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A multiscale second order model for the interaction between AV and traffic flows: analysis and existence of solutions

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Abstract

Autonomous vehicles (AV) offer new avenues for transportation applications and call for a new understanding of traffic dynamics when both regular cars and AV coexist. Many works looked at this question with a microscopic approach where both the AVs and the regular cars are modelled as ODEs. Here, present a second order model describing the interaction between a macroscopic traffic flow described by PDEs and autonomous vehicles (AV) described by ODEs. This model is inspired by recent development on moving bottlenecks for scalar conservation laws and is intrinsically multiscale. We give an analysis of this model and we show the existence of weak solutions with bounded variations.

Keywords:Traffic model, PDE-ODE systems, conservation laws, Autonomous Vehicle, weak solutions

AMS classification: 35L52; 35L70

Introduction

Traffic flows have been a natural subject of many mathematical studies for roughly as long as cars exist. This can be explained by the tremendous interest of understanding traffic dynamics, both from a theoretical and practical point of view. The interest of having a continuous macromodel compared to a micromodel comes from the curse of dimensionality and the unbearable computational weight when it comes to costly applications such as control design with a large number of cars. Many celebrated macromodels were derived in the last decades, as for instance the historical LWR [18, 21]. But traffic flow modeling has come to a new rise with the development of second order models. Among the most celebrated one can cite the Paye-Whitham model [20, 25] and the ARZ [2] model (see also its generalizations GARZ [10] and CGARZ [11]). As they consist of hyperbolic equations, the most natural framework to consider is entropic BV solutions. Since then, there have been hundreds of studies of these models that are relatively well understood. The advent of AVs in the last few years changed the story. It calls for a deep understanding of the interaction between AV and traffic flows, as AVs are soon to be a reality that cannot be ignored. This

question comes with a first modelling difficulty: how to look at the interaction between something intrinsically microscopic with something intrinsically macroscopic? While some studies exist on the effect of AV on traffic flows (see for instance [4] or [15]), an answer compatible with a macroscopic theory was given only recently in [5, 6] (see also [12]) motivated by the following observation: when a particular vehicle is in a traffic flow it has only two possibilities: it can either follow the flow or go slower and in this case it has a local effect on the flow. Therefore understanding the interaction between AVs and traffic flow is very much linked to understanding the effect of a moving bottleneck. While several works looked at the moving bottleneck problem in macromodels [13, 19] it is only recently that some studies looked directly at the effect of a moving bottleneck whose dynamic is given either by the flow or by the limiting speed that it imposes. In [5, 6, 8], Delle Monache and Goatin derived a flux condition that couples a moving bottleneck with a traffic flow model by a LWR equation. The resulting system consists of a PDE representing the bulk traffic coupled with an ODE representing the moving bottleneck where the state of the ODE acts on the PDE through the flux condition and the state of the PDE is involved in the ODE. In [12] the authors illustrated that this framework could adapt to the interaction of the AV and a first order model traffic flow. This system was proved to have BV solutions (coherent in some sense with the reality) [16] and then to be well-posed [17]. Its application to the control of traffic flow was later studied in [12]. Two things are worth to be noted: first, even though this model does not differentiate between lanes, it is intrinsically multilane since a moving bottleneck only reduces the flow but do not stop it. This means that the model takes into account the fact that the particles of the flow can get around the bottleneck, for instance by overtaking. Second, and this is one of the most surprising features of this class of model: the flux constraint can be in contradiction with the usual entropy condition. This means that entropic solutions are not the right framework anymore. Even though this may come as a surprise, it can easily be understood in practice: a single car can have a macroscopic effect on the system, for instance if a particular car decides to stop in the middle of the traffic flow it will create a wave with a low density of car propagating in front of it, and a wave with a large density of cars propagating behind it. This is a nonentropic shock. This also means that many of the classical tools that were developed for hyperbolic equations are useless here. Consequently analysing this system raised new and non-incremental difficulties.

Unfortunately first order models cannot grasp important physical phenomena as stop and go waves. Besides being interesting in themselves, stop and go waves (or jam) are paramount as they are at the origin of high fuel consumptions (and therefore CO2 emissions) in congested traffic. They occur because the uniform flow steady-state becomes unstable when the traffic is congested enough. Both theoretical and experimental studies suggests In recent years AVs have been seen as a new disruptive way of controlling traffic flow to dampen stop-and go waves with the hope of reducing by up to 40% the fuel consumption and CO2 emission in congested flows [23, 14]. It is therefore important to come up with a richer model capturing stop and go waves in order to understand the interaction between traffic flows and AVs. In this paper we propose a second-order ODE/PDE model consisting of a GARZ system coupled to a "Delle Monache Goatin-like" condition generalized and adapted to a second order model. Just like for the first order model, the correct solutions are not entropic anymore, leading to potential nonclassical shocks and the impossibility to apply the classical entropic theory. The main difficulties consist in dealing with the new family of wave (so-called second family [24]) arising from the second order model. In particular, trying to apply directly the strategy of the first-order model would fail as, even in the entropic parts of the solution, the density of cars can both discontinuously increase or decrease because of this new family of wavefronts. It is worth noting that in [24] the authors already looked at a similar second order model based on ARZ equations and managed to define and characterize precisely the Riemann problem and dealt with the numerical analysis. However, there did not address the

question of existence of solutions outside of the framework of the Riemann problem. The second order model we propose here can be seen as a generalization of their model and the analysis we carry could be directly applied to their model as well. Finally, another difficulty consists in defining the right notion of two dimensional weak-solution when the solution can be non entropic at the moving constraint. Concerning this last problem, one can mention the remarkable paper [1] where the authors consider a second-order model with a flux condition at a fixed boundary and where the same type of question appears. This paper is organized as follows: in Section 1, we describe the GARZ model coupled with a moving constraint, the different families of solutions, the properties of the model and the wave-front tracking approximation. In Section 2 we discuss the definition of a weak solution for this ODE/PDE system and we state the main result, namely the existence of a solution to the Cauchy Problem. Finally in Section 3, we prove this result in three parts: we first prove the convergence of the wave-front tracking approximation to a well defined BV state and an absolutely continuous function for the location of the moving bottleneck; then we show that this BV function is indeed a weak solution of the PDE; finally we show that the location of the moving bottleneck is a solution of the ODE.

1 GARZ model with a moving constraint

1.1 A strongly coupled PDE-ODE

This paper deals with the following strongly coupled PDE-ODE

$$\partial_t \rho + \partial_x \left(\rho\, V(\rho, w)\right) = 0, \tag{1.1a}$$

$$\partial_t (\rho w) + \partial_x \left(\rho w\, V(\rho, w)\right) = 0, \tag{1.1b}$$

$$\rho\left(t, y(t)\right)\left(V(\rho(t, y(t)), w(t, y(t))) - \dot{y}(t)\right) \leqslant F_\alpha(w(t, y(t)), \dot{y}(t)), \tag{1.1c}$$

$$\dot{y}(t) = \min\left(V_b, V(\rho(t, y(t)), w(t, y(t)))\right) \tag{1.1d}$$

$$\rho(0, \cdot) = \rho_0(\cdot) \text{ and } \dot{y}(0) = y_0 \tag{1.1e}$$

The PDE (1.1a)-(1.1b), introduced in [9], models the evolution of vehicular traffic and the ODE (1.1d) represents the trajectory of a slow moving vehicle. The ODE influences the PDE through the moving constraint on the flux (1.1d).

The function $\rho \in [0, \rho_{max}]$ is the traffic density, $w \in [w_{\min}, w_{\max}]$ is a particular feature of each driver and $V(\rho, w)$ is the speed of cars. For every driver with feature $w \in [w_{\min}, w_{\max}]$, $F_\alpha(w)$ is defined by

$$F_\alpha(w, \dot{y}) = \alpha(w) \max_{\rho \in (0, \rho_{\max})} F(\rho, w, \dot{y}),$$

with $\alpha \in C^1([w_{\min}, w_{\max}]; (0, 1))$ the reduction rate of the road capacity for a driver with feature w and where

$$F(\rho, w, \dot{y}) = \rho(V(\rho, w) - \dot{y}). \tag{1.2}$$

$F_\alpha(w, \dot{y})$ represents the maximum flux at the position of the slow moving vehicle.

Our goal is to prove the existence of solution of (1.1). To that end, we impose the following requirements on the velocity $V(\rho, w)$ and the associated flow rate function $f(\rho, w) = \rho V(\rho, w)$:

A1. $(p, w) \mapsto V(\rho, w)$ is $C^2\left([0, \rho_{max}] \times [w_{\min}, w_{\max}]\right)$.

A2. $V(\rho, w) \geqslant 0$ for any $(\rho, w) \in [0, \rho_{max}] \times [w_{\min}, w_{\max}]$, vehicles never drive backwards on the road.

A3. $V(0, w) = w$ for any $w \in [w_{\min}, w_{\max}]$, w is each driver's speed on an empty road.

A4. $\frac{\partial^2 Q}{\partial \rho^2}(\rho, w) < 0$ for any $(\rho, w) \in [0, \rho_{max}] \times [w_{\min}, w_{\max}]$. In particular, $\frac{\partial V}{\partial \rho}(\rho, w) < 0$.

A5. $\frac{\partial V}{\partial w}(\rho, w) > 0$, more a driver will be fast on a empty road more he will be fast on a stuffed road.

A6. $V(\rho_{\max}, w) = 0$, at maximal density $\rho_{\max}$, the speed of each driver is zero.

A7. $V_b < w_{\min}$, the maximum speed of the slow moving vehicle is slower than the minimum speed of each driver.

Throughout the paper we will use the following notations. Let $w \in [w_{\min}, w_{\max}]$, $\check{\rho}(w)$ and $\hat{\rho}(w)$ with $\check{\rho}(w) < \hat{\rho}(w)$ denote the two solutions of the equation $F_\alpha(w) + V_b\rho = \rho V(\rho, w)$ and $\rho^*(w)$ is the solution of $V_b\rho = \rho V(\rho, w)$. Since $f(\cdot, w)$ is stricly concave, $\check{\rho}(w)$, $\hat{\rho}(w)$ and $\rho^*(w)$ are well-defined. We denote by $\rho_c(w)$ the maximum point of $F(\rho, w)$. Thus, $\frac{\partial f(\rho_c(w), w)}{\partial \rho} = V_b$. Moreover, $V(\check{\rho}(w), w)$ is denoted by $\check{v}(w)$ and $V(\hat{\rho}(w), w)$ is denoted by $\hat{v}(w)$. We notice that $\check{\rho}_w < \rho_c(w) < \hat{\rho}_w < \rho^*_w$. The function $\sigma((\rho_l, w), (\rho_r, w)) := \frac{f(\rho_l, w) - f(\rho_r, w)}{\rho_- l \rho_r}$ represents the Rankine-Hugoniot speed of the 1-wave $((\rho_L, w), (\rho_r, w))$. For any BV function f, $f(x)$ will always refer to the right limit, meaning that $f(x)$ stands for $f(x+) = \lim_{y \to x,\ y > x} f(y)$. The left limit will be denoted $f(x-)$, given by $f(x-) := \lim_{y \to x,\ y < x} f(y)$.

1.2 The GARZ model

We recall some basic properties associated to the PDE (1.1a)-(1.1b). Away from the vacuum, the characteristic speeds of the GARZ model are $\lambda_1(\rho, w) = V(\rho, w) + \rho \frac{\partial V}{\partial \rho}(\rho, w)$ and $\lambda_2(\rho, w) = V(\rho, w)$. The first one is genuinely non linear and the second one is linearly degenerate. From $A.4$, $\lambda_1(\rho, w) < \lambda_2(\rho, w)$ for every $\rho > 0$. We list the waves that we will use for solving the Riemann problem associated to (1.1a) and (1.1b).

- *First family wave (1-wave)*: a wave connecting a left state (ρ_l, w_l) with a right state (ρ_r, w_r) such that $w_l = w_r$.
- *Second family wave (2-wave)*: a wave connecting a left state (ρ_l, w_l) with a right state (ρ_r, w_r) such that $V(\rho_l, w_l) = V(\rho_r, w_r)$.
- *Vacuum wave (V-wave)*: a wave connecting a left state (ρ_l, w_l) with a right state (ρ_r, w_r) such that $\rho_l = \rho_r = 0$ and $w_l < w_r$. Moreover, the speed of a V-wave s verifies $w_l < s < w_r$.

From *A5.* there exists an application $R : \{(v, w) \backslash\, 0 \leqslant v \leqslant w,\ w_{\min} \leqslant w \leqslant w_{\max}\} \to [0, \rho_{\max}]$ such that

$$\rho = R(v, w) \text{ with } V(\rho, w) = v. \tag{1.3}$$

We consider a Riemann problem associated to (1.1a) and (1.1b) with two states $U_l := (\rho_l, w_l)$ and $U_r = (\rho_r, w_r)$.

- If $w_l < w_r$ and $\rho_r < R(V(\rho_l, w_l), w_r)$ then U_l is connected to $U_{m_1} := (0, w_l)$ by a 1-wave, U_{m_1} is connected to $U_{m_2} := (0, V(U_r))$ by a V-wave and U_{m_2} is connected to U_r by a 2-wave.
- Otherwise, U_l is connected to $U_m := (R(V(\rho_r, w_r), w_l), w_l) \in [0, \rho_{max}] \times [w_{min}, w_{max}]$ by a 1-wave and U_m is connected to U_r by a 2-wave. Note that in the special case where $\rho_r = R(V(\rho_l, w_l), w_r)$, then U_l is directly connected to U_r by a 2-wave.

From now on, $\mathcal{RS}$ denotes the classical Riemann solver for the system of conservation laws in (1.1a)-(1.1b). Moreover, $\mathcal{RS}_\rho((\rho^l, w^l), (\rho^r, w^r)(\cdot))$, $\mathcal{RS}_w((\rho_l, w_l), (\rho_r, w_r)(\cdot))$ denotes respectively the ρ and w components of the classical solution $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(\cdot))$.

1.3 The constrained Riemann Problem

We consider the constrained Riemann problem for system (1.1) with initial data

$$\rho(0,x) = \begin{cases} \rho_l & \text{if } x < 0, \\ \rho_r & \text{if } x > 0, \end{cases} \quad \text{and} \quad w(0,x) = \begin{cases} w_l & \text{if } x < 0, \\ w_r & \text{if } x > 0, \end{cases} \quad \text{and} \quad y(0) = 0. \tag{1.4}$$

for all states $(\rho_l, w_l), (\rho_r, w_r) \in \{(v,w) \backslash 0 \leqslant v \leqslant w,\ w_{\min} \leqslant w \leqslant w_{\max}\}$. Fix $(\rho_l, w_l), (\rho_r, w_r) \in \{(v,w) \backslash 0 \leqslant v \leqslant w,\ w_{\min} \leqslant w \leqslant w_{\max}\}$. The constrained Riemann solver $\mathcal{RS}^\alpha$ for (1.1) is defined as follows.

i If $f(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(V_b)) > F_\alpha(w_l) + V_b \mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_r)(V_b))$, then

$$\begin{aligned} &\mathcal{RS}^\alpha\left((\rho_l, w_l), (\rho_r, w_r)\right)(x/t) = \begin{cases} \mathcal{RS}\left((\rho_l, w_l), (\hat{\rho}(w_l), w_l)\right)(x/t) & \text{if} \quad x < y(t), \\ \mathcal{RS}\left((\check{\rho}(w_l), w_l), (\rho_r, w_r)\right)(x/t) & \text{if} \quad x > y(t), \end{cases} \\ &y(t) = V_b t. \end{aligned} \tag{1.5}$$

ii If $f(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(V_b)) \leqslant F_\alpha(w_l) + V_b \mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_r)(V_b))$ and $V_b < V(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(V_b)))$, then

$$\begin{aligned} &\mathcal{RS}^\alpha\left((\rho_l, w_l), (\rho_r, w_r)\right)(x/t) = \mathcal{RS}\left((\rho_l, w_l), (\rho_r, w_r)\right)(x/t), \\ &y(t) = V_b t. \end{aligned} \tag{1.6}$$

iii If $V_b \geqslant V(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(V_b)))$, then

$$\begin{aligned} &\mathcal{RS}^\alpha\left((\rho_l, w_l), (\rho_r, w_r)\right)(x/t) = \mathcal{RS}\left((\rho_l, w_l), (\rho_r, w_r)\right)(x/t), \\ &y(t) = V(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)(V_b)))t. \end{aligned} \tag{1.7}$$

Note that in the first case, the traffic is influenced by the slow vehicle which travels with its own velocity. In the second case, the slow vehicle doesn't influence the traffic flow. The third case refers to a situation in which the traffic is so heavy that the slow vehicle has to adapt its speed.

Remark 1. *The function F_α is taken at $w = w_l$ in the Riemann solver to avoid the slow vehicle trajectory to interact with a 2-wave or a V-wave in front.*

1.4 Wave-Front Tracking Approximate Solution and TV type functional

The initial data ρ_0 and w_0 are approximated by piecewise constant functions denoted by ρ_0^n and w_0^n using an appropriate sampling (see [16, Section 2.2]). Solving the associated constrained Riemann problem for (1.1a)-(1.1b) with initial data $((\rho_0^n, w_0^n), y_0)$, the solution, denoted by $((\rho^n, w^n), y^n)$, can be prolonged until a first time t_1 is reached, when two wave-fronts interact. Note that the slow moving vehicle trajectory $y^n(\cdot)$ is regarded as a wavefront with speed given by (1.5), (1.6) or (1.7) depending on the value of (w^n, ρ^n) around y_0. We recall that, in the wave-front tracking method, the centered rarefaction waves are replaced by piecewise constant rarefaction fans. Thus, $\rho^n(t_1, \cdot)$ and $w^n(t_1, \cdot)$ are still piecewise constant functions, the corresponding Riemann problems with initial data $((\rho^n(t_1, \cdot), w^n(t_1, \cdot)), y^n(t_1))$ can again be approximately solved within the class of piecewise constant functions and so on. Thus, $y^n(\cdot)$ is the solution of

$$\begin{cases} \dot{y}(t) = \min\left(V_b, (\rho^n(t, y(t)), w^n(t, y(t)))\right), \\ y(0) = y_0, \end{cases} \qquad x \in \mathbb{R}. \tag{1.8}$$

where the couple $(\rho^n(t), w^n(t))$ is an approximate solution of (1.1a)-(1.1b).

2 The Cauchy Problem with Moving Constraints

In this section, we consider the Cauchy problem for the model (1.1) with initial data $(\rho_0, w_0) : \mathbb{R} \to [0, \rho_{\max}] \times [w_{\min}, w_{\max}]$ and $y_0 \in \mathbb{R}$. Before stating the main result, we introduce the definition of solution to the constrained Cauchy problem (1.1).

Definition 1. *The couple*

$$((\rho, w), y) \in C^0\left([0, +\infty[; L^1_{loc}(\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]\right) \times W^{1,1}_{loc}\left([0, +\infty[; \mathbb{R}\right)$$

is a solution to (1.1) if

i. the function (ρ, w) is a weak solution of (1.1a)-(1.1b), i.e for all $\varphi \in C^1_c(\mathbb{R}^2, \mathbb{R})$;

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \rho \left[\partial_t \varphi + V(\rho, w)\partial_x \varphi\right] \begin{pmatrix} 1 \\ w \end{pmatrix} dx\, dt + \int_{\mathbb{R}} \rho_0 \begin{pmatrix} 1 \\ w_0 \end{pmatrix} \varphi(0, x) = 0 \tag{2.1}$$

ii. The function ρ is an entropy admissible solution of (1.1a)-(1.1b), i.e for every $k \in [0, V(0, w_{\max})]$, for all $\varphi \in C^1_c(\mathbb{R}^2, \mathbb{R}_+)$, it holds

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathcal{E}_k(v(t,x), w(t,x))\partial_t \varphi + Q_k(v(t,x), w(t,x))\partial_x \varphi dx\, dt \\ &+ \int_{\mathbb{R}} \mathcal{E}_k(v_0, w_0)\varphi(0, x)\, dx \\ &+ \int_{\mathbb{R}_+} R(v(t, y(t)), w(t, y(t)))(v(t, y(t)) - \dot{y}) \left[\frac{k - \dot{y}}{F_\alpha(w(t, y(t)))} - 1\right]^+ \varphi(t, y(t)) dt \geqslant 0, \end{aligned} \tag{2.2}$$

where $v, w = (V(\rho, w), w)$ and

$$\mathcal{E}_k(v, w) = \begin{cases} 0 \text{ if } v \leqslant k \\ 1 - \frac{R(v,w)}{R(k,w)}, \quad \text{if } v > k, \end{cases} \tag{2.3}$$

$$\mathcal{Q}_k(v, w) = \begin{cases} 0 \text{ if } v \leqslant k \\ k - \frac{R(v,w)v}{R(k,w)}, \quad \text{if } v > k. \end{cases} \tag{2.4}$$

The entropy pairs $(\mathcal{E}_k, \mathcal{Q}_k)$ are taken from [1]. Here the term of the third line of (2.2) compensate for the potential non-classical shocks that would occur at $y(t)$. Note that all other non-classical shocks are prohibited with this condition. Besides, the second term of (2.2) also ensures that any solution maximize the flux when non-classical shock occurs, i.e. condition (1.1c) becomes an equality. This is similar to the result of [1, Section 3], and is shown in Appendix G.

iii. For a.e $t \in \mathbb{R}_+$, $\dot{y}(t) = \min\left(V_b, V(\rho(t, y(t)), w(t, y(t)))\right)$ or for every $t \in \mathbb{R}^+$,

$$y(t) = y_0 + \int_0^t \min\left(V_b, V(\rho(t, y(t)+), w(t, y(t)+))\right) ds\,. \tag{2.5}$$

iv. the constraint in (1.1c) is verified, in the sense that for a.e. $t \in \mathbb{R}^+$

$$\lim_{x \to y(t)\pm} \rho\,(t, x)\left(V(\rho(t, x), w(t, x)) - \dot{y}(t)\right) - F_\alpha(w, \dot{y}) \leqslant 0\,; \tag{2.6}$$

We can now state the main result of the paper:

Theorem 1. *We assume that $V_b < w_{\min}$. Let $y_0 \in \mathbb{R}$ and $(\rho_0, w_0) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}])$. Then the Cauchy problem (1.1) admits a solution $(\rho, w) \in \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}])$ in the sense of Definition 1.*

We construct a sequence of approximate solutions $((\rho^n, w^n), y^n)_{n\in\mathbb{N}}$ of (1.1) using a wave-front tracking algorithm, and we will prove its convergence. We deduce that the limit (ρ, w) satisfies Definition 1 i. ii. and iv. from the convergence of (ρ^n, w^n). We study the behavior of $(\rho^n, V(\rho^n, w^n))$ around the point $(t, y^n(t))$ in order to prove that the limit (ρ, w, y) verifies Definition 1 iii..

3 Proof of Theorem 1

Let $((\rho^n, w^n), y^n)$ be an approximate solution of (1.1) constructed by the wave-front tracking method described in Section 1.4. For a function $f \in C^0(\mathbb{R}_+, BV(\mathbb{R}))$ we will sometimes drop the notation in time and denote $f(x)$ instead of $f(t, x)$ to lighten the statements. The same will be done for continuous function and we will denote y^n instead of $y^n(t)$ and y instead of $y(t)$ when there is no ambiguity.

