

Arnold's symplectization of Picard-Lefschetz theory, braid group action on Stokes data and homological mirror symmetry for Fano manifolds.  
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## Arnold's proposal (end of 80's): study Lefschetz thimbles and vanishing cycles as lagrangian submanifolds

- ▶ Picard-Lefschetz theory  $\subset$  Lefschetz's: study topology of complex manifolds iteratively via holomorphic map  $f : X \rightarrow \mathbb{P}^1$ ,  $f^{-1}(pt) \subset X$
- ▶ Recall, Arnold's proposal that there exists natural symplectization of Morse theory has led via his fixed points conjecture  $\rightarrow$  Floer homology, quantum cohomology etc.
- ▶ Arnold proposed to put also the Picard-Lefschetz theory into symplectic setting, namely vanishing cycles and Lefschetz thimbles should be lagrangian submanifolds and the monodromy diffeomorphisms should be symplectomorphisms.

## Stokes data of quantum cohomology (Ceccotti-Vafa'93, Dubrovin'94)

- ▶ (FM "1st structure connection" )  $\nabla_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} + z^{-2}A + z^{-1}B$  - connection with pole of order 2 at 0 on trivial vector bundle  $H^*(X) \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$
- ▶ where  $A = (c_1(T_X^*) *_{q} \bullet)$  is the quantum multiplication by canonical class and  $B|_{H^k(X)} = \frac{1}{2}(k-d), d = \dim_{\mathbb{C}} X$  is the grading operator
- ▶ Example  $X = \mathbb{P}^{n-1}$ , in the standard basis

$$A = \begin{pmatrix} 0 & & & & nq \\ n & 0 & & & \\ & \ddots & \ddots & & \\ & & & n & 0 \end{pmatrix}, B = \begin{pmatrix} \frac{1-n}{2} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{n-1}{2} \end{pmatrix}$$

## Stokes data, upper-triangular matrix

- ▶  $\frac{\partial}{\partial w} - (U + \frac{1}{w}V)$  where  $U = \text{diag}(u_1, \dots, u_n)$  and  $V^t = -V$ ,  $w = z^{-1}$  has irregular singularity at  $\infty$  and its monodromy is given by Stokes data
- ▶ Stokes rays  $R_{ij} = \{\text{Re}(w(u_i - u_j)) = 0, w \neq 0\}$ , or  $w = ir(\bar{u}_i - \bar{u}_j)$ ,  $r > 0$ ,  $u_i \neq u_j$
- ▶  $\Pi_r: \varphi - \pi - \epsilon < \arg(w) < \varphi + \epsilon$ ,  
 $\Pi_l: \varphi - \epsilon < \arg(w) < \varphi + \pi + \epsilon$ ,
- ▶ where  $\varphi$  is such that  $\{\arg w = \varphi\} \cap R_{ij} = \emptyset$
- ▶ Fix a formal asymptotic solution at  $\infty: Y_{\text{formal}} = (C_0 + C_1/w + \dots) \exp(-wU)$
- ▶ There exists unique analytic solutions  $Y_{r/l}(w)$  on  $\Pi_r/\Pi_l$  having the asymptotic development  $Y_{\text{formal}}$
- ▶ Then  $Y_l = Y_r S$  on  $\varphi - \epsilon < \arg(w) < \varphi + \epsilon$
- ▶  $Y_r = Y_l S_-$  on  $\varphi + \pi - \epsilon < \arg(w) < \varphi + \pi + \epsilon$
- ▶  $S_- = S^t$ ,  $S_{ij} \neq 0$  only if  $R_{ij} \subset \Pi_l$ ,  $S_{ii} = 1$
- ▶ After reordering of  $u_i$ ,  $S$  is an upper-triangular matrix, with natural braid group action

## Upper-triangular matrix of exceptional collection in derived categories

- ▶ It was noticed by several people (Cecotti-Vafa, Zaslow...) at around 93 that the Stokes matrices obtained in such way, say, for  $X = \mathbb{P}^{n-1}$  coincides(!) with the upper-triangular matrix of the bilinear form

$$\chi(E, F) = \sum_k (-1)^k \text{Ext}^k(E, F)$$

in the basis given by particular orthogonal collection of exceptional objects in the derived category  $D^b \text{Coh}(\mathbb{P}^{n-1})$  (on the same variety  $X$ )

- ▶  $E_j = \mathcal{O}(j-1), j = 1, \dots, n$   $S_{ij} = \binom{n}{j-i} j \geq i$

