Ergodicity of stochastic Cahn-Hilliard equations with logarithmic potentials driven by degenerate or nondegenerate noises

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Abstract

We study the asymptotic properties of the stochastic Cahn-Hilliard equation with the logarithmic free energy by establishing different dimension-free Harnack inequalities according to various kinds of noises. The main characteristics of this equation are the singularities of the logarithmic free energy at 1 and -1 and the conservation of the mass of the solution in its spatial variable. Both the space-time colored noise and the space-time white noise are considered. For the highly degenerate space-time colored noise, the asymptotic log-Harnack inequality is established under the so-called essentially elliptic conditions. And the Harnack inequality with power is established for non-degenerate space-time white noise.

Keywords: Stochastic Cahn-Hilliard equation, asymptotic log-Harnack inequality, Harnack inequality with power, logarithmic free energy, essentially elliptic condition.

2010 Mathematics Subject Classification: Primary 60H15, 60H10; Secondary 60H07, 37L40, 47G20.

1 Introduction

The Cahn-Hilliard equation is initially introduced to describe the phase separation in a binary alloy comprising two species when the temperature is quenched from high temperature to low one [10, 11, 12] and now plays a very important role in

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material science, tumor growth, population dynamics, thin films and so on. The deterministic Cahn-Hilliard equation has been extensively studied after [11], see [24, 30] for the case of the polynomial free energy and [9, 16] for the case of the logarithmic free energy (see (1.1) below for such energy). The phase separation, spinodal decomposition and nucleation are also studied by many researchers, see [6, 7, 8, 27, 30] for instance. We refer the reader to [28] and references therein for more studies on the deterministic case.

On the other hand, in the presence of thermal fluctuations, a noise term is naturally required and now the stochastic Cahn-Hilliard equation is commonly accepted for modeling. There are many articles which have been devoted to the mathematical study of the stochastic Cahn-Hilliard equation with a polynomial free energy [1, 2, 13, 14]. On the other hand, in applications, the solution of the Cahn-Hilliard equation is explained as the rescaled density of atoms or concentration of one of material's components which takes values in $[-1, 1]$. The polynomial free energy can not ensure that the solution satisfies the constraint and usually the logarithmic free energy can remedy such problem. However, different from the deterministic case [16], for the stochastic case, owing to the impact of noise, the logarithmic free energy is not strong enough to prevent the solution from exiting $[-1, 1]$. To study it, reflection measures are required, see [17, 22].

From now on, let us introduce the stochastic Cahn-Hilliard equation with the logarithmic free energy. Let $\lambda > 0$ and define f by

$$f(u) = \begin{cases} +\infty, & u \leq -1, \\ \log\left(\dfrac{1-u}{1+u}\right) + \lambda u, & u \in (-1, 1), \\ -\infty, & u \geq 1. \end{cases} \tag{1.1}$$

Let $(W(t))_{t\geq 0}$ be a cylindrical Wiener process on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We formally consider the stochastic Cahn-Hilliard equation with singular nonlinearity and double reflections

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\theta) = -\dfrac{1}{2}\dfrac{\partial^2}{\partial\theta^2}\left\{\dfrac{\partial^2 u}{\partial\theta^2}(t,\theta) + f(u(t,\theta)) + \eta_-(t,\theta) - \eta_+(t,\theta)\right\} \\ \qquad\qquad + B\dot{W}(t,\theta), \quad t > 0,\ \theta \in (0,1), \\ u(t,0) = u(t,1) = \dfrac{\partial^3 u}{\partial x^3}(t,0) = \dfrac{\partial^3 u}{\partial x^3}(t,1) = 0,\ t \geq 0, \\ \displaystyle\int_0^t\int_0^1 (1+u(t,\theta))\eta_-(dtd\theta) = \int_0^t\int_0^1 (1-u(t,\theta))\eta_+(dtd\theta) = 0, \\ u(0,\theta) = x(\theta),\ \theta \in (0,1), \end{cases} \tag{1.2}$$

where the solution $u(t, x) \in [-1, 1]$ $a.s.$, is usually explained as the concentration of one species with respect to the other, η_-, η_+ are two non-negative random measures and B denotes some operator which be stated clearly in Sections 2 and 3 respectively. It is well-known that the Cahn-Hilliard equation can be regarded as a gradient

system in $H^{-1}(0,1)$ with the logarithmic free energy, which is called Ginzburg-Landau free energy

$$\mathcal{E}(u) = \int_0^1 \left(\frac{1}{2} |\nabla u(\theta)|^2 + F(u(\theta)) \right) d\theta,$$

where F denotes the primitive of $-f$ with $F(0) = 0$, that is

$$F(u) = (1+u)\log(1+u) + (1-u)\log(1-u) - \frac{\lambda}{2}u^2, \ u \in (-1,1).$$

Note that for $\lambda > 2$, F denotes a double well potential, which is important in application.

The stochastic PDE with reflecting measures like (1.2) is one kind of random obstacle problems [40], which is first studied for stochastic reaction-diffusion equations [19]. Such equation has been used to model the fluctuations for $\nabla\phi$ interface models on a hard wall with or without conservation of the area [20, 39] and hence it has attracted many researchers' attention. But, different from the stochastic reaction-diffusion equation, due to the lack of the maximum principle for the double Laplacian, there are few researches on stochastic Cahn-Hilliard equations with reflecting terms, see [18] for the case without nonlinear term f, [21] for the case of logarithmic and negative power nonlinear terms with only one reflection at 0. The stochastic Cahn-Hilliard equation (1.2), the main object of this paper, is studied mainly in [17] and [22] for different noises.

Roughly speaking, the main goal is to establish various dimension-free Harnack inequalities for the Markov semigroup associated with (1.2) driven by two kinds of noises and then study ergodic behavior of the solution and others properties. The dimension-free Harnack inequality is initially introduced in [32] by F.-Y. Wang to study the log-Sobolev inequality of a diffusion process on Riemannian manifolds and then it becomes as a very powerful and effective tool to the study of various important properties of diffusion semigroups or semigroup relative to stochastic (functional) partial differential equations, such as, Li-Yau type heat kernel bound, hypercontractivity, ultracontractivity, strong Feller property, estimates on the heat kernels and Varadhan type small time asymptotics [4, 15, 26, 31, 33, 35, 41].

Although recently dimension-free Harnack inequalities and their applications have also been studied for stochastic reaction-diffusion equations with reflections [29, 37, 41], to our best knowledge, there is no publications on stochastic Cahn-Hilliard equations. Therefore, in the paper, we intend to the study on the dimension-free Harnack inequalities for the Markov semigroup generated by the solutions of (1.2) perturbed by two different noises Then we study other important properties of the Markov semigroup obtained as corollary of Harnack inequalities

According to the characteristics of noises, both the asymptotic log-Harnack inequality and the Harnack inequality with power will be considered. More precisely, we first study the asymptotic log-Harnack inequality for the Markov semigroup relative to (1.2) driven by the highly degenerate colored noise under the so-called essentially elliptic conditions, see [22] and [23]. The asymptotic log-Harnack inequality is initially introduced in [38] with an application to stochastic 2D Navier-Stokes

equations. The most important property of the asymptotic log-Harnack inequality is that the asymptotic strong Feller property introduced in [23] can be deduced from it. Hence, it has been established for various stochastic (partial) differential equations, see [5] for stochastic systems with infinite memory and see [25] for 3D Leray-α model.

However, as far as we know, there is no publication on the asymptotic log-Harnack inequality for stochastic Cahn-Hilliard equations like (1.2), even for stochastic reaction-diffusion equations with reflections. Since the degenerate noise is considered, as explained in [22] and [23], it seems impossible to obtain the strong Feller property. On the other hand, it is well-known that the log-Harnack inequality or the Harnack inequality with power implies the strong Feller property, see Theorem 1.4.1 [35]. Therefore, it seems impossible for us to establish the log-Harnack inequality in this case, and also the Harnack inequality with power. Instead of such strong inequalities, we will show the Markov semigroup associated with (1.2) satisfies the asymptotic log-Harnack inequality, which is a weaker version of dimension-free Harnack inequalities. Although the asymptotic strong Feller property for (1.2) has been proved in [22], we give a new proof of the asymptotic strong Feller property under a weaker condition and cover partially the corresponding result obtained in [22].

The second purpose of this paper is to establish the Harnack inequality with power and then in particular, the log-Harnack inequality, for the Markov semigroup corresponding to (1.2) with $B = (-\Delta)^{\frac{1}{2}}$. It is known that in this case, the average of the solution $u(t)$ in its spatial variable is conservative in time [17]. But, the conservation of the average makes it more difficult to investigate the dimension-free Harnack inequalities via coupling by change of measures than the well-studied cases of stochastic partial differential equations drive by additive noises [26, 29, 33, 36, 41]. To overcome it, we make use of the strategy initially introduced in [34], in which the stochastic finite differential equation driven by multiplicative noise is investigated.

Let us now introduce some notations, which will be used throughout this paper. We denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the canonical inner product and the norm of $L^2(0,1)$ respectively. Let A be the realization of $\frac{\partial^2}{\partial\theta^2}$ with homogeneous Neumann boundary condition in $L^2(0,1)$, that is, $Au = \frac{\partial^2 u}{\partial\theta^2}$ for any $u \in D(A) := \{u \in H^2(0,1) :\ u'(0) = u'(1) = 0\}$. It is known that A is self-adjoint in $L^2(0,1)$ with a complete orthonormal system $\{e_n\}_{n=0}^{\infty}$ in $L^2(0,1)$, which satisfies $e_0(\theta) \equiv 1, e_n(\theta) = \sqrt{2}\cos(n\pi\theta), n = 1, 2, \cdots$ and $Ae_n = -(n\pi)^2 e_n, n = 0, 1, \cdots$.

