf{Z}_\bullet$  indicates the choice of condition with  $\mathbf{Z}_{\text{DtN}}^\ell$  giving the reference  
 635 one. The relative errors are shown in Figure 11.

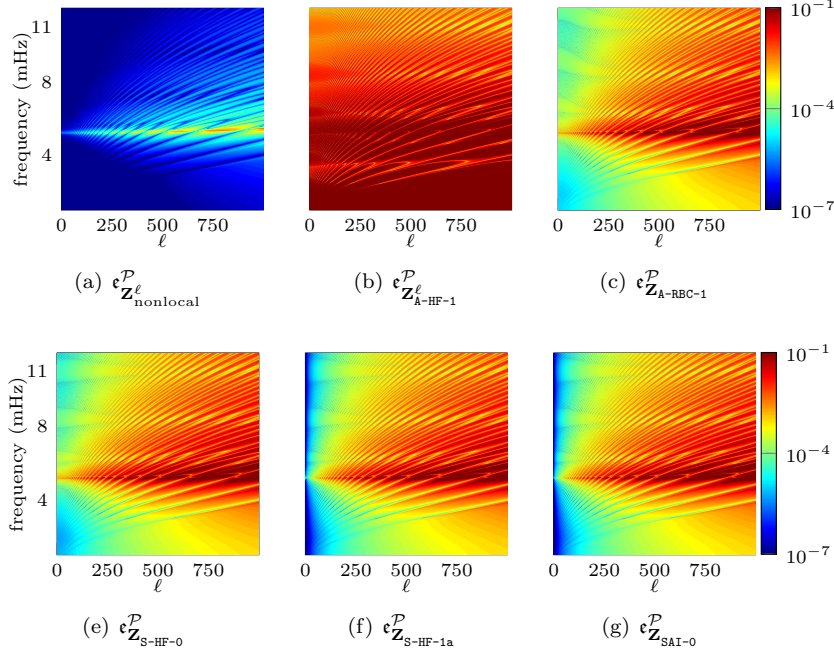
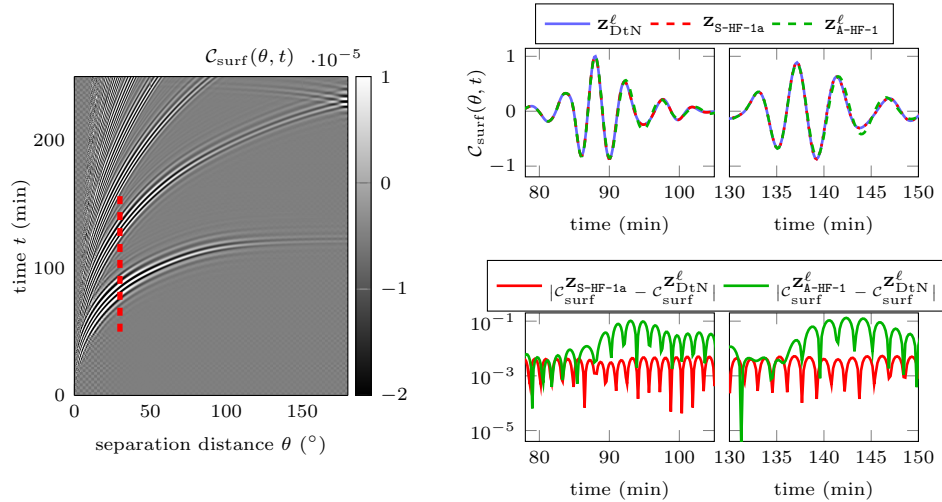


FIG. 11. Relative error (5.5) for the power spectrum associated with the model *S+Atmo* depending on the choice of radiation boundary condition.

