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Strict Lyapunov functions for consensus under directed connected graphs

Elena Panteley Antonio Loria Srikant Sukumar

Abstract—It is known that for consensus of systems interconnected under a general directed graph topology a necessary and sufficient condition for consensus is that there exist at least one rooted spanning tree. In this paper we present an original statement of linear algebra that serves to characterise the spanning-tree condition for directed graphs in terms of a Lyapunov equation involving the graph's Laplacian. Our results apply to the case of systems described by simple first and second order integrators. As a result, we provide strict Lyapunov functions that ensure, via direct constructive proof, global exponential stability of the consensus manifold.

Index Terms—Consensus, Lyapunov functions, synchronisation, linear systems, multiagent systems.

I. INTRODUCTION

Consensus has been extensively studied for linear multiagent systems and in particular, for first-order integrators under various perspectives involving tools of stability and algebraic graph theories —see *e.g.*, [1]–[4]. Algebraic graph theory was first used in [2] to analyse consensus of first-order integrators interconnected in undirected (as well as balanced) graphs and in [3], where the authors proved for directed graphs that consensus can be achieved if and only if the graph of interconnections contains a rooted spanning tree. In a similar spirit, by the *eigenvalue analysis method* consensus is analysed by characterising the distribution of the eigenvalues of the corresponding state matrices —see *e.g.* [3] for the single integrators case and [5]–[8] for double-integrators. These results are extended to the case of more complex linear second-order dynamics dynamics, *e.g.*, in [9], [10]. In [7] a linear change of coordinates is used to transform the networked systems dynamics into a reduced-order system and the consensus problem is recasted into one of stability analysis. In [11] Rumiantsev's stability theory with respect to part of the variables is used.

The most notorious stability-based methods are those that appeal to the construction of Lyapunov functions. At the exception of some simpler cases, this is in general a challenging task since, most often, the consensus problem is recasted into one of stability of a manifold, as opposed to the more common stability-of-the-equilibrium theory. For

instance, for an undirected graph the Laplacian matrix is symmetric, so a simple quadratic Lyapunov function may be used —*idem* for balanced graphs [2].

Lyapunov-function design for directed, *strongly connected*, graphs is more complex. It is based on the concepts of *M-matrices* and their irreducibility property, as well as Perron-Frobenius Theorem [12]. In particular, properties of strongly connected graphs guarantee that the left eigenvector of the corresponding Laplacian matrix, associated to the zero eigenvector has all positive elements. Using this property, a quadratic Lyapunov function, $V(x) := x^\top P x$ may be constructed in which P is diagonal and is composed of the elements of the left eigenvector. Such Lyapunov functions were introduced in [13], see also [14]–[16] and [17] for a review on Lyapunov-based analysis of consensus for single integrators interconnected under strongly connected graphs.

In the general case of directed graphs with a rooted spanning tree, unlike for strongly connected graphs, the left eigenvalue of the Laplacian matrix can have zero elements, so the previous construction method cannot be used in this case. This makes the study of consensus on such graphs much more involved. To the best of our knowledge, Lyapunov functions for directed graphs with a rooted spanning tree have been proposed only in [18] and [19]. In [18, Proposition 1] a Lyapunov function that measures the distance to the consensus set is constructed and it is established that it is non-increasing along the trajectories of the system; then, La Salle's invariance principle is invoked to assert consensus. In [19, Lemma 1] a quadratic Lyapunov function depending on the the synchronisation errors is designed; akin to the one proposed in [18] the Lyapunov function proposed in [19, Lemma 1] measures the distance to the consensus set, but in synchronisation-errors space. Furthermore, this Lyapunov function is strict (has negative definite derivative).

In this paper we contribute with a Lyapunov approach for the study of the consensus problem for the systems over general networks containing a directed spanning tree. In order to highlight the main idea, we restrict our presentation to one of the simplest cases, the continuous-time, leaderless consensus problem for single and double integrator systems with fixed communication topology and in absence of communication delays. Extensions to more complex scenarios will be reported elsewhere.