## Upper-triangular matrix of exceptional collection (cont'd)

- ▶ Also after applying the braid group, the matrix  $S^\sigma$  coincides (!) with the matrix of the bilinear form  $\chi$  for the collection of exceptional  $E_1^\sigma, \dots, E_n^\sigma$  where the action of braid group on exceptional collections was previously described in the theory of exceptional collections in the 80s. It was believed that similar coincidence should hold for other Fano manifolds

## Laplace transform, integral solutions

- ▶ The Laplace transform  $\tilde{\nabla}_{\frac{\partial}{\partial z}} = -z \frac{\partial}{\partial z} + A \frac{\partial}{\partial z} + B$  has regular singularities at  $\infty$  and at  $z = u_i$
- ▶ The solutions to  $\nabla_{\frac{\partial}{\partial z}} \Psi = 0$  are given by exponential integrals (Givental'93)
- ▶  $\psi_n = z^{(1-n)/2} \int_{\gamma_{z,q}} \exp\left(\frac{1}{z} f(x)\right) \prod \frac{dx_i}{x_i}$ ,  
 $f(x) = x_1 + \dots + x_{n-1} + \frac{q}{x_1 \dots x_{n-1}}$
- ▶  $\psi_i = \left(uq \frac{\partial}{\partial q}\right)^{n-i} \psi_n$ ,  $\Psi = \sum_i \psi_i p^i$  where  $p^0, \dots, p^{n-1}$  is the standard basis in  $H^*(\mathbb{P}^{n-1})$
- ▶ And  $\gamma_{z,q}$  are the standard relative cycles  $\text{Re}(f/u) \rightarrow -\infty$  in  $(\mathbb{C}^*)^{n-1}$  (Lefschetz thimbles)
- ▶ Solutions to  $\tilde{\nabla}_{\frac{\partial}{\partial z}} \tilde{\Psi}$  are similarly given by the fiber integrals over the corresponding vanishing cycles

## monodromy, upper-triangular matrix for the regular connection

- ▶ Stokes data of  $\nabla_{\frac{\partial}{\partial z}}$ , or monodromy of integrals over Lefschetz thimbles, can be computed by Picard-Lefschetz theory
- ▶ The Stokes matrix  $S$  coincides with upper half of the intersection pairing matrix on the vanishing cycles

- ▶ Arnold, Varchenko, Gusein-Zade, page 37:

Пусть  $S$  — матрица пересечений особенности  $f$  в каком-нибудь базисе,  $L$  — матрица формы Зейферта (или оператора  $\text{Var}^{-1}$ ) этой особенности в том же базисе,  $H$  — матрица оператора классической монодромии  $h_*$ ,  $H^{(r)}$  — матрица оператора  $h_*^{(r)}$  в двойственном базисе. Теоремы 4, 5 и 6 означают, что  $S = -L + (-1)^n L^T$ ,  $H = (-1)^n L^{-1} L^T$ ,  $H^{(r)} = (-1)^n L^T L^{-1}$  (значок  $T$  означает транспонированную матрицу). Если  $\{\Delta_i\}$  — отмеченный базис исчезающих циклов, то в нем матрица  $L$  является верхней треугольной, а матрица  $L^T$  — нижней треугольной. Таким образом, в отмеченном базисе



## Arnold's earlier proposal for symplectic Picard-Lefschetz theory

- ▶ As a student of Arnold during the MSU years, I've heard several times starting from 1990 this Arnold's proposal for symplectic Picard-Lefschetz theory
- ▶ From the above calculation via Picard-Lefschetz theory which I did in 1996 we see that we get the natural correspondence Lefschetz thimbles  $\longleftrightarrow$  exceptional objects such that the pairing  $\chi$  coincides (!) with the upper half of the intersection pairing on Lefschetz thimbles
- ▶ I realized that the coincidence of the two upper-triangular matrices together with the braid group actions could be explained using the setting of Arnold proposal for the Lefschetz thimbles to be Lagrangian manifolds, if one assumes an extension of the Homological Mirror Symmetry for Fano manifolds, in this Arnold's setting

## Arnold's symplectic Picard-Lefschetz and HMS

- ▶ The only problem was that the pairing on thimbles was symmetric or antisymmetric and one only needs the upper-half of this matrix.
- ▶ I rushed to my PhD adviser Maxim Kontsevich with this idea and he proposed next day to modify the pairing on Lefschetz thimbles via a small twist of the plane, say, in clockwise order.
- ▶ From this the extension of homological mirror symmetry conjecture for Fano manifolds was born, together with other developments involving “Fukaya-Seidel” category, application of matrix factorization categories to mirror symmetry etc...

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