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On presentations for mapping class groups of orientable surfaces via Poincaré's Polyhedron theorem and graphs of groups

Lluís Bacardit

Abstract

The mapping class group of an orientable surface with one boundary component, S, is isomorphic to a subgroup of the automorphism group of the fundamental group of S. We call these subgroups *algebraic mapping class groups*. An algebraic mapping class group acts on a space called *ordered Auter space*. We apply Poincaré's Polyhedron theorem to this action. We describe a decomposition of ordered Auter space. From these results, we deduce that the algebraic mapping class group of S is a quotient of the fundamental group of a graph of groups with, at most, two vertices and, at most, six edges. Vertex and edge groups of our graph of groups are mapping class groups of orientable surfaces with one, two or three boundary components. A presentation for the mapping class group of S can be obtained by adding, at most, 24 relations to the fundamental group of our graph of groups.

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Key words. Mapping class groups, presentations, automorphism groups, Auter space.

1 Introduction

Presentations for mapping class groups of orientable surfaces were obtained by different authors after a seminal paper by Hatcher and Thurston [7], where a method to deduce presentations was exposed although presentations themselves were not given. Wajnryb [12] followed Hatcher and Thurston's method, with improvements introduced by Harer [5], to obtain explicit presentations for mapping class groups of orientable surfaces of finite genus without punctures and one boundary component. Matsumoto [9] interpreted some of the relations in Wajnryb's presentations as equalities between centralizers in Artin groups. From

Wajnryb's presentations, Gervais [4] deduced presentations for surfaces without punctures and a finite number of boundary components. Labruère and Paris [8] generalized Matsumoto's presentation to orientable surfaces with a finite number of punctures and a finite number of boundary components.

The Mapping class group of an orientable surface with exactly one boundary component can be identified with a subgroup of the automorphism group of the fundamental group of the surface. We call these subgroups *algebraic mapping class groups*. Before Hatcher and Thurson's paper appeared, McCool [10], [11] had proved that algebraic mapping class groups are finitely presentable.

In [2], we followed Armstrong, Forrest and Vogtmann [1] to obtain an algorithm which gives presentations for algebraic mapping class groups. That algorithm gives explicit presentations, although a large number of generators and relations are needed. We did not find an argument to sensibly reduce the number of generators and relations.

In the present paper, we obtain a new algorithm which gives presentations for algebraic mapping class groups. Although this new algorithm gives generators and relations, our presentations are described as quotients of fundamental groups of graph of groups. For a genus 0 surface with at least three punctures, the graphs of groups has one vertex and one edge; and, one relation needs to be added. For a surface without punctures and genus at least 2, the graph of groups has one vertex and two edges. For a genus 2 surface without punctures, five relations need to be added. For a surface with genus at least 3 without punctures, six relations need to be added. For a surface with genus at least 1 and at least one puncture, the graph of groups has two vertices and no more than six edges; no more than 24 relations need to be added. We only give explicit presentations for some simple cases. Although we do not give explicit presentations in general, a presentation for the mapping class group of an orientable surface with one boundary components can be obtained by our method and an inductive argument. The obtained presentation has a reasonable number of generators and relations.

As in [2], we consider an action of algebraic mapping class groups on a space analogous to Auter space, called *ordered Auter space*. In [2], the Degree theorem by Hatcher and Vogtmann [6] was applied to obtain a 2-dimensional complex from which the algorithm was deduced. In the present paper, the new algorithm is deduced from Poincaré's Polyhedron theorem applied to the action of algebraic mapping class groups on ordered Auter space. Since ordered Auter space is a manifold, Poincaré's Polyhedron theorem can be applied. Recall Auter space is not a manifold. Hence, Poincaré's Polyhedron theorem cannot be applied to the action of the automorphism group of a free group on Auter space. We describe a decomposition of ordered Auter space which gives the vertices of our graph of groups. By refining once this decomposition, we obtain the edges of our graph of groups. And, by refining the decomposition a second time, we obtain the relations that need to be added to the fundamental group of the graph of groups.

The vertex groups of the graph of groups are algebraic mapping class group of an orientable surface with one or two boundary components. Presentations for the vertex groups of the graph of groups can be computed by our method either by induction or by our method applied one level deeper. The edge groups of the graph of groups are algebraic mapping class group of an orientable surface with one, two or three boundary components.

The outline of the paper is the following. Section 2 contains notations and definitions. Section 3 recovers some results which are needed to apply Poincaré's Polyhedron theorem, and, contains the algorithm obtained by applying Poincaré's Polyhedron theorem to the action of algebraic mapping class groups on ordered Auter space. Section 4 contains the description of a subcomplex of ordered Auter space which decomposes ordered Auter space. Section 5 describes a subcomplex of ordered Auter space which is a homotopy retract of ordered Auter space. The latter subcomplex is more suitable for being decomposed. Section 6 describes a decomposition of the subcomplex of ordered Auter space described in Section 5. Section 7 gives an inductive argument in two steps: generators and relations. Section 8 describes our presentations for algebraic mapping class groups in terms of graphs of groups. Section 9 contains examples.

2 Notation and Definitions

This section contains notation and definitions used through the paper.

2.1 Notation. Let X be a set. The cardinality of X will be denoted $|X|$.

Let G be a group. For g, $h \in G$, we write $[g,h] = g^{-1}h^{-1}gh$, the commutator of g and h; and, $g^h = h^{-1}gh$, the conjugated of g by h. We write $[g] = \{g^h \mid h \in G\}$, the conjugacy class of g in G.

Let $\mathrm{Aut}(G)$ be the group of automorphisms of G, and, let $\mathrm{Out}(G)$ be the automorphism group of G modulo inner automorphisms.

We fix an integer $n \geq 1$, and, non-negative integers g, p and b such that $n = 2g + p + b - 1 \geq 1$.