Our contributions can be summarised as follows. First, we propose a linear transformation mapping the nodes coordinates space into the error coordinates', thereby recasting the consensus problem into one of stability of a set. Second, we derive a necessary and sufficient condition for consensus in

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terms of a graph-dependent Lyapunov-like matrix equation. Furthermore, applying the general result to the double-integrator systems we get simple sufficient criteria in terms of the controllers gains and the network Laplacian. Our paper provides a basis for constructing Lyapunov functions for consensus of systems in the general case of connected graphs which have a rooted spanning tree. Hence, we lift a stumbling block for the design of Lyapunov functions for multi-agent systems; this facilitates both the control design and the stability analysis for cooperative control of networked systems.

The rest of the paper is organised as follows. In the next section we recall some material on graph theory. In Section III we present an original statement on matrix stability, which is fundamental to our main results. The latter, in turn, are presented in Section IV. Finally, we conclude with some closing remarks in Section V.

II. PRELIMINARIES

We start by presenting few basic facts from algebraic graph theory that will be used later in the paper. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a directed graph composed of a set of nodes, $\mathcal{V} = \{1, \dots, N\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a non-negative weighted adjacency matrix $A = [a_{ij}] \in R^{N \times N}$. An edge $(i, j) \in \mathcal{E}$ if and only if the j th agent receives information from the i th agent. The adjacency matrix A is such that an element $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$, otherwise $a_{ij} = 0$.

We say that a directed graph contains a rooted spanning tree if there exists a node from which there exists a path formed by graph edges that connects it to all the nodes of the graph. A directed graph is strongly connected if there exists a directed path from every node to every other node.

The (non-symmetric) Laplacian matrix $L = [l_{ij}] \in R^{N \times N}$, associated to the adjacency matrix A and the graph \mathcal{G} is defined as $l_{ij} = -a_{ij}$ for all $i \neq j$ and $l_{ii} = \sum_{i=1, j \neq i}^N a_{ij}$. Let I_N denote the $N \times N$ identity matrix, $\mathbf{1}_N \in R^N$ denote the vector of all ones, and $\mathbf{0}_N \in R^N$ denote the vector of all zeros. Then, we have $L\mathbf{1}_N = \mathbf{0}$. We define all the eigenvalues of L as $\lambda_i(L)$, $i \in \{1, \dots, N\}$ sometimes dependence on L will be omitted when it is clear from the context. Furthermore, we recall the following.

Lemma 1 [3] The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree. \square

III. ON THE GENERALISED LYAPUNOV EQUATION

In this section we present an original statement of linear algebra. It has interest of its own since it generalises the well-known Lyapunov-equation characterisation of a Hurwitz matrix to matrices having one equal-to-zero eigenvalue. In control systems theory, it has particular importance in the context of consensus analysis of networked systems and it is

the founding block of our main results, which are presented in the next section.

Let $L \in R^{N \times N}$ be a matrix having the following property.

Assumption 1 The matrix $L \in R^{N \times N}$ has exactly one eigenvalue equal to zero and $N - 1$ eigenvalues with positive real parts. That is,

$$\lambda_1(L) = 0, \quad \Re\{\lambda_i(L)\} > 0 \quad \forall i \in \{2, \dots, N\}. \quad \square$$

We now present a Lyapunov characterisation of the property previously described.

Lemma 2 Consider a matrix $L \in R^{N \times N}$ satisfying Assumption 1. Then, for any $Q = Q^\top > 0$ and $\alpha > 0$, there exists a matrix $P(\alpha) \in R^{N \times N}$, $P = P^\top > 0$ such that

$$PL + L^\top P = Q - \alpha[Pv_r v_\ell^\top + v_\ell v_r^\top P], \quad (1)$$

where v_ℓ and $v_r \in R$ are, respectively, the left and right eigenvectors of the matrix L corresponding to the eigenvalue that is equal to zero. \square

Remark 1 Notice that class of matrices satisfying Assumption 1 is larger than that of Laplacian matrices since no assumptions on the sign of the off-diagonal elements of the matrix L are imposed. In particular, if L corresponds to a Laplacian matrix then $v_r = \mathbf{1}_N$, but this is generally not the case. \bullet