3.1 Convergence of the wave-front tracking approximate solution

Lemma 3.1. *Let $((\rho_0, w_0), y_0) \in \mathbf{BV}((\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]) \times \mathbb{R}$ and $((\rho^n, w^n), y^n)$ be a wave-front approximate solution of (1.1). Then, there exists $C > 0$ such that, for any $t \in \mathbb{R}_+$,*

$$TV(w^n(t, \cdot)) + TV(V(\rho^n(t, \cdot), w^n(t, \cdot)) \leqslant C. \tag{3.1}$$

To prove Lemma 3.1, we introduce the TV type functional $\Gamma(\cdot)$ defined in (3.4). $\Gamma(\cdot)$ may vary only at times t when two waves interact or a wave hits the slow vehicle trajectory. After a meticulous analysis of wave interactions, we will show that $\Gamma(t) \leqslant \Gamma(0)$ which gives bounds on the total variation of $w^n(t, \cdot)$ and $V(\rho^n(t, \cdot), w^n(t, \cdot))$.

Lemma 3.2. *$w \mapsto V(\check{\rho}(w), w)$ and $w \mapsto V(\hat{\rho}(w), w)$ are continuously differentiable functions.*

Proof. Since $(p, w) \mapsto V(\rho, w)$ is $C^2([0, \rho_{max}] \times [w_{\min}, w_{\max}])$, it is enough to prove that $w \mapsto \check{\rho}(w)$ and $w \mapsto \hat{\rho}(w)$ are continuously differentiable functions. Since $\alpha \in C^1([w_{\min}, w_{\max}]; (0, 1))$ and $(p, w) \mapsto V(\rho, w)$ is $C^2([0, \rho_{max}] \times [w_{\min}, w_{\max}])$, we have $\check{\rho}(w) < \rho_c(w) < \hat{\rho}(w)$ and $(\rho, w) \mapsto F_\alpha(w, \dot{y}) + V_b w - f(p, w) \in C^2([0, \rho_{max}] \times [w_{\min}, w_{\max}])$. Moreover, $V_b - \frac{\partial f}{\partial \rho}(\check{\rho}(w), w) < 0$ and $V_b - \frac{\partial f}{\partial \rho}(\hat{\rho}(w), w) > 0$. Thus, applying implicit function Theorem, we deduce that $w \mapsto \check{\rho}(w)$ and $w \mapsto \hat{\rho}(w)$ are at least continuously differentiable. □

We introduce the two positive constant $C_{\check{\rho}}$ and $C_{\hat{\rho}}$ defined by

$$C_{\check{\rho}} = \sup_{w \in [w_{\min}, w_{\max}]} \frac{d}{dw} V(\check{\rho}(w), w) > 0, \tag{3.2}$$

$$C_{\hat{\rho}} = \sup_{w \in [w_{\min}, w_{\max}]} \frac{d}{dw} V(\hat{\rho}(w), w) > 0. \tag{3.3}$$

Using Lemma 3.2, $C_{\check{\rho}} < \infty$ and $C_{\hat{\rho}} < \infty$. Let $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$. For a.e. $t > 0$, we define the TV type functional

$$\Gamma(t) = TV(w^n(t, \cdot)) + TV(V(\rho^n(t, \cdot); w^n(t, \cdot))) + \gamma(t) + CTV(w^n(\cdot, y^n(\cdot)), [t, +\infty)), \tag{3.4}$$

where γ is given by

$$\gamma(t) = \begin{cases} -2|\hat{v}(w^n(t,y(t)-)) - \check{v}(w^n(t,y(t))),| & \text{if } \begin{cases} w^n(t,y_n(t)-) = w^n(t,y_n(t)), \\ \rho^n(t,y_n(t)-) = \hat{\rho}(w^n(t,y(t)-)), \\ \rho^n(t,y_n(t)) = \check{\rho}(w^n(t,y(t)-)), \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

The term γ is added to control the behavior of $TV(\rho, w)$ when a 1-wave interacts with slow vehicle trajectory creating or canceling a non classical shock as in [7]. We introduce the term $TV(w^n(\cdot, y^n(\cdot)), [t, +\infty))$ to deal with the cases where a 2-wave or a V-wave interact with the slow vehicle trajectory creating or canceling a non classical shock.

Lemma 3.3. *Let* $((\rho_0, w_0), y_0) \in \mathbf{BV}((\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]) \times \mathbb{R}$ *and* $((\rho^n, w^n), y^n)$ *be a wave-front approximate solution of* (1.1)*. Then, there exists* $C > 0$ *such that, for any* $t \in \mathbb{R}_+$*,*

$$\Gamma(t) \leqslant \Gamma(0) \tag{3.5}$$

The proof of Lemma 3.3 is postponed in Appendix A.

Lemma 3.4. *The total variation* $t \mapsto TV(w^n(t))$ *is constant. That is to say,* $TV(w^n(t)) = TV(w_0^n)$*.*

Proof. Considering a Riemann problem at time t associated to (1.1a) and (1.1b) with two states $U_l := (\rho_l, w_l)$ and $U_r = (\rho_r, w_r)$:

- If $w_l < w_r$ and $\rho_r < R(w_l, w_r)$ then U_l is connected to $U_{m_1} := (0, w_l)$ by a 1-wave, U_{m_1} is connected to $U_{m_2} := (0, V(U_r))$ by a V-wave and U_{m_2} is connected to U_r by a 2-wave (see Section 1). In this case, $TV(w(t^+)) - TV(w(t^-)) = w_r - V(U_r) + V(U_r) - w_l - w_r + w_l = 0$.
- Otherwise, U_l is connected to $U_m := (R(V(\rho_r, w_r), w_l), w_l) \in [0, \rho_{max}] \times [w_{min}, w_{max}]$ by a 1-wave and U_m is connected to U_r by a 2-wave. In this case, we have immediately $TV(w(t^+)) - TV(w(t^-)) = 0$.

Moreover, a non classical shock is a wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$.Thus, the variation of w is not modified. We conclude that the total variation $t \mapsto TV(w(t))$ is constant. □

Lemma 3.5. *For every* $t \in \mathbb{R}_+^*$*,*

$$TV(w^n(\cdot, y^n(\cdot)), [t, +\infty)) \leqslant TV(w_0^n)$$

Proof. The quantity $w^n(\cdot, y^n(\cdot))$ is modified only when the slow vehicle trajectory interacts with a V-wave or a 2-wave. From Lemma 3.4, a interaction between two waves or a wave and the slow vehicle trajectory does not modify the total variation of w. We claim that the slow vehicle trajectory never interacts with a second family wave and a vacuum wave from the left.

- We assume that a second family wave $((\rho_l, w_l), (\rho_r, w_r))$ interacts with the slow vehicle trajectory from the right. The speed of the second family wave is $V(\rho_l, w_l) = V(\rho_r, w_r)$ and the speed of the slow vehicle is $\min(V_b, V(\rho_l, w_l))$. We have $\min(V_b, V(\rho_l, w_l)) \leqslant V(\rho_l, w_l)$. Thus, the speed of the slow vehicle is slower than the speed of a second family wave, whence the contradiction.
- We assume that a vacuum wave $((0, w_l), (0, w_r))$ interacts with the slow vehicle trajectory from the right. The speed of the vacuum wave s verifying $w_l < s < w_r$ and the maximum speed of the slow vehicle is V_b. Since, $V_b < w_{\min}$ the speed of the slow vehicle is slower than the speed of a vacuum wave, whence the contradiction.

Thus, the slow vehicle trajectory interacts at most once with a V-wave or 2-wave. We conclude that $TV(w^n(\cdot, y^n(\cdot)), [0, +\infty)) \leqslant TV(w_0^n(\cdot), (-\infty, y^0])$. More precisely, for every $t \in \mathbb{R}_+^*$, there exists $x \in (-\infty, y_0)$ such that

$$TV(w^n(\cdot, y^n(\cdot)), [t, +\infty)) \leqslant TV(w_0^n(\cdot), (-\infty, x]) \leqslant TV(w_0^n)$$

□

Proof of Lemma 3.1. From Lemma 3.3, we have

$$\begin{aligned} TV(w^n(t,\cdot)) + TV(v(\rho^n(t,\cdot), w^n(t,\cdot))) \\ &\leqslant TV(w_0^n) + TV(v(\rho_0^n, w_0^n)) + \gamma(0) - \gamma(t) + CTV(w^n(\cdot, y^n(\cdot)), [0,t)), \\ &\leqslant TV(w_0^n) + TV(v(\rho_0^n, w_0^n)) + 2w_{\max} + CTV(w^n(\cdot, y^n(\cdot)), [0,t)). \end{aligned}$$

Since $(w_0^n, v_0^n) \in \mathbf{BV}((\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}])$ and from Lemma 3.5, there exists $C > 0$ such that

$$TV(w^n(t,\cdot)) + TV(V(\rho^n, w^n)(t,\cdot)) \leqslant C.$$

We are now ready to prove the convergence of the sequence of approximate solutions $((\rho^n, w^n), y^n)$.

Lemma 3.6. *Let $((\rho^n, w^n), y^n)$ be an approximate solution of (1.1) constructed by the wave-front tracking method described in Section 1.4. Then, up to a subsequence, we have the following convergences*

$$(\rho^n, w^n) \to (\rho, w), \qquad \text{in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]), \tag{3.6a}$$

$$y^n(\cdot) \to y(\cdot), \qquad \text{in } L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}), \tag{3.6b}$$

$$\dot{y}^n(\cdot) \to \dot{y}(\cdot), \qquad \text{in } L^1_{loc}(\mathbb{R}_+; \mathbb{R}), \tag{3.6c}$$

for some $(\rho, w) \in C^0(\mathbb{R}_+; L^1(\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]))$ and $y \in W^{1,1}_{loc}(\mathbb{R}_+; \mathbb{R})$ with Lipschitz constant V_b. Moreover, there exists $C > 0$ such that $TV(\rho(t,\cdot)) < C$ and $TV(w(t,\cdot)) < C$ for all $t \geqslant 0$.

Proof. Let $(\rho_1, \rho_2) \in [0, \rho_{\max}]$ such that $\rho_1 = R(V(\rho_1, w_1), w_1)$ and $\rho_2 = R(V(\rho_2, w_2), w_2)$ where $R \in C^2([0, w_{\max}] \times [w_{\min}, w_{\max}])$ is defined in (1.3). Then there exists $C > 0$ such that

$$|\rho_1 - \rho_2| \leqslant C\left(|V(\rho_1, w_1) - V(\rho_2, w_2)| + |w_1 - w_2|\right). \tag{3.7}$$

From (3.7), we deduce that

$$TV(\rho^n(t,\cdot)) \leqslant C\left(TV(\rho^n(t,\cdot), w^n(t,\cdot)) + TV(w^n(t,\cdot)\right). \tag{3.8}$$

From (3.1) and (3.8), there exists $C > 0$ (independent of n) such that, for every $t \geqslant 0$,

$$TV(w^n(t,\cdot)) < C \text{ and } TV(\rho^n(t,\cdot)) < C. \tag{3.9}$$

Combining (3.9) with the finite wave speed propagation, we show that

$$\int_{\mathbb{R}} |\rho^n(t,x) - \rho^n(s,x)| dx \leqslant L|t-s| \quad \text{for all } t, s \geqslant 0,$$

and

$$\int_{\mathbb{R}} |w^n(t,x) - w^n(s,x)| dx \leqslant L|t-s| \quad \text{for all } t, s \geqslant 0,$$

for some L independent of n. Helly's Theorem, see [3, Theorem 2.4], implies the existence of $(\rho, w) \in C^0\left(\mathbb{R}_+; L^1\left(\mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}]\right)\right)$ and a subsequence of $(\rho^n, w^n)_n$, which for simplicity we denote again by $(\rho^n, w^n)_n$. This implies that (3.6a) holds and there exists $C > 0$ such that $TV(\rho(t,\cdot)) < C$ and $TV(w(t,\cdot)) < C$ for all $t \geqslant 0$.

Fix $T > 0$. From (1.8), we deduce that

$$0 \leq \dot{y}^n(t) \leq V_b \tag{3.10}$$

for a.e. $t > 0$ and $n \in \mathbb{N} \setminus \{0\}$. Using Ascoli Theorem [22, Theorem 7.25], there exists a function $y \in C^0\left([0,T]; \mathbb{R}\right)$ and a subsequence of $(y^n)_n$, which for simplicity we denote again by $(y_n)_n$, such that y^n converges to y uniformly in $C^0\left([0,T]; \mathbb{R}\right)$. By the arbitrariness of T, (3.6b) holds. Moreover y is a Lipschitz continuous function with V_b as a Lipschitz constant.

To prove that $\dot{y}^n(\cdot) \to \dot{y}(\cdot)$ in $L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$, we show that $TV(\dot{y}^n; [0,T])$ is uniformly bounded for any $T \geqslant 0$. Since $\|\dot{y}^n\|_{L^\infty(0,T)} \leqslant V_b$, it is sufficient to estimate the positive variation of $\dot{y}^n$ over $[0,T]$, denoted by $PV(\dot{y}^n; [0,T])$. More precisely, we have

$$TV(\dot{y}^n; [0,T]) \leqslant 2PV(\dot{y}^n; [0,T]) + \|\dot{y}^n\|_{L^\infty}. \tag{3.11}$$

Since $\dot{y}^n$ satisfies (1.8), the speed of the MB cannot increase by interactions with a second family waves. Thus, from [16], the speed of the MB is increased only by interactions with rarefaction waves coming from the right of the MB trajectory. Since all rarefaction shocks start at $t = 0$, we have $PV(\dot{y}^n; [0,T]) \leqslant TV(\rho_0)$. From (3.11), we deduce that

$$TV(\dot{y}^n; [0,T]) \leqslant 2TV(\rho_0) + V_b,$$

whence (3.6c) by arbitrariness of T. □

3.2 The limit (ρ, w, y) satisfies Definition 1 i. ii. and iv.

Let us start by proving Definition i. and ii.. As $(\rho^n, w^n) \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is a weak solution of (1.1a)-(1.1b), it satisfies Definition 1. This implies that for any test function $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \rho^n \left[\partial_t \varphi + V(\rho^n, w^n) \partial_x \varphi\right] \begin{pmatrix} 1 \\ w^n \end{pmatrix} dx\, dt + \int_{\mathbb{R}} \rho_0^n \varphi(0,x) \begin{pmatrix} 1 \\ w_0^n \end{pmatrix} dx = 0. \tag{3.12}$$

Using the convergence proved in the previous section, and Lemma 3.6 we know that $(\rho^n, w^n) \to (\rho, w)$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}; [0, \rho_{\max}] \times [w_{\min}, w_{\max}])$ and therefore we can pass to the limit in (3.12) and get Definition i.. We get Definition ii. similarly as one can define $v^n = v(\rho^n, w^n)$ and therefore $v^n \to v$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}; [0, V(0, w_{\max})] \times [w_{\min}, w_{\max}])$. Noting that (ρ^n, w^n) satisfies for any $k \in [0, V(w_{\max})]$ and $\varphi \in C_c(\mathbb{R}^2, \mathbb{R}_+)$,

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathcal{E}_k(v^n(t,x), w^n(t,x)) \partial_t \varphi + Q_k(v^n(t,x), w^n(t,x)) \partial_x \varphi dx\, dt \\ &+ \int_{\mathbb{R}} \mathcal{E}_k(v_0^n, w_0^n) \varphi(0,x) dx\, dt \\ &+ \int_{\mathbb{R}_+} R(v^n, w^n)(v^n - \dot{y}^n) \left[\frac{k - \dot{y}^n(t)}{F_\alpha(w^n, \dot{y}^n)} - 1\right]^+ \varphi(t, y^n(t)) dt \geqslant 0, \end{aligned} \tag{3.13}$$

we therefore get

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathcal{E}_k(v(t,x),w(t,x))\partial_t\varphi + Q_k(v(t,x),w(t,x))\partial_x\varphi dx\, dt \\
&+\int_{\mathbb{R}} \mathcal{E}_k(v_0,w_0)\varphi(0,x)dx\, dt \\
&+\int_{\mathbb{R}_+} R(v,w)(v-\dot{y})\left[\frac{k-\dot{y}^n(t)}{F_\alpha(w,\dot{y})}-1\right]^+ \varphi(t,y(t))dt \geqslant 0,
\end{aligned} \tag{3.14}$$

The convergence of the last line being obtained by dominated convergence together with the convergence of $R(v^n,w^n)(v^n-\dot{y}^n)\ F_\alpha(w^n,\dot{y}^n)$ that is given in Appendix F. We now show that (ρ,y) satisfies Definition iv.. Let $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$, we denote

$$\begin{aligned}
\rho(t_0,x_0+) &= \lim_{x\to x_0, x>x_0}\rho(t_0,x), \\
\rho(t_0,x_0-) &= \lim_{x\to x_0, x<x_0}\rho(t_0,x),
\end{aligned} \tag{3.15}$$

which exists given that $\rho\in C^0([0,T],L^1_{loc}(\mathbb{R},[0,\rho_{\max}]))$. We define similarly $w(t_0,x_0+)$ and $w(t_0,x_0-)$ and the analogous quantities for (ρ^n,w^n). The idea will be to recover Definition iv. by using (ii.) with appropriate test functions. Let $\psi\in C_c^\infty((0,T)\times\mathbb{R},\mathbb{R}_+)$. As ψ has compact support in its space variable, there exists $a\in\mathbb{R}$ such that for any $(t,x)\in\mathrm{supp}(\psi)$, $x>a$. This implies that there exists $\varepsilon_0>0$ for any $x\leqslant a+2\varepsilon_0$ and any $t\in(0,T)$, $\psi(t,x)=0$. We now set $\varepsilon\in(0,\varepsilon_0)$. We define $\phi_{1,\varepsilon}\in C_c^1((0,T),\mathbb{R}_+)$ such that $\phi_{1,\varepsilon}\equiv 1$ on $[2\varepsilon,T-2\varepsilon]$. This function $\phi_{1,\varepsilon}$ is an approximation of the constant function equal to 1. Similarly we define $\phi^n_{2,\varepsilon}\in C_c^1((0,T)\times(-\infty,y^n(t)))$ such that

$$\phi^n_{2,\varepsilon} = \begin{cases} 0 \text{ if } x\leqslant a \\ 1 \text{ if } x\in(a+\varepsilon,y^n(t)-2\varepsilon) \\ \frac{y^n(t)-x}{\varepsilon}-1 \text{ if } x\in(y^n(t)-2\varepsilon,y^n(t)-\varepsilon) \\ 0 \text{ if } x\geqslant y^n(t)-\varepsilon. \end{cases} \tag{3.16}$$

Therefore $\phi^n_{2,\varepsilon}$ is an approximation of a stair function equal to 1 before $y^n(t)$ and 0 after. Setting the test function $\varphi=\psi\phi^n_{2,\varepsilon}$ and applying Definition i., one has

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho^n\left[\partial_t\psi\phi^n_{2,\varepsilon}+\psi\frac{\dot{y}^n}{\varepsilon}(1_{x\in(y^n(t)-2\varepsilon,y^n(t)-\varepsilon)})\right. \\
&\left.+V(\rho^n,w^n)\phi^n_{2,\varepsilon}\partial_x\psi + V(\rho^n,w^n)\psi\left(\frac{-1}{\varepsilon}\right)1_{x\in(y^n(t)-2\varepsilon,y^n(t)-\varepsilon)}\right]\begin{pmatrix}1\\ w^n\end{pmatrix}dx\, dt = 0,
\end{aligned} \tag{3.17}$$

where we used that $\psi(t,x)=0$ for $t=0$ or $x\in(-\infty,a+2\varepsilon)$. This gives

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho^n\left[\partial_t\psi\phi^n_{2,\varepsilon}+V(\rho^n,w^n)\phi^n_{2,\varepsilon}\partial_x\psi\right]\begin{pmatrix}1\\ w^n\end{pmatrix}dx\, dt \\
&=\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho^n\left[V(\rho^n,w^n)-\dot{y}^n\right]\psi(\frac{1}{\varepsilon}1_{x\in(y^n(t)-2\varepsilon,y^n(t)-\varepsilon)})\begin{pmatrix}1\\ w^n\end{pmatrix}dx\, dt.
\end{aligned} \tag{3.18}$$

As ρ^n, w^n, $\dot{y}$ are all L^1_{loc} functions and ψ is C^∞ with compact support we can use Dominated Convergence Theorem and let $\varepsilon \to 0$

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{-\infty}^{y^n(t)} \rho^n \left[\partial_t \psi + V(\rho^n, w^n)\partial_x \psi\right] \begin{pmatrix} 1 \\ w^n \end{pmatrix} dx\, dt \\
&= \int_0^T \rho^n(t, y^n(t)-) \left[V(\rho^n(t, y^n(t)-), w(t, y^n(t)-)) - \dot{y}^n(t)\right] \psi(t, y^n(t)) \begin{pmatrix} 1 \\ w^n(t, y^n(t)-) \end{pmatrix} dx\, dt.
\end{aligned} \tag{3.19}$$

The same can be done with (ρ, w, y) instead of (ρ^n, w^n, y^n) by defining $\phi_{2,\varepsilon}$ just as $\phi^n_{2,\varepsilon}$ but with $y(t)$ instead of y^n and one has

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{-\infty}^{y(t)} \rho \left[\partial_t \psi + V(\rho, w)\partial_x \psi\right] \begin{pmatrix} 1 \\ w \end{pmatrix} dx\, dt \\
&= \int_0^T \rho(t, y(t)-) \left[V(\rho(t, y(t)-), w(t, y(t)-)) - \dot{y}(t)\right] \psi(t, y(t)) \begin{pmatrix} 1 \\ w(t, y(t)-) \end{pmatrix} dx\, dt.
\end{aligned} \tag{3.20}$$

As (ρ^n, w^n, y^n) satisfies Definition iv. and ψ only takes values in $\mathbb{R}_+$, one has, taking the first component of (3.19)

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{-\infty}^{y^n(t)} \rho^n \left[\partial_t \psi + V(\rho^n, w^n)\partial_x \psi\right] dx\, dt \\
&\leqslant \int_0^T F_\alpha(\omega^n(t, y^n(t)), \dot{y}^n)\psi(t, y^n(t)) dx\, dt.
\end{aligned} \tag{3.21}$$

Note that, from Lemma 3.6, $\dot{y}^n \to \dot{y} \in L^1_{loc}(0, T, \mathbb{R})$ and there exists $C > 0$ independent of n such that

$$TV(w^n) < C \tag{3.22}$$

which implies that again using dominated convergence and the convergence of $F_\alpha(w^n, \dot{y}^n)$ Therefore one can use dominated convergence again and get

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\mathbb{R}_+} \int_{-\infty}^{y^n(t)} \rho^n \left[\partial_t \psi + V(\rho^n, w^n)\partial_x \psi\right] dx\, dt = \int_{\mathbb{R}_+} \int_{-\infty}^{y(t)} \rho \left[\partial_t \psi + V(\rho, w)\partial_x \psi\right] dx\, dt \\
&\lim_{n\to+\infty} \int_0^T F_\alpha(\omega^n(t, y^n(t)-), \dot{y}^n)\psi(t, y^n(t)) dx\, dt = \int_0^T F_\alpha(\omega(t, y(t)-), \dot{y})\psi(t, y(t)) dx\, dt.
\end{aligned} \tag{3.23}$$

Using (3.20) and (3.21) this means that

$$\begin{aligned}
&\int_0^T \rho(t, y(t)-) \left[V(\rho(t, y(t)-), w(t, y(t)-)) - \dot{y}(t)\right] \psi(t, y(t) dx\, dt \\
&\leqslant \int_0^T F_\alpha(\omega(t, y(t)-), \dot{y})\psi(t, y(t)) dx\, dt.
\end{aligned} \tag{3.24}$$

One can do exactly similarly to get

$$\begin{aligned}
&\int_0^T \rho(t, y(t)+) \left[V(\rho(t, y(t)+), w(t, y(t)+)) - \dot{y}(t)\right] \psi(t, y(t)) dx\, dt \\
&\leqslant \int_0^T F_\alpha(\omega(t, y(t)+), \dot{y})\psi(t, y(t)) dx\, dt.
\end{aligned} \tag{3.25}$$

As ψ was an arbitrary function with compact support in $(0, T) \times \mathbb{R}$ one has exactly (2.6) and (ρ, w, y) satisfies Definition iv..