For any $\gamma \in \mathbb{R}$, let us define $(-A)^{\frac{\gamma}{2}}u = \sum_{n=1}^{\infty}(n\pi)^{\gamma}u_n e_n$ for any $u = \sum_{n=0}^{\infty} u_n e_n$ with its domain

$$V_\gamma = D\left((-A)^{\frac{\gamma}{2}}\right) := \left\{u = \sum_{n=0}^{\infty} u_n e_n :\ \sum_{n=0}^{\infty}(n\pi)^{2\gamma}u_n^2 < \infty\right\}.$$

It will be endowed with the norm $\|u\|_\gamma = (|u|_\gamma^2 + \bar{u}^2)^{\frac{1}{2}}$. Hereafter, $\bar{u}$ denotes the average of u, that is $\bar{u} = \langle u, e_0\rangle$, and $|u|_\gamma$ denotes the seminorm, that is, $|u|_\gamma = \left|(-A)^{\frac{\gamma}{2}}u\right| = (\sum_{n=1}^{\infty}(n\pi)^{2\gamma}u_n^2)^{\frac{1}{2}}$. In addition, we will set $(u,v)_\gamma = \langle(-A)^{\frac{\gamma}{2}}u, (-A)^{\frac{\gamma}{2}}v\rangle$, which is the semiscalar product. For simplicity of notation, we set $\mathbf{H} = V_{-1}$ through-

out this paper. Let us also denote by $\mathbf{H}^c$ the affine space $\mathbf{H}^c = \{u \in \mathbf{H} : \ \bar{u} = c\}$. It is easy to check that $\mathbf{H}^c$ is a Polish space with the metric inherited form $\mathbf{H}$.

The remainder of this paper is organized as follows. In Section 2, the asymptotic log-Harnack inequality for (1.2) driven by highly degenerate noise is established by using the asymptotic coupling method and as its application, the asymptotic heat kernel estimate and the asymptotic irreducibility are mainly stated. In Section 3, the Harnack inequality with power and the log-Harnack inequality for (1.2) with $B = (-\Delta)^{\frac{1}{2}}$ are obtained and some important applications also are described as example.

2 Asymptotic log-Harnack inequality for the case of highly degenerate colored noise

In this section, we intend to establish the asymptotic log-Harnack inequality relative to (1.2) driven by highly degenerate colored noise, which is studied in [22], and then as its application, the asymptotic strong Feller property, the asymptotic gradient estimate and the asymptotic heat kernel estimate are studied. Moreover, our results can be partially applied to the (1.2) with the double-well potential F.

Let us recall the definition of the asymptotic log-Harnack inequality precisely based on [5, 38]. Let (E, d) be a Polish space and let $B_b(E)$ be the family of bounded measurable functions on E. We denote by $\|\phi\|_\infty$ the uniform norm of $\phi \in B_b(E)$. For a function ϕ on E, we denote by $|\nabla\phi|(x)$ its local Lipschitz constant at x, that is,

$$|\nabla\phi|(x) = \limsup_{y\to x} \frac{|\phi(x) - \phi(y)|}{d(x, y)}.$$

In addition, here and in the sequel, $\|\nabla\phi\|_\infty = \sup_{x\in E} |\nabla\phi|(x)$.

Definition 2.1. Let $(P_t)_{t\geq 0}$ be a Markov semigroup on (E, d). It is called that $(P_t)_{t\geq 0}$ satisfies an asymptotic log-Harnack inequality if there exist two non-negative functions $\Phi(\cdot, \cdot)$ on $E \times E$ and $\Psi(\cdot, \cdot, \cdot)$ on $[0, \infty) \times E \times E$ satisfying $\Psi(\cdot, \cdot, \cdot) \to 0$ as $t \to \infty$ such that

$$P_t \log \phi(y) \leq \log P_t\phi(x) + \Phi(x, y) + \Psi(t, x, y)\|\nabla \log \phi\|_\infty, \ t > 0$$

holds for any $x, y \in E$ and any positive $\phi \in B_b(E)$ with $\|\nabla \log \phi\|_\infty < \infty$.

Thanks to the Jensen inequality, it is natural to set $\Phi(x, x) = \Psi(t, x, x) = 0$ for any $t \geq 0$ and $x \in E$. It is known that one of the important applications of the asymptotic log-Harnack inequality is that it implies the asymptotic strong Feller property, see Proposition 1.6 [38] or Theorem 2.1 [5]. For the reader's convenience, let us recall the definition of the asymptotic strong Feller property according to the original paper [23]. For a pseudo-metric d_p on E and two probability measures μ_1, μ_2 on E, let us define the transportation cost $\|\mu_1 - \mu_2\|_{d_p}$ by

$$\|\mu_1 - \mu_2\|_{d_p} = \inf_{\mu\in\mathcal{C}(\mu_1,\mu_2)} \int_{E\times E} d_p(x, y)\mu(dx, dy),$$

where $\mathcal{C}(\mu_1, \mu_2)$ denotes the collection of all probability measures on $E \times E$ with marginals μ_1 and μ_2. We say that $\{d_n\}_{n=1}^{\infty}$ is a totally separating system of pseudo-metrics for E if for any $m < n$ and $x, y \in E$, $d_m(x, y) \leq d_n(x, y)$, and for any $x \neq y$ $\lim_{n\to\infty} d_n(x, y) = 1$.

Definition 2.2 (Definition 3.1 [23])**.** The Markov semigroup $(P_t)_{t\geq 0}$ on (E, d) is said to be asymptotically strong Feller at point $x \in E$ if there exist a totally separating system of pseudo-metrics $\{d_n\}_{n=1}^{\infty}$ for E and a positive sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$\inf_{B\in\mathcal{B}_x} \limsup_{n\to\infty} \sup_{y\in B} \|P_{t_n} 1_B(x) - P_{t_n} 1_B(y)\|_{d_n} = 0,$$

where $\mathcal{B}_x$ denotes the family of all open sets including x. In addition, if this property holds for any $x \in E$, then $(P_t)_{t\geq 0}$ is said to be asymptotically strong Feller.

Let us now explain our main goal of this section in detail. More precisely, we intend to establish the asymptotic log-Harnack inequality for the Markov semigroup associated with one of the limits of the sequence $\{u^n\}_{n=1}^{\infty}$ of the solutions of the stochastic partial differential equation studied in [22]

$$\begin{cases} \dfrac{\partial u^n}{\partial t}(t,\theta) = -\dfrac{1}{2}\dfrac{\partial^2}{\partial\theta^2}\left\{\dfrac{\partial^2 u^n}{\partial\theta^2}(t,\theta) - p_n(u^n(t,\theta)) + \lambda u^n(t,\theta)\right\} \\ \qquad\qquad + B\dot{W}(t,\theta), \quad t > 0,\ \theta \in (0,1), \\ u^n(t,0) = u^n(t,1) = \dfrac{\partial^3 u^n}{\partial\theta^3}(t,0) = \dfrac{\partial^3 u^n}{\partial\theta^3}(t,1) = 0,\ t \geq 0, \\ u^n(0,\theta) = x(\theta),\ \theta \in (0,1), \end{cases} \tag{2.1}$$

where

$$p_n(u) = 2\sum_{i=0}^{n} \frac{u^{2i+1}}{2i+1},\ u \in \mathbb{R}$$

is a non-decreasing $(2n+1)$-degree polynomial. It is easy to show that $-p_n(u) + \lambda u$ converges to $f(u)$ for $u \in (-1, 1)$.

In this part, we will assume that B is a Hilbert-Schmidt operator from $L^2(0,1)$ to $\mathbf{H}$, which it is equivalent to the fact that $B(-A)^{-1}B^*$ is a trace class on $L^2(0,1)$. Indeed,

$$\begin{aligned} \|B\|_{\mathcal{L}_{HS}}^2 &= \sum_{n=0}^{\infty} \|Be_n\|_{-1}^2 = \sum_{n=0}^{\infty} \left\langle (-A)^{-\frac{1}{2}} Be_n, (-A)^{-\frac{1}{2}} Be_n \right\rangle \\ &= \sum_{n=0}^{\infty} \left\langle B^*(-A)^{-1}Be_n, e_n \right\rangle = \mathrm{Tr}(B(-A)^{-1}B^*), \end{aligned}$$

where $\|\cdot\|_{\mathcal{L}_{HS}}^2$ denotes the norm of the Hilbert-Schmidt operator from $L^2(0,1)$ to $\mathbf{H}$, B^* denotes the adjoint operator of B and Tr denotes the trace of an operator on $L^2(0,1)$. In the following, we set $\mathrm{Tr}_{-1} = \mathrm{Tr}(B(-A)^{-1}B^*)$. In addition, to consider the ergodic property, we assume

(A1): $B^*e_0 = 0$.

Remark 2.1. To study the ergodic property, **(A1)** is necessary. In fact, it is easy to show that $\overline{u^n}(t) = \bar{x} + \langle B^* e_0, W(t)\rangle$. Thus, if **(A1)** fails, then there cannot be have a stationary solution. There is no fixed mass c and there is no invariant measure on $\mathbf{H}^{\mathbf{c}}$.

Using the notations introduced in Section 1, the SPDE (2.1) can be rewritten in its abstract form as below.

$$\begin{cases} du^n(t) = -\dfrac{1}{2}A\big\{Au^n(t) - p_n(u^n(t)) + \lambda u^n(t)\big\}dt + BdW(t), \quad t > 0, \\ u^n(0) = x. \end{cases} \tag{2.2}$$

It is known that for each $n \in \mathbb{N}$, (2.1) has a unique mild (or weak) solution u^n satisfying $u^n \in C([0,\infty); H) \cap L^{2n+2}((0,\infty)\times(0,1))$ *a.s.*, see [14] or [22]. We also know that the average of $u^n(t)$ is conservative, that is, $\overline{u^n}(t) = \bar{x}$ *a.s.* because of the assumption **(A1)**.

Hence, we know that (2.1) develops in the affine space $\mathbf{H}^c$ if the average $\bar{x}$ of the initial datum x equals to c.

For each $c \in \mathbb{R}$, let denote by $(P_t^{n,c})_{t\geq 0}$ the Markov semigroup determined by (2.1), that is,

$$P_t^{n,c}\phi(x) = \mathbb{E}[\phi(u^n(t;x))],\ t \geq 0, x \in \mathbf{H}^c, \phi \in B_b(\mathbf{H}^c),$$

Here and in the sequel, to specify the initial value x, we use $u^n(t;x)$ to denote the solution of (2.2).

