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# Fixed point stacks under groups of multiplicative type

Matthieu Romagny

January 6, 2021

**Abstract.** We prove that if a group scheme of multiplicative type acts on an algebraic stack with affine, finitely presented diagonal then the stack of fixed points is algebraic. For this, we extend two theorems of [SGA3.2] on functors of subgroups of multiplicative type, and functors of homomorphisms from a group of multiplicative type.

We fix a base scheme  $S$ . For applications in various domains of algebraic geometry and notably in enumerative geometry, it is useful to know that an  $S$ -algebraic stack (in the sense of Artin) acted on by a group scheme of multiplicative type has an algebraic stack of fixed points. This was mentioned in [Ro05], Remark 3.4 but the case of general Artin stacks was not considered there. Most recently Gross, Joyce and Tanaka developed in [GJT20] a framework for a theory for wall-crossing of enumerative invariants in  $\mathbb{C}$ -linear abelian categories. Provided the abelian category satisfies certain axioms, these authors explain how a general mechanism would yield invariants counting semi-stable objects for weak stability conditions and their wall-crossing formulae. One of these axioms is for the stack to be algebraic, and a first choice candidate is the fixed point category of the compactly supported sheaves on some quasi-projective manifold (toric surface or Calabi-Yau fourfold) which classifies torus equivariant compactly supported coherent sheaves.

In the present note we prove such a result of algebraicity for fixed point stacks under groups of multiplicative type, answering a question of Arkadij Bojko:

**Theorem 1** *Let  $X \rightarrow S$  be an algebraic stack with affine, finitely presented diagonal. Let  $G \rightarrow S$  be a finitely presented group scheme of multiplicative type acting on  $X$ . Then the fixed point stack  $X^G \rightarrow S$  is algebraic, and the morphism  $X^G \rightarrow X$  is representable by schemes, separated and locally of finite presentation.*

The proof of this soon reduces to the following statement, which generalizes [SGA3.2], Exp. XI, Th. 4.1 and Cor. 4.2 to non-smooth  $H$ , thus giving an answer to Exp. XI, Rem. 4.3.

**Theorem 2** *Let  $H$  be an affine, finitely presented  $S$ -group scheme.*

- (1) *The functor  $\text{Submt}(H)$  of subgroups of multiplicative type of  $H$  is representable by an  $S$ -scheme which is separated and locally of finite presentation. Moreover, each  $S$ -quasi-compact closed subscheme of  $\text{Submt}(H)$  is affine over  $S$ .*
- (2) *Let  $G$  be a finitely presented  $S$ -group scheme of multiplicative type. Then the functor of group scheme homomorphisms  $\text{Hom}(G, H)$  is representable by an  $S$ -scheme which is separated and locally of finite presentation. Moreover, each  $S$ -quasi-compact closed subscheme of  $\text{Hom}(G, H)$  is affine over  $S$ .*

Here are some comments on these results and their proof. Firstly, it should be emphasized that such statements are very special to groups of multiplicative type and should not be expected for more general affine group schemes. They are made possible by the very strong rigidity properties of groups of multiplicative type.

Secondly, we observe that Theorem 2 is proved in [SGA3.2] when  $H$  is smooth or more generally when  $H$  is a closed subgroup of a smooth group scheme  $H'$  (see Exp. XI, Rem. 4.3). This is sufficient to handle Theorem 1 in the case where  $X = [Y/H']$  is a quotient stack by a smooth group scheme, for

then the inertia stack of  $X$  (on which rests the representability of  $X^G \rightarrow X$ , see Section 1) embeds in  $H'$ , locally on  $X$ . However, for the general case another strategy is needed. What we do is that we use the density of finite flat subgroup schemes in groups of multiplicative type to reduce to a situation where we can apply Grothendieck's theorem on representability of unramified functors.

We finish this introduction with a remark of a more historical tone. It has been delightful to read and use the representability results of [SGA3.2] and [Mu65]. This left the author with the impression to follow a path across the 1960s, seeing in the soil the seeds of what would ultimately become Artin's criteria for representability by an algebraic space: the "auxiliary results of representability" of [SGA3.2], Exp. XI, § 3 and Grothendieck's representability theorem for unramified functors. May the reader share this pleasure.

The table of contents after the acknowledgements describes the plan of the article.

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## 1 Algebraicity of fixed points stacks

In this section, we show how to derive Theorem 1 from Theorem 2; the rest of the article will be devoted to the proof of Theorem 2. Let  $X^G$  be the stack of fixed points as described in [Ro05], Prop. 2.5. Recall that the sections of  $X^G$  over an  $S$ -scheme  $T$  are the pairs  $(x, \{\alpha_g\}_{g \in G(T)})$  composed of an object  $x \in X(T)$  and a collection of isomorphisms  $\alpha_g : x_T \rightarrow g^{-1} x_T$  satisfying the cocycle condition  $\alpha_{gh} = h^{-1} \alpha_g \circ \alpha_h$ :

$$\begin{array}{ccc}
 & h^{-1} x_U & \\
 \alpha_h \nearrow & & \searrow h^{-1} \alpha_g \\
 x_U & \xrightarrow{\alpha_{gh}} & (gh)^{-1} x_U.
 \end{array}$$

(The  $\alpha_g$  here is the  $\alpha_{g^{-1}}$  of [Ro05]; shifting notation simplifies the multiplication of the group functor  $K$  below.) It is enough to prove that the morphism  $X^G \rightarrow X$  is representable by schemes. For this we fix a point  $x : T \rightarrow X$  and we study the fibred product

$$F_G := X^G \times_X T.$$

This is the functor whose values on a  $T$ -scheme  $U$  are the collections of isomorphisms  $\{\alpha_g\}_{g \in G}$ . In order to prove that  $F_G$  is representable by a scheme, we introduce the functor  $K$  defined by

$$K(U) = \{(g, \alpha); g \in G(U) \text{ and } \alpha : x_U \xrightarrow{\sim} g^{-1} x_U \text{ an isomorphism}\}.$$

The formula  $(g, \alpha) \cdot (h, \beta) := (gh, h^{-1}\alpha \circ \beta)$  defines a law of multiplication on  $K$ :

$$x_U \xrightarrow{\beta} h^{-1}x \xrightarrow{h^{-1}\alpha} (gh)^{-1}x_U.$$

The element  $(g, \alpha) = (1, \text{id}_x)$  is neutral for this law. This makes  $K$  a group functor. Moreover, there is a morphism of group functors  $K \rightarrow G$ ,  $(g, \alpha) \mapsto g$ . This is representable, affine and finitely presented, because for each  $g : U \rightarrow G$  the fibre product  $K \times_G U$  is the functor  $\text{Isom}_U(x_U, g^{-1}x_U)$  which is affine and finitely presented, by assumption on  $X$ . Since  $G$  is also affine and finitely presented, we see that  $K$  is an affine  $T$ -group scheme of finite presentation.