Let S be a genus g orientable surface with p punctures and b boundary components. We consider homeomorphisms of S which fix the set of punctures and whose restriction to each boundary component is the identity. Hence, such an homeomorhism respects the orientation of S. The *mapping class group* of S, denoted $\mathcal{M}(S)$, is the group of isotopy classes of such homeomorphisms, where every isotopy leaves fixed the set of punctures and its restriction to each boundary component is the identity. Since two orientable surfaces with the same genus g, the same number of punctures p and the same number of boundary components b, are homeomorphic; their mapping class groups are isomorphic. Hence, we also denote $\mathcal{M}(S)$ by $\mathcal{M}_{g,p,b}$, and we say that $\mathcal{M}_{g,p,b}$ is the mapping class group of a genus g surface with p punctures and b boundary components.

We denote by $\pi_1(S)$ the fundamental group of S. For $(p,\, b) = (0,\, 0)$, the Dehn-Nielsen-Baer theorem states that $\mathcal{M}_{g,0,0}$ is isomorphic to an index 2 subgroup of $\mathrm{Out}(\pi_1(S))$, the group of outer automorphisms of $\pi_1(S)$. For $(p,\, b) \neq (0,\, 0)$, $\pi_1(S)$ is a free group of rank $n = 2g + p + b - 1$, and, a generalization of the Dehn-Nielsen-Baer theorem states that $\mathcal{M}_{g,p,b}$ is isomorphic to a subgroup of $\mathrm{Out}(\pi_1(S))$.

For $b \geq 1$, it can be deduced from Dehn-Nielsen-Baer theorem that $\mathcal{M}_{g,p,b}$ is isomorphic to a subgroup of $\mathrm{Aut}(\pi_1(S) * \langle c_1,\, c_2,\, \ldots,\, c_{b-1} \,|\rangle)$ denoted $\mathcal{AM}_{g,p,b}$. The following notation is needed in order to define $\mathcal{AM}_{g,p,b}$.

We consider the following presentation for $\pi_1(S)$

$$F_{g,p,b} = \langle x_1,\, y_1,\, x_2,\, y_2,\, \ldots,\, x_g,\, y_g,\, t_1,\, t_2,\, \ldots,\, t_p,\, z_1,\, z_2,\, \ldots,\, z_b \mid w_{g,p,b} \rangle,$$

where $w_{g,p,b} = [x_1,\, y_1][x_2,\, y_2] \ldots [x_g,\, y_g] t_1 t_2 \ldots t_p z_1 z_2 \ldots z_b$. Since $b \geq 1$, $F_{g,p,b}$ is a free group of rank $n = 2g + p + b - 1$ with basis x_i, y_i for $1 \leq i \leq g$, t_k for $1 \leq k \leq p$, and, z_l for $1 \leq l \leq b - 1$.

For $1 \leq l \leq b$, let $*_l$ be a point in the l-th boundary component of S. From a topological point of view, the presentation $F_{g,p,b}$ for $\pi_1(S)$ can be interpreted as follows: $w_{g,p,b}$ represents a loop based at $*_b$ around the b-th boundary component of S, t_i represents a loop based at $*_b$ around the i-th puncture of S for $1 \leq i \leq p$ and b_l represents a loop based at $*_b$ around the l-th boundary component of S for $1 \leq l \leq b-1$. To distinguish between punctures and boundary components, we introduce c_l, for $1 \leq l \leq b - 1$, which represents an arc from $*_b$ to $*_l$ such that $z_l^{c_l}$ represents a loop based at $*_l$ around the l-th boundary component. $\square$

The following definition of algebraic mapping class group appears in [3], with slightly different notation.

2.2 Definition. We denote by $\mathcal{AM}_{g,p,b}$ the subgroup of

$$\mathrm{Aut}(F_{g,p,b} * \langle c_1,\, c_2,\, \ldots,\, c_{b-1} \,|\rangle)$$

consisting of automorphisms φ of $F_{g,p,b} * \langle c_1,\, c_2,\, \ldots,\, c_{b-1} \,|\rangle$ such that the following conditions hold:

(a) The subgroup $F_{g,p,b}$ is invariant under φ.

(b) The set of conjugacy classes $\{[t_1^{-1}], [t_2^{-1}], \ldots, [t_p^{-1}]\}$ is fixed by φ.

(c) The words $z_l^{c_l}$, for $1 \leq l \leq b - 1$, and the generator z_b are fixed by φ.

(d) For $1 \leq l \leq (b-1)$, the image of c_l under φ lies inside the right coset $F_{g,p,b} \cdot c_l$ of $F_{g,p,b} \backslash (F_{g,p,b} * \langle c_1, c_2, \ldots, c_{b-1} \,|\rangle)$.

We call $\mathcal{AM}_{g,p,b}$ the *algebraic mapping class group* of an orientable genus g surface with p punctures and b boundary components. $\square$

We will use the following definition for combinatorial graph.

2.3 Definitions. A *combinatorial graph* Γ is a three-tuple $(V(\Gamma), E(\Gamma), V^*(\Gamma))$ such that $V(\Gamma)$ and $E(\Gamma)$ are disjoint sets, called the *vertex set* and the *edge set*, respectivebly; and $V^*(\Gamma) = \{v^* \mid v \in V(\Gamma)\}$ where v^* is defined as follows. Let $\overline{E}(\Gamma)$ be a set disjoint from $E(\Gamma)$ and let $\bar{\cdot} : E(\Gamma) \to \overline{E}(\Gamma)$ be a bijection which extends to an involution $\bar{\cdot} : E(\Gamma) \cup \overline{E}(\Gamma) \to E(\Gamma) \cup \overline{E}(\Gamma)$. For every $v \in V$, v^* is a subset of $E(\Gamma) \cup \overline{E}(\Gamma)$ such that $V^*(\Gamma) = \{v^* \mid v \in V(\Gamma)\}$ is a partition of $E(\Gamma) \cup \overline{E}(\Gamma)$; that is, $v_1^* \cap v_2^* = \emptyset$ if $v_1 \neq v_2$, and $\bigcup_{v \in V(\Gamma)} v^* = E(\Gamma) \cup \overline{E}(\Gamma)$.