The following theorem is summarized from Proposition 3.3 and Theorem 4.1 [22].

Theorem 2.1. *Under all of the above assumptions, for any $c \in (-1,1)$, the following results hold.*
(i) *There exists a subsequence $\{n_k\}$ and a Markov semigroup $(P_t^c)_{t\geq 0}$ such that*

$$\lim_{k\to\infty} P_t^{c,n_k}\phi(x) = P_t^c\phi(x)$$

holds for any $x \in \mathbf{H}^c$ and any $\phi \in B_b(\mathbf{H}^c)$.
(ii) *$(P_t^c)_{t\geq 0}$ has an invariant probability measure $\tilde{\mu}^c$.*

In the following, we will fix a converging subsequence $P_t^{n_k,c}$ stated in Theorem 2.1. For simplicity, we will still use the notation $P_t^{n,c}$ and $u^n(t)$ instead of $P_t^{n_k,c}$ and $u^{n_k}(t)$. Let us denote by $u(t;x)$ the limit process of $u^{n_k}(t)$, which is the Markov process associated with $(P_t^c)_{t\geq 0}$. Formally speaking, the sequence $\{u^n\}_{n=1}^\infty$ converges to the solution of (1.2), see [22]. But any limit of $\{u^n\}_{n=1}^\infty$ cannot be characterized as a solution of SPDEs, see Section 5, [22] for more information. Here we show that the invariant measure $\tilde{\mu}^c$ is exponentially integrable.

Theorem 2.2. *Let $c \in (-1,1)$ and suppose the assumptions in Theorem 2.1 hold. For any $\varsigma > 0$ satisfying $\pi^4 > 2\varsigma\|B^*\|^2$, where $\|B^*\|$ denotes the operator norm of B^*, then the invariant measure $\tilde{\mu}^c$ satisfies the exponential integrability*

$$\tilde{\mu}^c\left(\exp(\varsigma|\cdot|_{-1}^2)\right) < \infty. \tag{2.3}$$

If further $\pi^4 > \lambda$, then $\tilde{\mu}^c$ is the unique invariant measure and for any Lipschitz continuous function $\phi \in B_b(\mathbf{H}^c)$,

$$|P_t^c\phi(x) - \tilde{\mu}^c(\phi)| \leq \|\nabla\phi\|_\infty \exp^{-(\pi^4-\lambda)t}\left(|x|_{-1} + \tilde{\mu}(|\cdot|_{-1})\right),\ x \in \mathbf{H}^c, t \geq 0. \quad (2.4)$$

Proof. According to the proof of Proposition 3.1 [22], we have that for each $n \in \mathbb{N}$, $|u^n(t)|_{-1},\ t \geq 0$ is a continuous semimartingale with its local martingale part

$$M^n(t) = 2\int_0^t \langle B^*u^n(s), dW(s)\rangle,\ t \geq 0. \quad (2.5)$$

Moreover, the estimate

$$d|u^n(t)|_{-1}^2 \leq \left(-|u^n(t)|_1^2 + P_c(\lambda)\right)dt + 2dM^n(t),\ t \geq 0\ a.s.$$

is proved in the proof of Proposition 3.1 [22], where $P_c(\lambda)$ is a positive constant depending on c, λ and Tr_{-1}, but independent of n.

Noting that $|x|_1 \geq \pi^2|x|_{-1}, x \in V_1$, from the above inequality, it follows that

$$d|u^n(t)|_{-1}^2 \leq \left(-\pi^4|u^n(t)|_{-1}^2 + P_c(\lambda)\right)dt + 2dM^n(t),\ t \geq 0\ a.s. \quad (2.6)$$

Let $\tau_m^n = \inf\{t \geq 0 : |u^n(t)|_{-1} \geq m\}, m \in \mathbb{N}$ be the sequence of stopping times. Then it is easy to show $\lim_{m\to\infty}\tau_m^n = \infty\ a.s.$ and $M^n(t \wedge \tau_m^n), t \geq 0$ is a square integrable continuous martingale. Applying the Itô's formula and using (2.5) and (2.6), we have

$$\begin{aligned} &d\exp(\varsigma|u^n(t)|_{-1}^2) \quad (2.7)\\ \leq&\varsigma\exp(\varsigma|u^n(t)|_{-1}^2)(-\pi^4|u^n(t)|_{-1}^2 + P_c(\lambda))dt\\ &+ 2\varsigma\exp(\varsigma|u^n(t)|_{-1}^2)dM^n(t) + 2\varsigma^2\exp(\varsigma|u^n(t)|_{-1}^2)|B^*u^n(t)|^2dt\\ \leq&\varsigma\exp(\varsigma|u^n(t)|_{-1}^2)\{(-\pi^4 + 2\varsigma\|B^*\|^2)|u^n(t)|_{-1}^2 + P_c(\lambda)\}dt\\ &+ 2\varsigma\exp(\varsigma|u^n(t)|_{-1}^2)dM^n(t),\ t \leq T \wedge \tau_m^n. \end{aligned}$$

Combining the fact that $\pi^4 > 2\varsigma\|B^*\|^2$ with (2.7), we obtain that

there exists a positive constant $K = K(\varsigma, \|B^*\|, P_c(\lambda))$ independent of m, n and t such that

$$\begin{aligned} d\exp(\varsigma|u^n(t)|_{-1}^2) \leq&\left\{K - \varsigma(\pi^4 - 2\varsigma\|B^*\|^2)\exp(\varsigma|u^n(t)|_{-1}^2)\right\}dt \quad (2.8)\\ &+ 2\varsigma\exp(\varsigma|u^n(t)|_{-1}^2)dM^n(t),\ t \leq T \wedge \tau_m^n. \end{aligned}$$

To choose the constant K in the above inequality, the following fundamental inequality is utilized:
For any fixed $a, b > 0$, there exists a constant $c = c(a, b) > 0$, such that

$$(-ax + b)e^x \leq -ae^x + c,\ x \geq 0.$$

Now, noting $\varsigma > 0$ and then taking $a = \pi^4 - \varsigma\|B^*\|^2$, $b = P_c(\lambda)$, we can choose the proper constant K.

Thus, integrating both sides of (2.8) form 0 to $T \wedge \tau_m^n$, taking expectations, bounding nonpositive term, we obtain that

$$\mathbb{E}\left[\int_0^{T\wedge\tau_m^n} \exp(\varsigma|u^n(t)|_{-1}^2)dt\right] \leq \frac{\exp(|x|_{-1}^2) + KT}{\varsigma(\pi^4 - 2\varsigma\|B^*\|^2)},$$

which by letting $m \to \infty$ gives that

$$\mathbb{E}\left[\int_0^{T} \exp(\varsigma|u^n(t)|_{-1}^2)dt\right] \leq \frac{\exp(|x|_{-1}^2) + KT}{\varsigma(\pi^4 - 2\varsigma\|B^*\|^2)}$$

for all $n \in \mathbb{N}$.

Recalling that we have fixed the converging subsequence and then letting $n \to \infty$, we have

$$\mathbb{E}\left[\int_0^{T} \exp(\varsigma|u(t)|_{-1}^2)dt\right] \leq \frac{\exp(|x|_{-1}^2) + KT}{\varsigma(\pi^4 - 2\varsigma\|B^*\|^2)},$$

which implies the desired result (2.3).

Let us now give the proof of (2.4). Under our assumptions, we can easily show the following 1-Lipschitz continuity of $(u(t))_{t\geq 0}$ on its initial data:

$$|u(t;x) - u(t;y)|_{-1} \leq \exp(-(\pi^4 - \lambda)t)|x - y|_{-1} \tag{2.9}$$

holds for any $x, y \in \mathbf{H^c}, t \geq 0$. Here we omit its proof and refer the reader to Lemma 2.5 below for a similar discussion. Since $\tilde{\mu}^c$ is invariant for P_t^c, for any Lipschitz continuous function $\phi \in B_b(\mathbf{H}^c)$, we have

$$\begin{aligned}
|P_t^c\phi(x) - \tilde{\mu}^c(\phi)| =& |P_t^c\phi(x) - \tilde{\mu}^c(P_t^c\phi)| \\
\leq& \int_{\mathbf{H^c}} |P_t^c\phi(x) - P_t^c\phi(y)|\tilde{\mu}^c(dy) \\
\leq& \|\nabla\phi\|_\infty \int_{\mathbf{H^c}} |u(t;x) - u(t;y)|_{-1}\tilde{\mu}^c(dy).
\end{aligned}$$

Consequently, we can easily complete the proof of (2.4) by (2.9) and (2.3). □

From now on, let us establish the asymptotic log-Harnack inequality for P_t^c under the following highly degenerate condition:

(A2): There exists a non-negative sequence $\{b_i\}_{i=1}^\infty$ such that $Bu = \sum_{i=1}^\infty b_i\langle u, e_i\rangle e_i$ and there exists a big enough integer N such that $b_i > 0,\ i = 1, 2, \cdots, N$ and

$$(N+1)^2\pi^2 > \lambda. \tag{2.10}$$

From this assumption, it follows that $\text{span}\{e_1, \cdots, e_N\} \subset \text{Rang}(B)$ and such setting is known as the so-called essentially elliptic condition, see Section 4.5 [23].
Let Π_l be the projector from $\mathbf{H}^c$ into the $(N+1)$-dimension space $\text{span}\{e_0, e_1, \cdots, e_N\}$, where N is the integer appearing in the above assumption **(A2)**. Moreover, we know

that B restricted on $\mathrm{span}\{e_1, e_2, \cdots, e_N\}$ is invertible and its inverse will be denoted by B^{-1}. Thus, the operator $B^{-1}A\Pi_l$ is well-defined from $\mathbf{H}^c$ to $\mathrm{span}\{e_1, e_2, \cdots, e_N\}$ and is bounded. Set

$$\alpha = \frac{1}{2}\min\left\{\pi^4, \left[(N+1)^2\pi^2 - \lambda\right](N+1)^2\pi^2\right\}.$$

Now we can formulate the main result of this section.