Remark 2 Notice also that since L is real, its eigenvectors (particularly v_ℓ and v_r) can always be chosen real —see [20, Section 3.2]. \bullet

Proof of Lemma 2. The proof is constructive, but before we proceed to construct the matrix P , we first consider certain properties of L , in particular, its Jordan decomposition, which has the form

$$L = U\Lambda U^{-1}, \quad (2)$$

where $U \in R^{N \times N}$ and $\Lambda \in R^{N \times N}$. Since under Assumption 1, L has one equal-to-zero eigenvalue,

$$\Lambda := \begin{bmatrix} 0 & \cdot \\ \cdot & \Lambda_1 \end{bmatrix}, \quad \Lambda_1 \in R^{N-1 \times N-1},$$

and the matrix Λ_1 is block diagonal, *i.e.*,

$$\Lambda_1 := \begin{bmatrix} \Lambda_{11} & \cdots & \\ \vdots & \ddots & \\ & & \Lambda_{1m} \end{bmatrix},$$

where for each $i \leq m$, Λ_{1i} is the Jordan block corresponding to the i th eigenvalue of L , with positive real part and of multiplicity q_i . Hence, $\Lambda_{1i} \in R^{q_i \times q_i}$, $\Re\{\lambda_i\} > 0$, and

$$\Lambda_{1i} := \begin{bmatrix} \Re\{\lambda_i\} & & \\ & \ddots & (*) \\ (*) & & \Re\{\lambda_i\} \end{bmatrix}.$$

We recall that the matrices U and U^{-1} in (2) are composed of the right and left eigenvectors of the matrix L respectively. Therefore, denoting by $U_1, U_1^\dagger \in \mathbb{R}^{N \times N-1}$ the last $N-1$ columns of U and U^{-1} , we can write these matrices as

$$U = [v_r \ U_1], \quad \text{and} \quad U^{-1} = \begin{bmatrix} v_\ell^\top \\ U_1^\dagger \end{bmatrix}. \quad (3)$$

On the other hand, for further development, we remark that

$$L = \lambda_1(L)v_r v_\ell^\top + U_1 \Lambda_1 U_1^\dagger = U_1 \Lambda_1 U_1^\dagger \quad (4)$$

since $\lambda_1(L) = 0$.

Now, for any $\alpha > 0$, define

$$R(\alpha) := \alpha v_r v_\ell^\top + U_1 \Lambda_1 U_1^\dagger. \quad (5)$$

Note that $R(0) = L$ and, like the Laplacian in (2), the matrix $R(\alpha)$ satisfies

$$R(\alpha) = U \Lambda_R U^{-1},$$

where

$$\Lambda_R := \begin{bmatrix} \alpha & \cdot \\ \cdot & \Lambda_1 \end{bmatrix}.$$

From this decomposition and the properties of the matrix Λ_1 it is clear that $\Re\{\lambda_i(R)\} > 0$ for all $i \leq N$. That is, $-R(\alpha)$ is Hurwitz. Therefore, for any $Q = Q^\top > 0$ and $\alpha > 0$ there exists $P = P^\top > 0$ such that

$$\begin{aligned} -PR(\alpha) - R(\alpha)^\top P &= -Q \\ \Leftrightarrow -P[\alpha v_r v_\ell^\top + L] - [\alpha v_r v_\ell^\top + L]^\top P &= -Q \\ \Leftrightarrow PL + L^\top P &= Q - \alpha[Pv_r v_\ell^\top + v_\ell v_r^\top P]. \end{aligned}$$

■

IV. CONSENSUS IN DIRECTED-GRAPH NETWORKS

A. First-order systems

Let us consider a network of first-order systems forming a directed graph, that is, let¹

$$\dot{x}_i = u_i, \quad x_i \in \mathbb{R}, \quad i \leq N \quad (6a)$$

$$u_i = -\sum_{j=1}^N a_{ij}(x_i - x_j), \quad a_{ij} \geq 0. \quad (6b)$$

Assume that there does not exist self-cycles, that is, that $a_{ii} = 0$ for all $i \leq N$. Then, as it is customary to rewrite the closed-loop system in compact form, let us introduce the Laplacian matrix,

$$L := \begin{bmatrix} \sum_{j=2}^N a_{1j} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & & & -a_{2N} \\ \vdots & \ddots & & \vdots \\ -a_{N1} & \cdots & \sum_{j=1}^{N-1} a_{Nj} & \end{bmatrix}. \quad (7)$$

¹For simplicity in the notation we assume that $x_i \in \mathbb{R}$, but the same statements hold for multivariable systems.