A *graph with a distinguished vertex* is a two-tuple (Γ, v_0) where Γ is a combinatorial graph and v_0 is a vertex of Γ.

A *fat graph with a distinguished vertex* is a three-tuple $(\Gamma, v_0, \mathrm{ord})$ where (Γ, v_0) is a graph with a distinguished vertex, and, ord is an order relation in v^* for each $v \in V(\Gamma)$, denoted $\mathrm{ord}(v^*)$, such that $\mathrm{ord}(v_0^*)$ is a linear order and, for $v \neq v_0$, $\mathrm{ord}(v^*)$ is a cyclic order.

Let $(\Gamma, v_0, \mathrm{ord})$ be a fat graph with a distinguished vertex. Suppose $V(\Gamma) = \{v_0, v_1, \ldots, v_q\}$, $|v_i^*| = r_i$ and

$$\mathrm{ord}(v_i^*) = (e_1^i, e_2^i, \ldots, e_{r_i}^i), \qquad \text{for } 0 \leq i \leq q. \tag{1}$$

Consider the following element and conjugacy classes of $\pi_1(\Gamma, v_0)$:

$$\begin{array}{rcll} w_0 & = & a_1^0 a_2^0 \cdots a_{l_0}^0 & \\ [w_i] & = & [a_1^i a_2^i \cdots a_{l_i}^i], & \text{for } 1 \leq i \leq p, \end{array} \tag{2}$$

where $a_1^0 = e_1^0$, $a_{l_0}^0 = \overline{e}_{r_0}^0$, the subsequence $(\overline{a}_j^0, a_{j+1}^0)$ appears in (1) for every $1 \leq j \leq (l_0 - 1)$, the subsequence $(\overline{a}_j^i, a_{j+1}^i)$ appears in (1) for every $1 \leq i \leq p$, $1 \leq j \leq l_i$ and subindices of a^i are modulo l_i. We require that every element of $E(\Gamma) \cup \overline{E}(\Gamma)$ appears exactly once in (2). We denote by $w(\Gamma, v_0, \mathrm{ord})$ the set $\{w_0, [w_1], [w_2], \ldots, [w_p]\}$. We denote

$$g = \frac{n - p}{2}.$$

We say that $(\Gamma, v_0, \mathrm{ord})$ is a fat graph with a distinguished vertex which has *genus g and p punctures*, or, a *(g, p)-fat graph with a distinguished vertex.* □

It can be seen that the genus g is a non-negative integer. See [2, Lemma 4.10].

2.4 Notation. Let $(\Gamma, v_0, \mathrm{ord})$ be a fat graph with a distinguished vertex. Suppose $V(\Gamma) = \{v_0, v_1, \ldots, v_q\}$, $|v_i^*| = r_i$ and

$$\mathrm{ord}(v_i^*) = (e_1^i, e_2^i, \ldots, e_{r_i}^i), \qquad \text{for } 0 \leq i \leq q.$$

To simplify notation, we will write

$$\begin{aligned}(\Gamma, v_0, \mathrm{ord}, \phi) &= \mathrm{ord}(v_0^*);\, \mathrm{ord}(v_1^*), \ldots, \mathrm{ord}(v_q^*) \\ &= (e_1^0, e_2^0, \ldots, e_{r_0}^0); (e_1^1, e_2^1, \ldots, e_{r_1}^1), \ldots, (e_1^q, e_2^q, \ldots, e_{r_q}^q).\end{aligned}$$

□

2.5 Example. Let $\Gamma = (V(\Gamma), E(\Gamma), V^*(\Gamma))$ be the combinatorial graph where

$$\begin{aligned}V(\Gamma) &= \{u_1, u_2, u_3, u_4\}, \\ E(\Gamma) &= \{e_1, e_2, \ldots, e_9\}, \\ u_1^* &= \{e_1, \overline{e}_1, e_2, e_3, e_4\}, \\ u_2^* &= \{e_5, e_6, \overline{e}_2, \overline{e}_3\}, \\ u_3^* &= \{e_7, e_8, \overline{e}_4, \overline{e}_5, \overline{e}_7\}, \\ u_4^* &= \{e_9, \overline{e}_6, \overline{e}_8, \overline{e}_9\}.\end{aligned}$$

Let (Γ, v_0) be the graph with the distinguished vertex $v_0 = u_2$ and let $(\Gamma, v_0, \mathrm{ord})$ be the fat graph with a distinguished vertex where

$$\begin{aligned}\mathrm{ord}(v_0^*) = \mathrm{ord}(u_2^*) &= (\overline{e}_2, e_6, \overline{e}_3, e_5), \\ \mathrm{ord}(u_1^*) &= (e_1, e_3, e_4, \overline{e}_1, e_2), \\ \mathrm{ord}(u_3^*) &= (e_7, e_8, \overline{e}_7, \overline{e}_4, \overline{e}_5,), \\ \mathrm{ord}(u_4^*) &= (\overline{e}_8, e_9, \overline{e}_6, \overline{e}_9).\end{aligned}$$

With notation above, we write

$$(\Gamma, v_0, \mathrm{ord}) = (\overline{e}_2, e_6, \overline{e}_3, e_5); (e_1, e_3, e_4, \overline{e}_1, e_2), (e_7, e_8, \overline{e}_7, \overline{e}_4, \overline{e}_5,), (\overline{e}_8, e_9, \overline{e}_6, \overline{e}_9).$$

Then $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2]\}$ where

$$\begin{aligned}w_0 &= \overline{e}_2 e_1 e_2 e_6 \overline{e}_9 \overline{e}_6 \overline{e}_3 e_4 \overline{e}_5; \\ [w_1] &= [\overline{e}_1 e_3 e_5 e_7 \overline{e}_4]; \\ [w_2] &= [\overline{e}_7 e_8 e_9 \overline{e}_8].\end{aligned}$$

Notice $\pi_1(\Gamma, v_0)$ is a free group of rank $n = 6$. Since $w(\Gamma, v_0, \mathrm{ord})$ has two cyclic words, we have $p = 2$ and $g = (n - p)/2 = (6 - 2)/2 = 2$. Hence, $(\Gamma, v_0, \mathrm{ord})$ is a $(2, 2)$-fat graph with a distinguished vertex. □

The following operations on graphs are well-known.