By the definitions,  $F_G$  is the functor of group-theoretic sections of  $K \rightarrow G$ . Because  $\text{Hom}(G, G)$  is unramified and separated ([SGA3.2], Exp. VIII, Cor. 1.5),  $F_G$  is an open and closed subfunctor of the functor  $\text{Hom}(G, K)$  of group scheme homomorphisms. This reduces us to Theorem 2, item (2).

## 2 Representability of the functor of homomorphisms: reductions

In Sections 2 to 4 we prove item (2) of Theorem 2: representability of the functor  $\text{Hom}(G, H)$ .

**2.1 Localization and use of the Density Theorem.** The question of representability is local on  $S$  for the étale topology, so we may assume that  $S$  is affine and  $G$  is split. Then it is a product  $G = N \times \mathbb{G}_m^r$  where  $N$  is finite diagonalizable. But for a product  $G = G_1 \times G_2$ , the functor  $\text{Hom}(G, H)$  is the closed subfunctor of the product  $\text{Hom}(G_1, H) \times \text{Hom}(G_2, H)$  composed of pairs of maps that commute. Using [SGA3.2], Exp. VIII, 6.5.b) we see that it is enough to consider the factors individually. By [SGA3.2], Exp. XI, Prop. 3.12.b) the functor  $\text{Hom}(N, H)$  is representable by an affine  $S$ -scheme of finite presentation. Hence we just have to handle the key case  $G = \mathbb{G}_m$ .

For a prime number  $\ell$  let  $S_\ell \subset S$  be the open subscheme where  $\ell$  is invertible. We can choose two distinct primes  $\ell, \ell'$  and write  $S = S_\ell \cup S_{\ell'}$ . Since the question of representability is local on  $S$ , it is enough to handle  $S_\ell$  and  $S_{\ell'}$  separately. In this way we reduce to the case where  $\ell \in \mathcal{O}_S^\times$ .

Also, since the functor is locally of finite presentation, we can reduce to the case where  $S$  (still affine) is of finite type over  $\text{Spec}(\mathbb{Z})$ .

For each  $n > 0$ , let  $\mu_{\ell^n}$  be the kernel of the endomorphism  $\ell^n : \mathbb{G}_m \rightarrow \mathbb{G}_m$ . By [SGA3.2], Exp. XI, Prop. 3.12.b) again, the functor  $\text{Hom}(\mu_{\ell^n}, H)$  is representable by an affine  $S$ -scheme of finite presentation. By restricting morphisms to the torsion subschemes, we have a map of functors :

$$\varphi : \text{Hom}(\mathbb{G}_m, H) \longrightarrow \lim_n \text{Hom}(\mu_{\ell^n}, H).$$

Here the target is representable by an affine scheme, as a projective limit of affine schemes. Moreover, the Density Theorem ([SGA3.2], Exp. IX, Théorème 4.7 and Remark 4.10) implies that  $\varphi$  is a monomorphism. We will prove that  $\varphi$  is representable by using Grothendieck's theorem on representation of unramified functors. Let  $T$  be an  $S$ -scheme and let  $\{u_n : \mu_{\ell^n, T} \rightarrow H_T\}$  be a collection of morphisms of  $T$ -group schemes. We want to prove that the fibred product

$$\text{Hom}(\mathbb{G}_m, H) \times_{\lim_n \text{Hom}(\mu_{\ell^n}, H)} T$$

is representable. For this we change our notation, renaming  $T$  as  $S$ , and we end up with the following setting.

**2.2 Summary of situation.** After these reductions, we start anew with:

- a prime number  $\ell$ ,
- a  $\mathbb{Z}[1/\ell]$ -algebra of finite type  $R$  and  $S = \text{Spec}(R)$ ,
- a finitely presented affine  $S$ -group scheme  $H$ ,
- a family of compatible morphisms of  $S$ -group schemes  $\{u_n : \mu_{\ell^n} \rightarrow H\}$ .

The compatibility of the  $u_n$ 's means that  $u_{n+1}$  extends  $u_n$  for each  $n$ . We study the functor

$$F(T) = \{\text{morphisms } f : \mathbb{G}_m \rightarrow H \text{ that extend the } u_n, n \geq 0\}.$$

The Density Theorem implies that  $F \rightarrow S$  is a monomorphism of functors. That is, in fact  $F(T) = \{f\}$  is the one-element set composed of the unique morphism  $f : \mathbb{G}_{m,T} \rightarrow H_T$  that extends the  $u_n$  if there is one, and  $F(T) = \emptyset$  otherwise. In particular  $F \rightarrow S$  is formally unramified. In order to prove the representability of  $F$  in Section 4, we will use Grothendieck's theorem on unramified functors, verifying the eight relevant axioms. For this we need some preparations which we do in the next section.

### 3 Descent along schematically dominant morphisms

Again let  $F$  be the functor of group homomorphisms  $f : \mathbb{G}_m \rightarrow H$  extending the maps  $u_n$ . The verification of Grothendieck's axioms for  $F$  will be based to a large extent on the following fact: the map  $F(T) \rightarrow F(T')$  is an isomorphism for all schematically dominant morphisms of schemes  $T' \rightarrow T$ . This is Lemma 3.5 below. Its proof will use a variation on the argument used to show that formal homomorphisms from a group scheme of multiplicative type to an affine group scheme are algebraic, see [SGA3.2] Exp. IX, § 7. It is the purpose of this subsection to settle this.

We work over a  $\mathbb{Z}[1/\ell]$ -algebra  $A$ . The argument in *loc. cit.* uses the natural embedding of the Hopf algebra  $A[\mathbb{G}_m]$ , which as a module is the direct sum  $A^{(\mathbb{Z})}$ , into the direct product  $A^{\mathbb{Z}}$ . In our context, the useful information we have comes from the embedding of  $A[\mathbb{G}_m]$  into the limit of the Hopf algebras of the  $\mu_{\ell^n}$ 's. We need to relate the two embeddings.