3.3 The limit (ρ, y) satisfies Definition iii.

We begin with some preliminary Lemmas.

Lemma 3.7. *Let $t \in \mathbb{R}_+ \backslash \{0\}$ being fixed and $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 < x_2$, suppose that no V-wave occurs in (x_1, x_2) and that there exists $c > 0$ (independent of n) such that $V(\rho^n(x_1), w^n(x_1)) + c \leqslant V(\rho^n(x_2-), w^n(x_2-))$. Suppose in addition that there is no non-classical shock occurring in (x_1, x_2). Then there exists $\beta > 0$ such that, for n large enough,*

$$|x_2 - x_1| \geqslant \beta t |V(\rho^n(x_2-), w^n(x_2-)) - V(\rho^n(x_1), w^n(x_1)))| \tag{3.26}$$

where β is a constant independent of t, x_1 and x_2.

This lemma gives a minimal distance of travel between two states of a wave-front tracking solution with different velocity when no V-waves and no non-classical shock occurs. First, a shock can only increase the velocity and a 2-wave cannot change it. Since by assumption no V-wave occur in (x_1, x_2), we deduce that the only possible waves to increase the velocity are rarefaction shocks and this imply a minimal distance. This lemma will be the basic tool for all the following analysis.

Proof. To prove this lemma, we first prove that if $x_1 < x_2$ and $V(\rho^n(x_1), w^n(x_1)) + c < V(\rho^n(x_2-), w^n(x_2-))$ for any n, there is a minimal amplitude of variation of ρ^n through rarefactions shocks. Then we conclude using Lemma 3.8. Obviously 2-waves cannot change the velocity so the only waves that can occur to change the velocity are shocks and rarefaction shocks. As there is a finite number of discontinuities we can define $K_n \in \mathbb{N}$ and $x_1 =: z_0 < z_1 < ... < z_{K_n-1} < z_{K_n} := x_2$ such that there are only rarefaction shocks between (z_k, z_{k+1}) for any $k \in \{0, ..., K_n - 1\}$. In other words z_k are the points where a shock or a 2-wave occurs, and this implies that $w^n(z_k) = w^n(z_{k+1}-)$ for any $k \in \{0, ..., K_n - 1\}$. Note that when a shock occurs in z_k, $w^n(z_k-) = w^n(z_k)$ and $\rho^n(z_k-) < \rho^n(z_k)$ which implies that $V(\rho^n(z_k-), w^n(z_{k-})) > V(\rho^n(z_k), w^n(z_k))$, as V is strictly decreasing. So shocks have only a negative influence when trying to reach $V(\rho^n(x_2-), w^n(x_2-))$ from $V(\rho^n(x_1), w^n(x_1))$. As 2-waves do not change the velocity, for any $k \in \{1, ..., K_n - 1\}$,

$$V(\rho^n(z_k-), w^n(z_k-)) \geqslant V(\rho^n(z_k), w^n(z_k)). \tag{3.27}$$

Besides, as $V(\cdot, w)$ is $C^1([0, \rho_{\max}])$ with $w \in [w_{\min}, w_{\max}]$ and $w^n(z_k) = w^n(z_{k+1}-)$ there exists a constant $c_0 > 0$ independent of n, ρ^n or w^n, such that

$$0 \leqslant V(\rho^n(z_{k+1}-), w^n(z_{k+1}-)) - V(\rho^n(z_k), w^n(z_k)) \leqslant c_0(\rho^n(z_{k+1}-) - \rho^n(z_k)). \tag{3.28}$$

Now, we use the following lemma

Lemma 3.8. *Let $t \in \mathbb{R}_+ \backslash \{0\}$ and $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 < x_2$, suppose that w^n is constant between (x_1, x_2) and there exists $c > 0$ (independent of n) such that $\rho^n(x_1) \geqslant \rho^n(x_2-) + c$. Suppose in addition that there is no non-classical shock occurring in (x_1, x_2). Then there exists $\beta' > 0$ such that*

$$|x_2 - x_1| > \beta' t |\rho^n(x_1) - \rho^n(x_2-)| \tag{3.29}$$

where β' is a constant independent of t, x_1 and x_2.

The proof of this Lemma fosters directly [16, Lemma 1] by observing that on (x_1, x_2), ρ^n is a wave-front tracking solution of a LWR model with flux f given by $f(\rho, v(\rho)) = \rho V(\rho, w)$, as w is constant, there is no V-wave or 2-wave and for n large enough $\rho^n(x_1) - \rho^n(x_2) - \rho_{\max} 2^{-n+1} \geqslant \frac{\rho^n(x_1) - \rho^n(x_2)}{2}$. ∎

Using that $w(z_k) = w(z_{k+1}-)$ and applying (3.28) becomes :

$$\beta' t(V(\rho^n(z_{k+1}-), w^n(z_{k+1}-)) - V(\rho^n(z_k), w^n(z_k))) \leqslant c_0|z_{k+1} - z_k|. \tag{3.30}$$

Thus overall, using (3.27) and (3.28),

$$\beta' t(V(\rho^n(x_2-), w^n(x_2-)) - V(\rho^n(x_1), w^n(x_1))) \leqslant c_0 \sum_{k=0}^{K_n-1} |z_{k+1} - z_k| = c_0|x_2 - x_1|, \tag{3.31}$$

and setting $\beta = \beta'/c_0 > 0$ ends the proof of Lemma 3.7. □

We now introduce a Lemma that gives the maximal amplitude of the variation of the velocity through a rarefaction shock

Lemma 3.9. *Let (ρ^n, w^n) be a solution constructed by a wave-front tracking algorithm. Suppose that a rarefaction shocks occur in $x_1 \in \mathbb{R}$, then there exists a constant C independent of x_1, n, ρ^n, w^n and depending only on V such that*

$$V(\rho^n(x_1), w^n(x_1)) - V(\rho^n(x_1-), w^n(x_1-)) \leqslant \frac{C_0\rho_{\max}}{2^n} \tag{3.32}$$

Proof. The proof is straightforward : if there is a rarefaction shock only ρ^n is discontinuous through the shock and by definition $(\rho^n(x_1) - \rho^n(x_1-)) \leqslant \rho_{\max}/2^n$. Then the result holds with $C_0 := \max_{(\rho,w)\in[0,\rho_{\max}]\times[w_{\min},w_{\max}]}(|\partial_\rho V|)$. □

We are now ready to prove that the limit (ρ, y) satisfies Definition iii.. First, as (ρ^n, w^n) are approximate solutions of the problem (1.1) and from Lemma 3.6, we can define $\mathcal{N}_0$ a negligible space such that for any $t \in \mathbb{R} \setminus \mathcal{N}_0$,

- $\lim_{n\to+\infty}(\rho^n(t,x), w^n(t,x)) = (\rho(t,x), w(t,x))$ for almost every $x \in \mathbb{R}$.
- $s \to y(s)$ is a differentiable function at time $s = t$,
- $\lim_{n\to+\infty} y^n(t) = y(t)$,
- $\dot{y}^n = \min(V_b, V(\rho^n(t, y^n(t)), w^n(t, y^n(t))))$ for any $n \in \mathbb{N}$.

More precisely, it suffices to show that

$$\lim_{n\to+\infty} \min(V_b, V(\rho^n(t, y^n(t)), w^n(t, y^n(t)))) = \min(V_b, V(\rho(t, y(t)), w(t, y(t)))). \tag{3.33}$$

We define in the following $\rho^+ := \lim_{x\to y(t)^+} \rho(t,x)$, $\rho^- := \lim_{x\to y(t)^-} \rho(t,x)$, and similarly $w^+ := \lim_{x\to y(t)^+} w(t,x)$ and $w^- := \lim_{x\to y(t)^-} w(t,x)$. It would be very tempting to proceed as was done in [16] to treat the LWR model, which can be seen as a simplified scalar version of this GARZ model, and separate the proof in the cases

i $(\rho_+, \rho_-) \in (\rho^*, \rho_{\max}]$, as no nonclassical shock can occur when $\rho > \rho^*$.

ii $(\rho_+, \rho_-) \in [0, \rho^*]$, as $V(\rho, w) \geqslant V_b$ when $\rho \leqslant \rho^*$ and thus the minimum of (3.33) is dominated by V_b.

iii $\rho_+ \leqslant \rho^* < \rho_-$ or $\rho_- \leqslant \rho^* < \rho_+$.

However, as it is, this is doomed to failure as, ρ^* depends on w which can be discontinuous as well and thus $\rho^*(w_+)$ would not have the same value than $\rho^*(w_-)$. Besides, approximating ρ and w by a wave-front tracking algorithm would give some (ρ^n, w^n) and it could be that even if (ρ^n, w^n) is close to (ρ, w) for some x, it could be instantaneously be brought away by a 2-wave, trying to show that ρ^n and w^n are everywhere close to ρ and w is vain.

To deal with this problem we observe a fact from the conservation equations : almost everywhere in time only two cases can occur $w_- = w_+$ or $V(\rho_+, w_+) = V(\rho_-, w_-)$. In other words, there exists a negligible space $\mathcal{N} \subset \mathbb{R}_+$ such that for any $t \in \mathbb{R}_+ \setminus \mathcal{N}$ the velocity $V(\rho(t,\cdot), w(t,\cdot))$ and the parameter $w(t,\cdot)$ cannot be discontinuous in $y(t)$ at the same time. This is shown in Appendix B. In the following we will suppose without loss of generality that $\mathcal{N}_0$ is contained in $\mathcal{N}$. Thus, for any $t \in \mathbb{R} \setminus \mathcal{N}_0$,

- $V(\rho(t, y(t)), w(t, y(t))) = V(\rho(t, y(t)-), w(t, y(t)-))$ or $w(t, y(t)) = w(t, y(t)-)$ almost everywhere

In spirit, the first case where $w_- = w_+$ is similar to the case of the simplified LWR model treated in [16], as the GARZ model would reduce to the LWR model is if w was constant on $\mathbb{R}$. A new difficulty would arise as the 2-waves and V-waves can induce more complicated behaviors. We will come back to that in Subsection 3.3.2.
The second case, however, is specific to GARZ model and is a new feature. Thus, overall there are two new difficulties : dealing with the influence of the 2-waves and V-waves in the first case and treating this second case. We will start by a lemma that can be applied in these two cases, and then deal with the two cases separately.

Lemma 3.10. *Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\varepsilon > 0$. Let $(\rho_+, \rho_-) \in ([0, \rho_{max}])^2$ and $(w_+, w_-) \in ([w_-, w_+])^2$. There exists $\delta > 0$ such that for $n \in \mathbb{N}$ large enough : if $x \in (\min(y^n, y) - \delta, \min(y^n, y))$ two cases can occur:*

$$\begin{aligned} &V(\rho^n(t,x), w^n(t,x)) \in \mathcal{B}_\varepsilon(V(\rho_-, w_-)), \\ \text{or } &V(\rho^n(t,x), w^n(t,x)) \in [V_b - 2\varepsilon, +\infty) \text{ and } V(\rho_-, w_-) \in [V_b - \varepsilon, +\infty). \end{aligned} \tag{3.34}$$

And if $x \in (\max(y^n, y), \max(y^n, y) + \delta)$,

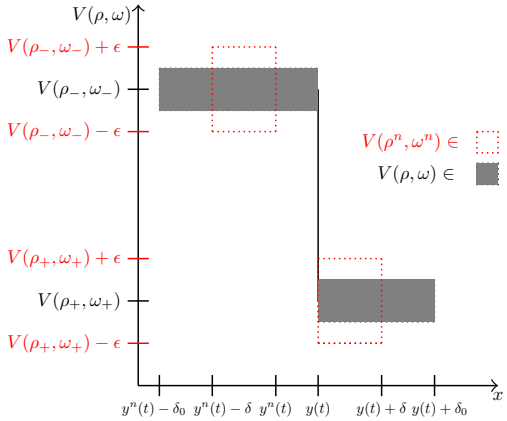
$$\begin{aligned} &V(\rho^n(t,x), w^n(t,x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)), \\ \text{or } &V(\rho^n(t,x), w^n(t,x)) \in [V_b - 2\varepsilon, +\infty) \text{ and } V(\rho_+, w_+) \in [V_b - \varepsilon, +\infty). \end{aligned} \tag{3.35}$$

where $\mathcal{B}_r(a)$ stands for the ball centered in a of radius ε

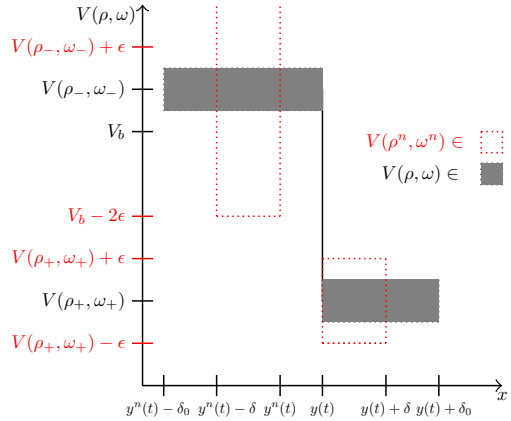
In other words, this lemma shows that in a short range before $\min(y^n, y)$ and after $\max(y^n, y)$, the approximated velocity $V(\rho^n, w^n)$ is everywhere close to $V(\rho, w)$ or is also close to be larger than V_b if $V(\rho, w)$ is. Note that this is not obvious as the convergence in BV of (ρ^n, w^n) gives only information almost everywhere and this is precisely all our problem : we need to obtain some convergence in a particular location $y(t)$. An illustration of Lemma 3.10 is given in Figure 1. We now prove this lemma.

Proof of Lemma 3.10. Let us assume without loss of generality that $x < \min(y^n, y)$ (the other case is similar by symmetry). From Lemma 3.6, for every $\epsilon > 0$, there exists $\delta_0 > 0$ such that $TV(\rho(t,\cdot), w(t,\cdot))_{|(\min(y^n,y)-\delta_0, \min(y^n,y))} < \frac{\epsilon}{2M}$ with M the Lipschitz constant of $V \in C^2([0, \rho_{\max}] \times [w_{\min}, w_{\max}])$. This implies that, for any $x \in (\min(y^n, y) - \delta_0, \min(y^n, y))$,

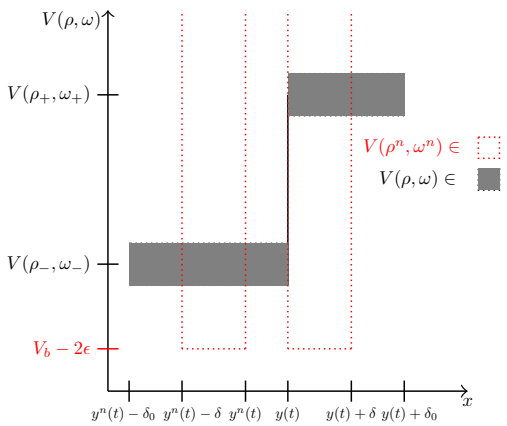
$$(\rho(t,x), w(t,x)) \in \mathcal{B}_{\varepsilon/2M}((\rho_-, w_-)). \tag{3.36}$$



$V(\rho_-,w_-) < V_b-\epsilon$ and $V(\rho_+,w_+) < V_b-\epsilon$



$V(\rho_-,w_-) \geqslant V_b-\epsilon$ and $V(\rho_+,w_+) < V_b-\epsilon$



$V(\rho_-,w_-) \geqslant V_b-\epsilon$ and $V(\rho_+,w_+) \geqslant V_b-\epsilon$

Figure 1: Illustration of Lemma 3.10; let $t\in\mathbb{R}_+\setminus\mathcal{N}$, $\varepsilon>0$, $(\rho_+,\rho_-)\in([0,\rho_{max}])^2$ and $(w_+,w_-)\in([w_-,w_+])^2$ with $y^n(t)<y(t)$. The approximate speed $V(\rho^n(t,\cdot),w^n(t,\cdot))$ over $[y^n(t)-\delta,y^n(t)]\cup[y(t),y(t)+\delta]$ belongs to the area surrounded by the dotted lines (...) and $\rho(t,\cdot)$ over $[y(t)-\delta_0,y(t)+\delta_0]$ belongs to the shaded zone.

We now treat the two cases $V(\rho_-,w_-)<V_b-\varepsilon$ and $V(\rho_-,w_-)\geqslant V_b-\varepsilon$ separately.

Step 1: if $V(\rho_-,w_-)<V_b-\varepsilon$. We now proceed by contradiction. Suppose by contradiction that for any $\delta>0$ and for any $n_0>0$ there exists $n_1\geqslant n_0$ and $x\in(\min(y^n,y)-\delta,\min(y^n,y))$ such that $V(\rho^{n_1}(x),w^{n_1}(x))\in\mathbb{R}\setminus\mathcal{B}_\varepsilon(V(\rho_-,w_-))$. Then by selecting $\delta=\delta_0/n$ we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$V(\rho^n(x_n),w^n(x_n))\in\mathbb{R}\setminus\mathcal{B}_\varepsilon(V(\rho_-,w_-)) \tag{3.37}$$

with

$$x_n\in(\min(y^n,y)-\delta,\min(y^n,y)) \text{ and } \lim_{n\to+\infty}x_n=y(t). \tag{3.38}$$

Then as we know that $(\rho^n, w^n) \to (\rho, w)$ in BV, we can build a sequence $(z_m)_{m\in\mathbb{N}}$ such that

$$z_m < y(t), \lim_{m\to+\infty} z_m = y(t) \text{ and } \lim_{n\to+\infty} (\rho^n, w^n)(z_m) = (\rho, w)(z_m). \tag{3.39}$$

Besides, from (3.36), $(\rho, w)(z_m) \in \mathcal{B}_{\varepsilon/2}((\rho_-, w_-))$. As $z_m \to y(t)$, we can use a diagonal argument to construct two sequences : (z_n^1) and (z_n^2) and $n_1 > n_0$ large enough such that for any $n \geqslant n_1$,

$$\begin{aligned}
&\lim_{n\to+\infty} z_n^1 = \lim_{n\to+\infty} z_n^2 = y(t),\\
&z_n^1 < x_n < z_n^2 < \min(y^n, y),\\
&(\rho^n, w^n)(z_n^1+) \in \mathcal{B}_{3\varepsilon/4M}(\rho_-, w_-) \text{ and } (\rho^n, w^n)(z_n^2-) \in \mathcal{B}_{3\varepsilon/4M}(\rho_-, w_-)\\
&V(\rho^n, w^n)(z_n^1+) \in \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)) \text{ and } V(\rho^n, w^n)(z_n^2-) \in \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)).
\end{aligned} \tag{3.40}$$

Note that we used the Lipschitz continuity of V for this last point. (3.37) and (3.40) imply that $V(\rho^n, w^n)$ has to reach $\mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-))$ from x_n to z_n^1 and from x_n to z_n^2. As n can be chosen large enough we suppose in addition that

$$|z_n^1 - \min(y^n, y)| < \frac{\beta t}{12}\varepsilon. \tag{3.41}$$

We denote $x_0 \in [z_n^1, x_n)$ the first point x such that $V(\rho^n(x-), w^n(x-))$ reaches $\mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-))$ from $V(\rho^n(x_n), w^n(x_n))$ (see Figure 2). In other words,

$$\begin{aligned}
&V(\rho^n(x_0-), w^n(x_0-)) \in \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)),\\
&V(\rho^n(x), w^n(x)) \in \mathbb{R} \setminus \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)),\ \forall x \in [x_0, x_n),
\end{aligned} \tag{3.42}$$

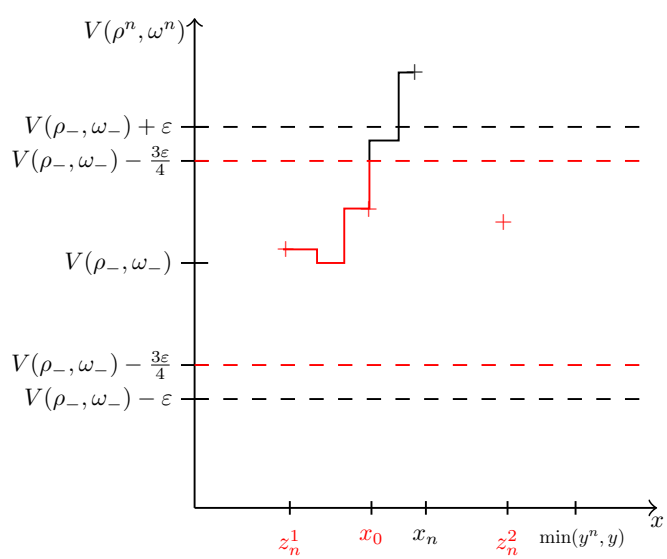


Figure 2: Illustration of $V(\rho^n, w^n)$ if $V(\rho_-, w_-) < V_b - \varepsilon$ and $V(\rho^n(x_n), w^n(x_n)) > V(\rho_-, w_-) + \epsilon$ with $z_n^1 < x_0 < x_n < z_n^2 < \min(y^n, y)$.