2.6 Definition. Let $(\Gamma, v_0, \mathrm{ord})$ be a fat graph with a distinguished vertex.

Let $f \in E(\Gamma) \cup \overline{E}(\Gamma)$ and $v_1, v_2 \in V(\Gamma)$, $v_1 \neq v_2$, such that $f \in v_1^*$, $\overline{f} \in v_2^*$. Suppose $v_2 \neq v_0$. We define the fat graph with a distinguished vertex

$(\Gamma^f, v_0^f, \mathrm{ord}^f)$ where

$$\begin{aligned} V(\Gamma^f) &= V(\Gamma) \cup \{u\} - \{v_1, v_2\}, \quad u \notin V(\Gamma); \\ v_1 \neq v_0 &\Rightarrow v_0^f = v_0; \\ v_1 = v_0 &\Rightarrow v_0^f = u; \\ E(\Gamma^f) \cup \overline{E}(\Gamma^f) &= E(\Gamma) \cup \overline{E}(\Gamma) - \{f, \overline{f}\}. \end{aligned}$$

For $v \in V(\Gamma) - \{v_1, v_2\}$, we define $\mathrm{ord}^f(v^*) = \mathrm{ord}(v^*)$. Suppose $\mathrm{ord}(v_i^*) = (e_1^i, e_2^i, \ldots, e_{r_i}^i)$, for $i = 1, 2$. Since $f \in v_1^*$, there exists $1 \leq k_1 \leq r_1$ such that $f = e_{k_1}^1$. Since $\overline{f} \in v_2^*$, there exists $1 \leq k_2 \leq r_2$ such that $\overline{f} = e_{k_2}^2$. We define

$$\begin{aligned} \mathrm{ord}^f(u^*) = &(e_1^1, e_2^1, \ldots, e_{k_1-1}^1, \\ &e_{k_2+1}^2, e_{k_2+2}^2, \ldots, e_{r_2}^2, e_1^2, e_2^2, \ldots, e_{k_2-1}, \\ &e_{k_1+1}^1, e_{k_1+2}^1, \ldots, e_{r_1}^1). \end{aligned}$$

We say that $(\Gamma^f, v_0^f, \mathrm{ord}^f)$ is obtained from $(\Gamma, v_0, \mathrm{ord})$ by *collapsing* the edge f. □

It is easy to see that collapsing an edge is well-defined and respects the genus and the number of punctures. See [2, Lemma 3.5, Lemma 4.5].

2.7 Definition. Let $u \in V(\Gamma)$ such that $|u^*| = r$. For $1 \leq k_1 < k_2 \leq r$, $(k_1, k_2) \neq (1, r)$, we define the fat graph with a distinguished vertex $(\Gamma^u, v_0^u, \mathrm{ord}^u)$ where

$$\begin{aligned} &V(\Gamma^u) = V(\Gamma) \cup \{v_1, v_2\} - \{u\}, \quad v_1,\, v_2 \notin V(\Gamma), \\ &u \neq v_0 \Rightarrow v_0^u = v_0, \\ &u = v_0 \Rightarrow v_0^u = v_1, \\ &E(\Gamma^u) = E(\Gamma) \cup \{f\}, \quad f \notin E(\Gamma) \cup \overline{E}(\Gamma). \end{aligned}$$

For $v \in V(\Gamma^u) - \{v_1, v_2\}$, we define $\mathrm{ord}^u(v^*) = \mathrm{ord}(v^*)$. Suppose $\mathrm{ord}(u^*) = (e_1, e_2, \ldots, e_r)$. We define

$$\begin{aligned} \mathrm{ord}^u(v_1^*) = &(e_1, e_2, \ldots, e_{k_1-1}, \\ &f, e_{k_2+1}, e_{k_2+2}, \ldots, e_r), \\ \mathrm{ord}(v_2^*) = &(\overline{f}, e_{k_1}, e_{k_1+1}, \ldots, e_{k_2}). \end{aligned}$$

We say that $(\Gamma^u, v_0^u, \mathrm{ord}^u)$ is obtained from $(\Gamma, v_0, \mathrm{ord})$ by *splitting* the vertex u. □

It is easy to see that splitting a vertex is well-defined and respects the genus and the number of punctures. See [2, Lemma 3.6, Lemma 4.8]. Often in the literature, splitting a vertex is called blowing up an edge.

2.8 Definition. Let $(\Gamma, v_0, \mathrm{ord}, \phi)$ be a four-tuple where $(\Gamma, v_0, \mathrm{ord})$ is a (g,p)-fat graph with a distinguished vertex such that:

(a) Γ is a metric graph with total volume 1;

(b) Γ is finite and connected without separating edges;

(c) $|v^*| \geq 3$ for every $v \in V(\Gamma) - \{v_0\}$ whereas $|v_0^*| \geq 2$;

(d) $\phi : \pi_1(\Gamma, v_0) \to F_{g,p,1}$ is an isomorphism called *the marking*.

We say that $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ is equivalent to $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ if there exists an isometry $h : \Gamma_1 \to \Gamma_2$ such that $h(v_{1,0}) = v_{2,0}$, if $\mathrm{ord}_1(v) = (e_1, e_2, \ldots, e_r)$ then $\mathrm{ord}_2(h(v)) = (h(e_1), h(e_2), \ldots, h(e_r))$ for every $v \in V(\Gamma_1)$, and the isomorphism $h_* : \pi_1(\Gamma_1, v_{1,0}) \to \pi_1(\Gamma_2, v_{2,0})$ satisfies $\phi_1 = \phi_2 \circ h_*$. □

It is easy to see that being equivalent is an equivalence relations among four-tuples $(\Gamma, v_0, \mathrm{ord}, \phi)$ as in Definition 2.8.