**3.1 The ind-scheme of  $\ell$ -power roots of unity.** We need some facts on the ind-scheme of  $\ell$ -power roots of unity:

$$\mu_{\ell^\infty} = \text{colim } \mu_{\ell^n}.$$

We consider its function algebra  $A[\mu_{\ell^\infty}] = \lim A[\mu_{\ell^n}]$  and the canonical injective morphism:

$$c : A[\mathbb{G}_m] \hookrightarrow A[\mu_{\ell^\infty}].$$

Contemplated from the right angle, this map is naturally isomorphic to the inclusion of  $A$ -modules  $A^{(\mathbb{Z})} \hookrightarrow A^{\mathbb{Z}}$  of the direct sum into the direct product. For later use, we need to make this precise. For each  $n \geq 0$  let  $\mu_{\ell^n}^* \subset \mu_{\ell^n}$  be the subscheme of primitive roots of unity; we have  $\mu_{\ell^n}^* = \text{Spec}(A[z]/(\Phi_n))$  where  $\Phi_n$  is the cyclotomic polynomial. Since  $\ell$  is invertible in the base ring, the  $\Phi_n$  are pairwise strongly coprime in  $A[z]$  and the factorization  $z^{\ell^n} - 1 = \Phi_0 \cdots \Phi_n$  gives rise to isomorphisms of algebras:

$$A[\mu_{\ell^n}] = \prod_{0 \leq i \leq n} \frac{A[z]}{(\Phi_i)}, \quad A[\mu_{\ell^\infty}] = \prod_{i \in \mathbb{N}} \frac{A[z]}{(\Phi_i)}.$$

**3.2  $\Phi$ -adic expansions.** Let  $n \geq 0$ . Let  $I_n := \{-\lfloor \varphi(\ell^n)/2 \rfloor, \dots, \varphi(\ell^n) - \lfloor \varphi(\ell^n)/2 \rfloor - 1\}$  where  $\varphi$  is Euler's totient function. The  $A$ -module  $C_n = \bigoplus_{i \in I_n} Az^i$  is finite free of rank  $\varphi(\ell^n) = \deg(\Phi_n)$ . It follows that the composition  $C_n \hookrightarrow A[z^{\pm 1}] \rightarrow A[z]/(\Phi_n)$  is an isomorphism of  $A$ -modules, and we can choose  $C_n$  as a module of representatives of residue classes for Euclidean division mod  $\Phi_n$ . As a result, for each Laurent polynomial  $P \in A[z^{\pm 1}]$  there are unique  $q \in A[z^{\pm 1}]$  and  $r \in C_n$  such that  $P = \Phi_n q + r$ . By writing such divisions :

$$\begin{aligned} P &= \Phi_0 q_0 + r_0 & (r_0 \in C_0) \\ q_0 &= \Phi_1 q_1 + r_1 & (r_1 \in C_1) \\ q_1 &= \Phi_2 q_2 + r_2 & (r_2 \in C_2) \end{aligned}$$

and so on, we reach a  $\Phi$ -adic expansion  $P = r_0 + r_1(z-1) + \dots + r_{d+1}(z^{\ell^d} - 1)$  for some  $d$ . This gives a direct sum decomposition by the submodules  $D_0 = A$  and  $D_n = C_n \cdot (z^{\ell^{n-1}} - 1)$  for all  $n \geq 1$ :

$$A[z^{\pm 1}] = \bigoplus_{n \geq 0} D_n.$$

**3.3 The map  $\theta_A : A^{\mathbb{Z}} \rightarrow A[\mu_{\ell^\infty}]$ .** As a module, the ring  $A[z^{\pm 1}]$  is the direct sum  $A^{(\mathbb{Z})}$ ; we embed it into the direct product  $A^{\mathbb{Z}}$ . Compatibly with the expression  $A^{(\mathbb{Z})} = \bigoplus_{n \geq 0} D_n$ , we have a direct product decomposition  $A^{\mathbb{Z}} = \prod_{n \geq 0} D_n$ . Also we view the algebra  $A[\mu_{\ell^\infty}]$  as the product  $\prod_{i \in \mathbb{N}} \frac{A[z]}{(\Phi_i)}$ . The restriction  $D_n \hookrightarrow A[z^{\pm 1}] \rightarrow A[z]/(\Phi_n)$  of the residue class map is an isomorphism; we abuse notation slightly by omitting it from the notation in defining the following map:

$$\begin{aligned} A^{\mathbb{Z}} = \prod_{n \in \mathbb{N}} D_n &\xrightarrow{\theta_A} A[\mu_{\ell^\infty}] = \prod_{n \in \mathbb{N}} \frac{A[z]}{(\Phi_n)} \\ (P_0, P_1, P_2, \dots) &\longmapsto (P_0, P_0 + P_1, P_0 + P_1 + P_2, \dots). \end{aligned}$$

**3.4 Lemma.** (1) *The map  $\theta_A$  is an isomorphism of  $A$ -modules and fits in a commutative diagram:*

$$\begin{array}{ccc} A^{\mathbb{Z}} & \xrightarrow[\sim]{\theta_A} & A[\mu_{\ell^\infty}] \\ & \swarrow \text{can} & \nearrow c \\ & A^{(\mathbb{Z})} = A[\mathbb{G}_m] & \end{array}$$

Here *can* is the canonical inclusion of the direct sum into the product.

(2) *The isomorphism  $\theta_A$  is functorial in  $A$ , that is, for each morphism of  $\mathbb{Z}[1/\ell]$ -algebras  $A \rightarrow B$  there is a cartesian diagram compatible with the maps  $\text{can}_A, \text{can}_B$  and  $c_A, c_B$ :*

$$\begin{array}{ccc} A[\mu_{\ell^\infty}] & \xrightarrow{\theta_A} & A^{\mathbb{Z}} \\ \downarrow & & \downarrow \\ B[\mu_{\ell^\infty}] & \xrightarrow{\theta_B} & B^{\mathbb{Z}} \end{array}$$

In particular, if  $A \rightarrow B$  is injective and  $P \in B^{(\mathbb{Z})}$ , we have  $\text{can}(P) \in A^{\mathbb{Z}}$  if and only if  $c(P) \in A[\mu_{\ell^\infty}]$ .

**Proof :** (1) Let  $P \in A[z^{\pm 1}]$  be written in the form  $P = P_0 + P_1 + \dots + P_d$  with  $P_i \in D_i$  for all  $i$ . The canonical injection  $\text{can} : \bigoplus D_n \hookrightarrow \prod D_n$  sends  $P$  to  $(P_0, P_1, P_2, \dots)$  while the map  $c$  sends  $P$  to

$(P_0, P_0 + P_1, P_0 + P_1 + P_2, \dots)$ . In other words,  $c(P) = \theta(\text{can}(P))$  as desired. The fact that  $\theta$  is an  $A$ -linear isomorphism follows from the fact that it is defined by a triangular unipotent matrix.

(2) The commutative diagram exists by the very construction; it is cartesian because  $\theta_A$  and  $\theta_B$  are isomorphisms.  $\square$

We can now prove that the objects of the functor  $F$  descend along schematically dominant morphisms. For the latter notion, we refer the reader to [EGA] IV<sub>3</sub>.11.10.