Step 2: *there cannot be any V-waves between x_0 and x_n.* Assume that a V-wave occurs between x_n and x_0, this implies that for some $x_1 \in [x_0, x_n]$, $\rho^n(x_1-) = \rho^n(x_1) = 0$. As there is a finite

number of discontinuities, we can assume that x_1 is the smallest $x \in [x_0, x_n]$ where a V-wave occurs. Thus $V(\rho^n(x_1-)), w^n(x_1-)) = w^n(x_1-) \geqslant w_{\min} > V_b$. From (3.42) and the fact that $V(\rho_-, w_-) \leqslant V_b - \varepsilon$, this implies that

$$V(\rho^n(x_1-)), w^n(x_1-)) \geqslant V(\rho^n(x_0-)), w^n(x_0-)) + \varepsilon/4, \tag{3.43}$$

and thus $x_0 < x_1$. Therefore the wave occuring in x_0 is not a V-wave, and it is not a 2-wave either as the velocity is discontinuous in x_0 from (3.42). As $x_n < \min(y^n, y(t))$ there cannot be any nonclassical shock. Hence, so the wave in x_0 is a rarefaction shock or a shock. A shock can only increase the density ρ^n and therefore decrease the velocity V, and a rarefaction shock can increase V at most by $C_0\rho_{\max}/2^n$ from Lemma 3.9. This implies that, for $n \in \mathbb{N}$, $C_0\rho_{\max}/2^n \leqslant \varepsilon/24$ and large enough,

$$V(\rho^n(x_0-), w^n(x_0-)) + \frac{\varepsilon}{24} \geqslant V(\rho^n(x_0), w^n(x_0)) \tag{3.44}$$

Thus, from (3.43) and (3.44), we deduce that

$$V(\rho^n(x_1-)), w^n(x_1-)) \geqslant V(\rho^n(x_0), w^n(x_0)) + 5\varepsilon/24. \tag{3.45}$$

By definition there is no V-wave and no non-classical shock between x_0 and x_1 and $x_1 > x_0$. This implies that we can use Lemma 3.7 and thus we have

$$\begin{aligned} |x_1 - x_0| &\geqslant \beta t |V(\rho^n(x_1-), w^n(x_1-)) - V(\rho^n(x_0), w^n(x_0))| \\ &\geqslant \tfrac{5\beta t\varepsilon}{24}. \end{aligned} \tag{3.46}$$

The inequalities (3.41), (3.46) and $z_n^1 \leqslant x_0 < x_1 < \min(y^n, y)$ lead to a contradiction. Thus there is no V-wave between x_0 and x_n.
Assumption (3.37) implies that

$$\begin{aligned} V(\rho^n(x_n), w^n(x_n)) &\geqslant V(\rho_-, w_-) + \varepsilon, \\ \text{or } V(\rho^n(x_n), w^n(x_n)) &\leqslant V(\rho_-, w_-) - \varepsilon. \end{aligned} \tag{3.47}$$

Step 1.2: (3.47) *doesn't hold.* Assume that $V(\rho^n(x_n), w^n(x_n)) \geqslant V(\rho_-, w_-) + \varepsilon$. Since $x_n < \min(y^n, y)$, there is no nonclassical shock in x_n. Moreover, as we just showed, there is no V-wave in x_n. Thus, from Lemma 3.9, for n large enough, we have

$$\begin{aligned} V(\rho^n(x_n-), w^n(x_n-)) &\geqslant V(\rho^n(x_n), w^n(x_n)) - \tfrac{\varepsilon}{24}, \\ &\geqslant V(\rho_-, w_-) + \varepsilon - \tfrac{\varepsilon}{24}. \end{aligned} \tag{3.48}$$

As there is no V-waves occuring in $[x_0, x_n]$, this means, using (3.42), (3.44) and (3.48) that we can apply Lemma 3.7 between x_0 and x_n similarly as previously and for $n \in \mathbb{N}$ large enough,

$$\begin{aligned} |x_n - x_0| &\geqslant \beta t |V(\rho^n(x_n-), w^n(x_n-)) - V(\rho^n(x_0), w^n(x_0))| \\ &\geqslant \tfrac{\beta t\varepsilon}{6}. \end{aligned} \tag{3.49}$$

The inequalities (3.41), (3.49) and $z_n^1 \leqslant x_0 < x_n < \min(y^n, y)$ lead to a contradiction. Therefore from (3.47)

$$V(\rho^n(x_n), w^n(x_n)) \leqslant V(\rho_-, w_-) - \varepsilon. \tag{3.50}$$

We will now show that this implies that there is no V-waves between x_n and z_n^2 either, and then we will get the final contradiction still using Lemma 3.7. Assume that there is a V-wave between x_n and z_n^2. This implies that there exists $x_3 \in [x_n, z_n^2]$ such that $\rho^n(x_3-) = \rho^n(x_3) = 0$. Without loss

of generality, as previously we can assume that x_3 is the smallest $x \in [x_n, z_n^2]$ such that a V-wave appears. Since $V_b < w_{\min}$, we know that

$$V(\rho^n(x_3-), w^n(x_3-)) = w(x_3-) > V_b. \tag{3.51}$$

Moreover, using that $V(\rho_-, w_-) < V_b - \varepsilon$ and (3.50), we deduce that

$$V(\rho^n(x_n), w^n(x_n)) \leqslant V(\rho_-, w_-) - \varepsilon < V_b - 2\epsilon. \tag{3.52}$$

Therefore, $x_3 > x_n$. By definition there is no V-wave occurring between x_n and x_3 and since $x_3 < \min(y^n, y)$, no nonclassical shock either. Thus we can use Lemma 3.7 with (3.51) and (3.52) to prove that

$$|x_3 - x_n| \geqslant 2\beta t\varepsilon \tag{3.53}$$

which gives again a contradiction with (3.41) and $z_n^1 < x_n < x_3 < \min(y^n, y)$. So no V-wave occurs between x_n and z_n^2. Then exactly as before we can use Lemma 3.7 with (3.40) and (3.50) to show that

$$|z_n^2 - x_n| \geqslant \frac{\beta t\varepsilon}{4}, \tag{3.54}$$

which is in contradiction with (3.41) and $z_n^1 < x_n < z_n^2 < \min(y^n, y)$. So overall both cases of (3.47) give a contradiction and this gives the result.

If $V(\rho_-, w_-) \geqslant V_b - \varepsilon$. We proceed again by contradiction. Suppose by contradiction that for any $\delta > 0$ and for any $n_0 > 0$ there exists $n_1 \geqslant n_0$ and $x \in (\min(y^n, y) - \delta, \min(y^n, y))$ such that $V(\rho^{n_1}(x), w^{n_1}(x)) \in \mathbb{R} \setminus [V_b - 2\varepsilon, +\infty)$. Then by selecting $\delta = \delta_0/n$ we can again construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$V(\rho^n(x_n), w^n(x_n)) \in \mathbb{R} \setminus [V_b - 2\varepsilon, +\infty) \tag{3.55}$$

with

$$x_n \in (\min(y^n, y) - \delta, \min(y^n, y)) \text{ and } \lim_{n\to+\infty} x_n = y(t). \tag{3.56}$$

As previously we can build (z_n^1) and (z_n^2) and $n_1 > n_0$ large enough such that for any $n \geqslant n_1$,

$$\begin{aligned}
&\lim_{n\to+\infty} z_n^1 = \lim_{n\to+\infty} z_n^2 = y(t),\\
&z_n^1 < x_n < z_n^2 < \min(y^n, y(t)),\\
&(\rho^n, w^n)(z_n^1+) \in \mathcal{B}_{3\varepsilon/4M}(\rho_-, w_-) \text{ and } (\rho^n, w^n)(z_n^2-) \in \mathcal{B}_{3\varepsilon/4M}(\rho_-, w_-),\\
&V(\rho^n, w^n)(z_n^1+) \in \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)) \text{ and } V(\rho^n, w^n)(z_n^2-) \in \mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-)),
\end{aligned} \tag{3.57}$$

with M the Lipschitz constant of $V \in C^2((0, \rho_{\max}) \times (w_{\min}, w_{\max}))$. As previously $V(\rho^n, w^n)$ has to reach $\mathcal{B}_{3\varepsilon/4}(V(\rho_-, w_-))$ from x_n to z_n^1 and from x_n to z_n^2. And we suppose again that $n \in \mathbb{N}$ is large enough such that

$$|z_n^1 - z_n^2| < \frac{\beta t}{12}\varepsilon. \tag{3.58}$$

Using (3.55) and (3.57), there exists $x_0 \in (x_n, z_n^2]$ such that

$$\begin{aligned}
&V(\rho^n(x_0), w^n(x_0)) \geqslant V_b - 3\varepsilon/2,\\
&V(\rho^n(x-), w^n(x-)) < V_b - 3\varepsilon/2, \ \forall x \in (x_n, x_0].
\end{aligned} \tag{3.59}$$

Note that there cannot be a V-wave in x_0 as $V(\rho^n(x_0-), w^n(x_0-)) < V_b$ from (3.59). The wave in x_0 is not a 2-wave either as the velocity V is discontinuous in this point and is not a nonclassical shock either as $z_n^2 < \min(y^n, y)$. Thus, from Lemma 3.9 and for $n \in \mathbb{N}$ large enough we have

$$V(\rho^n(x_0-), w^n(x_0-)) \geqslant V(\rho^n(x_0), w^n(x_0)) - \frac{\varepsilon}{24}. \tag{3.60}$$

We will show now that there cannot be any V-waves between x_n and x_0. Assume by contradiction that a V-wave occurs between x_n and x_0, this implies that for some $x_1 \in (x_n, x_0]$, $V(\rho^n(x_1-), w^n(x_1-)) > V_b$, which is in contradiction with (3.59), thus no V-wave occurs between x_0 and x_n. Therefore, using (3.55), (3.59) and (3.60), we can apply Lemma 3.7 and for $n \in \mathbb{N}$ large enough,

$$|x_0 - x_n| \geqslant \frac{11\beta t\varepsilon}{24}, \tag{3.61}$$

which is in contradiction with (3.58) and $z_n^1 < x_n < x_0 < \min(y^n, y)$ which ends the result. □

We will start by this second case $V(\rho_+, w_+) = V(\rho_-, w_-)$.

3.3.1 Case $V(\rho_+, w_+) = V(\rho_-, w_-)$.

Lemma 3.11. *Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\varepsilon > 0$. Assume that $V(\rho_-, w_-) = V(\rho_+, w_+)$, then for $n \in \mathbb{N}$ large enough if $x \in (\min(y^n(t), y(t)), \max(y^n(t), y(t)))$,*

- *If $V(\rho_+, w_+) < V_b - \varepsilon/2$, then*

$$V(\rho^n(t,x), w^n(t,x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.62}$$

- *If $V(\rho_+, w_+) \geqslant V_b - \varepsilon/2$, then*

$$V(\rho^n(t,x), w^n(t,x)) \in [V_b - 2\varepsilon, +\infty). \tag{3.63}$$

Proof of Lemma 3.11. Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\varepsilon > 0$.

If $V(\rho_+, w_+) < V_b - \varepsilon/2$ Suppose by contradiction that for any $n_0 \geqslant 0$ there exists $n \geqslant n_0$ such that there exists $x_n \in (\min(y^n, y), \max(y^n, y))$ such that

$$V(\rho^n(x_n), w^n(x_n)) \in \mathbb{R} \setminus \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.64}$$

As n_0 can be chosen large enough, we suppose that

$$|y^n - y| < \min\left(\frac{t\beta\varepsilon}{4}, \min_{w \in [w_{\min}, w_{\max}]}(\rho^*(w))\right), \tag{3.65}$$

which is possible as $\min_{w \in [w_{\min}, w_{\max}]}(\rho^*(w)) > 0$. We will obtain the contradiction by using that $(\rho^n(x_n), w^n(x_n))$ has to connect to the left part of the solution in $(\min(y^n, y) - \delta, \min(y^n, y))$ and to the right part of the solution in $(\max(y^n, y), \max(y^n, y) + \delta)$, and then using the information we have on those two parts of the solution from Lemma 3.10. Let first look at what happens for (ρ^n, w^n) at $\min(y^n, y)$: Since by assumption $V(\rho_-, w_-) = V(\rho_+, w_+) < V_b - \frac{\varepsilon}{2}$, we have $V(\rho_-, w_-) < V_b - \frac{\varepsilon}{6}$. Applying Lemma 3.10 with $\varepsilon = \frac{\varepsilon}{6}$, for n_0 large enough,

$$V(\rho^n(\min(y^n, y)-), w^n(\min(y^n, y)-)) \in \mathcal{B}_{\varepsilon/6}(V(\rho_-, w_-)) = \mathcal{B}_{\varepsilon/6}(V(\rho_+, w_+)). \tag{3.66}$$

As $V(\rho_+, w_+) + \varepsilon/6 < V_b$ this means that

$$\rho^n(\min(y^n, y)-) > \rho^*(w^n(\min(y^n, y)-)) \geqslant \hat{\rho}_\alpha(w^n(\min(y^n, y)-)). \tag{3.67}$$

Thus there cannot be any nonclassical shock or V-wave at $\min(y^n, y)$ and therefore there can only be a 2-wave, a rarefaction shock or a shock, which implies, together with (3.66), that

$$\begin{aligned} V(\rho^n(\min(y^n, y)), w^n(\min(y^n, y))) &\leqslant V(\rho^n(\min(y^n, y)-), w^n(\min(y^n, y)-)) + \frac{C_0\rho_{\max}}{2^n} \\ &\leqslant V(\rho_+, w_+) + \frac{\varepsilon}{6} + \frac{C_0\rho_{\max}}{2^n}, \end{aligned} \tag{3.68}$$

where C_0 given by Lemma 3.9 is the maximal amplitude of the velocity through of a rarefaction shock. As n can be chosen large enough, we assume that

$$\frac{C_0\rho_{\max}}{2^n} < \frac{\varepsilon}{6} \tag{3.69}$$

Thus, from (3.68),

$$V(\rho^n(\min(y^n, y)), w^n(\min(y^n, y))) \leqslant V(\rho_+, w_+) + \frac{\varepsilon}{3}. \tag{3.70}$$

Similarly on the right-hand side, one gets

$$V(\rho^n(\max(y^n, y)-), w^n(\max(y^n, y)-)) \geqslant V(\rho_+, w_+) - \frac{\varepsilon}{3}. \tag{3.71}$$

Now we know to which admissible values $(\rho^n(x_n), w^n(x_n))$ has to connect on the left and on the right and we will obtain a contradiction. We can now proceed exactly similarly as in the proof of Lemma 3.10. From (3.64) and (3.66), there exists $x_1 \in [\min(y^n, y), x_n]$ such that

$$\begin{aligned} &V(\rho^n(x_1-), w^n(x_1-)) \in \mathcal{B}_{\varepsilon/3}(V(\rho_+, w_+)) \\ &V(\rho^n(x), w^n(x)) \in \mathbb{R} \setminus \mathcal{B}_{\varepsilon/3}(V(\rho_+, w_+)), \ \forall x \in [x_1, x_n], \end{aligned} \tag{3.72}$$

We will now show that no V-wave can occur between x_1 and x_n by contradiction. Assume there exists $x_2 \in [x_1, x_n]$ such that $\rho^n(x_2-) = \rho^n(x_2) = 0$. As there is a finite number of discontinuities, we can assume that x_2 is the smallest $x \in [x_1, x_n]$ where a V-wave occurs. Therefore

$$V(\rho^n(x_2-), w^n(x_2-)) = w^n(x_2-) \geqslant w_{min} > V_b > V(\rho_+, w_+) + \frac{\varepsilon}{2}. \tag{3.73}$$

which implies, together with (3.72), $V(\rho^n(x_2-), w^n(x_2-)) > V(\rho^n(x_1-), w^n(x_1-))$. Thus, $x_2 \neq x_1$ and no V-wave can occur in $[x_1, x_2)$. The wave occurring in x_1 is not a 2-wave either as the velocity is discontinuous in x_1 by construction. Moreover, a non-classical shock cannot occur at x_1 since there cannot be a non-classical shock at $\min(y^n, y)$ as seen in (3.67) and $x_1 \in [\min(y^n, y), x_n]$ with $x_n < \min(y^n, y)$. Thus, the wave in x_1 is either a rarefaction wave or a shock. From Lemma 3.9 and (3.69), for n large enough,

$$V(\rho^n(x_1-), w^n(x_1-)) + \frac{\varepsilon}{6} \geqslant V(\rho^n(x_1), w^n(x_1)) \tag{3.74}$$

Thus, we can apply Lemma 3.7 with (3.72), (3.73) and (3.74). Thus,

$$\begin{aligned} |x_2 - x_1| &\geqslant \beta t |V(\rho^n(x_2-), w^n(x_2-)) - V(\rho^n(x_1), w^n(x_1))| \\ &\geqslant \tfrac{\beta t \varepsilon}{6}. \end{aligned} \tag{3.75}$$

which is in contradiction with (3.65) and $\min(y^n, y) \leqslant x_1 < x_2 < \max(y^n, y)$. So there is no V-waves between x_1 and x_n. From (3.64)

$$\begin{aligned} V(\rho^n(x_n), w^n(x_n)) &\geqslant V(\rho_+, w_+) + \varepsilon \\ \text{or } V(\rho^n(x_n), w^n(x_n)) &\leqslant V(\rho_+, w_+) - \varepsilon. \end{aligned} \tag{3.76}$$

The exact same proof as in the proof of Lemma 3.10 **Step 2** can be done to show that (3.76) doesn't hold which gives the contradiction and ends the proof of Lemma 3.11 in the case $V(\rho^+, w^+) < V_b - \varepsilon/2$.

If $V(\rho^+, w^+) \geqslant V_b - \varepsilon/2$. There are two possible situations : if there is a no nonclassical shock at $\min(y^n, y)$ and at $\max(y^n, y)$, then the proof is exactly the same as the previous case and for any $x \in (\min(y^n, y), \max(y^n, y))$,

$$V(\rho^n(x), w^n(x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)) \subset [V_b - 2\varepsilon, +\infty). \tag{3.77}$$

If there is a nonclassical shock occurring at $\min(y^n, y)$ or at $\max(y^n, y)$. Suppose by contradiction that for any $n_0 \geqslant 0$ there exists $n \geqslant n_0$ such that there exists $x_n \in (\min(y^n, y), \max(y^n, y))$ such that

$$V(\rho^n(x_n), w^n(x_n)) \in \mathbb{R} \setminus [V_b - 2\varepsilon, +\infty). \tag{3.78}$$

Suppose that the nonclassical shock occurs in $\min(y^n, y)$. It means that $\min(y^n, y) = y^n$ and thus no shock occurs on the right at $\max(y^n, y) = y$ so a similar proof as previously can be done. Roughly speaking, from Lemma 3.10 with $\varepsilon = \frac{\varepsilon}{2}$ we have $V(\rho^n(y), w^n(y)) \geqslant V_b - \frac{3\varepsilon}{2}$. Since there is not a nonclassical shock at y then only a rarefaction shock or a V-wave can increase V. If a V-wave occur at y then $V(\rho^n(y-), w^n(y-)) > V_b$. As n can be chosen large enough, we assume that $\frac{C_0 \rho_{\max}}{2^n} < \frac{\varepsilon}{6}$. Thus, from Lemma 3.9 for n large enough, $V(\rho^n(y-), w^n(y-)) \geqslant V_b - \frac{5\varepsilon}{3}$. As previously, we show that only shocks, rarefactions and 2-shocks are allowed over $[x_n, y]$. Thus, we have

$$\begin{aligned} |y - x_n| &\geqslant \beta t |V(\rho^n(y-), w^n(y-)) - V(\rho^n(x_n), w^n(x_n))| \\ &\geqslant \tfrac{\beta t \varepsilon}{3}, \end{aligned} \tag{3.79}$$

which leads to a contradiction using (3.65) with $y^n \leqslant x_n < y$. Suppose now a non-classical shock appears on the right at $\max(y^n, y)$, this implies that $\max(y^n, y) = y^n$ and that

$$\begin{aligned} \rho^n(\max(y^n, y)-) &= \hat{\rho}_\alpha(w(\max(y^n, y)-)), \\ V(\rho^n(\max(y^n, y)-), w(\max(y^n, y)-)) &\geqslant V_b. \end{aligned} \tag{3.80}$$

Therefore, using this together with (3.78), there exists $x_5 \in (x_n, \max(y^n, y)]$ such that

$$\begin{aligned} &V(\rho^n(x_5), w^n(x_5)) \geqslant V_b, \\ &V(\rho^n(x), w^n(x)) < V_b, \ \forall x \in [x_n, x_5). \end{aligned} \tag{3.81}$$

Note that in fact $x_5 < \max(y^n, y)$ as

$$V(\rho^n(x_5-), w^n(x_5-)) < V_b < V(\rho^n(\max(y^n, y)-), w(\max(y^n, y))). \tag{3.82}$$

Now, we can fix n_0 large enough such that

$$|y^n - y| < C\beta\varepsilon. \tag{3.83}$$

We show once again that there cannot be any V-wave between x_n and x_5 as if there existed $x_6 \in [x_n, x_5]$ such that a V-wave occurs, then, as $V(\rho^n(x_6-), w^n(x_6-)) = V(0, w^n(x_6-)) > V_b$

this implies that $x_5 > x_6$, but $V(\rho^n(x_6), w^n(x_6)) = V(0, w^n(x_6)) > V_b$ and this would be in contradiction with (3.81). Therefore there is no V-wave between x_n and x_5 and no nonclassical shocks either (as $x_5 < y^n$). This implies that

$$V(\rho^n(x_5-), w^n(x_5-)) \geqslant V(\rho^n(x_5), w^n(x_5)) - \frac{C_0 \rho_{\max}}{2^n}, \tag{3.84}$$

where C_0 is given by Lemma 3.9. Thus there exists n large enough such that

$$V(\rho^n(x_5-), w^n(x_5-)) - V(\rho^n(x_n), w^n(x_n)) \geqslant \frac{\varepsilon}{2}, \tag{3.85}$$

and we can apply Lemma 3.7 to get

$$|x_5 - x_n| \geqslant \frac{\beta t \varepsilon}{2}, \tag{3.86}$$

which is in contradiction with (3.83). Hence, for any $x \in (\min(y^n, y), \max(y^n, y))$,

$$V(\rho^n(x), w^n(x)) \in [V_b - 2\varepsilon, +\infty). \tag{3.87}$$

This ends the proof of Lemma 3.11. □

We can now prove the result in the case $V(\rho_-, w_-) = V(\rho_+, w_+)$. Let $\varepsilon > 0$, from Lemma 3.10 wtih $\varepsilon = \varepsilon/2$ and Lemma 3.11, and using the fact that $V(\rho_-, w_-) = V(\rho_+, w_+)$, there exists $n_0 > 0$ such that for any $n \geqslant n_0$ and any $x \in (\min(y^n(t), y(t)) - \delta, \max(y^n(t), y(t)) + \delta) \setminus \{y(t), y^n(t)\}$, if $V(\rho_+, w_+) < V_b - \frac{\varepsilon}{2}$

$$V(\rho^n(x), w^n(x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)), \tag{3.88}$$

and if $V(\rho_+, w_+) \geqslant V_b - \frac{\varepsilon}{2}$,

$$V(\rho^n(x), w^n(x)) \geqslant V_b - \varepsilon. \tag{3.89}$$

As this is true for any $x \in (\min(y^n(t), y(t)) - \delta, \max(y^n(t), y(t)) + \delta) \setminus \{y(t), y^n(t)\}$, this is true at the right limit of $y^n(t)$, namely $y_n(t)^+$ and hence this implies that

$$\min(V_b, V(\rho_+, w_+)) - \varepsilon \geqslant \min(V(\rho^n(y^n(t)+), w^n(y^n(t)+)), V_b) \leqslant \min(V_b, V(\rho_+, w_+)) + \varepsilon. \tag{3.90}$$

As $\varepsilon > 0$ was chosen arbitrarily (even though n_0 depends on ε), this gives directly

$$\lim_{n \to +\infty} \min(V_b, V(\rho^n(y^n(t)), w^n(y^n(t)))) = \min(V(\rho_+, w_+), V_b), \tag{3.91}$$

which is the result we aim at showing. We now move on to the next subsection and the case $w_- = w_+$.

3.3.2 Case $w_- = w_+$

In spirit, this case is similar to the case treated in [16] with the LWR model as we can now divide the situation in three cases :

i $(\rho_-, \rho_+) \in [0, \rho^*(w)]^2$, where $w = w_+ = w_-$

ii $(\rho_-, \rho_+) \in (\rho^*(w), \rho_{max}]^2$

iii $\rho_- \geqslant \rho^*(w) > \rho_+$ or $\rho_+ \geqslant \rho^*(w) > \rho_-$.