Theorem 2.3. *Suppose the assumptions* **(A1)**-**(A2)** *are satisfied. Then, for any* $c \in (-1, 1)$*, the Markov semigroup* $(P_t^c)_{t\geq 0}$ *satisfies the asymptotic log-Harnack inequality. More precisely, we have that*

$$\begin{aligned} P_t^c \log\phi(y) \leq & \log P_t^c\phi(x) + \frac{\lambda}{8\alpha}(1-\exp(-2\alpha t))\|B^{-1}A\Pi_l\|_{op}^2|x-y|_{-1}^2 \\ & + \exp(-\alpha t)\|\nabla\log\phi\|_\infty|x-y|_{-1},\ t>0 \end{aligned} \tag{2.11}$$

holds for any $x, y \in \mathbf{H}^c$ *and any positive* $\phi \in B_b(\mathbf{H}^c)$ *with* $\|\nabla\log\phi\|_\infty < \infty$*, where* $\|B^{-1}A\Pi_l\|_{op}$ *denotes the operator norm of* $B^{-1}A\Pi_l$ *from* $\mathbf{H}^c$ *to the* N*-dimensional space* $\mathrm{span}\{e_1, e_2, \cdots, e_N\}$*.*

The proof of Theorem 2.3 will be stated after Lemma 2.5 below. Here let us first state some applications of Theorem 2.3. As we have stated, the asymptotic strong Feller property can be immediately deduced from Theorem 2.3. Moreover, thanks to Theorem 2.1 [5], many other important properties of $P_t^c\phi$, such as its gradient estimate, asymptotic heat kernel estimate and asymptotic irreducibility, can be deduced.

Corollary 2.4. *Under the assumptions of Theorem 2.3, for any* $c \in (-1, 1)$ *the following assertions hold:*
(i) $(P_t^c)_{t\geq 0}$ *is asymptotically strong Feller.*
(ii) *For any Lipschitz continuous function* $\phi \in B_b(\mathbf{H}^c)$*,*

$$|\nabla P_t^c\phi| \leq \left(\frac{\lambda}{4\alpha}\right)^{\frac{1}{2}}\|B^{-1}A\Pi_l\|_{op}\sqrt{P_t^c\phi^2 - (P_t^c\phi)^2} + \|\nabla\phi\|_\infty\exp(-\alpha t).$$

(iii) *For any non-negative* $\phi \in B_b(\mathbf{H}^c)$ *with* $\|\phi\|_\infty < \infty$ *and all* $x \in \mathbf{H}^c$*,*

$$\limsup_{t\to\infty} P_t^c\phi(x) \leq \log\left(\frac{\tilde{\mu}^c(\exp\phi)}{\int_{\mathbf{H}^c}\exp(-\frac{\lambda}{8\alpha}\|B^{-1}A\Pi_l\|_{op}^2|x-y|_{-1}^2)\tilde{\mu}^c(dy)}\right),$$

where $\tilde{\mu}^c$ *the invariant probability measure of* P_t^c*.*
(iv) *Suppose for some* $x \in \mathbf{H}^c$ *and a measurable set* $A \subset \mathbf{H}^c$*,* $\liminf_{t\to\infty} P_t^c(x, A) > 0$ *holds. Then, for any* $y \in \mathbf{H}^c$ *and* $\epsilon > 0$

$$\liminf_{t\to\infty} P_t^c(y, A_\epsilon) > 0,$$

where A_ϵ *denotes the* ϵ*-neighborhood of* A *in* $\mathbf{H}^c$*.*

Proof. For any $x, y \in \mathbf{H}^c$, let us set

$$\Phi(x,y) = \frac{\lambda}{8\alpha}\|B^{-1}A\Pi_l\|_{op}^2|x-y|_{-1}^2$$

and

$$\Psi(t,x,y) = \exp(-\alpha t)|x-y|_{-1}.$$

Then, it is clear that

$$\lim_{y\to x}\frac{\Phi(x,y)}{|x-y|_{-1}^2} = \frac{\lambda}{8\alpha}\|B^{-1}A\Pi_l\|_{op}^2$$

and

$$\lim_{y\to x}\frac{\Psi(t,x,y)}{|x-y|_{-1}} = \exp(-\alpha t).$$

Hence, the conditions in Theorem 2.1 (1) [5] are satisfied, and consequently (i) and (ii) can be shown by Theorem 2.1 (1) [5].

On the other hand, (iii) and (iv) are the direct results from Theorem 2.1 (2) and (4) [5] respectively. □

Remark 2.2. (i) By analogy to the proof of Proposition 1.6 [38], we can also show that for any Lipschitz continuous function $\phi \in B_b(\mathbf{H}^c)$,

$$|\nabla P_t^c\phi|(x) \le \left(\frac{\lambda}{4\alpha}\right)^{\frac{1}{2}}\|B^{-1}A\Pi_l\|_{op}\|\phi\|_\infty + 2\|\nabla\phi\|_\infty\exp(-\alpha t), \tag{2.12}$$

which is a sufficient condition for the asymptotical strong Feller property, see Proposition 3.12 [23]. Although the asymptotic strong Feller property has been proved in Proposition 4.3 [22], the estimate like (2.12) has not been proved. So a new proof is given for Proposition 4.3 [22] by our result.
(ii) From the asymptotical strong Feller property, it follows that any two different ergodic invariant measures must have disjoint topological supports, see Theorem 3.16 [23].
(iii) The uniqueness of invariant measures of P_t^c is proved by showing the asymptotical strong Feller property and weakly topological irreducibility in [22]. From the proof of Theorem 2.3, we see that "N" in the assumption **(A2)** for the asymptotical strong Feller property can be chosen a little smaller than that in Proposition 4.3 [22] (because of the factor π^2) since their assumption was not completely optimal. In addition, the uniqueness of invariant measures can be also easily shown by (2.4) under the assumption of Theorem 2.2.

Theorem 2.3 will be proved using the asymptotic coupling by change of measures. Let first us construct the asymptotic coupling. Let us consider the coupling stochastic partial differential equation

$$\begin{cases} dv^n(t) = -\frac{1}{2}A\big\{Av^n(t) - p_n(v^n(t)) + \lambda\Pi_h v^n(t) + \lambda\Pi_l u^n(t)\big\}dt \\ \qquad\qquad + BdW(t), \quad t>0, \\ v^n(0) = y, \end{cases} \tag{2.13}$$

where $\Pi_h = I - \Pi_l$. By the similar arguments to (2.2), one can show that (2.13) has a unique mild solution v^n such that $v^n \in C([0,\infty); H) \cap L^{2n+2}((0,\infty)\times(0,1))$ *a.s.* Furthermore, we know that the mass of $v^n(t)$ is conservative in $t \geq 0$ by considering the assumption **(A1)**.

Lemma 2.5. *The solution $u^n(t;x)$ of (2.2) and the solution $v^n(t;y)$ of (2.13) are asymptotically coupling in the following sense:*

$$|u^n(t;x) - v^n(t;y)|_{-1} \leq \exp(-\alpha t)|x-y|_{-1},\ x, y \in \mathbf{H}^c. \tag{2.14}$$

Proof. By the density of $L^2(0,1)$ in $\mathbf{H}$, it is enough for us to show (2.14) holds for any $x, y \in L^2(0,1)$ whenever $\bar{x} = \bar{y} = c$. For simplicity of notations, we write $u^n(t)$ for $u^n(t;x)$ and respectively $v^n(t)$ for $v^n(t;y)$ in the following.

Let $X^n(t) = u^n(t) - v^n(t)$. Then it is clear that $X^n(t)$ satisfies

$$\begin{cases} dX^n(t) = -\dfrac{1}{2}A\big\{AX^n(t) - [p_n(u^n(t)) - p_n(v^n(t))] + \lambda\Pi_h X^n(t)\big\}dt, \\ X^n(0) = x - y. \end{cases} \tag{2.15}$$

Let us first point out that $\overline{X^n}(t) = 0$ for any $t \geq 0$ by the conservative properties of $u^n(t)$ and $v^n(t)$, which will be used below.

Without loss of generality, we assume the integer $K > N$ and let us set

$$X^{n,K}(t) = \sum_{k=0}^{K} \langle u^n(t) - v^n(t), e_k\rangle e_k.$$

Then it is known that $X^{n,K}(t) \in D(A)$ *a.s.* Therefore, by (2.15) and the spectral property of the operator A,

$$\begin{aligned} \frac{d}{dt}|X^{n,K}(t)|_{-1}^2 =& \langle AX^{n,K}(t), X^{n,K}(t)\rangle - \langle p_n(u^n(t)) - p_n(v^n(t)), X^{n,K}(t)\rangle \\ &+ \lambda\langle \Pi_h X^{n,K}(t), X^{n,K}(t)\rangle \\ =& -|X^{n,K}(t)|_1^2 - \langle p_n(u^n(t)) - p_n(v^n(t)), X^{n,K}(t)\rangle \\ &+ \lambda\langle \Pi_h X^{n,K}(t), X^{n,K}(t)\rangle. \end{aligned} \tag{2.16}$$

Let us note that for any $u \in V_1$ with $\bar{u} = 0$,

$$|u|_1^2 \geq \pi^2|\Pi_l u|^2 + (N+1)^2\pi^2|\Pi_h u|^2.$$

Recalling that $\overline{X^n}(t) = 0$ and noting the increasing property of p_n, then by (2.16), we obtain that

$$\frac{d}{dt}|X^{n,K}(t)|_{-1}^2 \leq -\pi^2|\Pi_l X^{n,K}(t)|^2 - \{(N+1)^2\pi^2 - \lambda\}|\Pi_h X^{n,K}(t)|^2. \tag{2.17}$$

Hence, using (2.10) in the assumption **(A2)** and combining (2.17) with the next relations

$$|\Pi_l u|^2 \geq \pi^2|\Pi_l u|_{-1}^2 \text{ and } |\Pi_h u|^2 \geq (N+1)^2\pi^2|\Pi_h u|_{-1}^2,\ u \in L^2(0,1),$$

we have that

$$\begin{aligned}\frac{d}{dt}|X^{n,K}(t)|_{-1}^2 \leq & -\pi^4|\Pi_l X^{n,K}(t)|_{-1}^2 - \{(N+1)^2\pi^2-\lambda\}(N+1)^2\pi^2|\Pi_h X^{n,K}(t)|_{-1}^2 \\ \leq & -2\alpha|X^{n,K}(t)|_{-1}^2.\end{aligned}$$

Finally, letting $K \to \infty$ in the above inequality, we have

$$\frac{d}{dt}|X^{n}(t)|_{-1}^2 \leq -2\alpha|X^{n}(t)|_{-1}^2,$$

which obviously implies the desired result (2.14). □

From now on, let us now formulate the proof of Theorem 2.3.