Then, defining $x := [x_1 \cdots x_N]^\top$, the system (6) may be written as

$$\dot{x} = -Lx, \quad x \in \mathbb{R}^N. \quad (8)$$

For (the trajectories of) this system, consensus is typically defined in terms of attractivity of the manifold

$$\mathcal{S}_x := \{x_i = x_j, \quad \forall i \neq j\} \quad (9)$$

and, as it is well known, it is reached if and only if the graph contains a rooted spanning tree. The following statement provides a characterisation reminiscent of the *Lyapunov equation* of this connectivity condition.

Proposition 1 Let \mathcal{G} be a directed graph of order N and let $L \in \mathbb{R}^{N \times N}$ be its associated, non-symmetric, Laplacian matrix. The following statements are equivalent:

- 1) the graph \mathcal{G} contains a directed spanning tree;
- 2) for any matrix $Q \in \mathbb{R}^{N \times N}$, $Q = Q^\top > 0$, and any real $\alpha > 0$, there exists matrix $P(\alpha) \in \mathbb{R}^{N \times N}$, $P = P^\top > 0$ such that

$$PL + L^\top P = Q - \alpha[P\mathbf{1}v_\ell^\top + v_\ell\mathbf{1}^\top P], \quad (10)$$

where $v_\ell \in \mathbb{R}$ is the left eigenvector of L corresponding to the zero eigenvalue. \square

Proof: 1) \Rightarrow 2): By assumption, the graph \mathcal{G} contains a directed spanning tree. Then, from Lemma 1 it follows that L has a unique zero eigenvalue and all others have positive real parts. That is, Assumption 1 holds. Thus, the second statement follows directly invoking Lemma 2 with $v_r = \mathbf{1}_N$.

2) \Rightarrow 1): Since L is a Laplacian matrix it may have either equal-to-zero eigenvalues or eigenvalues with positive real part —see e.g. [21]. We prove the implication by *reductio ad absurdum*. Assume that statement 2) holds, but the matrix L has two equal-to-zero eigenvalues, so, in view of Lemma 1, statement 1) does not hold. As established in [22, Theorem 4], the null eigenvalue of L is always semisimple, *i.e.*, its algebraic and geometric multiplicities coincide. Therefore, the matrix L has two distinct pairs of (right and left) eigenvectors, $(v_{l1}, \mathbf{1})$ and (v_{l2}, v_{r2}) , corresponding to the equal-to-zero eigenvalues.

Now, the Jordan decomposition of the matrix L has the form $L = U \Lambda U^{-1}$, where

$$\Lambda := \begin{bmatrix} 0_{2 \times 2} & \cdot \\ \cdot & \Lambda_1 \end{bmatrix}, \quad \Lambda_1 \in \mathbb{R}^{N-2 \times N-2}$$

and, as in (3), the matrix U can be decomposed as

$$U = [\mathbf{1} \ v_{r2} \ U_1], \quad U^{-1} = \begin{bmatrix} v_{l1}^\top \\ v_{l1}^\top \\ U_1^\dagger \end{bmatrix}$$

where $U_1, U_1^\dagger \in \mathbb{R}^{N \times N-2}$ correspond to the last $N-2$ columns of U and U^{-1} .