However, the possibility of 2-waves and V-waves complicate a lot this analysis and, in particular, we cannot try to confine ρ^n and w^n close to the set $[\rho_-, \rho_+] \times \{w\}$ as the 2-waves makes it doomed to failure : (ρ^n, w^n) could be much outside this set and become instantaneously close using a 2-wave. Instead, the idea will be to confine $V(\rho^n, w^n)$ and thus neutralise the 2-waves. In this aim, we show the following lemma

Lemma 3.12. *Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\varepsilon > 0$. Assume that $w_- = w_+$, and let w denote this value, then for $n \in \mathbb{N}$ large enough : if $x \in (\min(y^n, y), \max(y^n, y))$*

- *if $(\rho_-, \rho_+) \in [0, \rho^*(w)]^2$, then*

$$V(\rho^n, w^n) \in [V_b - \varepsilon, +\infty) \tag{3.92}$$

- *if $(\rho_-, \rho_+) \in (\rho^*(w), \rho_{max}]^2$, then*

$$V(\rho^n, w^n) \in [V(\max(\rho_-, \rho_+), w) - \varepsilon, V(\min(\rho_-, \rho_+), w) + \varepsilon] \tag{3.93}$$

- *if $\rho_- \geqslant \rho^*(w) > \rho_+$ or $\rho_+ \geqslant \rho^*(w) > \rho_-$, then*

$$V(\rho^n, w^n) \in [V(\max(\rho_-, \rho_+), w) - \varepsilon, V(\min(\rho_-, \rho_+), w) + \varepsilon] \cup [V_b - \varepsilon, +\infty) \tag{3.94}$$

The proof is very similar to the proof of Lemmas 3.10 and 3.11 and will be given in the Appendix C. As expected this lemma shows how we can confine $V(\rho^n, w^n)$ to the set of speed generated by $[\rho_-, \rho_+] \times \{w\}$. In two cases this confinement in imperfect as $V(\rho^n, w^n)$ could also lie in $[V_b, \infty)$. This is due to the presence of nonclassical shock and the fact that the solution is not entropic. However, this will not be a problem in the following. Indeed, what we aim to show overall is formally the following $\min(V(\rho^n(y^n(t)+), w^n(y^n(t)+)), V_b) \to \min(V(\rho_+, w_+), V_b)$ so any value above V_b will be canceled by the min operator. We can now prove the result in each of the three cases listed above :

- If $(\rho_-, \rho_+) \in [0, \rho^*(w)]^2$, then using Lemmas 3.10 and 3.12, for any $\varepsilon > 0$ there exists $\delta > 0$ and $n_1 > 0$ such that for any $n \geqslant n_1$ and any $x \in (\min(y^n(t), y(t)) - \delta, \max(y^n(t), y(t)) + \delta) \setminus \{y^n(t), y(t)\}$

$$V(\rho^n(x), w^n(x)) \in \mathcal{B}_\varepsilon(V(\rho_-, w)) \cup \mathcal{B}_\varepsilon(V(\rho_+, w)) \cup [V_b - \varepsilon, +\infty), \tag{3.95}$$

  where $w = w_+ = w_-$. But as $(\rho_-, \rho_+) \in [0, \rho^*(w)]^2$, then both $V(\rho_-, w)$ and $V(\rho_+, w)$ are above V_b. This implies that

$$V(\rho^n(x), w^n(x)) \in [V_b - \varepsilon, +\infty), \tag{3.96}$$

  hence

$$V_b - \varepsilon \leqslant \min(V_b, V(\rho^n(y^n(t)), w^n(y^n(t)))) \leqslant V_b. \tag{3.97}$$

  Since $V_b = \min(V_b, V(\rho_+, w_+))$, as $\rho^+ \leqslant \rho^*(w)$, and as ε was chosen arbitrarily we have using (3.97)

$$\lim_{n \to +\infty} \min(V(\rho^n(y^n(t)+), w^n(y^n(t)+)), V_b) = \min(V(\rho_+, w_+), V_b), \tag{3.98}$$

  which is the result we aimed at.

- If $(\rho_-,\rho_+)\in(\rho^*(w),\rho_{\max}]^2$ we can now proceed as in [16], provided that we look at 1-waves and find a way to ignore 2-waves. Let us first exclude a simple case : if $y^n(t)\geqslant y(t)$ for an infinite set of index, then extracting a subsequence we can assume that $y^n(t)\geqslant y(t)$ for any $n\geqslant n_1$. $(\rho_-,\rho_+)\in(\rho^*(w),\rho_{\max}]^2$ implies that $V(\rho_+,w)<V_b$ and $V(\rho_-,w)<V_b$. If $V(\rho_+,w)<V_b-\epsilon$ then from Lemma 3.10 we have, for any $x\in(y^n(t),y^n(t)+\delta)$,

$$V(\rho^n(x),w^n(x))\in\mathcal{B}_\varepsilon(V(\rho_+,w)). \tag{3.99}$$

If $V_b-\varepsilon\leqslant V(\rho_+,w)<V_b$ then (3.99) holds or $V(\rho^n(x),w^n(x))\in[V_b-2\varepsilon,\infty)$. Thus, (3.98) holds. So we can assume from now on that $y^n(t)<y(t)$ on an infinite set of index, and extracting a subsequence we can assume that $y^n(t)<y(t)$ for any $n\geqslant n_1$. We claim that

$$V(\rho_-,w)\geqslant V(\rho_+,w). \tag{3.100}$$

To prove it, we argue by contradiction similarly to the proof of Lemma 3.10. Roughly speaking, we assume that $V(\rho_-,w)<V(\rho_+,w)$ then for ε small enough $V(\rho_-,w)+13\varepsilon/6<V(\rho_+,w)$ and $V(\rho_+,w)+\varepsilon<V_b$. Therefore, from Lemmas 3.10 and 3.12, that there is no non-classical shock and V-wave in $(\min(y^n(t),y(t))-\delta,\max(y^n(t),y(t))+\delta)$ and we have to connect $\rho^n(t,\min(y^n(t),y(t))-)\in[V(\rho_-,w)-\varepsilon,V(\rho_-,w)+\varepsilon]$ to $\rho^n(t,\max(y^n(t),y(t))+)\in[V(\rho_+,w)-\varepsilon,V(\rho_+,w)+\varepsilon]$. This leads to a contradiction using that $y^n(t)\to y(t)$ as n tends to ∞ and $V(\rho_+,w)-\varepsilon-V(\rho_-,w)+\varepsilon>\frac{\varepsilon}{6}$.

Let us define the triangle $\mathcal{T}_0$ by

$$\mathcal{T}_0:=\{(s,x)\in[t,t_f)\times(w_{\max}(s-t)+y^n(t)-\delta,\partial_\rho f(\rho_{\max},w_{\max})(s-t)+y(t)+\delta)\}, \tag{3.101}$$

and where t_f is the closing point of the triangle defined by

$$t_f=\frac{y(t)-y^n(t)+2\delta}{w_{\max}-\partial_\rho f(\rho_{\max},w_{\max})}. \tag{3.102}$$

Let us also define $t_{y^n}>t$ the time at which $y^n(s)$ gets out of the triangle, i.e.

$$\begin{aligned}&(s,y^n(s))\in\mathcal{T}_0,\ \forall s\in[t,t_{y^n}),\\&(t_{y^n},y^n(t_{y^n}))\notin\mathcal{T}_0.\end{aligned} \tag{3.103}$$

Obviously $t_{y^n}\leqslant t_f$. Using the triangle $\mathcal{T}_0$ we will start from a point $\xi^n(t)$ after $y^n(t)$, where for any $x\in(\xi^n(t),y(t)+\delta(t))$, $V(\rho^n(t,x),w^n(t,x))\in\mathcal{B}_\varepsilon(V(\rho_+,w_+))$ and propagate to delimit a region (s,x) where $V(\rho^n(t,x),w^n(t,x))\in\mathcal{B}_\varepsilon(V(\rho_+,w_+))$. Then we will show that there exists a time $t_n>t$ above which $y^n(s)$ is in this region which implies that after this time, $V(\rho^n(t,y^n(t)),w^n(t,y^n(t)))\in\mathcal{B}_\varepsilon(V(\rho_+,w_+))$. Finally, we will show that this time t_n goes to 0 when n goes to $+\infty$ which will imply that for any $s>t$ and for n large enough $V(\rho^n(s,y^n(s)),w^n(s,y^n(s)))\in\mathcal{B}_\varepsilon(V(\rho_+,w_+))$ and by letting n goes to infinity will give the result (we will come back to this last point). This is summarized rigorously in the following lemma :

Lemma 3.13. *Let $t\in\mathbb{R}_+\setminus\mathcal{N}$ and $\varepsilon>0$. Assume that $(\rho_-,\rho_+)\in(\rho^*(w),\rho_{\max}]$ with $\rho_-\neq\rho_+$, that ε is small enough such that $\min(\rho_-,\rho_+)-\varepsilon>\rho^*(w)$, and that $y^n<y(t)$ for any $n\geqslant n_1$. Let $\delta>0$ be given by Lemma 3.10. Then for any $n\geqslant n_1$, there exists $t_{\xi^n}>t$ and a piecewise linear function ξ^n such that*

$$(s,\xi^n(s))\in\mathcal{T}_0,\ \forall s\in[t,t_{\xi^n}), \tag{3.104}$$

and for any $(s,x) \in \{[t, t_{\xi^n}) \times \mathbb{R} | x > \xi^n(s)\} \cap \mathcal{T}_0$

$$V(\rho^n(s,x), w^n(s,x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.105}$$

Besides, if we denote t_{y^n} *the time at which* $y^n(\cdot)$ *exits the triangle, there exists* $c > 0$ *independent of* n *such that* $\min(t_{y^n}, t_{\xi^n}) - t \geqslant c$ *and there exists* $t_n > 0$ *such that* $\xi(t_n) = y^n(t_n)$ *and* $\lim_{n\to+\infty} t_n = t$.

We will show this lemma right after, but first we show that we indeed get the result with this Lemma. If $\rho_- = \rho_+$ then, from Lemma 3.10 and Lemma 3.12,

$$V(\rho_+, w) - \varepsilon \leqslant \min(V_b, V(\rho^n(y^n(t)), w^n(y^n(t)))) \leqslant \min(V_b, V(\rho_+, w) + \epsilon).$$

Therefore, as ε was chosen arbitrarily, (3.98) holds. Assuming that $\rho_- \neq \rho_+$. Let $s \in (t, t+c)$ where c is given in Lemma 3.13, there exists $n_2 > n_1$ such that for any $n \geqslant n_2$, $t_n < s$. Hence

$$V(\rho^n(s, y^n(s)), w^n(s, y^n(s))) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.106}$$

Besides, as (ρ^n, w^n, y^n) satisfies (1.8) we have

$$y^n(s) - y^n(t) = \int_t^s \min(V_b, V(\rho^n(\tau, y^n(\tau)), w^n(\tau, y^n(\tau)))d\tau. \tag{3.107}$$

This implies with (3.106) that

$$\min(V_b, V(\rho_+, w_+)) - \varepsilon \leqslant \frac{y^n(s) - y^n(t)}{s-t} \leqslant \min(V_b, V(\rho_+, w_+)) + \varepsilon. \tag{3.108}$$

Using the convergence of y^n to y we have

$$\min(V_b, V(\rho_+, w_+)) - \varepsilon \leqslant \frac{y(s) - y(t)}{s-t} \leqslant \min(V_b, V(\rho_+, w_+)) + \varepsilon, \tag{3.109}$$

and from the definition of $\mathcal{N}_0$, y is differentiable in t, thus, using the fact that $\varepsilon > 0$ can be arbitrarily small.

$$\dot{y} = \min(V_b, V(\rho_+, w_+)), \tag{3.110}$$

which ends the result in the case where $(\rho_-, \rho_+) \in (\rho^*(w), \rho_{max}]^2$.

Proof of Lemma 3.13. This lemma is the analogous of Lemma 3.6 in [16] and the proof will be similar, with two differences : other types of waves than 1-waves can appear, and an interaction between two waves can result in several output wavefronts. To neutralize these effects we will consider $V(\rho^n, w^n)$ instead of ρ^n and w^n and when tracing the wavefront forward in time we will ignore the 2-waves and follow the wavefronts of 1-waves. The structure of the proof is illustrated in Figure 3.

Let $t \in \mathbb{R} \setminus \mathcal{N}$. Since $(\rho_-, \rho_+) \in (\rho^*(w), \rho_{\max}]$ with $\rho_- \neq \rho_+$ and (3.100), for $\varepsilon > 0$ small enough $V(\rho_+, w) < V(\rho_-, w) < V_b - \varepsilon$. Then, from the definition of $\mathcal{T}_0$ given in (3.101), for any $(s,x) \in \mathcal{T}_0$, we claim that

$$V(\rho^n(s,x), w^n(s,x)) \in \{V(\rho_+, w) - \epsilon, V(\rho_-, w) + \epsilon\}, \tag{3.111}$$

Indeed, at $s = t$ this is true from Lemma 3.10 and Lemma 3.12 as $x \in (y^n(t) - \delta, y(t) + \delta)$, and for $s \in (t, t_f)$ so no wave can reach the triangle $\mathcal{T}_0$ from outside given its definition.

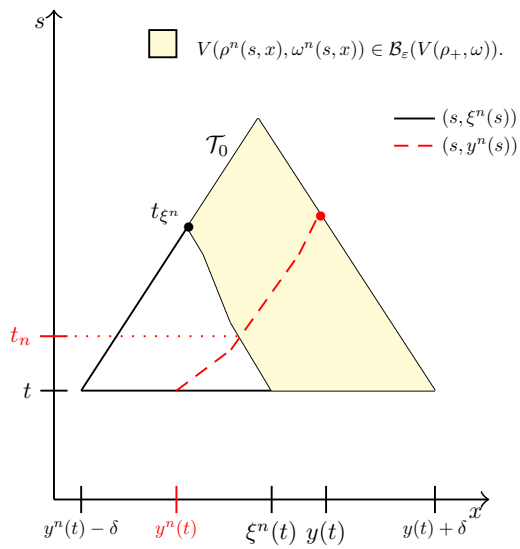


Figure 3: $(\rho_-,\rho_+) \in (\rho^*(w),\rho_{\max}]$ with $\rho_- \neq \rho_+$ with $y^n(t) < y(t)$, $n \geqslant n_1$

Thus, observing all the possible interactions described by Section 1, the set of accessible velocity at time s is included in the set of accessible velocity at time t, which gives (3.111). Note that, (3.111) implies that, for any $(s,x) \in \mathcal{T}_0$, $V(\rho^n(s,x),w^n(s,x)) < V_b$. Thus, a non-classical shock and a V-wave cannot appear in $\mathcal{T}_0$. Let us now denote by $N(t,n)$ the number of discontinuity points of the speed $V(\rho^n(t,\cdot),w^n(t,\cdot))$ on $[y^n(t)-\delta,y(t)+\delta]$ and x_j^n these discontinuity points. From Lemma 3.10 and Lemma 3.12 and $\rho_- \neq \rho_+$, there exists $j_0 \in \{1,...,N(t,n)\}$ such that $x_{j_0}^n \in [y^n(t),y(t)]$ satisfies

$$\begin{aligned} &V(\rho^n(t,x_{j_0}^n-),w^n(t.x_{j_0}^n-)) \in \mathbb{R} \setminus \mathcal{B}_\varepsilon(V(\rho_+,w)) \\ &V(\rho^n(t,x_j^n),w^n(t,x_j^n)) \in \mathcal{B}_\varepsilon(V(\rho_+,w)),\ \forall j \geqslant j_0 \text{ such that } x_j^n < y(t)+\delta, \end{aligned} \tag{3.112}$$

Note that, since $\rho_- \neq \rho_+$, for ε small enough, $\mathcal{B}_\varepsilon(V(\rho_-,w)) \cap \mathcal{B}_\varepsilon(V(\rho_+,w)) = \emptyset$.

$$V(\rho^n(t,x_j^n),w^n(t,x_j^n)) \in \mathcal{B}_\varepsilon(V(\rho_+,w_+)). \tag{3.113}$$

This comes directly from Lemma 3.10 and the fact that (ρ^n,w^n) is piecewise constant. Note that there cannot be a V-wave at x_0 from (3.113) and the fact that $V(\rho_+,w_+)+\varepsilon \leqslant V_b$. We construct now the piecewise constant function ξ^n that we are going to track forward in time : let $\xi^n(t) = x_{j_0}^n$ and

$$\dot{\xi}(s) = \sigma\left(\rho^n(t,x_{j_0}^n-),R(V(\rho^n(t,x_{j_0}^n),w^n(t,x_{j_0}^n)),w^n(t,x_{j_0}-))\right),\ \text{for } s \in [t,s_1], \tag{3.114}$$

where the term at the right of (3.114) is the speed of the 1-wave created at $x_{j_0}^n$ and s_1 is defined as the minimum between the time when ξ^n exits the triangle $\mathcal{T}_0$ and the time when it interacts or attains another wavefront. Note that actually this cannot be a V-wave or a non-classical shock as otherwise there would exists (s,x) in $\mathcal{T}_0$ such that $V(\rho^n(s,x),w^n(s,x)) > V_b$ and this contradicts with (3.111). Assuming that ξ^n interacts with another wavefront at s_1. In that case, ξ^n can only interact with a 2-wave or a shock, or a rarefaction shock. Therefore, there is exactly one 1-wave generated by this interaction. So, we define, for any $\in [s_1,s_2]$,

$$\dot{\xi}(s) = \sigma\left(\rho^n(s_1,\xi^n(s_1)-),R(V(\rho^n(s_1,\xi^n(s_1)),w^n(s_1,\xi^n(s_1))),w^n(s_1,\xi^n(s_1)-))\right), \tag{3.115}$$

where the term at the right of (3.115) is the speed of the 1-wave generated by the interaction and s_2 is the minimum between the time when ξ^n exits the triangle $\mathcal{T}_0$ and the time when it interacts with another wavefront with a discontinuous velocity. We then define s_3,...,s_k and so on. As ξ sees only shocks and rarefaction shocks, the rest of the proof is similar to [16, Appendix A]. Let t_{ξ^n} (resp. t_{y^n}) be the time when $\xi^n(\cdot)$ (resp. $y^n(\cdot)$) exits the triangle $\mathcal{T}_0$. From (3.113) and by construction of $\xi^n(\cdot)$, for any $(s,x) \in \{[t, t_{\xi^n}) \times \mathbb{R} | x \geqslant \xi^n(t)\} \cap \mathcal{T}_0$

$$V(\rho^n(s,x), w^n(s,x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.116}$$

Thus, together with (3.111), for any $s \in (0, t_{\xi^n})$

$$\begin{aligned} &\dot{\xi}^n(s) \leqslant \max_{y\in[w_{\min},w_{\max}]} \sigma(f(R(V(\rho_-,w)+\varepsilon,y),y), f(R(V(\rho_+,w)+\varepsilon,y),y)), \\ &\min_{w\in[w_{\min},w_{\max}]} \partial_\rho f(R(V(\rho_+-\varepsilon,w),w)) \leqslant \dot{\xi}^n(s), \end{aligned} \tag{3.117}$$

and for any $s \in (0, t_{y^n})$

$$V(\rho_+, w) - \varepsilon \leqslant \dot{y}^n(s) \leqslant V(\rho_-, w) + \varepsilon \tag{3.118}$$

Using (3.101), (3.117) and (3.118), there exits $c > 0$ (independent of n) such that $\min(t_{y^n}, t_{\xi^n}) \geqslant t + c$. From (3.117) and (3.118), for every $s \in (t, t+c)$,

$$\dot{y}^n(s) - \dot{\xi}^n(s) \geqslant V(\rho_-, w_0) + \varepsilon - \sigma(f(R(V(\rho_-,w_0)+\varepsilon,w_0),w_0), f(R(V(\rho_+,w_0)+\varepsilon,w_0),w_0)), \tag{3.119}$$

with $w_0 = \operatorname{argmax}_{w\in[w_{\min},w_{\max}]} \sigma(f(R(V(\rho_-,w)+\varepsilon,w),w), f(R(V(\rho_+,w)+\varepsilon,w),w))$. Therefore, for every $s \in (t, t+c)$,

$$\dot{y}^n(s) - \dot{\xi}^n(s) > 0. \tag{3.120}$$

Using (3.120), $\lim_{n\to\infty} y^n(t) = y(t)$ and $y^n(t) \leqslant \xi^n(t) \leqslant y(t)$, $y^n(\cdot)$ interacts with $\xi^n(\cdot)$ at time $t_n \in (t, t+c)$ and $t_n \to 0$ as $n \to \infty$. □

- If $\rho_- \geqslant \rho^*(w) > \rho_+$ or $\rho_+ \geqslant \rho^*(w) > \rho_-$. As previously let us first exclude a simple case : if $y^n(t) \geqslant y(t)$ for an infinite set of index, then extracting a subsequence we can assume that $y^n(t) \geqslant y(t)$ for any $n \geqslant n_1$. From Lemma 3.10 we have, for any $x \in (y^n(t), y^n(t)+\delta)$,

$$\begin{aligned} &V(\rho^n(x), w^n(x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)), \\ &\text{or } V(\rho^n(x), w^n(x)) \in [V_b - 2\varepsilon, +\infty) \text{ and } V(\rho_+, w_+) \in [V_b - \varepsilon, +\infty). \end{aligned} \tag{3.121}$$

This implies that

$$\min(V_b, V(\rho_+, w_+)) - 2\varepsilon \leqslant \min(V_b, V(\rho^n(t, y^n(t)), w^n(t, y^n(t)))) \leqslant \min(V_b, V(\rho_+, w_+)) + \varepsilon. \tag{3.122}$$

As $\varepsilon > 0$ was chosen arbitrarily provided n is large enough, this implies that

$$\lim_{n\to+\infty} \min(V_b, V(\rho^n(t, y^n(t)), w^n(t, y^n(t)))) = \min(V_b, V(\rho_+, w_+)), \tag{3.123}$$

and ends the results. So we can assume from now on that $y^n(t) < y(t)$ on an infinite set of index, and extracting a subsequence we can assume that $y^n(t) < y(t)$ for any $n \geqslant n_1$. We will now use the following Lemma to eliminate one more case

Lemma 3.14. *Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$, and suppose that $\rho_- > \rho^*(w) \geqslant \rho_+$ or $\rho_+ > \rho^*(w) \geqslant \rho_-$ for any $n \geqslant n_1$. Then the only possible case is $\rho_+ > \rho^*(w) \geqslant \rho_-$.*

This Lemma is proven in Appendix D.

We have the following Lemma

Lemma 3.15. *Let $t \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\varepsilon > 0$. Assume that $\rho_+ \geqslant \rho^*(w) > \rho_-$, ε is small enough such that $V(\rho_+, w_+) - \varepsilon < V_b$ and $y^n(t) < y(t)$ for any $n \geqslant n_1$. Let $\delta > 0$ be given by Lemma 3.10. Then for any $n \geqslant n_1$, there exists $t_{\xi^n} > t$ and a piecewise constant function ξ^n such that*

$$(s, \xi(s)) \in \mathcal{T}_0, \ \forall s \in [t, t_{\xi^n}), \tag{3.124}$$

and for any $(s, x) \in \{[t, t_{\xi^n}) \times \mathbb{R} | x > \xi^n(s)\} \cap \mathcal{T}_0$

$$V(\rho^n(s, x), w^n(s, x)) \in \mathcal{B}_\varepsilon(V(\rho_+, w_+)). \tag{3.125}$$

Besides, if we denote t_{y^n} the time at which $y^n(\cdot)$ exits the triangle, there exists $c > 0$ independent of n such that $\min(t_{y^n}, t_{\xi^n}) - t \geqslant c$ and there exists $t_n > 0$ such that $\xi(t_n) = y^n(t_n)$ and $\lim_{n \to +\infty} t_n = t$.

This proof is analogous to the proof of Lemma 3.13 and will be given in the Appendix E.