Auter space is an analogous of Outer space for graphs with a distinguished vertex. The following definition is an analogous of Auter space for four-tuples $(\Gamma, v_0, \mathrm{ord}, \phi)$ as in Definition 2.8.

2.9 Definition. We denote by $\mathrm{ord}\mathbb{A}_{g,p}$ the space of equivalent classes of four-tuples $(\Gamma, v_0, \mathrm{ord}, \phi)$ as in Definition 2.8 where the isomorphism $\phi : (\Gamma, v_0) \to F_{g,p,1}$ and $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$ satisfy

$$\phi(w_0) = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p = w_{g,p,1} z_1^{-1},$$
$$\{[\phi(w_1)],\, [\phi(w_2)], \ldots, [\phi(w_p)]\} = \{[t_1^{-1}],\, [t_2^{-1}], \ldots, [t_p^{-1}]\}.$$

Let $\mathfrak{p}$ be a point in $\mathrm{ord}\mathbb{A}_{g,p}$ represented by $(\Gamma, v_0, \mathrm{ord}, \phi)$. Suppose $|E(\Gamma)| = k+1$. Varying the length of the edges of Γ defines an open k-simplex $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ of $\mathrm{ord}\mathbb{A}_{g,p}$. Let Δ^k be the standard open k-dimensional simplex of $\mathbb{R}^{k+1}$. Then Δ^k parametrizes the k-simplex $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ by saying that $(\Gamma_s, v_0, \mathrm{ord}, \phi) \in \sigma(\mathfrak{p})$ is the point in $\mathrm{ord}\mathbb{A}_{g,p}$ such that the length of the edges of Γ_s equal the barycentric coordinates of $s \in \Delta^k$. Since a non-trivial isometry of Γ permutes some edges of Γ, such an isometry gives a non-trivial element of $H_1(\Gamma)$. Hence, a non-trivial isometry changes ϕ in $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ and the parametrization above is a bijection.

Some faces of $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ belong to $\mathrm{ord}\mathbb{A}_{g,p}$. Let $f \in E(\Gamma)$, $v_1, v_2 \in V(\Gamma)$, $v_1 \neq v_2$, such that $f \in v_1^*$ and $\overline{f} \in v_2^*$. We can collapse f to obtain a new graph $(\Gamma^f, v_0^f, \mathrm{ord}^f)$ with metric induced by the metric of $(\Gamma, v_0, \mathrm{ord})$ scaled such that the total volume is one. There exists a quotient map $p : (\Gamma, v_0, \mathrm{ord}) \to (\Gamma^f, v_0^f, \mathrm{ord}^f)$. Since $p_* : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma^f, v_0^f)$ is an isomorphism, there exists a point $\mathfrak{p}^f = (\Gamma^f, v_0^f, \mathrm{ord}^f, \phi^f)$ in $\mathrm{ord}\mathbb{A}_{g,p}$ where $\phi = \phi^f \circ p_*$. We say that $\sigma(\mathfrak{p}^f) = \sigma(\Gamma^f, v_0^f, \mathrm{ord}^f, \phi^f)$ is a face of $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$. Faces of $\sigma(\mathfrak{p}^f)$

are faces of $\sigma(\mathfrak{p})$. We cannot collapse an edge which is incident to a unique vertex, i.e. $v_1 = v_2$. Hence, some faces of $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ are missing. In particular, $\mathrm{ord}\mathbb{A}_{g,p}$ is not a simplicial complex.

The closure of $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$, denoted $\mathrm{cl}(\sigma(\mathfrak{p})) = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$, is the union of $\sigma(\mathfrak{p})$ and all its faces. Notice $\mathrm{cl}(\sigma(\mathfrak{p}))$ is parametrized by the closure of Δ^k with some faces missing. The topology of $\mathrm{cl}(\sigma(\mathfrak{p}))$ is induced by this parametrization. Since $\mathrm{ord}\mathbb{A}_{g,p}$ can be seen as the disjoint union of all these simplices $\mathrm{cl}(\sigma(\mathfrak{p}))$ after identifying equivalent points in the sense of Definition 2.8, $\mathrm{ord}\mathbb{A}_{g,p}$ has the quotient topology of the union of all these simplices $\mathrm{cl}(\sigma(\mathfrak{p}))$.

Thus, $\mathrm{ord}\mathbb{A}_{g,p}$ is a topological space with a complex structure with open simplices $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$. Since the fat graph with a distinguished vertex $(\Gamma, v_0, \mathrm{ord})$ has a finite number of vertices and every vertex can be split in a finite number of different manners, there exists a finite number of simplices of $\mathrm{ord}\mathbb{A}_{g,p}$ which have $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ as a face. Since the fat graph with a distinguished vertex $(\Gamma, v_0, \mathrm{ord})$ has a finite number of edges, $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ has a finite number of faces. Hence, the complex structure of $\mathrm{ord}\mathbb{A}_{g,p}$ is locally finite. □

2.10 Remark. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ be a point in $\mathrm{ord}\mathbb{A}_{g,p}$. By an Euler characteristic argument, the condition $|v^*| \geq 3$ for all $v \in V(\Gamma) - \{v_0\}$ whereas $|v_0^*| \geq 2$ in Definition 2.8, implies $|E(\Gamma)| \leq 6g + 3p - 2 = 3n - 2$. Recall $n = 2g + p$. Hence, $\mathrm{ord}\mathbb{A}_{g,p}$ is a complex of dimension $6g + 3p - 3 = 3n - 3$. □

$\mathrm{Out}(F_n)$ acts on Outer space by "changing" the marking, and $\mathrm{Aut}(F_n)$ acts on Auter space in a similar manner. We define the same action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$.