Next, let us consider the matrix $R = L + \alpha \mathbf{1} v_\ell^\top$. Using the Jordan decomposition above, this matrix can be presented as

$$R := \begin{bmatrix} \alpha & 0 & \cdot \\ 0 & 0 & \cdot \\ \cdot & \cdot & \Lambda_1 \end{bmatrix}, \quad \Lambda_1 \in \mathbb{R}^{N-2 \times N-2}.$$

Clearly R is not positive definite since one of its eigenvalues is equal to zero. Therefore, there exists a matrix $Q = Q^\top$ for which there does not exist a matrix $P = P^\top$ such that $-PR - R^\top P = Q$. This, in view of the definition of the matrix R contradicts statement 2). ■

Next, using Proposition 1 we show how to construct a strict Lyapunov function for the problem of consensus involving first-order systems interconnected under a directed graph. Our Lyapunov function's domain is the space of synchronisation errors, that is, it is based on the fact that consensus may also be defined in terms of attractivity of the manifold

$$\mathcal{S} := \{e = 0\} \quad (11)$$

where

$$e := x - \mathbf{1} v_\ell^\top x, \quad (12)$$

$v_\ell \in \mathbb{R}^N$ and $\mathbf{1} \in \mathbb{R}^N$, $\mathbf{1} := [1 \ \dots \ 1]^\top$, denote, respectively, the left and right eigenvectors corresponding to the null eigenvalue of L , $\lambda_1(L) = 0$. Then, $v_\ell^\top x$ may be regarded as a weighted average of x and $e = 0$ if and only if $x_i = v_\ell^\top x$ for all $i \leq N$, which implies that $x \in \mathcal{S}$. The advantage of this characterisation of the consensus problem is that it may be recasted as a problem of stability analysis of the dynamical system —cf. [19],

$$\dot{e} = -Le. \quad (13)$$

Thus, relying on Proposition 1 and Lemma 2, our next statement provides a constructive proof (based on Lyapunov's direct method) of exponential stability of the manifold \mathcal{S} , which implies (but is not equivalent to) consensus.

Proposition 2 Consider the system (6) (equivalently system (12)-(13)). If the interconnection graph contains a rooted spanning tree then, for any $Q = Q^\top > 0$ there exists $P = P^\top > 0$ such that

$$V(e) = e^\top P e \quad \text{and} \quad \dot{V}(e) = -e^\top Q e. \quad (14)$$

Consequently, the manifold $\{e = 0\}$ is globally exponentially stable. □

Proof: Let $Q = Q^\top > 0$ and $\alpha > 0$ be arbitrarily fixed. Since by assumption the graph contains a rooted spanning tree, by Proposition 1, there exists $P = P^\top > 0$ such that (10) holds. Consider the function $V : \mathbb{R}^N \rightarrow \mathbb{R}_{\geq 0}$, $V(e) := e^\top P e$. Its total derivative along the trajectories of (13) yields, using (1),

$$\dot{V}(e) = -e^\top Q e + \alpha e^\top [P \mathbf{1} v_\ell^\top + v_\ell \mathbf{1}^\top P] e. \quad (15)$$

On the other hand, after (12), we have

$$P \mathbf{1} v_\ell^\top e = P \mathbf{1} v_\ell^\top [I - \mathbf{1} v_\ell^\top] x$$

$$= P [\mathbf{1} v_\ell^\top - \mathbf{1} v_\ell^\top \mathbf{1} v_\ell^\top] x$$

and, in view of the fact that $v_\ell^\top \mathbf{1} = 1$, the terms in the brackets above cancel out. Therefore, $P \mathbf{1} v_\ell^\top e = 0$ and, consequently, (15) is equivalent to

$$\dot{V}(e) = -e^\top Q e. \quad (16)$$

The result follows invoking classical Lyapunov theory. ■

Proposition 2 guarantees the existence of a Lyapunov function, via which exponential consensus may be established. As a matter of fact, such Lyapunov function may be constructed by solving a simple Lyapunov equation. More precisely, we have the following inverse statement.