Proof of Theorem 2.3. Let us set

$$\xi(t) = \xi^n(t) := \frac{\lambda}{2}B^{-1}A\Pi_l(u^n(t)-v^n(t)),\ t \geq 0.$$

Although $\xi^n(t)$ depends on n, we will omit the superscript n, because uniform estimates on n can be shown as below.

By Lemma 2.5, it goes that

$$\begin{aligned}|\xi(t)| \leq & \frac{\lambda}{2}\|B^{-1}A\Pi_l\|_{op}|u^n(t)-v^n(t)|_{-1} \\ \leq & \frac{\lambda}{2}\|B^{-1}A\Pi_l\|_{op}\exp(-\alpha t)|x-y|_{-1}.\end{aligned} \tag{2.18}$$

Therefore, by the Novikov condition, we have that

$$M(t) = \exp\left(\int_0^t \langle \xi(s), dW(s)\rangle - \frac{1}{2}\int_0^t |\xi(s)|^2 ds\right)$$

is a real-valued martingale and then by the Girsanov theorem,

$$\widetilde{W}(t) = W(t) - \int_0^t \xi(s)ds,\ t \geq 0$$

is a cylindrical Wiener process on $L^2(0,1)$ under the probability $\tilde{\mathbb{P}}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M(t).$$

According to the definition of $\xi(t)$, we point out that $M(t), \tilde{W}(t)$ and $\tilde{\mathbb{P}}$ are depending on n. For our goal, uniform estimates on n should be established.

Now by using the stochastic processes $(\widetilde{W}(t))_{t\geq 0}$ and $(\xi(t))_{t\geq 0}$, the coupling equation (2.13) can be rewritten as

$$\begin{cases} dv^n(t) = -\frac{1}{2}A\big\{Av^n(t) - p_n(v^n(t)) + \lambda v^n(t)\big\}dt + Bd\widetilde{W}(t), \quad t>0, \\ v^n(0) = y. \end{cases} \tag{2.19}$$

In particular, by the uniqueness in law of the solution of (2.13), it is known that the distribution of $v^n(t)$ under $\tilde{\mathbb{P}}$ is same as that of $u^n(t;y)$ under $\mathbb{P}$.

We first note that for any positive $\phi \in B_b(\mathbf{H}^c)$ with $\|\nabla \log \phi\|_\infty < \infty$,

$$\begin{aligned} P_t^{n,c} \log \phi(y) =& \mathbb{E}^{\tilde{\mathbb{P}}} \left[\log \phi(v^n(t))\right] \\ =& \mathbb{E}^{\tilde{\mathbb{P}}}[\log \phi(u^n(t;x))] + \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\log \phi(v^n(t))\right] - \mathbb{E}^{\tilde{\mathbb{P}}} \left[\log \phi(u^n(t;x))\right] \right\} \\ :=& I_1^n(t) + I_2^n(t), \end{aligned} \tag{2.20}$$

where $\mathbb{E}^{\tilde{\mathbb{P}}}$ denotes the expectation with respect to $\tilde{\mathbb{P}}$.

Using the definition of $\tilde{\mathbb{P}}$ and the martingale property of $(M(t))_{t\geq 0}$, we have that

$$\begin{aligned} I_1^n(t) =& \mathbb{E}[M(t) \log \phi(u^n(t;x))] \\ \leq& \mathbb{E}[M(t) \log M(t)] - \mathbb{E}[M(t)] \log \mathbb{E}[M(t)] + \mathbb{E}[M(t)] \log \mathbb{E}[\phi(u^n(t;x))] \\ =& \mathbb{E}[M(t) \log M(t)] + \mathbb{E}[M(t)] \log \mathbb{E}[\phi(u^n(t;x))] \\ =& \mathbb{E}[M(t) \log M(t)] + \log P_t^{n,c} \phi(x), \end{aligned} \tag{2.21}$$

where the Young inequality

$$\mathbb{E}[XY] \leq \mathbb{E}[X \log X] - \mathbb{E}[X] \log E[X] + \mathbb{E}[X] \log \mathbb{E}[e^Y] \tag{2.22}$$

for any non-negative random variables $X, Y \geq 0$ $a.s.$ with $\mathbb{E}[X] > 0$ has be used for the second line; see Lemma 2.4 [3] for its proof.

On the other hand, using (2.18), we deduce that

$$\begin{aligned} \mathbb{E}[M(t) \log M(t)] =& \mathbb{E}^{\tilde{\mathbb{P}}}[\log M(t)] \\ =& \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^t \langle \xi(s), dW(s) \rangle - \frac{1}{2} \int_0^t |\xi(s)|^2 ds \right] \\ =& \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^t \langle \xi(s), d\widetilde{W}(s) \rangle + \frac{1}{2} \int_0^t |\xi(s)|^2 ds \right] \\ =& \frac{1}{2} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^t |\xi(s)|^2 ds \right] \\ \leq& \frac{\lambda}{4} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^t \|B^{-1} A \Pi_l\|_{op}^2 \exp(-2\alpha s) |x-y|_{-1}^2 ds \right] \\ =& \frac{\lambda}{8\alpha} (1 - \exp(-2\alpha t)) \|B^{-1} A \Pi_l\|_{op}^2 |x-y|_{-1}^2. \end{aligned}$$

Hence, plugging this estimate into (2.21), we have

$$I_1^n(t) \leq \frac{\lambda}{8\alpha} (1 - \exp(-2\alpha t)) \|B^{-1} A \Pi_l\|_{op}^2 |x-y|_{-1}^2 + \log P_t^{n,c} \phi(x). \tag{2.23}$$

Let us now give the required estimate for $I_2(t)$, which is easier. In fact, by Lemma 2.5, we have that

$$\begin{aligned} |I_2^n(t)| \leq& \|\nabla \log \phi\|_\infty \mathbb{E}^{\tilde{\mathbb{P}}}[|u^n(t) - v^n(t)|_{-1}] \\ \leq& \exp(-\alpha t) \|\nabla \log \phi\|_\infty |x-y|_{-1}. \end{aligned} \tag{2.24}$$

Inserting (2.23) and (2.24) into (2.20), we see that for any $n \in \mathbb{N}$

$$P_t^{n,c} \log \phi(y) \leq \log P_t^{n,c}\phi(x) + \frac{\lambda}{8\alpha}(1-\exp(-2\alpha t))\|B^{-1}A\Pi_l\|_{op}^2|x-y|_{-1}^2 + \exp(-\alpha t)\|\nabla \log \phi\|_\infty |x-y|_{-1},\ t>0$$

holds for any $x, y \in \mathbf{H}^c$ and any positive $\phi \in B_b(\mathbf{H}^c)$ with $\|\nabla \log \phi\|_\infty < \infty$.

Consequently, noting that $\|B^{-1}A\Pi_l\|_{op}$ is independent of n and then using Theorem 2.1, we can obtain the desired result (2.11) by letting $n \to \infty$. Therefore, the proof of Theorem 2.3 is completed. □

3 Harnack inequality for the case of nondegenerate space-time white noise

In this section, we will intend to study the properties of the Markov semigroup generated by the SPDE (1.2) for the special case of $B = \frac{d}{d\theta}$ (or equivalently $B = (-A)^{\frac{1}{2}}$, see Remark 3.1 below) with its domain $H^1(0,1)$, which is studied in [17]. Let us recall the definition of solution of (1.2) according to Definition 1.1 [17].

Definition 3.1. Let the initial datum x be a continuous function defined on $[0,1]$ with its values in $[-1,1]$, i.e., $x \in C([0,1];[-1,1])$.
(1) The quadruplet $(u(\cdot), \eta_+, \eta_-, W)$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}; \mathbb{P})$ is said to be a weak solution of (1.2) with its initial value x if all of the following conditions are satisfied:
(i) The stochastic process $u(\cdot) \in C((0,T]\times[0,1];[-1,1]) \cap C([0,1];\mathbf{H})$ $a.s.$ with $u(0) = x$, and $f(u) \in L^1([0,T]\times[0,1])$ $a.s.$ for any $T > 0$.
(ii) η_+ and η_- are two positive random measures on $[0,\infty)\times[0,1]$ satisfying the following property:

$$\eta_\pm([\delta, T]\times[0,1]) < \infty \ a.s. \ \text{ for all } \delta \in (0,T] \ \text{ and } T > 0.$$

(iii) $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $L^2(0,1)$. Moreover, the initial value x is independent of $(W(t))_{t\geq 0}$ and the stochastic process $(u(t), W(t))_{t\geq 0}$ is $(\mathcal{F}_t)$-adapted.
(iv) For all $\phi \in D(A^2)$ and $0 < \delta < t$,

$$\begin{aligned}\langle u(t), \phi\rangle =& \langle u(\delta), \phi\rangle - \frac{1}{2}\int_\delta^t \langle u(s), A^2\phi\rangle ds - \frac{1}{2}\int_\delta^t \langle f(u(s)), A\phi\rangle ds \\ &- \frac{1}{2}\int_\delta^t\int_0^1 A\phi(\theta)\eta_+(dsd\theta) + \frac{1}{2}\int_\delta^t\int_0^1 A\phi(\theta)\eta_-(dsd\theta) \\ &+ \int_\delta^t \langle B^*\phi, dW(s)\rangle \ \ a.s.\end{aligned} \tag{3.1}$$

(v) The contact properties $\mathrm{supp}(\eta_+) \subset \{(t,\theta) \in [0,\infty)\times[0,1] :\ u(t,\theta) = +1\}$ and $\mathrm{supp}(\eta_-) \subset \{(t,\theta) \in [0,\infty)\times[0,1] :\ u(t,\theta) = -1\}$ hold almost surely, that is,

$$\int_0^\infty\int_0^1 (1-u(t,\theta))\eta_+(dtd\theta) = \int_0^\infty\int_0^1 (1+u(t,\theta))\eta_-(dtd\theta) = 0 \ a.s.$$

(2) A weak solution $(u(\cdot), \eta_+, \eta_-, W)$ is said to be a strong one if the stochastic process $(u(t))_{t\geq 0}$ is adapted to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by $(W(t))_{t\geq 0}$.