636 We see that the nonlocal boundary condition provides the most accurate results,  
 637 and that the maximum error is on a line that corresponds to the cut-off frequency, in  
 638 particular for high-degree modes. On the other hand,  $\mathbf{Z}_{\text{A-HF-1}}^\ell$  gives the worst results,  
 639 while all other conditions give very similar errors, with a slight advantage towards  
 640  $\mathbf{Z}_{\text{SAI-0}}$  and  $\mathbf{Z}_{\text{S-HF-1a}}$ . We further provide the means of the relative errors:

$$\begin{aligned}
 \text{mean}(\epsilon_{\mathbf{Z}_{\text{nonlocal}}^\ell}^{\mathcal{P}}) &= 1 \times 10^{-5}, & \text{mean}(\epsilon_{\mathbf{Z}_{\text{A-HF-1}}^\ell}^{\mathcal{P}}) &= 4.53 \times 10^{-1}, \\
 \text{mean}(\epsilon_{\mathbf{Z}_{\text{S-HF-0}}^\ell}^{\mathcal{P}}) &= 9.26 \times 10^{-3}, & \text{mean}(\epsilon_{\mathbf{Z}_{\text{A-RBC-1}}^\ell}^{\mathcal{P}}) &= 9.28 \times 10^{-3}, \\
 \text{mean}(\epsilon_{\mathbf{Z}_{\text{S-HF-1a}}^\ell}^{\mathcal{P}}) &= 9.10 \times 10^{-3}, & \text{mean}(\epsilon_{\mathbf{Z}_{\text{SAI-0}}^\ell}^{\mathcal{P}}) &= 9.10 \times 10^{-3}.
 \end{aligned}$$

642 **5.3.2. Radiation boundary conditions for time-distance diagram.** We  
 643 compute the time-distance diagram using the exact  $\mathbf{Z}_{\text{DtN}}^\ell$  and compare with two high-  
 644 frequency approximations of the nonlocal RBC coefficient:  $\mathbf{Z}_{\text{S-HF-1a}}^\ell$  and  $\mathbf{Z}_{\text{A-HF-1}}^\ell$ . The  
 645 first approximation is obtained with an expansion in  $k^{-1}$  while the latter in terms  
 646 of  $\sigma^{-1}$ , cf. [8]. In addition,  $\mathbf{Z}_{\text{A-HF-1}}^\ell$  depends on  $\ell$  but not  $\mathbf{Z}_{\text{S-HF-1a}}^\ell$ , meaning that the  
 647 latter is much more convenient to implement in 3D. Figure 12 shows the time-distance  
 648 diagram for  $C_{\text{surf}}(\theta, t)$  where  $\theta = \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2$  is the angle between the two observation points  
 649  $\hat{\mathbf{r}}_1$  and  $\hat{\mathbf{r}}_2$ . We first compute  $C_{\text{surf}}(\theta, \omega)$  using (4.18) with a frequency  $h_\omega = 5$   $\mu\text{Hz}$  and  
 650 then apply the inverse Fourier transform using (4.19) with  $N_t = 2^{15}$ . We show the  
 651 section at  $\theta = 30^\circ$  for the different RBC, together with the difference with respect to  
 652 the reference (using  $\mathbf{Z}_{\text{DtN}}^\ell$ ) solution.



(a) Time-distance diagram with the DtN (b) Line at  $\theta = 30^\circ$  (top) and difference between the boundary conditions (bottom).

FIG. 12. Time-distance diagram representing the expectation value of the cross-covariance as a function of separation distance and time: two-dimensional diagram and the section for  $\theta = 30^\circ$ , indicated by the red dashed lines on the left panel.

653 We see that in Figure 12(a), the time-distance diagram shows different skips  
 654 corresponding to waves going directly from  $\hat{r}_1$  to  $\hat{r}_2$ , or being reflected at the surface  
 655 once or several times. We can visually observe the difference in the solutions between  
 656  $\mathbf{Z}_{\text{DtN}}^\ell$  and  $\mathbf{Z}_{\text{A-HF-1}}^\ell$ , which differ in phase and amplitude. The condition  $\mathbf{Z}_{\text{S-HF-1a}}$  gives  
 657 the best result (that is, the closest to  $\mathbf{Z}_{\text{DtN}}^\ell$ ) and we have between 2 to 3 orders of  
 658 magnitude difference in the accuracy. Thus, working with the wavenumber  $k$  leads to  
 659 an improved performance compared to using  $\sigma/c_\infty$ .