**3.5 Lemma.** *Let  $T' \rightarrow T$  be a morphism of  $S$ -schemes and assume that either:*

- (1)  $T' \rightarrow T$  is schematically dominant, or
- (2)  $T = \text{Spec}(A)$  is affine,  $T' = \coprod_i \text{Spec}(A_i)$  is a disjoint sum of affines, with  $A \rightarrow \prod_i A_i$  injective.

*Then the map  $F(T) \rightarrow F(T')$  is bijective.*

The result is easier when  $T' \rightarrow T$  is quasi-compact, but the general case will be crucial for us.

**Proof :** (1) Since  $F(T)$  has at most one point, the map  $F(T) \rightarrow F(T')$  is injective and it is enough to prove that it is surjective. By fpqc descent of morphisms, this holds when  $T' \rightarrow T$  is a covering for the fpqc topology. Applying this remark with chosen Zariski covers  $\coprod T_i \rightarrow T$  and  $\coprod_{i,j} T'_{ij} \rightarrow \coprod_i T_i \times_T T' \rightarrow T'$ , we see that the vertical maps in the following commutative square are bijective:

$$\begin{array}{ccc} F(T) & \longrightarrow & F(T') \\ \downarrow \wr & & \downarrow \wr \\ \prod F(T_i) & \longrightarrow & \prod F(T'_{ij}). \end{array}$$

Choosing  $T = \text{Spec}(A)$  and  $T'_{ij} = \text{Spec}(A'_{ij})$  affine, the assumption that  $T' \rightarrow T$  is schematically dominant implies that  $A \rightarrow \prod A'_{ij}$  is injective. This way we reduce to case (2).

(2) We start with an element of  $F(T')$ , i.e. a family of morphisms of  $A_i$ -group schemes  $f_i : \mathbb{G}_{m, A_i} \rightarrow H_{A_i}$  each of which extends the morphisms  $u_n : \mu_{n, A_i} \rightarrow H_{A_i}$ ,  $n \geq 0$ . For simplicity, in the sequel we write again  $f_i : \mathcal{O}_H \otimes A_i \rightarrow A_i[z^{\pm 1}]$  and  $u_n : \mathcal{O}_H \rightarrow R[z]/(z^{\ell^n} - 1)$  the corresponding comorphisms of Hopf algebras; this should not cause confusion. For each  $R$ -algebra  $A$  we also write  $u_{\infty, A} : \mathcal{O}_H \otimes A \rightarrow A[\mu_{\ell^\infty}]$  for the product of the  $u_{n, A}$ . These fit in a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_H \otimes A_i & & \\ f_i \downarrow & \searrow^{u_{\infty, A_i}} & \\ A_i[z^{\pm 1}] & \xrightarrow{c_{A_i}} & A_i[\mu_{\ell^\infty}]. \end{array}$$

We now reduce to the case where  $A$  and the  $A_i$  are noetherian. For this let  $L$  resp.  $L_i$  be the image of  $R \rightarrow A$ , resp. of  $R \rightarrow A_i$ . Being quotients of  $R$ , the rings  $L$  and  $L_i$  are noetherian. Moreover, since  $A \rightarrow \prod A_i$  is injective then so is  $L \rightarrow \prod L_i$ . Since the  $u_n$  are defined over  $R$  hence over  $L$ , we have a commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_H \otimes L_i & & & & \\ \downarrow & \searrow^{f_{i, L_i}} & & \searrow^{u_{\infty, L_i}} & \\ \mathcal{O}_H \otimes L_i & & L_i[z^{\pm 1}] & \xrightarrow{c_{L_i}} & L_i[\mu_{\ell^\infty}] \\ & \searrow^{f_i} & \downarrow & & \downarrow \\ & & A_i[z^{\pm 1}] & \xrightarrow{c_{A_i}} & A_i[\mu_{\ell^\infty}]. \end{array}$$

Since the lower right square is cartesian, there is an induced dotted arrow. In this way we see that  $f_i$  is actually defined over  $L_i$ . So replacing  $A$  (resp.  $A_i$ ) by  $L$  (resp.  $L_i$ ), we obtain the desired reduction.

Let  $\widehat{A} := \prod_i A_i$ . Taking products over  $i$ , we build a commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_H \otimes A & \xrightarrow{u_{\infty, A}} & A[\mu_{\ell\infty}] \\
\downarrow & \searrow \text{dotted} & \downarrow \\
\mathcal{O}_H \otimes \widehat{A} & & \widehat{A}[\mu_{\ell\infty}] \\
\downarrow & & \downarrow \\
\prod_i (\mathcal{O}_H \otimes A_i) & \xrightarrow{\prod f_i} \prod_i (A_i[z^{\pm 1}]) \xleftarrow{\prod c_{A_i}} & \widehat{A}[\mu_{\ell\infty}]
\end{array}$$

Henceforth we set  $C := \mathcal{O}_H \otimes A$  and we write  $\Phi_0$  the dotted composition in the diagram above. What the diagram shows is that

$$C \xrightarrow{\Phi_0} \prod_i (A_i[z^{\pm 1}]) \xleftarrow{\prod c_{A_i}} \widehat{A}[\mu_{\ell\infty}]$$

factors through  $A[\mu_{\ell\infty}]$ . According to Lemma 3.4(2) applied with  $B = \widehat{A}$ , this implies that

$$C \xrightarrow{\Phi_0} \prod_i (A_i[z^{\pm 1}]) \xleftarrow{\prod \text{can}_{A_i}} \widehat{A}^{\mathbb{Z}}$$

factors through  $A^{\mathbb{Z}}$ , providing a map  $\Phi : C \rightarrow A^{\mathbb{Z}}$ . From the diagrams expressing the fact that the  $f_i$  respect the comultiplications, taking products over  $i$ , we obtain a commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\Phi} & A^{\mathbb{Z}} \\
\downarrow & & \downarrow \\
C \otimes_A C & \xrightarrow{\Phi \otimes \Phi} & A^{\mathbb{Z}} \otimes_A A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z} \times \mathbb{Z}}
\end{array}$$

Let  $g \in C$  and write  $\Phi(g) = (a_m)_{m \in \mathbb{Z}}$ . Since  $A$  is noetherian, Lemme 7.2 of [SGA3.2], Exp. IX is applicable and shows that only finitely many of the  $a_m$  are nonzero, that is  $\Phi(g) \in A[z^{\pm 1}]$ . Therefore  $\Phi$  gives rise to a map  $f : C \rightarrow A[z^{\pm 1}]$ . The fact that  $f$  respects the comultiplication of the Hopf algebras follows immediately by embedding  $A[z^{\pm 1}] \otimes A[z^{\pm 1}]$  into  $\widehat{A}[z^{\pm 1}] \otimes \widehat{A}[z^{\pm 1}]$  where the required commutativity holds by assumption. The fact that  $f$  respects the counits is equally clear.  $\square$

## 4 Representability of the functor of homomorphisms: proof

We come back to the setting of 2.2 and we proceed to show that  $F \rightarrow S$  is representable; this will complete the proof of item (2) of Theorem 2. For this, we use Grothendieck's theorem on representation of unramified functors, which we begin by recalling.