2.11 Definition. $\mathcal{AM}_{g,p,1}$ acts on $\mathrm{ord}\mathbb{A}_{g,p}$ via

$$\varphi \cdot \mathfrak{p} = \varphi \cdot (\Gamma, v_0, \mathrm{ord}, \phi) = (\Gamma, v_0, \mathrm{ord}, \varphi \circ \phi).$$

where φ is an element of $\mathcal{AM}_{g,p,1}$ and $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ is a point in $\mathrm{ord}\mathbb{A}_{g,p}$. □

3 Poincaré's Polyhedron theorem

First, we recall some results from [2]. Some of these results are directly transferred from Outer space [6]. Then, we apply Poincaré's Polyhedron theorem to the action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$ given in Definition 2.11. Since Poincaré's Polyhedron theorem can be applied in different contexts under suitable hypothesis, we give a proof of Poincaré's Polyhedron theorem in our context. Finally, we give two examples.

3.1 Remark. $\mathrm{ord}\mathbb{A}_{g,p}$ is connected. See [2, Remark 4.20 (d)] □

3.2 Lemma. *$\mathrm{ord}\mathbb{A}_{g,p}$ is simply connected.*

See [2, Remark 4.20 (g)].

3.3 Lemma. *The action of $\mathcal{AM}_{g,p,1}$ on ord$\mathbb{A}_{g,p}$ is free. Hence, $\mathcal{AM}_{g,p,1}$ is isomorphic to the fundamental group of $\mathcal{AM}_{g,p,1}\backslash ord\mathbb{A}_{g,p}$.*

See [2, Lemma 6.3, Remark 6.2].

Notice we replaced ordS$\mathbb{A}_{g,p}$ in [2, Remark 4.20(g), Lemma 6.3] by ord$\mathbb{A}_{g,p}$. Since ordS$\mathbb{A}_{g,p}$, called the spine of ord$\mathbb{A}_{g,p}$, is a homotopy retract of ord$\mathbb{A}_{g,p}$, this replacement can be done.

3.4 Lemma. *The ordered Auter space ord$\mathbb{A}_{g,p}$ is a $(6g+3p-3)$-dimensional manifold with boundary of dimension $6g+3p-4$.*

See [2, Lemma 4.15]. We recall some facts in the proof of [2, Lemma 4.15] which will be used in the statement and proof of Poincaré's Polyhedron theorem.

3.5 Remark. Let $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ be a $(6g+3p-3)$-dimensional simplex of ord$\mathbb{A}_{g,p}$ and let f be an edge of Γ. By collapsing f we obtain a codimension one face of $\sigma(\mathfrak{p})$ denoted $\sigma(\mathfrak{p}^f) = \sigma(\Gamma^f, v_0^f, \mathrm{ord}^f, \phi^f)$. Notice $(\Gamma^f, v_0^f, \mathrm{ord}^f)$ has a vertex v such that either $v \neq v_0^f$ and $|v^*| = 4$, or, $v = v_0^f$ and $|v^*| = 3$. In both cases, v can be split to obtain two different graphs. One of these graphs is $(\Gamma, v_0, \mathrm{ord})$ and the other one is a graph which may have a separating edge. Hence, $\sigma(\mathfrak{p}^f)$ is either an interior face or a boundary face as follows.

(a) If there are exactly two $(6g+3p-3)$-dimensional simplices of ord$\mathbb{A}_{g,p}$ which have $\sigma(\mathfrak{p}^f)$ as a face, then $\sigma(\mathfrak{p}^f)$ lies in the interior of ord$\mathbb{A}_{g,p}$. We say that $\sigma(\mathfrak{p}^f)$ is an interior face.

(b) If $\sigma(\mathfrak{p})$ is the unique $(6g+3p-3)$-dimensional simplex of ord$\mathbb{A}_{g,p}$ which has $\sigma(\mathfrak{p}^f)$ as a face, then $\sigma(\mathfrak{p}^f)$ lies in the boundary of ord$\mathbb{A}_{g,p}$. We say that $\sigma(\mathfrak{p}^f)$ is a boundary face.

□

We are interested in interior codimension one faces of $(6g+3p-3)$-dimensional simplices of ord$\mathbb{A}_{g,p}$.

3.6 Definition. We denote by $\mathbb{P}_{g,p} \subseteq \mathrm{ord}\mathbb{A}_{g,p}$ a fundamental domain for the action of $\mathcal{AM}_{g,p,1}$ on ord$\mathbb{A}_{g,p}$. □

Notice $\mathbb{P}_{g,p}$ is the finite union of the closure of $(6g+3p-3)$-dimensional simplices of ord$\mathbb{A}_{g,p}$. Hence, $\mathbb{P}_{g,p}$ is a polyhedron.

The following theorem is Poincaré's Polyhedron theorem applied to the action of $\mathcal{AM}_{g,p,1}$ on ord$\mathbb{A}_{g,p}$. See Definition 2.11.

3.7 Theorem. *The group $\mathcal{AM}_{g,p,1}$ is finitely presented.*

The group $\mathcal{AM}_{g,p,1}$ has generators represented by interior codimension one faces of $(6g+3p-3)$-dimensional simplices of $\mathbb{P}_{g,p}$.

There are three sets of relations of $\mathcal{AM}_{g,p,1}$ which we describe.

Let $\sigma(\mathfrak{p}_1)=\sigma(\Gamma_1, v_{1,0}, ord_1, \phi_1)$ and $\sigma(\mathfrak{p}_2)=\sigma(\Gamma_2, v_{2,0}, ord_2, \phi_2)$ be two $(6g+3p-3)$-dimensional simplices of $ord\mathbb{A}_{g,p}$ and $e_1 \in E(\Gamma_1)$, $f_2 \in E(\Gamma_2)$ such that

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{f_2}, v_{2,0}^{f_2}, ord_2^{f_2}, \phi_2^{f_2}) = \sigma(\mathfrak{p}_2^{f_2}). \tag{3}$$

Hence, $\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\mathfrak{p}_2^{f_2})$ is an interior codimension one face of $ord\mathbb{A}_{g,p}$ obtained by either collapsing e_1 in $(\Gamma_1, v_{1,0}, ord_1)$ or collapsing f_2 in $(\Gamma_2, v_{2,0}, ord_2)$. Suppose $\sigma(\mathfrak{p}_1)$ lies in $\mathbb{P}_{g,p}$. Then $\sigma(\mathfrak{p}_1^{e_1})$ represents a generator of $\mathcal{AM}_{g,p,1}$.