A Proof of Lemma 3.3

Since $\Gamma(\cdot)$ may vary only at times t when two waves interact or a wave hits the slow vehicle trajectory, we will consider different types of interactions separately. It is not restrictive to assume that at any interaction time $t = t$ either two waves interact or a wave hits the slow vehicle trajectory (and not multiple interactions). We describe wave interactions by the type of the involved waves, see Section 1. To simplify the notations, a i-wave interacting with a j-wave from the left is represented by i-j with $i, j \in \{1, 2, V\}$. If a i-wave interacts with a j-wave producing a k-wave and a wave of a l-family, we write i-j/ k-l. Here the symbol "/" divides the waves before and after the interaction. Let us start with some basic results

Lemma A.1. *The interactions V-V, V-2, 2-V and 2-2 cannot occur.*

Proof. We argue by contradiction.

i We assume V-wave $((0, w_l), (0, w_m))$ interacts with the V-wave $((0, w_m), (0, w_r))$. By definition of vacuum waves, we have $w_l < w_m < w_r$. Thus, the speed of the V-wave $((0, w_l), (0, w_m))$ is slower than the speed of the V-wave $((0, w_m), (0, w_r))$, whence the contradiction.

ii We assume V-wave $((0, w_l), (0, w_m))$ interacts with the 2-wave $((0, w_m), (\rho_r, w_r))$. We have $w_l < w_m = V(0, w_m) = V(\rho_r, w_r)$. Thus, the speed of the V-wave $((0, w_l), (0, w_m))$ is slower than the speed of the 2-wave $((0, w_m), (\rho_r, w_r))$, whence the contradiction.

iii We assume 2-wave $((\rho_l, w_l), (0, w_m))$ interacts with the V-wave $((0, w_m), (0, w_r))$. We have $V(\rho_l, w_l) = V(0, w_m) = w_m < w_r$. Thus, the speed of the 2-wave $((\rho_l, w_l), (0, w_m))$ is slower than the speed of the V-wave $((0, w_m), (0, w_r))$, whence the contradiction.

iv We assume 2-wave $((\rho_l, w_l), (\rho_m, w_m))$ interacts with the 2-wave $((\rho_m, w_m), (\rho_r, w_r))$. We have $V(\rho_l, w_l) = V(\rho_m, w_m)$ and $V(\rho_m, w_m) = V(\rho_r, w_r)$. Thus, the speed of the 2-wave $((\rho_l, w_l), (\rho_m, w_m))$ is equal to the speed of the 2-wave $((\rho_m, w_m), (\rho_r, w_r))$. We conclude that no interaction occurs, whence the contradiction.

□

For the classical collision between two waves (away from the slow vehicle trajectory), we have the following result:

Lemma A.2. *Assume that the wave* $((\rho_l, w_l), (\rho_m, w_m))$ *interacts with the wave* $((\rho_m, w_m), (\rho_r, w_r))$ *at the point* (t, x) *with* $t > 0$ *and* $x \in \mathbb{R}$. *Then* $\Gamma(t+) \leq \Gamma(t-)$. *The possible interactions are:* 2-1/1-2, 2-1/1-V-2, 2-1/V-2, 1-1/1, V-1/V-2, V-1/1-2.

Next, we introduce the following notations:

- $FV-$*wave*: a wave denoting the slow vehicle trajectory without discontinuity in (ρ^n, w^n). The notation is to indicate a fictitious wave.
- $NF-$*wave*: a wave denoting the slow vehicle trajectory with discontinuity in (ρ^n, w^n). The notation is to indicate a non fictitious wave. A NF-wave is decomposed into three different waves:
  - $NC-$*wave*: a wave $((\rho_l, w_l), (\rho_r, w_r))$ denoting the slow vehicle trajectory verifying $w_l = w_r$, $\rho_l = \hat{\rho}(w_l)$ and $\rho_r = \check{\rho}(w_l)$. The notation is to indicate a non classical shock.
  - $1C-$*wave*: A $1C-$wave is decomposed into two waves:
    - $1Ca-$*wave*: a wave $((\rho_l, w_l), (\rho_r, w_r))$ denoting the slow vehicle trajectory verifying $w_l = w_r$ and $\sigma((\rho_l, w_l), (\rho_r, w_r)) = V_b$ with $\rho_l \in [0, \check{\rho}(w_l)]$ and $\rho_r \in [\hat{\rho}(w_l), 1]$.
    - $1Cb-$*wave*: a wave$((\rho_l, w_l), (\rho_r, w_r))$ denoting the slow vehicle trajectory verifying $w_l = w_r$, $\rho_l = 0$ and $\rho_r > \rho^*(w_l)$.

    The notation $1C$ is to indicate a first family classical shock
  - $2C-$*wave*: a wave$((\rho_l, w_l), (\rho_r, w_r))$ denoting the slow vehicle trajectory verifying $V(\rho_l, w_l) = V(\rho_r, w_r)$ and $\rho_l \geqslant \rho^*(w_l)$ and $\rho_r \geqslant \rho^*(w_r)$. The notation is to indicate a second family classical shock.

A.1 Interactions with a $FV-$wave.

Lemma A.3, Lemma A.4, Lemma A.5 and Lemma A.6 deal with interactions between a i-wave with $i \in \{1, 2, V\}$ and a $FV-$wave. Since $\min(V_b, V(\rho, w)) \leqslant V(\rho, w)$ and $V_b < w_{\min}$, the interactions FV-2 and FV-V can not occur. We only consider the interactions FV-1, 1-FV, 2-FV and V-FV.

Lemma A.3. *Assume that the the* $FV-$*wave interacts with the first family wave* $((\rho_l, w_l), (\rho_r, w_r))$ *at the point* (t, x) *with* $t > 0$ *and* $x \in \mathbb{R}$. *We have the following cases:*

i No wave is produced, FV-1/1-FV. *Then* $\Delta\Gamma(t) = 0$.

ii The interaction is FV-1/NC-1. *Then* $\Delta\Gamma(t) \leqslant 0$.

Proof. In this situation $\rho_l V(\rho_l, w_l) \leqslant F_\alpha(w_l) + V_b \rho_l$ and $w_l = w_r$. We have two different cases.

i In this case, the first family wave hits the slow vehicle trajectory and no new wave is created. It means that $\rho_r V(\rho_r, w_r) \leqslant F_\alpha(w_l) + V_b \rho_r$. Thus, we have:

$$\Delta\Gamma(t) = 0.$$

ii In this case, the first family wave $((\hat{\rho}(w_l), w_l), (\rho_r, w_l))$ hits the slow vehicle trajectory producing a $NC-$wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$ and a first family wave $((\hat{\rho}(w_l), w_l), (\rho_r, w_l))$, with

positive speed. It means that $\rho_r V(\rho_r, w_r) > F_\alpha(w_r) + V_b\rho_r$ and $\hat{v}(w_l) \leqslant V(\rho_r, w_r) \leqslant \check{v}(w_l)$. We have

$$\begin{aligned}
\Delta TV(w) &= 0,\\
\Delta TV(v) &= |V(\rho_r, w_r) - \check{v}(w_l)| + |\check{v}(w_l) - \hat{v}(w_l)| - |V(\rho_r, w_r) - \hat{v}(w_l)|\\
&= 2|V(\rho_r, w_r) - \check{v}(w_l)|,\\
\Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) &= 0.
\end{aligned}$$

Since $\gamma(t^+) - \gamma(t^-) = -2|\hat{v}(w_l) - \check{v}(w_l)|$ and $2|V(\rho_r, w_r) - \check{v}(w_l)| \leqslant 2|\hat{v}(w_l) - \check{v}(w_l)|$, we conclude that

$$\Delta\Gamma(t) = 2|V(\rho_r, w_r) - \check{v}(w_l)| - 2|\hat{v}(w_l) - \check{v}(w_l)| \leqslant 0.$$

□

Lemma A.4. *Assume that the first family wave $((\rho_l, w_l), (\rho_r, w_r))$ interacts with the $FV-$wave at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. We have the following cases:*

i No wave is produced, 1-FV/FV-1. Then $\Delta\Gamma(t) = 0$.

ii The interaction is 1-FV/1-NC. Then $\Delta\Gamma(t) \leqslant 0$,

Proof. In this situation $\rho_r V(\rho_r, w_r) \leqslant F_\alpha(w_r) + V_b\rho_r$ and $w_l = w_r$. We have two different cases.

i In this case, the first family wave hits the slow vehicle trajectory and no new wave is created. It means that $\rho_l V(\rho_l, w_l) \leqslant F_\alpha(w_l) + V_b\rho_l$. Thus, we have

$$\Delta\Gamma(t) = 0.$$

ii In this case, the first family wave hits the slow vehicle trajectory producing a first family wave $((\rho_l, w_l), (\hat{\rho}(w_l), w_l))$ and a $NC-$wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$. It means that $\rho_l V(\rho_l, w_l) > F_\alpha(w_l) + V_b\rho_l$ and we have $(\rho_r, w_r) = (\hat{\rho}(w_l), w_l)$ and $\hat{v}(w_l) \leqslant V(\rho_l, w_l) \leqslant \check{v}(w_l)$. We have

$$\begin{aligned}
\Delta TV(w) &= 0,\\
\Delta TV(v) &= |\check{v}(w_l) - \hat{v}(w_l)| + |\hat{v}(w_l) - V(\rho_l, w_l)| - |V(\rho_l, w_l) - \check{v}(w_l)|\\
&= 2|V(\rho_l, w_l) - \hat{v}(w_l)|,\\
\Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) &= 0.
\end{aligned}$$

Since $\gamma(t^+) - \gamma(t^-) = -2|\hat{v}(w_l) - \check{v}(w_l)|$ and $2|V(\rho_l, w_l) - \check{v}(w_l)| \leqslant 2|\hat{v}(w_l) - \check{v}(w_l)|$, we conclude that

$$\Delta\Gamma(t) = 2|V(\rho_l, w_l) - \hat{v}(w_l)| - 2|\hat{v}(w_l) - \check{v}(w_l)| \leqslant 0.$$

□

Lemma A.5. *Assume that the second family wave $((\rho_l, w_l), (\rho_r, w_r))$ interacts with the $FV-$wave at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. We have the following cases:*

i No wave is produced, 2-FV/FV-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.

ii The interaction is 2-FV/1-NC-1-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.

Proof. i In this case, the second family wave hits the slow vehicle trajectory and no new wave is created. It means that $\rho_l V(\rho_l, w_l) \leqslant F_\alpha(w_l) + V_b\rho_l$. Thus, we have:

$$\Delta\Gamma(t) = -C|w_l - w_r|.$$

ii In this case, the second family wave hits the slow vehicle trajectory producing a first family wave $((\rho_l, w_l), (\hat{\rho}(w_l), w_l))$, a $NFV-$wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$, a first family wave with positive speed $((\check{\rho}(w_l), w_l), (\rho_l, w_l))$ and a second family wave $((\rho_l, w_l), (\rho_r, w_r))$. It means that $\rho_l V(\rho_l, w_l) > F_\alpha(w_l) + V_b \rho_l$. We have

$$\begin{aligned}
\Delta TV(w) &= |w_r - w_l| - |w_r - w_l| = 0, \\
\Delta TV(v) &= |V(\rho_r, w_r) - V(\rho_l, w_l)| + |V(\rho_l, w_l) - \check{v}(w_l)| + |\hat{v}(w_l) - \check{v}(w_l)| \\
&\quad + |\hat{v}(w_l) - V(\rho_l, w_l)| - |V(\rho_l, w_l) - V(\rho_r, w_r)| \\
&= 2|\hat{v}(w_l) - \check{v}(w_l)|.
\end{aligned}$$

Since $\gamma(t^+) - \gamma(t^-) = -2|\hat{v}(w_l) - \check{v}(w_l)|$ and $TV(w(\cdot, y(\cdot)), [t, +\infty)) = -|w_l - w_r|$, we conclude that

$$\Delta\Gamma(t) = -C|w_l - w_r|.$$

□

Lemma A.6. *Assume that the V-wave $((0, w_l), (0, w_r))$ with $w_l < w_r$ interacts with the $FV-$wave at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. We have a unique case: no wave is produced, V-FV/FV-V. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.*

Proof. The vacuum wave hits the slow vehicle trajectory and no new wave is created. It means that $\rho_l V(\rho_l, w_l) = 0 \leqslant F_\alpha(w_l) + V_b \rho_l$. Thus, we have immediately:

$$\Delta\Gamma(t) = -C|w_l - w_r|.$$

□

A.2 Interactions with a $NC-$wave.

Lemma A.7, Lemma A.8, Lemma A.9 deal with interactions between a i-wave with $i \in \{1, 2, V\}$ and a NC-wave. Since $0 \neq \check{\rho}(w) < \hat{\rho}(w)$, $\min(V_b, V(\rho, w)) \leqslant V(\rho, w)$ and $V_b < w_{\min}$, the interactions NC-2, NC-V and V-NC can not occur. We only consider the interactions NC-1, 1-NC and 2-NC.

Lemma A.7. *Assume that the first family wave $((\rho_l, w_l), (\hat{\rho}(w_l), w_l))$ interacts with the NC-wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$ at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The only interaction is 1-NC/FV-1. Then $\Delta\Gamma(t) = 0$.*

Proof. We have $\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_l), w_l))(V_b) = (\rho_l, w_l)$ and $\rho_l \in (0, \check{\rho}(w_l))$. The only possible interaction is 1-NC/FV-1. Since $V(\rho_l, w_l) > \check{v}(w_l)$, we have:

$$\begin{cases}
\Delta TV(w) & = 0, \\
\Delta TV(v) & = |\check{v}(w_l) - V(\rho_l, w_l)| - |\check{v}(w_l) - \hat{v}(w_l)| - |\hat{v}(w_l) - V(\rho_l, w_l)| \\
& = -2|\check{v}(w_l) - \hat{v}(w_l)|, \\
\Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) & = 0, \\
\gamma(t^+) - \gamma(t^-) & = 2|\check{v}(w_l) - \hat{v}(w_l)|.
\end{cases}$$

We conclude that

$$\Delta\Gamma(t) = 0.$$

□

Lemma A.8. *Assume that the a NC-wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$ from the left interacts with the first family wave $((\check{\rho}(w_l), w_l), (\rho_r, w_l))$ at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The only interaction is NC-1/1-FV. Then $\Delta\Gamma(t) = 0$.*

Proof. We have $\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_l), w_l))(V_b) = (\rho_r, w_l)$ and $\rho_r \in (\hat{\rho}(w_l), \rho_{\max}]$. The only possible interaction is NC-1/1-FV. Since $V(\rho_r, w_r) := V(\rho_r, w_l) < \hat{v}(w_l)$, we have:

$$\begin{cases} \Delta TV(w) & = 0, \\ \Delta TV(v) & = |V(\rho_r, w_r) - \hat{v}(w_l)| - |V(\rho_r, w_r) - \check{v}(w_l)| - |\check{v}(w_l) - \hat{v}(w_l)| \\ & = -2|\check{v}(w_l) - \hat{v}(w_l)|, \\ \Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) & = 0, \\ \gamma(t^+) - \gamma(t^-) & = 2|\check{v}(w_l) - \hat{v}(w_l)|. \end{cases}$$

We conclude that

$$\Delta\Gamma(t) = 0.$$

□

Lemma A.9. *Assume that the second family wave $((\rho_l, w_l), (\hat{\rho}(w_r), w_r))$ interacts with the $NC-$wave $((\hat{\rho}(w_r), w_r), (\check{\rho}(w_r), w_r))$ at the point (t, x) with $t \geqslant 0$ and $x \in \mathbb{R}$. We have the following cases:*

i No wave is produced, 2-NC/1-FV-2, 2-NC/FV-1-2 or 2-NC/FV-1-V-2. Then $\Delta\Gamma(t) = 0$.

ii The interaction are 2-NC/1-NC-1-2 and 2-NC/1-NC-1-V-2. Then $\Delta\Gamma(t) \leqslant 0$.

Proof. Since $V(\rho_l, w_l) = V(\hat{\rho}(w_r), w_r) < V(\check{\rho}(w_r), w_r)$, we have $\rho_l > R(\check{v}(w_r), w_l)$. Thus,

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = \begin{cases} (R(\check{v}(w_r), w_l), w_l) & \text{if} \quad \check{v}(w_r) < V(\rho_c(w_l), w_l), \\ (\rho_l, w_l) & \text{if} \quad V(\rho_c(w_l), w_l) < \hat{v}(w_r), \\ (\rho_c(w_l), w_l) & \text{if} \quad \hat{v}(w_r) \leqslant V(\rho_c(w_l), w_l) \leqslant \check{v}(w_r). \end{cases} \tag{A.1}$$

i $\hat{v}(w_r) < \check{v}(w_r) \leqslant \hat{v}(w_l) < \check{v}(w_l)$: from (A.1) and using $\check{v}(w_r) < V(\rho_c(w_l), w_l)$,

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = (R(\check{v}(w_r), w_l), w_l).$$

Since $\check{v}(w_r) \leqslant \hat{v}(w_l)$, $\hat{\rho}(w_l) \leqslant R(\check{v}(w_r), w_l)$. Thus, no wave is produced and the only interaction is 2-NC/1-FV-2; The second family wave hits the NC-wave producing a first family wave $((\rho_l, w_l), (\rho_r, w_r))$ and a second family wave $((R(\check{v}(w_r), w_l), w_l), (\check{\rho}(w_r), w_r))$. Since $V(\rho_l, w_l) = \hat{v}(w_r)$ and $V(R(\check{v}(w_r), w_l), w_l) = \check{v}(w_r)$ we have

$$\begin{cases} \Delta TV(w) & = 0, \\ \Delta TV(v) & = |\check{v}(w_r) - \hat{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)| = 0, \\ \Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) & = -C|w_l - w_r|, \\ \gamma(t^+) - \gamma(t^-) & = 2|\hat{v}(w_r) - \check{v}(w_r)| \end{cases} \tag{A.2}$$

We conclude that

$$\Delta\Gamma(t) = 2|\hat{v}(w_r) - \check{v}(w_r)| - C|w_l - w_r|.$$

Using $\hat{v}(w_r) < \check{v}(w_r) \leqslant \hat{v}(w_l)$, $|\hat{v}(w_r) - \check{v}(w_r)| \leqslant |\hat{v}(w_r) - \hat{v}(w_l)|$. From Lemma 3.2, we conclude that

$$|\hat{v}(w_r) - \check{v}(w_r)| \leqslant C_{\hat{\rho}}|w_l - w_r|.$$

Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

$$\Delta\Gamma(t) \leqslant 0.$$

ii $\hat{v}(w_l) < \check{v}(w_l) \leqslant \hat{v}(w_r) < \check{v}(w_r)$: from (A.1) and using $V(\rho_c(w_l), w_l) < \hat{v}(w_r)$,

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = (\rho_l, w_l).$$

Since $\check{v}(w_l) \leqslant \hat{v}(w_r)$, we have $\rho_l \leqslant \check{\rho}(w_l)$. Thus, no waves is produced and the possible interactions are 2-NC/FV-1-2 and 2-NC/FV-1-V-2;

- Case 2-NC/FV-1-2: (A.2) holds and we conclude that

$$\Delta\Gamma(t) = 2|\hat{v}(w_r) - \check{v}(w_r)| - C|w_l - w_r|.$$

Using $\check{v}(w_l) \leqslant \hat{v}(w_r) < \check{v}(w_r)$, $|\hat{v}(w_r) - \check{v}(w_r)| \leqslant |\check{v}(w_r) - \check{v}(w_l)|$. From Lemma 3.2, we conclude that

$$|\hat{v}(w_r) - \check{v}(w_r)| \leqslant C_{\check{\rho}}|w_l - w_r|.$$

Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

$$\Delta\Gamma(t) \leqslant 0.$$

- Case 2-NC/FV-1-V-2: The second family wave hits the NC-wave producing a first family wave $((\rho_l, w_l), (0, w_l))$, a vacuum wave $((0, w_l), (0, \check{v}(w_r)))$ and a second family wave $((0, \check{v}(w_r)), (\check{\rho}(w_r), w_r))$. From Section 1, since a vacuum occurs then $\hat{v}(w_r) = V(\rho_l, V(\rho_l, w_l)) \leqslant V(0, w_l) = w_l$ and $w_l \leqslant \check{v}(w_r)$. Thus, we have

$$\begin{aligned}\Delta TV(w) &= |w_r - \check{v}(w_r)| + |\check{v}(w_r) - w_l| - |w_r - w_l| = 0,\\ \Delta TV(v) &= |\check{v}(w_r) - w_l| + |w_l - \check{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)| = 0.\end{aligned}$$

Thus, (A.2) holds and we conclude that

$$\Delta\Gamma(t) = 2|\hat{v}(w_r) - \check{v}(w_r)| - C|w_l - w_r|.$$

Using $\check{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_r)$, $|\hat{v}(w_r) - \check{v}(w_r)| \leqslant |\check{v}(w_r) - \check{v}(w_l)|$. From Lemma 3.2, we conclude that

$$|\hat{v}(w_r) - \check{v}(w_r)| \leqslant C_{\check{\rho}}|w_l - w_r|.$$

Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

$$\Delta\Gamma(t) \leqslant 0.$$

iii Case $\hat{v}(w_r) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_r) \leqslant \check{v}(w_l)$: from (A.1)

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = \begin{cases} (R(\check{v}(w_r), w_l), w_l) & \text{if} \quad \check{v}(w_r) < V(\rho_c(w_l), w_l) \\ (\rho_c(w_l), w_l) & \text{if} \quad \check{v}(w_r) \geqslant V(\rho_c(w_l), w_l) \geqslant \hat{v}(w_l) \geqslant \hat{v}(w_r) \end{cases}$$

Since $\hat{v}(w_l) \leqslant \check{v}(w_r) \leqslant \check{v}(w_l)$ we have $R(\check{v}(w_r), w_l) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. Thus, $\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. The only interaction is 2-NC/1-NC-1-2; the second family wave hits the NC-wave producing a first family wave $((\rho_l, w_l), (\hat{\rho}(w_l), w_l))$, a NC-wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$, a first family wave $((\check{\rho}(w_l), w_l), (R(\check{v}(w_r), w_l), w_l))$ and a second family wave $((R(\check{v}(w_r), w_l), w_l), (\check{\rho}(w_r), w_r))$. Since $V(\rho_l, w_l) = \hat{v}(w_r)$ and $V(R(\check{v}(w_r), w_l), w_l) = \check{v}(w_r)$ we have

$$\begin{cases} \Delta TV(w) & = 0, \\ \Delta TV(v) & = |\check{v}(w_r) - \check{v}(w_l)| + |\check{v}(w_l) - \hat{v}(w_l)| + |\hat{v}(w_l) - \hat{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)|, \\ \Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) & = -C|w_l - w_r|, \\ \gamma(t^+) - \gamma(t^-) & = -2|\hat{v}(w_l) - \check{v}(w_l)| + 2|\hat{v}(w_r) - \check{v}(w_r)|. \end{cases} \tag{A.3}$$

Since $\hat{v}(w_r) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_r) \leqslant \check{v}(w_l)$, we get $\Delta TV(v) = 2|\check{v}(w_l) - \check{v}(w_r)|$ and $\gamma(t^+) - \gamma(t^-) = -2|\check{v}(w_l) - \check{v}(w_r)| + 2|\hat{v}(w_l) - \hat{v}(w_r)|$. We conclude that

$$\Delta\Gamma(t) = 2|\hat{v}(w_r) - \hat{v}(w_l)| - C|w_l - w_r|.$$

From Lemma 3.2, we get

$$|\hat{v}(w_r) - \hat{v}(w_l)| \leqslant C_{\hat{\rho}}|w_l - w_r|.$$

Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

$$\Delta\Gamma(t) \leqslant 0.$$

iv Case $\hat{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_l) \leqslant \check{v}(w_r)$: from (A.1)

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = \begin{cases} (\rho_l, w_l) & \text{if} \quad \hat{v}(w_r) > V(\rho_c(w_l), w_l) \\ (\rho_c(w_1), w_1) & \text{if} \quad \hat{v}(w_r) \leqslant V(\rho_c(w_l), w_l) \leqslant \check{v}(w_l) \leqslant \hat{v}(w_r) \end{cases}$$

Since $\hat{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_l)$, we have $\rho_l \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. Thus, $\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. Thus, the possible interactions are 2-NC/1-NC-1-2 and 2-NC/1-NC-1-V-2;