**4.1 Representation of unramified functors: statement.** In what follows, all affine schemes  $\text{Spec}(A)$  that appear are assumed to be  $S$ -schemes, and we write  $F(A)$  instead of  $F(\text{Spec}(A))$ .

**Theorem 3 (Grothendieck [Mu65])** *Let  $S$  be a locally noetherian scheme and  $F$  a set-valued contravariant functor on the category of  $S$ -schemes. Then  $F$  is representable by an  $S$ -scheme which is locally of finite type, unramified and separated if and only if Conditions  $(F_1)$  to  $(F_8)$  below hold.*

$(F_1)$  *The functor  $F$  is a sheaf for the fpqc topology.*



- (F<sub>2</sub>) The functor  $F$  is locally of finite presentation; that is, for all filtering colimits of rings  $A = \operatorname{colim} A_\alpha$ , the map  $\operatorname{colim} F(A_\alpha) \rightarrow F(A)$  is bijective.
- (F<sub>3</sub>) The functor  $F$  is effective; that is, for all noetherian complete local rings  $(A, m)$ , the map  $F(A) \rightarrow \varprojlim F(A/m^k)$  is bijective.
- (F<sub>4</sub>) The functor  $F$  is homogeneous; more precisely, for all exact sequences of rings  $A \rightarrow A' \rightrightarrows A' \otimes_A A'$  with  $A$  local artinian,  $\operatorname{length}_A(A'/A) = 1$  and trivial residue field extension  $k_A = k_{A'}$ , the diagram  $F(A) \rightarrow F(A') \rightrightarrows F(A' \otimes_A A')$  is exact.
- (F<sub>5</sub>) The functor  $F$  is formally unramified.
- (F<sub>6</sub>) The functor  $F$  is separated; that is, it satisfies the valuative criterion of separation.

For the last two conditions we let  $A$  be a noetherian ring,  $N$  its nilradical,  $I$  a nilpotent ideal such that  $IN = 0$ ,  $T = \operatorname{Spec}(A)$ ,  $T' = \operatorname{Spec}(A/I)$ . We assume that  $T$  is irreducible and we call  $t$  its generic point.

- (F<sub>7</sub>) Assume moreover that  $A$  is complete one-dimensional local with a unique associated prime. Then any point  $\xi' : \operatorname{Spec}(A/I) \rightarrow F$  such that

$$\xi'_t : \operatorname{Spec}((A/I)_t) \longrightarrow \operatorname{Spec}(A/I) \longrightarrow F$$

can be lifted to a point  $\xi^* : \operatorname{Spec}(A_t) \rightarrow F$ , can be lifted to a point  $\xi : \operatorname{Spec}(A) \rightarrow F$ .

- (F<sub>8</sub>) Assume that  $\xi' : \operatorname{Spec}(A/I) \rightarrow F$  is such that

$$\xi'_t : \operatorname{Spec}((A/I)_t) \longrightarrow \operatorname{Spec}(A/I) \longrightarrow F$$

can not be lifted to any subscheme of  $\operatorname{Spec}(A_t)$  which is strictly larger than  $\operatorname{Spec}((A/I)_t)$ . Then there exists a nonempty open set  $W \subset T$  such that for all open subschemes  $W_1 \subset T$  contained in  $W$ , the restriction  $\xi'_{|W_1} : W_1 \rightarrow F$  (with  $W_1 = W_1 \times_T T'$ ) can not be lifted to any subscheme of  $W_1$  which is strictly larger than  $W_1$ .

We now start to verify the conditions one by one.

**4.2 Conditions** (F<sub>1</sub>), (F<sub>4</sub>), (F<sub>5</sub>), (F<sub>6</sub>), (F<sub>7</sub>). We begin by checking the conditions which turn out easy in our case.

- (F<sub>1</sub>) This follows from fpqc descent, see e.g. [SGA1], Exp. VIII, Th. 5.2.
- (F<sub>4</sub>) Since  $A \rightarrow A'$  is injective, by Lemma 3.5 the map  $F(A) \rightarrow F(A')$  is a bijection. This gives a statement which is much stronger than the (F<sub>4</sub>) in the theorem.
- (F<sub>5</sub>) Since  $F \rightarrow S$  is a monomorphism, it is formally unramified.
- (F<sub>6</sub>) Since  $F \rightarrow S$  is a monomorphism, it is separated.
- (F<sub>7</sub>) Since  $A$  has a unique associated prime, the map  $\operatorname{Spec}(A_t) \rightarrow \operatorname{Spec}(A)$  is schematically dominant. Hence by Lemma 3.5, the point  $\xi^* : \operatorname{Spec}(A_t) \rightarrow F$  automatically extends to  $\operatorname{Spec}(A)$ .

**4.3 Condition** (F<sub>2</sub>). Let  $\mathcal{F} := \operatorname{Hom}(\mathbb{G}_m, H)$  be the functor of all morphisms of group schemes  $\mathbb{G}_m \rightarrow H$  — that is, not just those that extend the collection  $u_n$ . It is standard that  $\mathcal{F}$  is locally of finite presentation (see [EGA] IV<sub>3</sub>.8.8.3). Should the affine scheme  $\varprojlim_n \operatorname{Hom}(\mu_{\ell^n}, H)$  be locally of finite type over  $S$ , it would follow that  $F \rightarrow S$  is locally of finite presentation ([EGA] IV<sub>1</sub>.1.4.3(v)). However this is not the case in general, and the verification of (F<sub>2</sub>) needs more work.

So let  $A = \operatorname{colim} A_\alpha$  be a filtering colimits of rings. We want to prove that  $\operatorname{colim} F(A_\alpha) \rightarrow F(A)$  is bijective. We look at the diagram

$$\begin{array}{ccc} \operatorname{colim} F(A_\alpha) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ \operatorname{colim} \mathcal{F}(A_\alpha) & \xrightarrow{\sim} & \mathcal{F}(A). \end{array}$$

Since  $\mathcal{F}$  is locally of finite presentation, the bottom row is an isomorphism. We deduce that the upper row is injective. We shall now prove that the upper row is surjective, and in fact that the diagram is cartesian.