The term $\langle f(u(s)), Ah\rangle$ appearing in the right hand side of (3.1) should be understood in a duality between L^1 and L^∞. In fact, it is assumed that $f(u(t)) \in L^1([0,T]\times[0,1])$ *a.s.* for any fixed T in (i). In addition, for the uniqueness of the solution, we mean the pathwise uniqueness, that is, for any two solutions $(u^i, \eta^i_+, \eta^i_-, W), i = 1, 2$ of (1.2) with same initial data defined on the same probability space with same W, then $(u^1, \eta^1_+, \eta^1_-) = (u^2, \eta^2_+, \eta^2_-)$ *a.s.*

Now let us summarize main results obtained in [17], which will be used in the following. For brevity, in this section, we will use the same notations introduced in Section 3. To emphasize the initial value, $u(t;x)$ or $u(t,\cdot;x)$ will be used according to purposes in the sequel.

Theorem 3.1. *For any $c \in (-1,1)$ and $x \in K := \{x \in L^2(0,1) : x \in [-1,1]\}$ with $\bar{x} = c$, the SPDE (1.2) has a unique strong solution $(u(\cdot;x); \eta_+, \eta_-, W)$. Moreover, the following hold:*
(i) *The mass of $u(t;x)$ is conservative in t, that is, $\bar{u}(t;x) = \bar{x}$ for all $t > 0$.*
(ii) *$(u(t;x); t \geq 0, x \in K \cap \mathbf{H}^c)$ is a $K \cap \mathbf{H}^c$-valued continuous Markov process and its associated Markov transition semigroup P^c_t is strong Feller on $\mathbf{H}^c$.*
(iii) *For each $c \in (-1,1)$,*

$$\nu^c(dx) = \frac{1}{Z^c} \exp\left(-\int_0^1 F(x(\theta))d\theta\right) 1_K(x)\mu_c(dx)$$

is the unique invariant measure of P^c_t, where μ^c denotes the Gaussian measure $N(ce_0, (-A)^{-1})$ and Z^c denotes the normalization constant.
(iv) *For any $k \in \mathbb{N}$ and $0 = t_0 < t_1 < t_2 < \cdots < t_k$, $(u^n(t_i;x))_{i=1}^k$ converges weakly to $(u(t_i;x))_{i=1}^k$ as $n \to \infty$. In particular, for any $\phi \in B_b(\mathbf{H}^c)$ and $t \geq 0$, we have $\lim_{n\to\infty} P^{n,c}_t \phi(x) = P^c_t \phi(x)$. Hereafter, $u^n(t;x)$ and $P^{n,c}_t$ denote the solution of (2.1) with $B = \frac{d}{d\theta}$ and its associated Markov semigroup.*

Remark 3.1. (i) $(-A)^{-1}$ appearing in (iii) denotes the inverse of $-A$ from L^2_0 to L^2_0. From Lemma 2.1 [18], it is known that μ^c is the distribution of the Gaussian process $(B(\theta) - \bar{B} + c)_{\theta\in[0,1]}$ on $C([0,1])$, where $(B(\theta))_{\theta\in[0,1]}$ denotes a standard Brownian motion and $\bar{B} = \int_0^1 B(\theta)d\theta$.
(ii) Noting that $\frac{d}{d\theta}\dot{W}(t,\theta)$ and $(-A)^{\frac{1}{2}}\dot{W}(t,\theta)$ have the same covariance structure, we see that it is equivalent for us to consider $B = (-A)^{\frac{1}{2}}$ in (1.2) instead of $\frac{d}{d\theta}$ and note that $(-A)^{\frac{1}{2}}$ is symmetric. So, for simplicity, we will consider $B = (-A)^{\frac{1}{2}}$ in the sequel and we know that Theorem 3.1 still holds.

Lemma 3.2. *Let $B = (-A)^{\frac{1}{2}}$. Then B is reversible on* $\mathrm{span}\{e_i : i = 1, 2, \cdots\}$ *and*

$$|B^{-1}z|^2 = |z|^2_{-1},\ z \in \mathbf{H}^0.$$

Proof. Recalling the definition of the operator A and the seminorm $|\cdot|_\gamma$, we can easily proof this lemma. □

The following is the main result of this section. Since the mass of the solution to (1.2) is required to be conserved, the well-known approaches used for the stochastic partial differential equation with additive noise, see [33, 36, 37, 41] for example, can not applied to our case. Moreover, the case of double-well potential is covered. To show our main result, we make use of the approach initially introduced in [34], in which the stochastic different equations with multiplicative noise is studied.

Theorem 3.3. *Suppose $\pi^2 > \lambda$. Then the Harnack inequality with power $p > 1$*

$$|P_t^c\phi|^p(y) \le P_t^c|\phi|^p(x)\exp\left\{\frac{p(\pi^2-\lambda)\pi^2|x-y|_{-1}^2}{2(p-1)(e^{(\pi^2-\lambda)\pi^2 t}-1)}\right\} \tag{3.2}$$

holds for any $\phi \in B_b(\mathbf{H}^c)$, $x, y \in K \cap \mathbf{H}^c$ and $t > 0$. In particular, the log-Harnack inequality

$$P_t^c \log\phi(y) \le \frac{(\pi^2-\lambda)\pi^2|x-y|_{-1}^2}{2(e^{(\pi^2-\lambda)\pi^2 t}-1)} + \log P_t^c\phi(x) \tag{3.3}$$

holds for any $0 < \phi \in B_b(\mathbf{H}^c)$, $x, y \in K \cap \mathbf{H}^c$ and $t > 0$.

Proof. Let us fix $T > 0$ and let $\gamma(t)$ be a continuously differentiable and strictly positive function on $[0, T)$ with $\gamma(T) = 0$, which be specified later. Let $\aleph$ denote the projection of $\mathbf{H}$ to $\mathrm{span}\{e_i : i = 1, 2, 3, \cdots\}$ and then consider the coupling stochastic partial differential equation

$$\begin{cases} dw^n(t) = -\frac{1}{2}A\big\{Aw^n(t) - p_n(w^n(t)) + \lambda w^n(t)\big\}dt + \frac{\aleph(u^n(t)-w^n(t))}{\gamma(t)}dt \\ \qquad\qquad + BdW(t), \quad t \in [0, T), \\ w^n(0) = y, \end{cases} \tag{3.4}$$

where $(u^n(t))_{t\ge 0}$ denotes the solution of (2.2) with $B = (-A)^{\frac{1}{2}}$.

Since $\aleph$ is a bounded linear operator, by following the arguments used in [14], one can show that for each initial value $y \in \mathbf{H}$, the SPDE (3.4) has a unique solution w^n up to the explosion time σ^n such that $w^n \in C([0, \sigma^n \wedge T); H) \cap L^{2n+2}((0, \sigma^n \wedge T) \times (0, 1))$ a.s., where $\sigma^n := \lim_{k\to\infty}\sigma_k^n$ with $\sigma_k^n = \inf\{t \in [0, T) : |w^n(t)|_{-1} \ge k\}$.

Moreover, the conservation of the average of $w^n(t)$ holds for $t \in [0, \sigma^n \wedge T)$. Indeed, considering the mild solution of (3.4), we have that for any $x \in L^2(0, 1)$ with $\bar{x} = c \in (-1, 1)$ and $t \le \sigma_k^n \wedge T$,

$$\begin{aligned} &\langle w^n(t), e_0\rangle \\ =&\langle e^{-\frac{1}{2}A^2 t}x, e_0\rangle + \int_0^t \left\langle Ae^{-\frac{1}{2}A^2(t-s)}\big[p_n(w^n(s)) - \lambda w^n(s)\big], e_0\right\rangle ds \\ &+ \int_0^t \left\langle e^{-\frac{1}{2}A^2(t-s)}\frac{\aleph(u^n(s)-w^n(s))}{\gamma(s)}, e_0\right\rangle ds + \int_0^t \left\langle Be^{-\frac{1}{2}A^2(t-s)}e_0, dW(s)\right\rangle. \end{aligned}$$

Now noting that $e^{-\frac{1}{2}A^2 t}e_0 = e_0$ and $Be_0 = \aleph e_0 = 0$, we obtain that

$$\langle w^n(t), e_0\rangle = \langle x, e_0\rangle = c, \ t \in [0, \sigma_k^n \wedge T),$$

which clearly implies our claim by the density of L^2 in $\mathbf{H}$. From now on, the proof will divided into three steps.

Step 1: The goal of this step is to construct a successful coupling up to time T. More precisely, we will show that $w^n(T;y) = u^n(T;x)$ holds almost surely under a probability measure equivalent to $\mathbb{P}$.