- Case 2-NC/1-NC-1-2: from (A.3) and using $\hat{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_l) \leqslant \check{v}(w_r)$, we get $\Delta TV(v) = 2|\hat{v}(w_l) - \hat{v}(w_r)|$ and $\gamma(t^+) - \gamma(t^-) = 2|\check{v}(w_l) - \check{v}(w_r)| - 2|\hat{v}(w_l) - \hat{v}(w_r)|$. We conclude that

  $$\Delta\Gamma(t) = 2|\check{v}(w_r) - \check{v}(w_l)| - C|w_l - w_r|.$$

  From Lemma 3.2, we get

  $$|\check{v}(w_r) - \check{v}(w_l)| \leqslant C_{\check{\rho}}|w_l - w_r|.$$

  Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

  $$\Delta\Gamma(t) \leqslant 0.$$

- Case 2-NC/1-NC-1-V-2: the second family wave hits the NC-wave producing a first family wave $((\rho_l, w_l), (\hat{\rho}(w_l), w_l))$, a NC-wave $((\hat{\rho}(w_l), w_l), (\check{\rho}(w_l), w_l))$, a first family wave $((\check{\rho}(w_l), w_l), (0, w_l))$ a vacuum wave $((0, w_l), (0, \check{v}(w_r)))$ and a second family wave $((0, \check{v}(w_r)), (\check{\rho}(w_r), w_r))$. From Section 1, since a vacuum occurs then $\hat{v}(w_r) = V(\rho_l, w_l) \leqslant V(0, w_l) = w_l$ and $w_l \leqslant \check{v}(w_r)$. Thus, we have

  $$\begin{aligned} \Delta TV(w) &= |w_r - \check{v}(w_r)| + |\check{v}(w_r) - w_l| - |w_r - w_l| = 0 \\ \Delta TV(v) &= |\check{v}(w_r) - w_l| + |w_l - \check{v}(w_l)| + |\check{v}(w_l) - \hat{v}(w_l)| + |\hat{v}(w_l) - \hat{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)| \\ &= |\check{v}(w_r) - \check{v}(w_l)| + |\check{v}(w_l) - \hat{v}(w_l)| + |\hat{v}(w_l) - \hat{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)|. \end{aligned}$$

  Thus, (A.3) holds and similarly to the case 2-NC/1-NC-1-V-2 above we have $\Delta\Gamma(t) \leqslant 0$.

v Case $\hat{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_r) \leqslant \check{v}(w_l)$: we have $R(\check{v}(w_r), w_l) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$ and $\rho_l \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. Thus, $\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. The only possible interaction is 2-NC/1-NC-1-2. From (A.3) and using $\hat{v}(w_l) \leqslant \hat{v}(w_r) \leqslant \check{v}(w_r) \leqslant \check{v}(w_l)$, we get $\Delta TV(v) = 2|\check{v}(w_l) - \check{v}(w_r)| + 2|\hat{v}(w_l) - \hat{v}(w_r)|$ and $\gamma(t^+) - \gamma(t^-) = -2|\check{v}(w_l) - \check{v}(w_r)| - 2|\hat{v}(w_l) - \hat{v}(w_r)|$. We conclude that

$$\Delta\Gamma(t) = -C|w_l - w_r| < 0.$$

vi Case $\hat{v}(w_r) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_l) \leqslant \check{v}(w_r)$: from (A.1)

$$\mathcal{RS}((\rho_l, w_l), (\check{\rho}(w_r), w_r))(V_b) = (\rho_c(w_l), w_l) \in (\check{\rho}(w_l), \hat{\rho}(w_l)).$$

Thus, the possible interactions are 2-NC/1-NC-1-2 and 2-NC/1-NC-1-V-2;

- Case 2-NC/1-NC-1-2: From (A.3) and using $\hat{v}(w_r) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_l) \leqslant \check{v}(w_r)$, we get $\Delta TV(v) = 0$ and $\gamma(t^+) - \gamma(t^-) = 2|\check{v}(w_l) - \check{v}(w_r)| + 2|\hat{v}(w_l) - \hat{v}(w_r)|$. We conclude that

$$\Delta\Gamma(t) = 2|\check{v}(w_l) - \check{v}(w_r)| + 2|\hat{v}(w_l) - \hat{v}(w_r)| - C|w_l - w_r|.$$

From Lemma 3.2, we get

$$|\check{v}(w_l) - \check{v}(w_r)| + |\hat{v}(w_l) - \hat{v}(w_r)| \leqslant (C_{\check{\rho}} + C_{\hat{\rho}})|w_l - w_r|.$$

Since $C := 2(C_{\hat{\rho}} + C_{\check{\rho}})$, we obtain

$$\Delta\Gamma(t) \leqslant 0.$$

- Case 2-NC/1-NC-1-V-2: From Section 1, since a vacuum occurs then $\hat{v}(w_r) = V(\rho_l, w_l) \leqslant V(0, w_l) = w_l$ and $w_l \leqslant \check{v}(w_r)$. Thus, we have

$$\Delta TV(w) = |w_r - \check{v}(w_r)| + |\check{v}(w_r) - w_l| - |w_r - w_l| = 0$$
$$\Delta TV(v) = |\check{v}(w_r) - \check{v}(w_l)| + |\check{v}(w_l) - \hat{v}(w_l)| + |\hat{v}(w_l) - \hat{v}(w_r)| - |\check{v}(w_r) - \hat{v}(w_r)|.$$

So, (A.3) holds and similarly to the case 2-NC/1-NC-1-2 above, we conclude that

$$\Delta\Gamma(t) \leqslant 0.$$

□

A.3 Interactions with a $1C-$wave.

Lemma A.10, Lemma A.11 and Lemma A.12 deal with interactions between a i-wave with $i \in \{1, 2, V\}$ and a $1Ca-$wave. Since $\rho^*(w) > 0$ and $\min(V_b, V(\rho, w)) \leqslant V(\rho, w)$, the interactions $1Ca$-V and $1Ca$-2 can not occur. We only consider the interactions 1-$1Ca$, $1Ca$-1, 2-$1Ca$ and V-$1Ca$.

Lemma A.10.

- *Assume that the first family wave $((\rho_l, w_l), (\rho_m, w_l))$ interacts with the $1Ca$-wave $(\rho_m, w_l), (\rho_r, w_l))$ from the left at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are 1-$1Ca/FV$-1 and 1-$1Ca$/1-FV. Then $\Delta\Gamma(t) = 0$.*
- *Assume that the $1Ca$-wave $((\rho_l, w_l), (\rho_m, w_l))$ interacts with the first family wave $(\rho_m, w_l), (\rho_r, w_l))$ from the left at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are $1Ca$-1/FV-1 and $1Ca$-1/1-FV. Then $\Delta\Gamma(t) = 0$.*

Proof.

- Since the first family wave interacts with the slow vehicle trajectory coming from the left, we have $\rho_l \in (0, \rho_r)$. Thus,

$$\mathcal{RS}((\rho_l, w_l), (\rho_r, w_l))(V_b) = \begin{cases} (\rho_l, w_l) & \text{if} \quad \rho_l \in (0, \rho_m), \\ (\rho_r, w_l) & \text{if} \quad \rho_l \in (\rho_m, \rho_r). \end{cases} \tag{A.4}$$

Using that $\rho_m \in [0, \check{\rho}(w_l)]$ and $\rho_r \in [\hat{\rho}(w_l), \rho^*(w_l)]$, we conclude that $\mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_l))(V_b) \in [0, \check{\rho}(w_l)] \cup [\hat{\rho}(w_l), \rho^*(w_l)]$; no waves is produced and the possible interactions are 1-$1Ca/FV$-1 and 1-$1Ca$/1-FV. From Lemma A.2, $\Delta\Gamma(t) = 0$.

- Since the first family interacts with the slow vehicle trajectory coming from the right, we have $\rho_r \in (\rho_l, \rho_{\max})$. Thus,

$$\mathcal{RS}((\rho_l, w_l), (\rho_r, w_l))(V_b) = \begin{cases} (\rho_l, w_l) & \text{if} \quad \rho_r \in (\rho_l, \rho_m) \\ (\rho_r, w_l) & \text{if} \quad \rho_r \in (\rho_m, \rho_{\max}) \end{cases} \tag{A.5}$$

  Using that $\rho_l \in [0, \check{\rho}(w_l)]$ and $\rho_r \in [\hat{\rho}(w_l), \rho^*(w_l)]$, we conclude that $\mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_l))(V_b) \in [0, \check{\rho}(w_l)] \cup [\hat{\rho}(w_l), \rho^*(w_l)]$; no waves is produced and the possible interactions are $1Ca$-$1/FV$-1 and $1Ca$-$1/1$-FV. From Lemma A.2, $\Delta\Gamma(t) = 0$.

□

Lemma A.11. *Assume that the second family wave $((\rho_l, w_l), (\rho_m, w_m))$ interacts with the $1Ca-$wave $((\rho_m, w_m), (\rho_r, w_r))$ at the point (t, x) with $t \geqslant 0$ and $x \in \mathbb{R}$. We have the following cases:*

i No wave is produced, 2-$1Ca/1$-FV-2, 2-$1Ca/FV$-1-2 or 2-$1Ca/FV$-1-V-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.

ii The interaction are 2-$1Ca/1$-NC-1-2 and 2-$1Ca/1$-NC-1-V-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.

Proof. We have $V(\rho_l, w_l) = V(\rho_m, w_m)$, $w_m = w_r$ and $\sigma((\rho_m, w_m), (\rho_r, w_r)) = V_b$. Moreover, $\rho_m \in [0, \check{\rho}(w_r)]$ and $\rho_r \in [\hat{\rho}(w_r), \rho^*(w_r)]$. Since $V(\rho_l, w_l) = V(\rho_m, w_m) > V(\rho_r, w_r) = V(R(V(\rho_r, w_r), w_l), w_l)$, we have $\rho_l < R(V(\rho_r, w_r), w_l), w_l)$. Thus,

$$\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = \begin{cases} (\rho_l, w_l) & \text{if} \quad V_b \leqslant \sigma(\rho_l, R(V(\rho_r, w_r), w_l)) \\ (R(\rho_r, w_l), w_l) & \text{if} \quad \sigma(\rho_l, R(V(\rho_r, w_r), w_l)) < V_b \end{cases} \tag{A.6}$$

i Case $\hat{v}(w_l) \leqslant \check{v}(w_l) \leqslant V(\rho_r, w_r) \leqslant V(\rho_m, w_m)$: in this case, $0 \leqslant \rho_l < R(V(\rho_r, w_r), w_l) \leqslant \check{\rho}(w_l)$ then $V_b \leqslant \sigma(\rho_l, R(V(\rho_r, w_r), w_l))$. Thus, we have $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = (\rho_l, w_l)$ and $\rho_l \in [0, \check{\rho}(w_l)]$. So, no wave is produced and the possible interactions are 2-$1Ca/FV$-1-2 and 2-$1Ca/FV$-1-V-2. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

ii Case $\hat{v}(w_l) \leqslant V(\rho_r, w_r) \leqslant \check{v}(w_l) \leqslant V(\rho_m, w_m)$: in this case $\rho_l \leqslant \check{\rho}(w_l) \leqslant R(V(\rho_r, w_r), w_l) \leqslant \hat{\rho}(w_l)$. Thus, $V_b \leqslant \sigma(\rho_l, R(\rho_r, w_l))$ and then $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = (\rho_l, w_l)$ with $\rho_l \in [0, \check{\rho}(w_l)]$. So, no wave is produced and the only possible interaction is 2-$1Ca/FV$-1-2. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

iii $V(\rho_r, w_r) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_l) \leqslant V(\rho_m, w_m)$: in this case $\rho_l \leqslant \check{\rho}(w_l) \leqslant \hat{\rho}(w_l) \leqslant R(V(\rho_r, w_r), w_l)$. From (A.6), $\mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_r))(V_b) \in [0, \check{\rho}(w_l)] \cup [\hat{\rho}(w_l), \rho_{\max}]$. No wave is produced and the possible interactions are 2-$1Ca/FV$-1-2 and 2-$1Ca/1$-FV-2. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

iv $V(\rho_r, w_r) \leqslant V(\rho_m, w_m) \leqslant \hat{v}(w_l) \leqslant \check{v}(w_l)$: in this case $\check{\rho}(w_l) \leqslant \hat{\rho}(w_l) \leqslant \rho_l \leqslant R(V(\rho_r, w_r), w_l)$. No wave is produced and the possible interactions are 2-$1Ca/FV$-1-2 and 2-$1Ca/1$-FV-2. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

v $V(\rho_r, w_r) \leqslant \hat{v}(w_l) \leqslant V(\rho_m, w_m) \leqslant \check{v}(w_l)$: in this case $\check{\rho}(w_l) \leqslant \rho_l \leqslant \hat{\rho}(w_l) \leqslant R(V(\rho_r, w_r), w_l)$. Thus, $\sigma(\rho_l, R(\rho_r, w_l)) \leqslant V_b$ and then, from (A.6), $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = (R(\rho_r, w_l), w_l)$ with $R(\rho_r, w_l) \in [\hat{\rho}(w_l), \rho_{\max}]$. No wave is produced and the only possible interaction is 2-$1Ca/1$-FV-2. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

vi $\hat{v}(w_l) \leqslant V(\rho_r, w_r) \leqslant V(\rho_m, w_m) \leqslant \check{v}(w_l)$: in this case $\check{\rho}(w_l) \leqslant \rho_l \leqslant R(V(\rho_r, w_r), w_l) \leqslant \hat{\rho}(w_l)$. From (A.6), $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) \in (\check{\rho}(w_l), \hat{\rho}(w_l))$. The only possible interaction is 2-$1Ca/1$-NC-1-2. We have

$$\begin{cases} \Delta TV(w) & = 0, \\ \Delta TV(v) & = |\check{v}(w_l) - V(\rho_r, w_r)| + |\check{v}(w_l) - \hat{v}(w_l)| \\ & \quad + |\hat{v}(w_l) - V(\rho_m, w_m)| - |V(\rho_m, w_m) - V(\rho_r, w_r)|, \\ \Delta TV(w(\cdot, y(\cdot)), [t, +\infty)) & = -C|w_r - w_l|, \\ \gamma(t^+) - \gamma(t^-) & = -2|\check{v}(w_l) - \hat{v}(w_l)|. \end{cases}$$

Since $\hat{v}(w_l) \leqslant V(\rho_r, w_r) \leqslant V(\rho_m, w_m) \leqslant \check{\rho}(w_l)$, $\Delta TV(v) = 2|\check{v}(w_l) - \hat{v}(w_l)|$. We conclude that

$$\Delta\Gamma(t) = -C|w_r - w_l|.$$

□

Lemma A.12. *Assume that the vacuum family wave $((0, w_l), (0, w_r))$ interacts with the $1Ca-$wave $((0, w_r), (\rho_r, w_r))$ at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are V-$1Ca/1Ca$-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.*

Proof. By definition of vacuum waves, $w_l < w_r$ and by definition of $1Ca$-wave, $\rho_r = \rho^*(w_r)$. The only possible interaction is V-$1Ca/1Ca$-2. Since $V(\rho^*(w_r), w_r) = V(\rho^*(w_l), w_l)$, we have immediately that $\Delta\Gamma(t) = -C|w_l - w_r|$.

□

Lemma A.13, Lemma A.14, Lemma A.15 deal with interactions between a i-wave with $i \in \{1, 2, V\}$ and a $1Cb-$wave. Since $\min(V_b, V(\rho, w)) \leqslant V(\rho, w)$ and $V_b < w_{\min}$, the interactions $1Cb$-2 and $1Cb$-V can not occur. We only consider the interactions $1Cb$-1, 1-$1Cb$, 2-$1Cb$ and V-$1Cb$.

Lemma A.13.
- *Assume that the first family wave $((\rho_l, w_l), (0, w_l))$ interacts with the $1Cb$-wave $(0, w_l), (\rho_r, w_l))$ from the left at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are 1-$1Cb/FV$-1 and 1-$1Cb/1$-FV. Then $\Delta\Gamma(t) = 0$.*
- *Assume that the $1Cb$-wave $((0, w_l), (\rho_m, w_l))$ interacts with the first family wave $(\rho_m, w_l), (\rho_r, w_l))$ from the left at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are $1Cb$-$1/FV$-1 and $1Cb$-1/1-FV. Then $\Delta\Gamma(t) = 0$.*

Proof.
- We have $\rho_l \in (0, \rho_r)$. Thus, using that $\rho_r > \rho^*(w_l)$, $\mathcal{RS}((\rho_l, w_l)(\rho_r, w_l))(V_b) = (\rho_r, w_r)$. Thus, no wave is produced and the only possible interaction is 1-$1Cb/1$-FV. From Lemma A.2, $\Delta\Gamma(t) = 0$.
- We have

$$\mathcal{RS}((0, w_l), (\rho_r, w_l))(V_b) = \begin{cases} (0, w_l) & \text{if} \quad \rho_r < \rho^*(w_l) \\ (\rho_r, w_l) & \text{if} \quad \rho_r > \rho^*(w_l) \end{cases} \tag{A.7}$$

We conclude that $\mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_l))(V_b) \in [0, \check{\rho}(w_l)] \cup [\hat{\rho}(w_l), \rho^*(w_l)]$; no waves is produced and the possible interactions are $1Cb$-$1/FV$-1 and $1Cb$-$1/1Cb$. From Lemma A.2, $\Delta\Gamma(t) = 0$

□

Lemma A.14. *Assume that the second family wave $((\rho_l, w_l), (\rho_m, w_m))$ interacts with the $1Cb-$wave $((0, w_r), (\rho_r, w_r))$ at the point (t, x) with $t \geqslant 0$ and $x \in \mathbb{R}$. No wave is produced, the only possible interaction is 2-$1Cb/1$-FV-2. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.*

Proof. Since the speed of the first family $((\rho_l, w_l), (R(V(\rho_r, w_r), w_l), w_l))$ is slower than the speed V_b, we have $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = (R(V(\rho_r, w_r), w_l), w_l)$. Besides, using that $V(R(V(\rho_r, w_r), w_l), w_l) = V(\rho_r, w_r)$ and $\rho_r > \rho^*(w_r)$, $R(V(\rho_r, w_r), w_l) > \rho^*(w_l)$. Thus, no wave is produced, the only possible interaction is 2-1Cb/1-FV-2. Then, from Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$

□

Lemma A.15. *Assume that the vacuum family wave $((0, w_l), (0, w_r))$ interacts with the $1Cb-$wave $((0, w_r), (\rho_r, w_r))$ at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The possible interactions are V-1Cb/1Cb-2C. Then $\Delta\Gamma(t) = -C|w_l - w_r|$.*

Proof. Since $w_l < w_r$ and $\rho_r > \rho^*(w_r)$ we have $V(\rho_r, w_r) \leqslant V(\rho^*(w_r), w_r) = V(\rho^*(w_l), w_l) \leqslant w_l$ Thus, the only possible interaction is V-1Cb/1-2C. Then, from Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$. □

A.4 Interactions with a $2C-$wave.

Using Lemma A.1, $\min(V_b, V(\rho, w)) \leqslant V(\rho, w)$ and $V_b < w_{\min}$, the interactions 1-2C, V-2C, 2C-V, 2C-2 and 2-2C can not occur. Thus, Lemma A.16 deals with only the interaction 2C-1.

Lemma A.16. *Assume that the a 2C-wave $((\rho_l, w_l), (\rho_m, w_r))$ interacts with the first family wave $((\rho_m, w_r), (\rho_r, w_r))$ at the point (t, x) with $t > 0$ and $x \in \mathbb{R}$. The two possible interactions are 2C-1/1-FV-2 and 2C-1/1-2C. Then $\Delta\Gamma(t) = 0$.*

Proof. We have $\rho_l \geqslant \rho^*(w_l)$ and $\rho_m \geqslant \rho^*(w_r)$.

- if $\rho_m > \rho_r$ (that is to say the first family is a shock) then $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = (\rho_r, w_l)$. Thus, the only possible interaction is 2C-1/1-2C and no wave is produced. Then, from Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.
- if $\rho_m < \rho_r$ the first family wave $((\rho_m, w_r), (\rho_r, w_r))$ is a rarefaction. Thus, $\rho_r \geqslant \hat{\rho}(w_r)$ and $R(V(\rho_r, w_r), w_l) \geqslant \hat{\rho}(w_l)$. Besides,

$$\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b) = \begin{cases} (R(V(\rho_r, w_r), w_l), w_l) & \text{if} \quad V_b < V(\rho_r, w_r) \\ (\rho_r, w_r) & \text{if} \quad V_b \geqslant V(\rho_r, w_r) \end{cases}$$

  Thus, no wave is produced. From Lemma A.2, $\Delta\Gamma(t) = -C|w_l - w_r|$.

□

B Proof that $V(\rho(t, y(t)+), w(t, y(t)+)) = V(\rho(t, y(t)-), w(t, y(t)-))$ or $w(t, y(t)+) = w(t, y(t)-)$ almost everywhere.