**4.4 Lemma.** *Let  $f : \mathbb{G}_{m,A} \rightarrow H_A$  be a morphism extending  $u_{n,A} : \mu_{\ell^n,A} \rightarrow H_A$  for all  $n \geq 0$ . Then there exists an index  $\alpha$  such that  $f$  descends to a map  $f_\alpha : \mathbb{G}_{m,A_\alpha} \rightarrow H_{A_\alpha}$  extending  $u_{n,A_\alpha}$  for all  $n \geq 0$ .*

**Proof :** Since  $G$  and  $H$  are finitely presented, the morphism  $f$  is defined at finite level, that is there exists an index  $\alpha$  and a morphism of  $A_\alpha$ -group schemes  $g : \mathbb{G}_{m,A_\alpha} \rightarrow H_{A_\alpha}$  whose pullback along  $\operatorname{Spec}(A) \rightarrow \operatorname{Spec}(A_\alpha)$  is  $f$ . Since the groups are affine, the morphism  $g$  is given by a map of rings  $g^\sharp : \mathcal{O}_H \otimes A_\alpha \rightarrow A_\alpha[z^{\pm 1}]$ . Fix a presentation  $\mathcal{O}_H = R[x_1, \dots, x_s]/(P_1, \dots, P_t)$ . Then :

- $u_n^\sharp$  is determined by the elements  $z_{n,j} := u_n^\sharp(x_j) \in R[z]/(z^{\ell^n} - 1)$  satisfying  $P_k(z_{n,1}, \dots, z_{n,s}) = 0$  for  $k = 1, \dots, t$ ,
- $g^\sharp$  is determined by the elements  $y_j = g^\sharp(x_j) \in A_\alpha[z^{\pm 1}]$  satisfying  $P_k(y_1, \dots, y_s) = 0$  for  $k = 1, \dots, t$ .

Moreover, saying that  $f$  extends  $u_{n,A}$  is just saying that  $z_{n,j} = \pi_n(y_j)$  in  $A[z]/(z^{\ell^n} - 1)$ , for all  $j$ , where

$$\pi_n : A_\alpha[z^{\pm 1}] \rightarrow A[z]/(z^{\ell^n} - 1)$$

is the projection. So we have to prove that we may enlarge the index  $\alpha$  in such a way that  $g$  extends  $u_{n,A_\alpha}$  for all  $n \geq 0$ .

For  $n_0 \geq 0$  an integer, let  $J_0 := \{-\lfloor \ell^{n_0}/2 \rfloor, \dots, \ell^{n_0} - \lfloor \ell^{n_0}/2 \rfloor - 1\}$  and  $E_0 = \bigoplus_{i \in J_0} R z^i$ . Then  $\pi_{n_0|E_0}$  is an isomorphism and for all  $n \geq n_0$  we can define

$$\chi_n = \pi_n \circ (\pi_{n_0|E_0})^{-1} : R[z]/(z^{\ell^{n_0}} - 1) \longrightarrow R[z]/(z^{\ell^n} - 1).$$

By base change, these objects are defined over any  $R$ -algebra. We choose  $n_0$  large enough so that  $E_0 \otimes_R A_\alpha$  contains the Laurent polynomials  $y_1, \dots, y_s$ .

In the present context, the condition that  $f$  extends all the maps  $u_{n,A} : G_{n,A} \rightarrow H_A$  is a finiteness constraint imposed by  $f$  on  $\{u_n\}$  (whereas in other places of our arguments it is best seen as a condition imposed by  $\{u_n\}$  on  $f$ ). Indeed, from the relations  $z_{n,j} = \pi_n(y_j)$  in  $A$ , we deduce that

$$z_{n,j} = \chi_n(z_{n_0,j}) \quad \text{in } A[z]/(z^{\ell^n} - 1) \quad \text{for all } n \geq n_0,$$

namely  $\chi_n(z_{n_0,j}) = (\pi_n \circ (\pi_{n_0|E_0})^{-1})(\pi_{n_0}(y_j)) = \pi_n(y_j) = z_{n,j}$ . We claim that we may increase  $\alpha$  to achieve that these equalities hold in  $A_\alpha[z]/(z^{\ell^n} - 1)$ , for all  $n \geq n_0$  and all  $j$ . In order to see this, note that the elements  $\delta_{n,j} := z_{n,j} - \chi_n(z_{n_0,j})$  are defined over  $R$ , and as we have just proved, they belong to the kernel of the morphism  $R[z]/(z^{\ell^n} - 1) \rightarrow A[z]/(z^{\ell^n} - 1)$ . Let  $I \subset R$  be the ideal generated by the coefficients of the expressions of  $\delta_{n,j}$  on the monomial basis, for varying  $n \geq n_0$  and  $j$ . Since  $R$  is noetherian,  $I$  is generated by finitely many elements. These elements vanish in  $A$ , hence they vanish in  $A_\alpha$  provided we increase  $\alpha$  a little, whence our claim.

The relations  $z_{n,j} = \pi_n(y_j)$  in  $A[z]/(z^{\ell^n} - 1)$  with  $j = 1, \dots, s$  and  $n \leq n_0$  being finite in number, we may increase  $\alpha$  so as to ensure that all of them hold in  $A_\alpha[z]/(z^{\ell^n} - 1)$ . Then for  $n \geq n_0$  we have

$$z_{n,j} = \chi_n(z_{n_0,j}) = \chi_n(\pi_{n_0}(y_j)) = \pi_n(y_j) \quad \text{in } A_\alpha[z]/(z^{\ell^n} - 1)$$

again. That is,  $g$  extends the maps  $u_{n,A_\alpha}$  for all  $n \geq 0$ . □

**4.5 Condition** ( $F_3$ ). Let  $(A, m)$  be a noetherian complete local ring. We want to prove that the map  $F(A) \rightarrow \lim F(A/m^k)$  is bijective. We write again  $\mathcal{F} := \text{Hom}(\mathbb{G}_m, H)$ . We look at the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & \lim F(A/m^k) \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\sim} & \lim \mathcal{F}(A/m^k). \end{array}$$