Proposition 3 (converse) Given a Laplacian $L \in \mathbb{R}^{N \times N}$ corresponding to that of a graph containing a directed spanning tree and given $Q = Q^\top$ and $\alpha > 0$, define the matrix $P = P^\top > 0$ as the solution of

$$[\alpha \mathbf{1} v_\ell^\top + L]^\top P + P [\alpha \mathbf{1} v_\ell^\top + L] = Q. \quad (17)$$

Then, such P and Q satisfy (14). □

Proof: The proof follows directly using the fact that

$$[\alpha \mathbf{1} v_\ell^\top + L] =: -R(\alpha)$$

is Hurwitz and invoking the (well-known) statement that (for any fixed $\alpha > 0$ and) for any $Q = Q^\top > 0$, the matrix $-R(\alpha)$ is Hurwitz if and only if $P = P^\top > 0$ is a solution of

$$[-R(\alpha)]^\top P + P [-R(\alpha)] = -Q. \quad (18)$$

■

B. Second order systems

We consider now, a consensus protocol for second-order systems which the state of each agent converges to a common constant. Consider systems of the form

$$\dot{x}_i = y_i \quad x_i, y_i \in \mathbb{R} \quad (19a)$$

$$\dot{y}_i = u_i, \quad i \leq N \quad (19b)$$

in closed loop with the consensus control law

$$u_i = -\gamma_1 \sum_{j=1}^N a_{ij} (x_i - x_j) - \gamma_2 y_i, \quad a_{ij} \geq 0 \quad (20)$$

where $\gamma_1, \gamma_2 > 0$. Now, let $y := [x_1^\top \ \dots \ x_N^\top]^\top$, similarly for $u \in \mathbb{R}^N$. Then, in compact form, the dynamics of system (19) in closed loop with

$$u = -\gamma_1 L x - \gamma_2 y \quad (21)$$

yields

$$\dot{x} = y \quad (22a)$$

$$\dot{y} = -\gamma_1 L x - \gamma_2 y. \quad (22b)$$

Now, for the purpose of analysis, define the synchronisation errors

$$e_x := x - \mathbf{1} v_\ell^\top x \quad e_y := y - \mathbf{1} v_\ell^\top y, \quad (23)$$

which satisfy the dynamical equations

$$\dot{e}_x = e_y \quad (24a)$$

$$\dot{e}_y = -\gamma_1 L e_x - \gamma_2 e_y. \quad (24b)$$

Proposition 4 Consider the system (24) and let L be the Laplacian of a graph containing a directed spanning tree. Let P be generated by Proposition 3 with $Q = I_N$ and an arbitrary choice of $\mu > 0$. Let γ_1 and $\gamma_2 > 0$ be such that

$$M = \begin{bmatrix} \varepsilon\gamma_1 & I - \varepsilon\gamma_2 P - 2\mu\gamma_1 L^\top \\ I - \varepsilon\gamma_2 P - 2\mu\gamma_1 L & \mu\gamma_2 I - 2\varepsilon P \end{bmatrix} \geq 0 \quad (25)$$

for some ε satisfying

$$\varepsilon \leq \frac{\sqrt{2\mu}}{|P|}, \quad (26)$$

where $|P|$ is the induced norm of P . Then, the set $\{(e_x, e_y) = (0, 0)\}$ is globally exponentially stable. \square

Proof: Consider the Lyapunov function candidate

$$V(e) = \frac{1}{2}|e_x|^2 + \varepsilon e_x^\top P e_y + \mu|e_y|^2, \quad (27)$$

which is positive definite, under condition (26). The total time derivative of V along the trajectories of (24) yields

$$\begin{aligned} \dot{V} &= e_x^\top e_y + \varepsilon e_y^\top P e_y + \varepsilon e_x^\top P [-\gamma_1 L e_x - \gamma_2 e_y] \\ &\quad - \mu[\gamma_1 L e_x + \gamma_2 e_y]^\top e_y - \mu e_y^\top [\gamma_1 L e_x + \gamma_2 e_y] \\ &= e_x^\top P [I - \varepsilon\gamma_2 P - 2\mu\gamma_1 L^\top] e_y - e_y^\top [2\mu\gamma_2 - \varepsilon P] e_y \\ &\quad - \frac{1}{2}\varepsilon\gamma_1 e_x^\top [PL + L^\top P] e_x. \end{aligned}$$