660 **6. Conclusion.** In this work, we have proposed a two-step algorithm to com-  
 661 pute efficiently and accurately the full outgoing modal Green's kernel for the scalar  
 662 wave equation in local helioseismology under spherical symmetry. The full Green's  
 663 kernel enables the computations of more realistic synthetic observables with varying  
 664 observation heights, leading to drastic differences. It gives a convenient framework to  
 665 study the origin of sources of excitation in the Sun. With its low cost, the algorithm  
 666 not only opens up new applications that are based on better synthetic representations  
 667 of the observations (e.g., with integrated quantities), but also has direct implications  
 668 in inverse problems for the determination of the solar interior. Finally, our compari-  
 669 son of RBC also paves the way towards three-dimensional discretizations, by singling  
 670 out the condition  $\mathbf{Z}_{\text{S-HF-1a}}$  for its independence of the harmonic mode, and its high  
 671 accuracy in approximating the outgoing solution.

672 **Appendix A. Power spectrum under Assumptions 4.2 and 4.4.**

673 Here, we show that the power spectrum takes the form given in (4.15), under  
 674 Assumptions 4.2 and 4.4. To find this relation, let us write

$$675 \quad (\text{A.1}) \quad \overline{L_\ell G_\ell(r_1, r)} = \delta(r_1 - r), \quad L_\ell G_\ell(r_2, r) = \delta(r_2 - r).$$

676 We then multiply the first equation by  $G_\ell(r_2, r)$ , the second one by  $\overline{G_\ell(r_1, r)}$  and

677 integrate to obtain

$$678 \quad (\text{A.2}) \quad \int_0^\infty \left( \overline{L_\ell G_\ell(r_1, r)} G_\ell(r_2, r) - L_\ell G_\ell(r_2, r) \overline{G_\ell(r_1, r)} \right) dr = G_\ell(r_2, r_1) - \overline{G_\ell(r_2, r_1)}.$$

679 Canceling out the identical terms leads to

$$680 \quad (\text{A.3a}) \quad G_\ell(r_2, r_1) - \overline{G_\ell(r_2, r_1)} = 4i\omega \int_0^\infty \frac{\gamma(r)}{c^2(r)} \overline{G_\ell(r_1, r)} G_\ell(r_2, r) dr$$

$$681 \quad (\text{A.3b}) \quad + \int_0^\infty \left( -\frac{d^2}{dr^2} \overline{G_\ell(r_1, r)} G_\ell(r_2, r) + \overline{G_\ell(r_1, r)} \frac{d^2}{dr^2} G_\ell(r_2, r) \right) dr.$$

683 Integrating by part twice, the second line of the equation is equal to zero and thus

$$684 \quad (\text{A.4}) \quad \text{Im}[G_\ell(r_2, r_1)] = 2\omega \int_0^\infty \frac{\gamma(r)}{c^2(r)} \overline{G_\ell(r_1, r)} G_\ell(r_2, r) dr.$$

685 Using (2.28) to come back to the Green's function of the original problem, we obtain

$$686 \quad (\text{A.5}) \quad \text{Im}[\mathcal{G}_\ell(r_2, r_1)] = 2\omega \int_0^\infty \frac{\gamma(r)}{\rho(r)c^2(r)} \overline{\mathcal{G}_\ell(r_1, r)} \mathcal{G}_\ell(r_2, r) dr.$$

687 Identifying this expression with (4.10), we see that the choice of this distribution in  
688 (4.14) leads to the simplified expression for the power spectrum (4.15).

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