From [SGA3.2], Exp. IX, Th. 7.1 we know that  $\mathcal{F}$  is effective, that is the bottom arrow is bijective. We deduce that the upper row is injective. We shall now prove that the upper row is surjective, and in fact that the diagram is cartesian. So let  $f_k : \mathbb{G}_{m,A/m^k} \rightarrow H_{A/m^k}$  be a collection of  $A/m^k$ -morphisms such that  $f_k$  extends  $u_{n,A/m^k} : \mu_{\ell^n, A/m^k} \rightarrow H_{A/m^k}$  for all  $n \geq 0$ , and let  $f : \mathbb{G}_{m,A} \rightarrow H_A$  be a morphism that algebraizes the  $f_k$ . We must prove that  $f$  extends  $u_{n,A}$ , for each  $n$ . For this let  $i_n : \mu_{\ell^n, A} \rightarrow \mathbb{G}_{m,A}$  be the closed immersion. The two maps  $f \circ i_n$  and  $u_n$  coincide modulo  $m^k$  for each  $k \geq 1$ , hence so do the morphisms of Hopf algebras

$$(f \circ i_n)^\sharp, u_n^\sharp : \mathcal{O}_H \otimes A \rightarrow A[z]/(z^{\ell^n} - 1).$$

Since  $A[z]/(z^{\ell^n} - 1)$  is separated for the  $m$ -adic topology, we deduce that  $(f \circ i_n)^\sharp = u_n^\sharp$  and hence  $f \circ i_n = u_n$ . This concludes the argument.

**4.6 Remark.** We could also appeal to the following more general result extending the injectivity part of [EGA] III<sub>1</sub>.5.4.1: *let  $(A, m)$  be a noetherian complete local ring, and  $S = \text{Spec}(A)$ . Let  $X, Y$  be  $S$ -schemes of finite type with  $X$  pure and  $Y$  separated. Let  $f, g : X \rightarrow Y$  be  $S$ -morphisms. If we have the equality of completions  $\hat{f} = \hat{g}$ , then  $f = g$ .* For the notion of a pure morphism of schemes we refer to Raynaud and Gruson [RG71] and Romagny [Ro12]. As to the proof of the italicized statement, by [EGA1<sub>new</sub>], 10.9.4 the morphisms  $f$  and  $g$  agree in an open neighbourhood of  $\text{Spec}(A/m)$ . Then the arguments in the proof of [Ro12], Lemma 2.1.9 apply verbatim.

**4.7 Condition** ( $F_8$ ). This condition will be verified with the help of the following lemma.

**4.8 Lemma.** *Let  $T$  be a scheme and  $T'$  a closed subscheme. Let  $\xi' : T' \rightarrow F$  be a point. Then there is a largest closed subscheme  $\mathcal{Z}_T \subset T$  such that  $\xi'$  extends to  $\mathcal{Z}_T$ . Moreover, its formation is Zariski local: if  $U \subset T$  is an open subscheme and  $U' = U \cap T'$ , we have  $\mathcal{Z}_T \cap U = \mathcal{Z}_U$ .*

**Proof :** Throughout, for all open subschemes  $U \subset T$  we write  $U' = U \cap T'$  and all closed subschemes  $Z$  of  $U$  such that  $\xi'_{|U'}$  extends to  $Z$  are implicitly assumed to contain  $U'$ . We proceed by steps.

Let  $U = \text{Spec}(A)$  be an affine open subscheme of  $T$ . Consider the family of all closed subschemes  $Z_\alpha = V(I_\alpha) \subset U$  to which  $\xi'_{|U'}$  extends. Consider the ideal  $I = \bigcap I_\alpha$  and define  $\mathcal{Z}_U = V(I)$ . Since the map  $A/I \rightarrow \prod A/I_\alpha$  is injective, applying Lemma 3.5, we see that  $\xi'_{|U'}$  extends to  $\mathcal{Z}_U$ . By its very definition the closed subscheme  $\mathcal{Z}_U$  is largest.

Let  $U, V$  be two affine opens of  $T$  with  $U \subset V$ . We claim that  $\mathcal{Z}_V \cap U = \mathcal{Z}_U$ . Indeed, since  $\xi'_{|V}$  extends to  $\mathcal{Z}_V$  then  $\xi'_{|U}$  extends to  $\mathcal{Z}_V \cap U$ , hence  $\mathcal{Z}_V \cap U \subset \mathcal{Z}_U$ . Conversely, let  $Z$  be the schematic image of  $\mathcal{Z}_U \rightarrow U \rightarrow V$ . The latter map being quasi-compact, the map  $\mathcal{Z}_U \rightarrow Z$  is schematically dominant. By Lemma 3.5 it follows that  $\xi'_{\mathcal{Z}_U}$  extends to  $Z$ . By maximality this forces  $Z \subset \mathcal{Z}_V$ , hence  $\mathcal{Z}_U \subset \mathcal{Z}_V \cap U$ .

Let  $U, V$  be arbitrary affine opens of  $T$ . We claim that  $\mathcal{Z}_U \cap V = \mathcal{Z}_V \cap U$ . Indeed, by the previous step, for all affine opens  $W \subset U \cap V$  we have  $\mathcal{Z}_U \cap V \cap W = \mathcal{Z}_W = \mathcal{Z}_V \cap U \cap W$ .

Let  $\mathcal{Z}_T$  be the closed subscheme of  $T$  obtained by glueing the  $\mathcal{Z}_U$  when  $U$  varies over all affine opens; thus  $\mathcal{Z}_T \cap U = \mathcal{Z}_U$  by construction. Now  $\xi'$  extends to  $\mathcal{Z}_T$ , because  $\xi'_{|U}$  extends to  $\mathcal{Z}_U$  for each  $U$ , and we can glue these extensions. Moreover  $\mathcal{Z}_T$  is maximal with this property, because if  $\xi'$  extends to some closed subscheme  $Z \subset T$  then for each affine  $U$  the element  $\xi'_{|U}$  extends to  $Z \cap U$ , hence  $Z \cap U \subset \mathcal{Z}_U = \mathcal{Z}_T \cap U$ , hence  $Z \subset \mathcal{Z}_T$ .

The fact that  $\mathcal{Z}_T \cap U = \mathcal{Z}_U$  for all open subschemes  $U$  follows by restricting to affine opens.  $\square$

In order to now verify Condition  $(F_8)$ , we write  $T = \text{Spec}(A)$  and  $T' = \text{Spec}(A/I)$ . The assumption that  $\xi'_t : \text{Spec}((A/I)_t) \rightarrow \text{Spec}(A/I) \rightarrow F$  does not lift to any subscheme of  $\text{Spec}(A_t)$  which is strictly larger than  $\text{Spec}((A/I)_t)$  means that the inclusion  $T' \subset \mathcal{Z}_T$  is an equality at the generic point. It follows that  $T' \cap W = \mathcal{Z}_T \cap W = \mathcal{Z}_W$  for some open  $W$ . Applying Lemma 4.8 to variable opens  $W_1 \subset W$ , we obtain  $T' \cap W_1 = \mathcal{Z}_T \cap W_1 = \mathcal{Z}_{W_1}$ , which shows that  $W$  fulfills the required condition and we are done.