To show it, let us set $Y^n(t) = u^n(t) - w^n(t),\ t \le \sigma_k^n \wedge T$ and let $R \in (0,T)$ be fixed. Then by the conservation of the mass, we have that $\overline{Y^n}(t) = 0$ whenever $x, y \in \mathbf{H}^c$ and $Y^n(t)$ satisfies

$$\begin{cases} dY^n(t) = -\dfrac{1}{2}A\big\{AY^n(t) - [p_n(u^n(t)) - p_n(w^n(t))] + \lambda Y^n(t)\big\}dt \\ \qquad\qquad - \dfrac{\aleph Y^n(t)}{\gamma(t)}dt, \quad t \in [0, \sigma_k^n \wedge r), \\ Y^n(0) = x - y. \end{cases} \tag{3.5}$$

Then, using the increasing property of p_n and $\overline{Y^n}(t) = 0$, we can deduce analogously to (2.17) that

$$\begin{aligned} d|Y^n(t)|_{-1}^2 \le & -|Y^n(t)|_1^2 dt + \lambda |Y^n(t)|^2 dt - \frac{2\langle (-A)^{-1}\aleph Y^n(t), Y^n(t)\rangle}{\gamma(t)}dt \\ = & -|Y^n(t)|_1^2 dt + \lambda |Y^n(t)|^2 dt - \frac{2|\aleph Y^n(t)|_{-1}^2}{\gamma(t)}dt \\ \le & -(\pi^2 - \lambda)|Y^n(t)|^2 dt - \frac{2|\aleph Y^n(t)|_{-1}^2}{\gamma(t)}dt \\ \le & -(\pi^2 - \lambda)\pi^2 |Y^n(t)|_{-1}^2 dt - \frac{2|Y^n(t)|_{-1}^2}{\gamma(t)}dt,\ t \in [0, \sigma_k^n \wedge R), \end{aligned} \tag{3.6}$$

where the assumption $\pi^2 > \lambda$ has been used for last inequality.

Hence, (3.6) and the chain rule give that

$$d\frac{|Y^n(t)|_{-1}^2}{\gamma(t)} \le -\frac{|Y^n(t)|_{-1}^2}{\gamma^2(t)}\big(\gamma'(t) + (\pi^2 - \lambda)\pi^2\gamma(t) + 2\big)dt,\ t \in [0, \sigma_k^n \wedge R], \tag{3.7}$$

where the strict positivity of $\gamma(t)$ has been used.

Now let us specify the function $\gamma(t)$. Let $\alpha \in (0,2)$ and $\gamma(t)$ be the unique solution of the ordinary differential equation

$$\gamma'(t) + (\pi^2 - \lambda)\pi^2\gamma(t) + 2 = \alpha$$

with $\gamma(T) = 0$, that is,

$$\gamma(t) = \frac{2-\alpha}{(\pi^2 - \lambda)\pi^2}\left(e^{(\pi^2-\lambda)\pi^2(T-t)} - 1\right),\ t \in [0,T]. \tag{3.8}$$

It is easy to testify that $\gamma(t), t \in [0,T]$ has all of the properties stated at the beginning of the proof.

By noting that $\alpha \in (0,2)$ and using (3.7), we easily see that

$$\int_0^t \frac{|Y^n(s)|_{-1}^2}{\gamma^2(s)}ds + \frac{|Y^n(t)|_{-1}^2}{\alpha\gamma(t)} \leq \frac{|x-y|_{-1}^2}{\alpha\gamma(0)},\ t \in [0, \sigma_k^n \wedge R]. \tag{3.9}$$

Let us define the stochastic process $N(t), t \in [0, \sigma^n \wedge R]$ by

$$\begin{aligned} N(t) = &\exp\left(- \int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s) - w^n(s))}{\gamma(s)}, dW(s) \right\rangle \right. \\ &\left. - \int_0^t \frac{|B^{-1}\aleph(u^n(t) - w^n(t))|^2}{2\gamma^2(s)}ds \right). \end{aligned} \tag{3.10}$$

Thanks to (3.9) and Lemma 3.2, we have that for all $x, y \in \mathbf{H}^c$ and $t \in [0, \sigma_k^n \wedge R]$

$$\int_0^t \frac{|B^{-1}\aleph(u^n(s) - w^n(s))|^2}{2\gamma^2(s)}ds \leq \frac{|x-y|_{-1}^2}{2\alpha\gamma(0)}. \tag{3.11}$$

Let us now define the stochastic process $\overline{W}(t),\ t \in [0, \sigma^n \wedge R]$ by

$$d\overline{W}(t) = dW(t) + \frac{B^{-1}\aleph(u^n(t) - w^n(t))}{\gamma(t)}dt. \tag{3.12}$$

Then by the Novikov condition and the Girsanov theorem, we know that $\overline{W}(t), t \in [0, \sigma_k^n \wedge R]$ is a cylindrical Wiener process on $L^2(0,1)$ under the probability measure $N(\sigma_k^n \wedge T)\mathbb{P}$.

By the definitions of $N(t)$ and $\overline{W}(t)$ and by noting (3.11), we have

$$\begin{aligned} &\log N(t) \\ =& - \int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s) - w^n(s))}{\gamma(t)}, d\overline{W}(s) \right\rangle + \int_0^t \frac{|B^{-1}\aleph(u^n(t) - w^n(t))|^2}{2\gamma^2(s)}ds \\ \leq& - \int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s) - w^n(s))}{\gamma(t)}, d\overline{W}(s) \right\rangle + \frac{|x-y|_{-1}^2}{2\alpha\gamma(0)},\ t \in [0, \sigma_k^n \wedge R]. \end{aligned}$$

Therefore, by taking the expectations of both sides of the above inequality with respect to $N(\sigma_k^n \wedge T)\mathbb{P}$, we obtain

$$\mathbb{E}\left[N(\sigma_k^n \wedge R) \log N(\sigma_k^n \wedge R)\right] \leq \frac{|x-y|_{-1}^2}{2\alpha\gamma(0)}. \tag{3.13}$$

Recalling that $R \in [0,T)$ is arbitrary, we have that $N(\sigma_k^n \wedge R), R \in [0,T)$ is uniformly integrable and

$$\sup_{R\in[0,T)} \sup_{k,n\in\mathbb{N}} \mathbb{E}\left[N(\sigma_k^n \wedge R) \log N(\sigma_k^n \wedge R)\right] \leq \frac{|x-y|_{-1}^2}{2\alpha\gamma(0)}. \tag{3.14}$$

Then by the martingale convergence theorem and the Doob optional sampling theorem, it follows that $N(t\wedge\sigma^n), t\in[0,T]$ is a martingale and by letting $k\to\infty$ in (3.13),

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[N(\sigma^n\wedge t)\log N(\sigma^n\wedge t)\right]\leq\frac{|x-y|_{-1}^2}{2\alpha\gamma(0)},\ t\in[0,T]. \tag{3.15}$$

In addition, we known that $(\overline{W}(t))$ is a cylindrical Wiener process on $L^2(0,1)$ under the probability measure $\mathbb{Q}:=N(\sigma^n\wedge T)\mathbb{P}$ up to time $\sigma^n\wedge R$. By (3.9), in fact we can show that for all $n\in\mathbb{N}$, $\sigma^n=T$ $\mathbb{Q}$-$a.s.$ Indeed, since $(u^n(t))_{t\geq0}$ is the global solution of (2.2), we see that $\tau_l^n=\inf\{t\geq0:|u^n(t)|_{-1}\geq l\}$ diverges to ∞ as $l\to\infty$. Noting that $\gamma(t)$ is decreasing with respect to $t\in[0,T]$ and

$$|Y^n(t\wedge\tau_k^n\wedge\sigma_{2k}^n)|_{-1}\geq k,$$

we have

$$\mathbb{E}^{\mathbb{Q}}\left[1_{\{\sigma_{2k}^n\leq t<\tau_k^n\}}\frac{|Y^n(t\wedge\tau_k^n\wedge\sigma_{2k}^n)|_{-1}^2}{\gamma(t\wedge\tau_k^n\wedge\sigma_{2k}^n)}\right]\geq\frac{k^2}{\gamma(0)}\mathbb{Q}(\sigma_{2k}^n\leq t<\tau_k^n). \tag{3.16}$$

On the other hand, by (3.9), it is known that the left hand of (3.16) is bounded from above by $\frac{|x-y|_{-1}^2}{\gamma(0)}$. Hence, letting now $k\to\infty$ in (3.16), we obtain

$$\mathbb{Q}(\sigma^n\leq t)=0,\ t\in[0,T),$$

which clearly implies $\mathbb{Q}(\sigma^n=T)=1$.

Consequently, in the sequel, we can write $d\mathbb{Q}=N(T)d\mathbb{P}$ and then we know that $(\overline{W}(t))_{t\in[0,T]}$ defined by (3.12) is a cylindrical Wiener process on $L^2(0,1)$ with respect to $\mathbb{Q}$.

Using the cylindrical Wiener process $(\overline{W}(t))_{t\in[0,T]}$, we easily see that the SPDE (3.4) can be rewritten as follows:

$$\begin{cases} dw^n(t)=-\dfrac{1}{2}A\big\{Aw^n(t)-p_n(w^n(t))+\lambda w^n(t)\big\}dt+Bd\overline{W}(t),\quad t\in[0,T),\\ w^n(0)=y\in\mathbf{H}^c, \end{cases} \tag{3.17}$$

Since under $\mathbb{Q}$, $\overline{W}(t),t\in[0,T]$ is a cylindrical Wiener process on $L^2(0,1)$, similarly to (2.2), we know that (3.17) has global unique solution $w^n\in C([0,T];H)\cap L^{2n+2}((0,T)\times(0,1))$. Moreover, the distribution of $w^n(t)$ under $\mathbb{Q}$ is same as that of $u^n(t;x)$ under $\mathbb{P}$ by the uniqueness in law of solutions. Therefore, by the equivalence of $\mathbb{Q}$ and $\mathbb{P}$, we know that (3.4) also has the global solution up to time T.