Now, akin to the proof of Proposition 2, we use (1) with $Q = I_N$ and the identity $P\mathbf{1}v_\ell^\top e_x = 0$ to obtain

$$e_x^\top [PL + L^\top P] e_x = -|e_x|^2.$$

In turn, in view of (25), we obtain

$$\dot{V}(e) \leq -\frac{1}{2}[\varepsilon\gamma_1|e_x|^2 + \mu\gamma_2|e_y|^2] \quad (28)$$

and the result follows invoking classical Lyapunov theory. \blacksquare

Corollary 1 Consider the system (24a), (24b) and let L be the Laplacian of a graph containing a directed spanning tree. The set \mathcal{S} is globally exponentially stable for the system (24a), (24b) if

$$\gamma_2 > 2\sqrt{2\bar{\lambda}_P + \gamma_1^3 \bar{\lambda}_L^2},$$

where $\bar{\lambda}_P$ and $\bar{\lambda}_L$ are constants such that $|P| \leq \bar{\lambda}_P$ and $|L| \leq \bar{\lambda}_L$. \square

Proof: In the definition of matrix M let us choose $\mu = \sqrt{\frac{\gamma_1}{\gamma_2}}$ and $\varepsilon = \frac{1}{\gamma_2}$, with this particular choice of constants matrix M takes the form

$$M = \begin{bmatrix} \frac{\gamma_1}{2\gamma_2} I & -2\gamma_1 \sqrt{\frac{\gamma_1}{\gamma_2}} L^\top \\ -2\gamma_1 \sqrt{\frac{\gamma_1}{\gamma_2}} L & \frac{1}{2}\sqrt{\gamma_1\gamma_2} I - \frac{1}{\gamma_2} P \end{bmatrix}. \quad (29)$$

To compute the eigenvalues of this matrix we use the Schur complement, in particular we use the following:

Fact. A $[2 \times 2]$ -block symmetric matrix,

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix},$$

is positive definite if A is invertible and matrices A and $C - B^\top A^{-1} B$ are positive definite. \bullet

We apply this statement with $A = \frac{\gamma_1}{2\gamma_2} I$, which is invertible and positive definite. It is left to verify positive-definiteness of the matrix $C - B^\top A^{-1} B$. Using (29) we obtain

$$\begin{aligned} C - B^\top A^{-1} B &= \frac{1}{2}\sqrt{\gamma_1\gamma_2} I - \frac{1}{\gamma_2} P - 8\gamma_1^2 L L^\top \\ &=: D. \end{aligned}$$

The matrix D is positive definite if the quadratic form $x^\top D x > 0$ for all $x \neq 0$. We have

$$\begin{aligned} x^\top D x &= \frac{1}{2} \left(\sqrt{\gamma_1\gamma_2} I - \frac{2}{\gamma_2} P - 16\gamma_1^2 L L^\top \right) x \\ &\geq \frac{1}{2} (\sqrt{\gamma_1\gamma_2} - 2\bar{\lambda}_P/\gamma_2 - 16\lambda_L^2 \gamma_1^2) |x|^2. \end{aligned}$$

Since we assume that $\gamma_2 > 1$, then

$$x^\top D x \geq \frac{1}{2} (\sqrt{\gamma_1\gamma_2} - 2\bar{\lambda}_P - 16\lambda_L^2 \gamma_1^2) |x|^2,$$

this inequality holds if $\sqrt{\gamma_1\gamma_2} \geq 2\bar{\lambda}_P + 16\lambda_L^2 \gamma_1^2$ or equivalently $\gamma_2 \geq (2\bar{\lambda}_P + 16\lambda_L^2 \gamma_1^2)^2 / \gamma_1$. \blacksquare

V. CONCLUSIONS

We have presented an original generalisation of the Lyapunov-equation characterisation of the Hurwitz property of a matrix. Our statement holds for matrices with one eigenvalue equal to zero. One of the many applications of this statement is the construction of strict Lyapunov functions for multiagent linear first and second-order systems interconnected under a directed graph containing a spanning tree. We are confident that our main results establish a solid basis for the study of consensus under directed graphs, considering more complex scenarios as that studied here. For instance, in the case of systems with delays, general linear and nonlinear systems and, even, heterogeneous systems. Such topics are subject of current and future research.

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