The first set of relations of $\mathcal{AM}_{g,p,1}$ is described by saying that if $\sigma(\mathfrak{p}_2)$ lies in $\mathbb{P}_{g,p}$, then the generator of $\mathcal{AM}_{g,p,1}$ represented by $\sigma(\mathfrak{p}_1^{e_1})$ is the identity.

The second set of relations of $\mathcal{AM}_{g,p,1}$ is described as follows. If $\sigma(\mathfrak{p}_2)$ does not lie in $\mathbb{P}_{g,p}$, then there exists a $(6g+3p-3)$-dimensional simplex $\sigma(\mathfrak{p}_3) = \sigma(\Gamma_3, v_{3,0}, ord_3, \phi_3) \subseteq \mathbb{P}_{g,p}$ which lies in the $\mathcal{AM}_{g,p,1}$-orbit of $\sigma(\mathfrak{p}_2)$. Hence, there exists $\psi \in \mathcal{AM}_{g,p,1}$ such that $\psi \cdot \sigma(\mathfrak{p}_2) = \sigma(\mathfrak{p}_3)$. And there exists an isometry $h : (\Gamma_2, v_{2,0}, ord_2) \to (\Gamma_3, v_{3,0}, ord_3)$ realising this equality. Let $f_3 = h(f_2) \in E(\Gamma_3)$. Then, by (3), the faces $\sigma(\mathfrak{p}_1^{e_1})$ and $\sigma(\mathfrak{p}_3^{f_3})$ are in the same $\mathcal{AM}_{g,p,1}$-orbit. The second set of relations is given by saying that the generator represented by the face $\sigma(\mathfrak{p}_3^{f_3})$ of $\sigma(\mathfrak{p}_3) \subseteq \mathbb{P}_{g,p}$ is the inverse of the generator represented by the face $\sigma(\mathfrak{p}_1^{e_1})$ of $\sigma(\mathfrak{p}_1) \subseteq \mathbb{P}_{g,p}$.

The third set of relations of $\mathcal{AM}_{g,p,1}$ correspond to codimension two simplices. Let $f_1 \in E(\Gamma_1) - \{e_1\}$ such that the face $\sigma(\mathfrak{p}_1^{f_1}) = \sigma(\Gamma_1^{f_1}, v_{1,0}^{f_1}, ord_1^{f_1}, \phi_1^{f_1})$ of $\sigma(\mathfrak{p}_1)$ is interior. Suppose $\{e_1, f_1\}$ are the edges of a forest in Γ_1. Equivalently, there exists the codimension two face $\sigma(\mathfrak{p}_1^{e_1,f_1}) = \sigma(\Gamma_1^{e_1,f_1}, v_{1,0}^{e_1,f_1}, ord_1^{e_1,f_1}, \phi_1^{e_1,f_1})$ obtained by collapsing either f_1 in $\Gamma_1^{e_1}$ or e_1 in $\Gamma_1^{f_1}$. We can move from $\sigma(\mathfrak{p}_1)$ to $\sigma(\mathfrak{p}_2)$ through the face $\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\mathfrak{p}_2^{f_2})$. Then, f_1 is identified with an edge e_2 of Γ_2. Since f_1 can be collapsed in $\Gamma_1^{e_1}$, $\{e_2, f_2\}$ are the edges of a forest in Γ_2. Let $\sigma(\mathfrak{p}_3) = \sigma(\Gamma_3, v_{3,0}, ord_3, \phi_3)$ be a simplex in $\mathbb{P}_{g,p}$ which lies in the $\mathcal{AM}_{g,p,1}$-orbit of $\sigma(\mathfrak{p}_2)$ with isometry $h : (\Gamma_2, v_{2,0}, ord_2) \to (\Gamma_3, v_{3,0}, ord_3)$. Let $e_3 = h(e_2)$, $f_3 = h(f_2)$. If $\sigma(\mathfrak{p}_3^{e_3}) = \sigma(\Gamma_3^{e_3}, v_{3,0}^{e_3}, ord_3^{e_3}, \phi_3^{e_3})$ is an interior face, we can apply the same procedure to the $(6g+3p-3)$-dimensional simplex $\sigma(\mathfrak{p}_3)$ instead of $\sigma(\mathfrak{p}_1)$, where e_1, f_1 are replaced by e_3, f_3, respectively. After a finite number of times repeating this procedure either we will obtain a boundary face or we will be back to $\sigma(\mathfrak{p}_1)$ with pair of edges $\{e_1, f_1\}$ and collapsing e_1. Each time we apply the procedure, we obtain a generator of $\mathcal{AM}_{g,p,1}$. The first time we obtain the generator corresponding to the face $\sigma(\mathfrak{p}_1^{e_1})$ of $\sigma(\mathfrak{p}_1)$, the second time we obtain the generator corresponding to the face $\sigma(\mathfrak{p}_3^{e_3})$ of $\sigma(\mathfrak{p}_3)$, etc. If at some point we obtain a boundary face, then we do not obtain a new relation. If we go back to $\sigma(\mathfrak{p}_1)$ with pair of edges $\{e_1, f_1\}$ and we have to collapse e_1; then we obtain a relation which is the word described by the procedure.

Proof. Let G be a group with presentation given in the statement. We have to prove that G is isomorphic to $\mathcal{AM}_{g,p,1}$. We will construct group homomorphisms $G \to \mathcal{AM}_{g,p,1}$ and $\mathcal{AM}_{g,p,1} \to G$ such that both composition are the identity.