**4.9 Conclusion of the proof.** Let  $G$  be a general finitely presented  $S$ -group scheme of multiplicative type  $G$ . Let  $G_n = \ker(n : G \rightarrow G)$  be the finite flat torsion subschemes; the limit  $L = \lim \text{Hom}(G_n, H)$  is an affine  $S$ -scheme. By reducing to the case  $G = \mathbb{G}_m$  and using the monomorphism  $\text{Hom}(G, H) \rightarrow L$ , we have thus proven that  $\text{Hom}(G, H)$  is representable by a scheme which is locally of finite presentation over  $L$  and over  $S$ .

To finish the proof of item (2) of Theorem 2, let  $F$  be  $\text{Hom}(G, H)$  viewed as an  $L$ -scheme. Let  $Y \subset F$  be an  $S$ -quasi-compact closed subscheme. Let  $Z \subset L$  be the schematic image of  $Y \rightarrow L$ . This is a closed subscheme and by quasi-compactness, the morphism  $Y \rightarrow Z$  is schematically dominant. It follows from Lemma 3.5 that  $F(Z) \rightarrow F(Y)$  is bijective, that is  $Y \rightarrow F$  factors uniquely through a map  $Z \rightarrow F$ . Since  $Z \hookrightarrow L$  is a closed immersion and  $F \rightarrow L$  is separated, then  $Z \rightarrow F$  is a closed immersion. By the same argument  $Y \rightarrow Z$  is a closed immersion; being also schematically dominant, it is an isomorphism. Thus  $Y$  is in fact a closed subscheme of  $L$ . Since  $L$  is affine over  $S$ , it follows that  $Y$  is affine over  $S$ .

## 5 Representability of the functor of subgroups

In this section we prove item (1) of Theorem 2. It would be possible to do this again using Grothendieck's theorem on unramified functors, with arguments very similar to those used to prove item (2). However, to save energy and spare the reader tedious repetitions, we reduce (1) to (2) by using more advanced technology. We write  $\text{Submt}(H)$  for the functor of multiplicative type subgroups of  $H$ .

**5.1 Disjoint sum decomposition.** The type  $M$  of a group of multiplicative type is locally constant ([SGA3.2], Exp. IX, Rem. 1.4.1). Although we will not need this, note that by looking at the inclusion  $G_s \subset H_s$  in a fibre, we see that only abelian groups  $M$  of finite type occur. Hence

$$\text{Submt}(H) = \coprod_M \text{Submt}_M(H)$$

where  $\text{Submt}_M(H)$  is the functor of multiplicative type subgroup schemes of  $H$  of type  $M$ . Thus it is enough to establish that  $\text{Submt}_M(H)$  is representable.

**5.2 Representability by an algebraic space.** Let  $D(M) = \text{Spec } \mathcal{O}_S(M)$  be the diagonalizable group of type  $M$  as in [SGA3.2], Exp. VIII. By item (2) of Theorem 2, the functor  $\text{Hom}(D(M), H)$  is representable by a scheme. It follows from [SGA3.2], Exp. IX, Cor. 6.6 that the subfunctor

$$\text{Mono}(D(M), H) \subset \text{Hom}(D(M), H)$$

of monomorphisms of group schemes is an open subscheme. Moreover, since  $D(M)$  is of multiplicative type and  $H$  is affine, by [SGA3.2], Exp. IX, Cor. 2.5 any monomorphism  $f : D(M) \rightarrow H$  is a closed immersion, inducing an isomorphism between  $D(M)$  and a closed subgroup scheme  $K \hookrightarrow H$ . By taking a monomorphism  $f$  to its image  $K$ , we obtain a morphism of functors:

$$\pi : \text{Mono}(G, H) \rightarrow \text{Submt}_M(H).$$

Let  $A = \text{Aut}(D(M))$  be the functor of automorphisms of  $D(M)$ ; this is isomorphic to the locally constant group scheme  $\text{Aut}(M)_S$ . It acts freely on  $\text{Mono}(D(M), H)$  by the rule  $af = f \circ a^{-1}$  for  $a \in \text{Aut}(D(M))$  and  $f \in \text{Mono}(D(M), H)$ . Let  $\text{Mono}(D(M), H)/A$  be the quotient sheaf; this is an algebraic space by Artin's Theorem, see [SP20], Tag 04S5. Since the morphism  $\pi$  is  $A$ -equivariant, it induces a morphism

$$i : \text{Mono}(D(M), H)/A \rightarrow \text{Submt}_M(H).$$

We claim that  $i$  is an isomorphism. It is enough to prove that it is an isomorphism of fppf sheaves:

- surjectivity: this can be checked étale-locally over  $S$ , so we can assume that the given subgroup scheme  $K \subset H$  is isomorphic to  $D(M)$  and then the canonical inclusion of  $D(M)$  into  $H$  is a monomorphism that provides a point of  $\text{Mono}(D(M), H)$  lifting  $K$ .
- injectivity: if  $f_i : D(M) \rightarrow H$  are two monomorphisms with the same image  $K$ , then  $f_2^{-1} \circ f_1 : D(M) \rightarrow K \rightarrow D(M)$  is an automorphism of  $D(M)$ . This settles the claim.

**5.3 Representability by a scheme.** To prove that  $\text{Submt}_M(H)$  is representable by a scheme, let  $L := \lim \text{Submt}_{M/nM}(H)$  be the limit of the functors of finite flat multiplicative type subgroups of type  $M/nM$ . Since  $\text{Submt}_{M/nM}(H)$  is representable and affine ([SGA3.2], Prop. 3.12.a), the functor  $L$  is an affine scheme. By mapping any subgroup  $G \hookrightarrow H$  to the collection of subgroups  $G_n \hookrightarrow H$  where  $G_n = \ker(n : G \rightarrow G)$ , we define a morphism of functors  $u : \text{Submt}_M(H) \rightarrow L$ . By the Density Theorem, this is a monomorphism. As  $\text{Submt}_M(H) \rightarrow S$  is locally of finite type, so is  $u$ . In particular  $u$  is a separated, locally quasi-finite morphism. By [SP20], Tag 0418 all such morphisms are representable by schemes, hence  $\text{Submt}_M(H)$  is a scheme. Finally, in order to prove that each  $S$ -quasi-compact closed subscheme of  $\text{Hom}(G, H)$  is affine over  $S$ , we proceed as in 4.9.

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Matthieu ROMAGNY, UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE  
 Email address: [matthieu.romagny@univ-rennes1.fr](mailto:matthieu.romagny@univ-rennes1.fr)