In this appendix we prove the following claim : there exists $\mathcal{N}$ a space of zero measure such that for any $t \in \mathbb{R}_+ \setminus \mathcal{N}$,

$$V(\rho(t, y(t)+), w(t, y(t)+)) = V(\rho(t, y(t)-), w(t, y(t)-)) \text{ or } w(t, y(t)+) = w(t, y(t)-). \tag{B.1}$$

The argument we use is very similar to the one generally used to derive the Rankine-Hugoniot condition, except that we would like it to be true along the curve $y(t)$ instead of along a shock curve. Let ξ be a continuous function on $\mathbb{R}_+$ and differentiable on $\mathbb{R}_+ \setminus \mathcal{N}_0$, where $\mathcal{N}_0$ is a set of

zero measure and $\dot{\xi} \in L^\infty(\mathbb{R}_+)$. As (ρ, w) is a solution to the weak equation (1.1), we have for any test function $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} \partial_t \phi(t,x)\rho(t,x) + \rho(t,x)V(\rho(t,x), w(t,x))\partial_x \phi(t,x) dtdx = 0, \\ \int_0^{+\infty} \int_{\mathbb{R}} \partial_t \phi(t,x)\rho(t,x)w(t,x) + \rho(t,x)w(t,x)V(\rho(t,x), w(t,x))\partial_x \phi(t,x) dtdx = 0. \end{aligned} \tag{B.2}$$

Let $\varepsilon > 0$, as $C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ can be seen as $C_c^\infty(\mathbb{R}_+; C_c^\infty(\mathbb{R}))$, it is therefore dense in $C_c^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$. We can therefore construct an approximation of $1_{[\xi(t)-\varepsilon, \xi(t)+\varepsilon]}(x)d(t)$ where $d \in C_c^\infty(\mathbb{R}_+)$ and $\mathbf{1}_X$ stands for the characteristic function of X, in the following way : define

$$\phi_n(t,x) = d(t) \int_{-\infty}^x n\chi(n(y - (\xi(t) - \varepsilon))) - n\chi(n(y - (\xi(t) + \varepsilon)))dy \tag{B.3}$$

where $\chi \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(\chi) \subset (-1, 1)$ and

$$\int_{\mathbb{R}} \chi(x)dx = 1. \tag{B.4}$$

Defined this way $\phi_n \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ and therefore using the first equation of (B.2) we have

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} \rho(t,x)nd(t)(-\dot{\xi}(t)) \left(\int_{-\infty}^x n\chi'(n(y - (\xi(t) - \varepsilon))) - n\chi'(n(y - (\xi(t) + \varepsilon)))dy \right) \\ + \rho(t,x)d'(s) \left(\int_{-\infty}^x n\chi(n(y - (\xi(t) - \varepsilon))) - n\chi(n(y - (\xi(t) + \varepsilon)))dy \right) \\ + d(s)\rho(t,x)V(\rho(t,x), w(t,x)) \left[n\chi(n(x - (\xi(t) - \varepsilon))) - n\chi(n(x - (\xi(t) + \varepsilon))) \right] dtdx = 0, \end{aligned} \tag{B.5}$$

Thus integrating the first terms and using a change of variable in terms of the middle line,

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} \rho(t,x)nd(t)(-\dot{\xi}(t)) \left[\chi(n(x - (\xi(t) - \varepsilon))) - \chi(n(x - (\xi(t) + \varepsilon))) \right] \\ + nd(s)\rho(t,x)V(\rho(t,x), w(t,x)) \left[\chi(n(x - (\xi(t) - \varepsilon))) - \chi(n(x - (\xi(t) + \varepsilon))) \right] dtdx \\ + \int_0^\infty \int_{\mathbb{R}} \rho(t,x)d'(s) \left(\int_{-\infty}^{nx} \chi(y - (\xi(t) - \varepsilon)) - \chi(y - (\xi(t) + \varepsilon))dy \right) dtdx = 0. \end{aligned} \tag{B.6}$$

Now, this is true for any $\varepsilon > 0$ so we can choose $\varepsilon = 1/n$, and we get

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} \rho(t,x)nd(t)(-\dot{\xi}(t)) \left[\chi(n(x - (\xi(t) - \frac{1}{n}))) - \chi(n(x - (\xi(t) + \frac{1}{n}))) \right] \\ + nd(s)\rho(t,x)V(\rho(t,x), w(t,x)) \left[\chi(n(x - (\xi(t) - \frac{1}{n}))) - \chi(n(x - (\xi(t) + \frac{1}{n}))) \right] dtdx \\ + \int_0^\infty \int_{\mathbb{R}} \rho(t,x)d'(s) \left(\int_{-\infty}^{nx} \chi(y - (\xi(t) - \frac{1}{n})) - \chi(y - (\xi(t) + \frac{1}{n}))dy \right) dtdx = 0. \end{aligned} \tag{B.7}$$

Dividing the first integral and performing again a change of variables in each of the divided parts, we have

$$\begin{aligned}\int_0^{+\infty}\Big[&\int_{\mathbb{R}}\rho(t,\frac{x}{n}+(\xi(t)-\frac{1}{n}))d(t)(-\dot{\xi}(t))\chi(x)dx-\int_{\mathbb{R}}\rho(t,\frac{x}{n}+(\xi(t)+\frac{1}{n}))d(t)(-\dot{\xi}(t))\chi(x)dx\\&+\int_{\mathbb{R}}d(s)\rho(t,\frac{x}{n}+(\xi(t)-\frac{1}{n}))V(\rho(t,\frac{x}{n}+(\xi(t)-\frac{1}{n})),w(t,\frac{x}{n}+(\xi(t)-\frac{1}{n})))\chi(x)dx\\&-\int_{\mathbb{R}}d(s)\rho(t,\frac{x}{n}+(\xi(t)+\frac{1}{n}))V(\rho(t,\frac{x}{n}+(\xi(t)+\frac{1}{n})),w(t,\frac{x}{n}+(\xi(t)+\frac{1}{n})))\chi(x)dx\Big]\,dt\\&+\int_0^{\infty}\int_{\mathbb{R}}\rho(t,x)d'(s)\left(\int_{-\infty}^{nx}\chi(y-(\xi(t)-\frac{1}{n}))-\chi(y-(\xi(t)+\frac{1}{n}))dy\right)dtdx=0.\end{aligned}\tag{B.8}$$

We will now justify that we can use a dominated convergence theorem in each of the integrals to conclude. Note first that as $\mathrm{supp}(\chi)\subset(-1,1)$, we have for the last integral

$$\int_{-\infty}^{nx}\chi(y-(\xi(t)-\frac{1}{n}))-\chi(y-(\xi(t)+\frac{1}{n}))dy=0 \text{ for } nx\in(-\infty,-1+\xi(t)-\frac{1}{n}]\cup[1+\xi(t)+\frac{1}{n},+\infty),\tag{B.9}$$

thus this function is compactly supported and, as (ρ,w) is bounded and $d'\in C_c^\infty(\mathbb{R}_+)$, we have by dominated convergence

$$\lim_{n\to+\infty}\int_0^{\infty}\int_{\mathbb{R}}\rho(t,x)d'(s)\left(\int_{-\infty}^{nx}\chi(y-(\xi(t)-\frac{1}{n}))-\chi(y-(\xi(t)+\frac{1}{n}))dy\right)dtdx=0.\tag{B.10}$$

Now, observe that as $\mathrm{supp}(\chi)\subset(-1,1)$, the integration bounds of the first four integrals in x in (B.8) can be limited to $(-1,1)$. Besides for any $x\in(-1,1)$, $x/n-1/n<0$ and $x+1/n>0$, thus, as $x\to(\rho(t,x),w(t,x))$ is a BV function with a left and right limit that exists for any $x\in\mathbb{R}$, we have for any $x\in(-1,1)$

$$\begin{aligned}\lim_{n\to+\infty}\rho(t,\frac{x}{n}+(\xi(t)\pm\frac{1}{n}))=&\rho(t,\xi(t)\pm)\text{ and}\\\lim_{n\to+\infty}\rho(t,\frac{x}{n}+(\xi(t)\pm\frac{1}{n}))V(\rho(t,\frac{x}{n}+(\xi(t)\pm\frac{1}{n})),w(t,\frac{x}{n}+(\xi(t)\pm\frac{1}{n})))=&\rho(t,\xi(t)\pm)V(\rho(t,\xi(t)\pm),w(t,\xi(t)\pm)).\end{aligned}\tag{B.11}$$

Therefore, as $(\rho(t,x),w(t,x))$ and $\dot{\xi}(t)$ are uniformally bounded function, and $d\in C_c^\infty(\mathbb{R}_+)$, we can apply the dominated convergence theorem and using also (B.10) we have

$$\int_0^{+\infty}d(t)\left(\dot{\xi}(t)[\rho(t,\cdot)]_{\xi(t)-}^{\xi(t)+}-[(\rho V(\rho,w))(t,\cdot)]_{\xi(t)-}^{\xi(t)+}\right)dt=0,\tag{B.12}$$

where $[f]_{a-}^{a+}$ denotes $\lim_{x\to a+}f-\lim_{x\to a-}f$. As this is true for any $d\in C_c^\infty(\mathbb{R}_+)$, then the function in the integrand is equal to 0 almost everywhere which implies that for almost every $t\in\mathbb{R}_+$, $\dot{\xi}(t)[\rho(t,\cdot)]_{\xi(t)-}^{\xi(t)+}=[(\rho V(\rho,w))(t,\cdot)]_{\xi(t)-}^{\xi(t)+}$. Doing similarly for the second equation of (B.2), there exists a space $\mathcal{N}$ of zero measure such that for all $t\in\mathbb{R}_+\setminus\mathcal{N}$

$$\begin{aligned}\dot{\xi}(t)[\rho(t,\cdot)]_{\xi(t)-}^{\xi(t)+}&=[(\rho V(\rho,w))(t,\cdot)]_{\xi(t)-}^{\xi(t)+},\\\dot{\xi}(t)[\rho(t,\cdot)w(t,\cdot)]_{\xi(t)-}^{\xi(t)+}&=[(\rho wV(\rho,w))(t,\cdot)]_{\xi(t)-}^{\xi(t)+}.\end{aligned}\tag{B.13}$$

Using the second equation in the first one and using that $[fg]_{a-}^{a+} = [f]_{a-}^{a+} g(a+) + f(a-)[g]_{a-}^{a+}$, we finally get that for any $t \in \mathcal{N}$

$$V(\rho(t,\xi(t)+), w(t,\xi(t)+)) = V(\rho(t,\xi(t)-), w(t,\xi(t)-)) \text{ or } w(t,\xi(t)+) = w(t,\xi(t)-). \tag{B.14}$$

Now, as y is continuous, differentiable up to a zero measure space and $y' \in \mathbb{R}_+$, we can take $\xi = y$ and this ends the proof of the claim.

C Proof of Lemma 3.12

D Proof of Lemma 3.14

In this Appendix we show Lemma 3.14

Proof. Suppose by contradiction that $\rho_- > \rho^*(w) \geqslant \rho_+$. Since $\rho_- > \rho^*(w)$, for ε small enough, we have $V(\rho_-, w) - 4\varepsilon < V_b$. From Lemma 3.10, for any $x \in (\min(y^n, y) - \delta, \min(y^n, y))$, $V(\rho^n(t,x), w^n(t,x)) \in \mathcal{B}_\varepsilon(\rho_-, w))$. In particular, $V(\rho^n(t, \min(y^n, y)-), w^n(t, \min(y^n, y)-)) < V_b - 3\varepsilon$. In $\min(y^n, y)$, we cannot have a V-wave or a non-classical shock since $V(\rho_-, w) < V_b$. Thus, only shocks and rarefaction waves are allowed at $\min(y^n, y)$. Using Lemma 3.9, for n large enough,

$$V(\rho^n(t, \min(y^n, y)+), w^n(t, \min(y^n, y)+)) < V_b - \frac{8\varepsilon}{3}. \tag{D.1}$$

From $\rho^*(w) \geqslant \rho_+$ and Lemma 3.10, for any $x \in (\max(y^n, y) - \delta, \max(y^n, y))$, $V(\rho^n(t,x), w^n(t,x)) \in [V_b - 2\epsilon, +\infty)$. In particular, $V(\rho^n(t, \max(y^n, y)-), w^n(t, \max(y^n, y)-)) > V_b - 2\varepsilon$. In $\max(y^n, y)$, a 1-wave, a 2-wave and and V-wave are allowed. Note that the speed of a V-wave and the speed of a non classical shock is greater than V_b. Thus, to decrease the speed from $V_b - 2\varepsilon$, the only possible way is to use a rarefaction wave. Therefore, from Lemma 3.9, for n large enough,

$$V(\rho^n(t, \max(y^n, y)-), w^n(t, \max(y^n, y)-)) > V_b - \frac{7\varepsilon}{3} \tag{D.2}$$

To go from $V_b - \frac{8\varepsilon}{3}$ to $V_b - \frac{7\varepsilon}{3}$ in $V(\rho^n, w^n)$, there are only shocks and rarefaction waves. From Lemma 3.9, (D.1) and (D.2), for n large enough,

$$\begin{aligned} |y(t) - y^n(t)| &\geqslant |V(\rho^n(\max(y^n, y)-), w^n(\max(y^n, y)-)) - V(\rho^n(\min(y^n, y)+), w^n(\min(y^n, y)+))| \\ &\geqslant \tfrac{t\beta\varepsilon}{3}. \end{aligned}$$

The convergence of y^n to y and the arbitrariness of ε leads to a contradiction which conclude the proof.

□

E Proof of Lemma 3.15

Proof. This proof is similar to the proof of Lemma 3.13. The main difference with Lemma 3.13 is that $\rho_- < V_b$ which means that there could be some interaction with the AV that result in a non-classical shock. Let $t \in \mathbb{R} \setminus \mathcal{N}$. Once again, given the definition of $\mathcal{T}_0$ a wavefront inside the triangle cannot interact with wavefronts out of the triangle. Let us denote by $N(t,n)$ the number of discontinuity points of the speed $V(\rho^n, \omega^n)$ on $[y^n(t) - \delta, y(t) + \delta]$ and x_j^n these points. As

$V(\rho+,w)+\varepsilon < V_b \leqslant V(\rho^-,w)$, from Lemma 3.10 and 3.12 , there exists j_0 and j_1 such that $x_{j_0} \in [y^n(t), y(t)]$ and $x_{j_1} \in [y^n(t), y(t)]$ and

$$\begin{aligned}
&V(\rho^n(t,x^n_{j_1}-), w^n(t,x^n_{j_1}-)) \in \mathbb{R} \setminus \mathcal{B}_\varepsilon(V(\rho_+,w)) \\
&V(\rho^n(t,x^n_j), w^n(t,x^n_j)) \in \mathcal{B}_\varepsilon(V(\rho_+,w)),\ \forall\, j \geqslant j_1 \\
&V(\rho^n(t,x^n_{j_0}-), w^n(t,x^n_{j_0}-)) \in [V_b - 2\varepsilon, +\infty), \\
&V(\rho^n(t,x^n_j), w^n(t,x^n_j)) \in [V(\rho_+,w)-\varepsilon, V_b),\ \forall\, j_0 \leqslant j \leqslant j_1.
\end{aligned} \tag{E.1}$$

Note that a priori x_{j_1} could be equal to x_{j_0} and x_{j_0} might not be unique. Now, we define similarly as previously the trajectories ξ^n_0 and ξ^n_1 by

$$\begin{aligned}
\xi^n_{j_0}(0) &= x^n_{j_0} \\
\dot{\xi}^n_0 &= \sigma(\rho^n(t,x^n_{j_0}-), R(V(\rho^n(t,x^n_{j_0}), w^n(t,x^n_{j_0})), w(t,x^n_{j_0}-))) \ \text{ on } [0,s_0],
\end{aligned} \tag{E.2}$$

where the right-hand side is the speed of the 1-wave propagating from the discontinuity in x_{j_0} (note that it cannot be a 2-wave since x_{j_0} is a discontinuity point of the velocity), and where s_0 is the minimum between the time where ξ_{j_0} exit the triangle or interact with another wavefront. We also set

$$\begin{aligned}
\xi^n_{j_1}(0) &= x^n_{j_1} \\
\dot{\xi}^n_{j_1} &= \sigma(\rho^n(t,x^n_{j_0}-), R(V(\rho^n(t,x^n_{j_0}), w^n(t,x^n_{j_0})), w(t,x^n_{j_0}-))) \ \text{ on } [0,s_1],
\end{aligned} \tag{E.3}$$

where s_1 is the minimum between the time where ξ_{j_1} exits the triangle or interact with another wavefront. Given that ξ_{j_0} does not interact with another wavefront before s_0 and ξ_{j_1} before s_1, we have that

$$\begin{aligned}
V(\rho^n(s,x), w^n(s,x)) &\in \mathcal{B}_\varepsilon(V(\rho_+,w)) \ \text{ for any } (s,x) \in \mathcal{T}_0 \cap \{\xi_1(s) \leqslant x\} \\
V(\rho^n(s,x), w^n(s,x)) &\in [V(\rho_+,w)-\varepsilon, V_b) \ \text{ for any } (s,x) \in \mathcal{T}_0 \cap \{\xi_0(s) \leqslant x \leqslant \xi_1(s)\}.
\end{aligned} \tag{E.4}$$

Assume that ξ_{j_1} interact with another wavefront at time s_1, then there are only three possibilities: a 2-wave, a shock, a rarefaction shock or a non-classical shock if $\xi_{j_0}(s_0) = y^n(t)$. If there is a 2-wave, a shock or a rarefaction shock there is again only one 1-wave generated by this interaction. If ξ_{j_1} interact with $y^n(t)$ at s_1, then, as $V(\rho(s_1)+, w(s_1)) < V_b$ from (E.4) the only possible case (Lemmas A.3, A.4, A.7 and A.8) is that the interaction give rise to a 1-wave and a FV-wave, which is a wave denoting the slow vehicle trajectory without discontinuity in (ρ^n, w^n). Note that $y^n(t)$ can only come from the left as $x_{j_1} \geqslant y^n(t)$ by definition. This implies that there is again only one 1-wave generated by this interaction. The same holds for $\xi_{j_0})(s_0)$ as $V(\rho(\xi_{j_0}(s_0)+), w(\xi_{j_0}(s_0))) \leqslant V_b$. Therefore we can follow this 1-wave and set

$$\begin{aligned}
\dot{\xi}_0(s) = \sigma(\rho^n(s_0,\xi^n_0(s_0)-), R(V(\rho^n(s_0,\xi^n_0(s_0)), w^n(s_0,\xi^n(s_1))), w^n(s_0,\xi^n_0(s_0)-)) \\
, \text{ for any } s \in [s_0, s_{0,2}],
\end{aligned} \tag{E.5}$$

where $s_{0,2}$ is the minimum between the time when ξ_0 exits the triangle and the time when ξ_0 interact with another wavefront. Similarly we set

$$\begin{aligned}
\dot{\xi}_1(s) = \sigma(\rho^n(s_1,\xi^n_1(s_1)-), R(V(\rho^n(s_1,\xi^n_1(s_1)), w^n(s_1,\xi^n_1(s_1))), w^n(s_1,\xi^n_1(s_1)-)), \\
\text{for any } s \in [s_1, s_{1,2}],
\end{aligned} \tag{E.6}$$

where the right-hand side is the speed of the 1-wave generated by the interaction and where $s_{1,2}$ is the minimum between the time when ξ_1 exits the triangle and the time when ξ_1 interact

with another wavefront. Again the only interactions are 2-waves, shocks, rarefaction shocks, or interactions with $y^n(t)$ that leads to a single 1-wave. We can then proceed similarly and define $s_{0,k}$ and $s_{1,k}$ for $k \geqslant 2$ until the times t_{ξ_0} and t_{ξ_1} where ξ_0 and ξ_1 respectively exit the triangle. Note that with this procedure, if ξ_0 and ξ_1 interact together at a time s, they merge and are identical for any time larger than s. Note also that y^n can only interact at most once with ξ_1, since y^n propagates with a speed larger than ξ_1. Indeed, for $s \geqslant t$ such that $(s, y^n(s)) \in \mathcal{T}_0$, from Lemma 3.10 and 3.12,

$$\min(V_b - 2\varepsilon, V(\rho^+, w) - \varepsilon) \leqslant \dot{y}^n(s) \leqslant V_b, \tag{E.7}$$

and, by construction, just like in the proof of Lemma 3.13

$$\dot{\xi}_1(s) \leqslant \max_{y \in [w_{min}, w_{\max}]} \sigma(f(R(V(\rho_-, w) + \varepsilon, y), y), f(R(V(\rho_+, w) + \varepsilon, y), y). \tag{E.8}$$

Therefore there exists a constant $d > 0$ depending on ε but independent of n such that

$$\dot{y}^n(t) - \dot{\xi}_1(s) \geqslant d. \tag{E.9}$$

Let us denote t_{y^n} the time at which y^n exits the triangle. Exactly as in the proof of Lemma 3.13 there exists c independent of n such that $\min(t_{\xi_1}, t_{y^n(t)}) \geqslant t + c$. Besides, from (E.9), if n is large enough,

$$\frac{y^n - y(t)}{d} \leqslant c, \tag{E.10}$$

and therefore there exists $t_n \leqslant \frac{y^n - y(t)}{d}$ such that $\xi_1^n(t_n) = y^n(t_n)$ and for any $s \in [t_n, t + c]$ $y^n(s) > \xi_1^n(s)$. Finally, from (E.10), $\lim_{b \to +\infty} t_n = 0$. Therefore Lemma 3.15 holds with $\xi = \xi_1$. □

F Convergence of $F_\alpha(w^n, \dot{y}^n)$ and $R(v^n, w^n)(v^n - \dot{y}^n)$

G The second term of (2.2) maximize the flux at nonclassical shocks

References

[1] Boris Andreianov, Carlotta Donadello, and Massimiliano Daniele Rosini. A second-order model for vehicular traffics with local point constraints on the flow. *Mathematical Models and Methods in Applied Sciences*, 26(04):751–802, 2016.

[2] A. Aw and Michel Rascle. Resurrection of" second order" models of traffic flow. *SIAM journal on applied mathematics*, 60(3):916–938, 2000.

[3] Alberto Bressan. *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem*, volume 20. Oxford University Press on Demand, 2000.

[4] LC Davis. Effect of adaptive cruise control systems on traffic flow. *Physical Review E*, 69(6):066110, 2004.

[5] Maria Laura Delle Monache and Paola Goatin. A front tracking method for a strongly coupled pde-ode system with moving density constraints in traffic flow. *Discrete & Continuous Dynamical Systems-S*, 7(3):435.

[6] Maria Laura Delle Monache and Paola Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *Journal of Differential equations*, 257(11):4015–4029, 2014.

[7] Maria Laura Delle Monache and Paola Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *Journal of Differential equations*, 257(11):4015–4029, 2014.

[8] Maria Laura Delle Monache and Paola Goatin. Stability estimates for scalar conservation laws with moving flux constraints. *Networks & Heterogeneous Media*, 12(2):245, 2017.

[9] Shimao Fan, Michael Herty, and Benjamin Seibold. Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model. *arXiv preprint arXiv:1310.8219*, 2013.

[10] SHIMAO FAN, MICHAEL HERTY, and BENJAMIN SEIBOLD. Comparative model accuracy of a data-fitted generalized aw-rascle-zhang model. *Networks & Heterogeneous Media*, 9(2), 2014.

[11] Shimao Fan, Ye Sun, Benedetto Piccoli, Benjamin Seibold, and Daniel B Work. A collapsed generalized aw-rascle-zhang model and its model accuracy. *arXiv preprint arXiv:1702.03624*, 2017.

[12] Mauro Garavello, Paola Goatin, Thibault Liard, and Benedetto Piccoli. A controlled multiscale model for traffic regulation via autonomous vehicles. 2019.

[13] Denos C Gazis and Robert Herman. The moving and "phantom" bottlenecks. *Transportation Science*, 26(3):223–229, 1992.

[14] Amaury Hayat, Xiaoqian Gong, Jonathan Lee, Sydney Truong, Sean McQuade, Nicolas Kardous, Alexander Keimer, Yiling You, Saleh Albeaik, Eugene Vinistky, Paige Arnold, Maria Laura Delle Monache, Alexandre Bayen, Benjamin Seibold, Jonathan Sprinkle, Dan Work, and Benedetto Piccoli. A holistic approach to the energy-efficient smoothing of traffic via autonomous vehicles. Technical report, November 2020. working paper or preprint, 2020.

[15] Florian Knorr, Daniel Baselt, Michael Schreckenberg, and Martin Mauve. Reducing traffic jams via vanets. *IEEE Transactions on Vehicular Technology*, 61(8):3490–3498, 2012.

[16] Thibault Liard and Benedetto Piccoli. On entropic solutions to conservation laws coupled with moving bottlenecks. Technical report, June 2019. working paper or preprint, 2019.

[17] Thibault Liard and Benedetto Piccoli. Well-posedness for scalar conservation laws with moving flux constraints. *SIAM Journal on Applied Mathematics*, 79(2):641–667, 2019.

[18] Michael James Lighthill and Gerald Beresford Whitham. On kinematic waves ii. a theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 229(1178):317–345, 1955.

[19] Gordon Frank Newell. A moving bottleneck. *Transportation Research Part B: Methodological*, 32(8):531–537, 1998.

[20] Harold J Payne. Model of freeway traffic and control. *Mathematical Model of Public System*, pages 51–61, 1971.

[21] Paul I Richards. Shock waves on the highway. *Operations research*, 4(1):42–51, 1956.

[22] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1964.

[23] Raphael E Stern, Shumo Cui, Maria Laura Delle Monache, Rahul Bhadani, Matt Bunting, Miles Churchill, Nathaniel Hamilton, Hannah Pohlmann, Fangyu Wu, Benedetto Piccoli, et al. Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments. *Transportation Research Part C: Emerging Technologies*, 89:205–221, 2018.

[24] Stefano Villa, Paola Goatin, and Christophe Chalons. Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model. *Discrete and Continuous Dynamical Systems - Series B*, 22(10):3921–3952, 2017.

[25] Gerald Beresford Whitham. *Linear and nonlinear waves*, volume 42. John Wiley & Sons, 2011.