From now on, we claim that the coupling of (2.2) and (3.4) is made successfully up to time T. Let τ denote the coupling time, that is,

$$\tau=\inf\{t\in[0,T]:u^n(t)=w^n(t)\text{ in }\mathbf{H}^c\}.$$

with the convention $\inf\emptyset=\infty$. Then we can show $\tau\leq T$ $a.s.$ by contradiction. In fact, if $\tau(\omega)>T$, then

$$\inf_{t\in[0,T]}|u^n(t,\omega)-w^n(t,\omega)|_{-1}^2$$

is strictly positive, since both u^n and w^n are continuous stochastic processes with values in $\mathbf{H}^c$. Hence, we obtain that the integral of $\frac{|u^n(t,\omega)-w^n(t,\omega)|^2_{-1}}{\gamma(t)}$ on [0,T] diverges, by noting that $\int_0^T \frac{1}{\gamma(t)}dt = \infty$. Therefore, we have that on the set $\{\tau > T\}$,

$$\int_0^T \frac{|Y^n(s)|^2_{-1}}{\gamma(s)}ds = \infty. \tag{3.18}$$

On the other hand, noting that $(\pi^2 - \lambda)\pi^2 > 0$, we obtain by (3.6) that

$$\int_0^t \frac{|Y^n(s)|^2_{-1}}{\gamma(s)}ds \leq \frac{|x-y|^2_{-1}}{2},\ t \in [0, \sigma_k^n \wedge R],$$

which contradicts with (3.18). Consequently, our claim is proved. In particular, we have

$$w^n(T;y) = u^n(T;x)\ \ \mathbb{Q}\text{-}a.s.$$

Based on the above preparations, this theorem can be shown in the usual way [35]. For the reader's convenience, we give the outline of the proof.
Step 2: Let us formulate the proof of (3.2). We first show for any $q > 1$,

$$\mathbb{E}[|N(t)|^q] \leq \exp\left\{\frac{(q-1)q|x-y|^2_{-1}}{2\alpha\gamma(0)}\right\},\ t \leq T. \tag{3.19}$$

By the definitions of $N(t)$ and $W(t)$, it follows easily that for any $q > 1$,

$$\begin{aligned}
\mathbb{E}[|N(t)|^q] =& \mathbb{E}^{\mathbb{Q}}[|N(t)|^{q-1}] \qquad (3.20)\\
=& \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-(q-1)\int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s)-w^n(s))}{\gamma(s)}, dW(s)\right\rangle\right.\right.\\
&\left.\left. -\ (q-1)\int_0^t \frac{|B^{-1}\aleph(u^n(t)-w^n(t))|^2}{2\gamma^2(s)}ds\right\}\right]\\
=& \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-(q-1)\int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s)-w^n(s))}{\gamma(s)}, d\overline{W}(s)\right\rangle\right.\right.\\
&\left.\left. +\ (q-1)\int_0^t \frac{|B^{-1}\aleph(u^n(t)-w^n(t))|^2}{2\gamma^2(s)}ds\right\}\right],\ t \leq T.
\end{aligned}$$

By (3.11), we have that

$$\sup_{t\in[0,T]} \exp\left\{\int_0^t \frac{|B^{-1}\aleph(u^n(s)-w^n(s))|^2}{2\gamma^2(s)}ds\right\} \leq \exp\left\{\frac{(q-1)q|x-y|^2_{-1}}{2\alpha\gamma(0)}\right\}. \tag{3.21}$$

Note that

$$\begin{aligned}
U(t) :=& \exp\left\{-(q-1)\int_0^t \left\langle \frac{B^{-1}\aleph(u^n(s)-w^n(s))}{\gamma(s)}, d\overline{W}(s)\right\rangle\right.\\
&\left. -(q-1)^2\int_0^t \frac{|B^{-1}\aleph(u^n(s)-w^n(s))|^2}{2\gamma^2(s)}ds\right\},\ t \leq T
\end{aligned}$$

is an exponential martingale under $\mathbb{Q}$. Then, by (3.21), we have

$$\begin{aligned}\mathbb{E}[|N(t)|^q] =&\mathbb{E}^{\mathbb{Q}}\left[U(t)\exp\left\{\int_0^t \frac{(q-1)q|B^{-1}\aleph(u^n(s)-w^n(s))|^2}{2\gamma^2(s)}ds\right\}\right]\\ \leq&\mathbb{E}^{\mathbb{Q}}\left[U(t)\sup_{t\in[0.T]}\exp\left\{\int_0^t \frac{(q-1)q|B^{-1}\aleph(u^n(s)-w^n(s))|^2}{2\gamma^2(s)}ds\right\}\right]\\ \leq&\exp\left\{\frac{(q-1)q|x-y|^2_{-1}}{2\alpha\gamma(0)}\right\}\mathbb{E}^{\mathbb{Q}}[U(t)]\\ =&\exp\left\{\frac{(q-1)q|x-y|^2_{-1}}{2\alpha\gamma(0)}\right\},\ t\leq T,\end{aligned}$$

where (3.11) has been used for the second inequality. Therefore, the proof of (3.19) is completed.

Let us now formulate the proof (3.2). According to the relation between $w^n(t;y)$ and $u^n(t;x)$, we have that for any $p>1$, any $\phi\in B_b(\mathbf{H})$ and any $x,y\in K\cap\mathbf{H}^c$

$$\begin{aligned}|P_T^{n,c}\phi|^p(y) =&|\mathbb{E}^{\mathbb{Q}}[\phi(w^n(T;y))]|^p\\ =&|\mathbb{E}^{\mathbb{Q}}[\phi(u^n(T;x))]|^p\\ =&|\mathbb{E}[N(T)\phi(u^n(T;x))]|^p\\ \leq&\mathbb{E}[N(T)^{\frac{p}{p-1}}]^{p-1}P_T^{n,c}|\phi|^p(x)\\ \leq&P_T^{n,c}|\phi|^p(x)\exp\left\{\frac{p|x-y|^2_{-1}}{2\alpha(p-1)\gamma(0)}\right\},\end{aligned}$$

where (3.19) with $q=\frac{p-1}{p}$ has been used for the last inequality.

Consequently, we can complete the proof of (3.2) by letting $\alpha=1$ and then $n\to\infty$ thanks to Theorem 3.1.

Step 3: Let us finally give the proof (3.3) in brief. By the definition of $\mathbb{Q}$, the Young inequality (2.22) and the estimate (3.15), it follows that

$$\begin{aligned}P_T^{n,c}\log\phi(y) =&\mathbb{E}^{\mathbb{Q}}[\log\phi(w^n(T;y))]\\ =&\mathbb{E}[N(T)\phi(u^n(T;x))]\\ \leq&\mathbb{E}[N(T)\log N(T)]+\log\mathbb{E}[\phi(u^n(T;x))]\\ \leq&\frac{|x-y|^2_{-1}}{2\alpha\gamma(0)}+\log P_T^{n,c}\phi(x).\end{aligned}\tag{3.22}$$

Recalling the representation of γ, see (3.8), and minimizing the first term in (3.22) with respect to $\alpha\in(0,2)$, we see that

$$P_T^{n,c}\log\phi(y)\leq\frac{(\pi^2-\lambda)\pi^2|x-y|^2_{-1}}{2(e^{(\pi^2-\lambda)\pi^2T}-1)}+\log P_T^{n,c}\phi(x).$$

Now thanks to Theorem 3.1, we can easily complete the proof of (3.3) with $t=T$ by letting $n\to\infty$ in the above inequality. □

Remark 3.2. If we consider $B = \frac{d}{d\theta}$ with $\mathrm{Dom}(B) = H^1(0,1)$ as that in the original paper [17], then we can show the following equation

$$|B^*(BB^*)^{-1}z|^2 = |z|_{-1}^2,\ z \in \mathbf{H}^0,$$

by noting that $BB^* = -A$. Thus, we can replace B in the definition of $N(t)$, see (3.10), by $B^*(BB^*)^{-1}$ and then obtain the same results as those in Theorem 3.3.

In addition, the method used in Theorem 3.3 can be also applied to the SPDE (1.2) with more general B instead of $B = \frac{d}{d\theta}$ or $B = (-A)^{\frac{1}{2}}$. In fact, if BB^* is reversible restricted on

$$\mathrm{span}\{e_n : n = 1, 2, \cdots\}$$

and

$$|B^*BB^*z| \le C|z|_{-1}, z \in \mathbf{H}$$

for some $C > 0$ and (i), (ii), (iv) in Theorem 3.1 hold, then the Harnack equalities similar as those in Theorem 3.3 can be established. For example, if there exists a strictly positive sequence $\{b_n\}_{n=1}^{\infty}$ such that $Be_n = b_n e_n, n = 1, 2, \cdots$ and the sequence $\{nb_n^{-1}\}_{n=1}^{\infty}$ is bounded, then B satisfies the assumptions stated above.

According to Theorem 1.4.1 [35], many important properties of P_t^c can be deduced from Theorem 3.3. For example, uniqueness of invariant probability measures can be easily known. As we stated in Theorem 3.1, the existence and uniqueness of invariant measures has been proved in [17] by a different approach. Here, it is valuable to know that it can be reproved as the application of Harnack inequalities. Moreover, we also know that P_t^c is absolutely continuous with respect to its invariant measure ν^c and the following results hold for the density $p^c(t,x,y)$ of P_t^c with respect to ν^c.

Corollary 3.4. *Under the assumptions of Theorem 3.3, the following heat kernel inequalities are fulfilled for any $t > 0, x, y \in \mathbf{H}^c$ and $p > 1$*

$$\int_{\mathbf{H}^c} p^c(t,x,z)\left\{\frac{p^c(t,x,z)}{p^c(t,y,z)}\right\}^{\frac{1}{p-1}}\nu^c(dz) \le \exp\left\{\frac{p(\pi^2-\lambda)\pi^2|x-y|_{-1}^2}{2(p-1)^2(e^{(\pi^2-\lambda)\pi^2 t}-1)}\right\},$$
$$\int_{\mathbf{H}^c} p^c(t,x,z)\log\frac{p^c(t,x,z)}{p^c(t,y,z)}\nu^c(dz) \le \frac{(\pi^2-\lambda)\pi^2|x-y|_{-1}^2}{2(e^{(\pi^2-\lambda)\pi^2 t}-1)}.$$

Acknowledgments

L. Goudenège was supported in part by the French National Research Agency (ANR) as leader of the SIMALIN project ANR-19-JCJC, and B. Xie was supported in part by Grant-in-Aid for Scientific Research (C) 16K05197 from Japan Society for the Promotion of Science (JSPS). The authors warmly thank the referee for invaluable comments and suggestions, which led to significant improvement in the presentation of the paper

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