Let x be a generator of G represented by the interior codimension 1 face $\sigma' \subseteq \mathbb{P}_{g,p}$. There exist two $(6g+3p-3)$-dimensional simplices $\sigma(\mathfrak{p}_1) = \sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, $\sigma(\mathfrak{p}_2) = \sigma(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ and edges $e_1 \in E(\Gamma_1)$, $f_2 \in E(\Gamma_2)$ such that $\sigma(\mathfrak{p}_1) \subseteq \mathbb{P}_{g,p}$ and $\sigma' = \sigma(\mathfrak{p}_1^{e_1}) = \sigma(\mathfrak{p}_2^{f_2})$. Since $\mathbb{P}_{g,p}$ is a fundamental domain of the action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$, there exists $\psi \in \mathcal{AM}_{g,p,1}$ such that $\psi \cdot \sigma(\mathfrak{p}_2) \subseteq \mathbb{P}_{g,p}$. We define the group homomorphism $G \to \mathcal{AM}_{g,p,1}$, $x \mapsto \psi$.

To see $G \to \mathcal{AM}_{g,p,1}$ is well-defined, we have to prove the three set of relations of G hold in $\mathcal{AM}_{g,p,1}$.

If $\sigma(\mathfrak{p}_2) \subseteq \mathbb{P}_{g,p}$, then ψ is the identity. Hence, the first set of relations of G holds.

Since $\psi \cdot \sigma(\mathfrak{p}_2) = \sigma(\mathfrak{p}_3) = \sigma(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3) \subseteq \mathbb{P}_{g,p}$, there exists an isometry $h : (\Gamma_2, v_{2,0}, \mathrm{ord}_2) \to (\Gamma_3, v_{3,0}, \mathrm{ord}_3)$. Let $h(f_2) = f_3$. Since $\sigma(\mathfrak{p}_3^{f_3})$ is an interior codimension one face of $\sigma(\mathfrak{p}_3) \subseteq \mathbb{P}_{g,p}$ and $\psi^{-1} \cdot \sigma(\mathfrak{p}_3^{f_3}) = \psi^{-1}\psi \cdot \sigma(\mathfrak{p}_2^{f_2}) = \sigma(\mathfrak{p}_1^{e_1})$, we see $\sigma(\mathfrak{p}_3^{f_3})$ represents x^{-1}. Hence, the second set of relations of G holds.

Let $x_k x_{k-1} \cdots x_1$ be a word given by applying the procedure in the third set of relations. Let $\psi_i \in \mathcal{AM}_{g,p,1}$ such that $x_i \mapsto \psi_i$. Then ψ_1 sends the interior codimension 1 face $\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\mathfrak{p}_2^{f_2})$ to $\sigma(\mathfrak{p}_3^{f_3})$. Notice $\psi_1 \cdot (\sigma(\mathfrak{p}_1^{e_1}) \cap \sigma(\mathfrak{p}_1^{f_1})) = \psi_1 \cdot (\sigma(\mathfrak{p}_2^{f_2}) \cap \sigma(\mathfrak{p}_2^{e_2})) = (\psi_1 \cdot \sigma(\mathfrak{p}_2^{f_2})) \cap (\psi_1 \cdot \sigma(\mathfrak{p}_2^{e_2})) = \sigma(\mathfrak{p}_3^{h(f_2)}) \cap \sigma(\mathfrak{p}_3^{h(e_2)}) = \sigma(\mathfrak{p}_3^{f_3}) \cap \sigma(\mathfrak{p}_3^{e_3})$. Next, we apply the procedure with $\sigma(\mathfrak{p}_1)$ replaced by $\sigma(\mathfrak{p}_3)$ and e_1, f_1 replaced by e_3, f_3, respectively. Hence, ψ_2 sends $\sigma(\mathfrak{p}_3^{e_3}) = \sigma(\mathfrak{p}_4^{f_4})$ to $\sigma(\mathfrak{p}_5^{f_5})$ and $\psi_2\psi_1 \cdot (\sigma(\mathfrak{p}_1^{f_1}) \cap \sigma(\mathfrak{p}_1^{e_1})) = \psi_2 \cdot (\sigma(\mathfrak{p}_3^{e_3}) \cap \sigma(\mathfrak{p}_3^{f_3})) = \sigma(\mathfrak{p}_5^{f_5}) \cap \sigma(\mathfrak{p}_5^{e_5})$. The procedure gives a relation if we go back to $\sigma(\mathfrak{p}_1)$ with pair of edges $\{e_1, f_1\}$, or equivalently, if $\psi_k \cdots \psi_2\psi_1 \cdot (\sigma(\mathfrak{p}_1^{e_1}) \cap \sigma(\mathfrak{p}_1^{f_1})) = \sigma(\mathfrak{p}_1^{f_1}) \cap \sigma(\mathfrak{p}_1^{e_1})$. In this case, since $\mathcal{AM}_{g,p,1}$ acts freely on $\mathrm{ord}\mathbb{A}_{g,p}$, $\psi_k \cdots \psi_2\psi_1$ is the identity.

Recall $\mathrm{ord}\mathbb{A}_{g,p}$ is locally finite. Since generators of G are represented by interior codimension 1 faces of $\mathbb{P}_{g,p}$, we see G is finitely generated. Since the number of relations of the first and second sets are bounded by the number of codimension 1 faces of $\mathbb{P}_{g,p}$, and, the number of relations of the third set is bounded by the number of codimension 2 faces of $\mathbb{P}_{g,p}$, we see G is finitely presented.

Let $\mathfrak{p}$ be a point such that $\sigma(\mathfrak{p})$ is a $(6g+3p-3)$-dimensional simplex and $\sigma(\mathfrak{p}) \subseteq \mathbb{P}_{g,p}$. Let $\varphi \in \mathcal{AM}_{g,p,1}$. Since $\mathrm{ord}\mathbb{A}_{g,p}$ is simply-connected and $\mathcal{AM}_{g,p,1}$ acts freely on $\mathrm{ord}\mathbb{A}_{g,p}$, we see φ is determined by any path $\gamma : [0,1] \to \mathrm{ord}\mathbb{A}_{g,p}$ from $\mathfrak{p}$ to $\varphi \cdot \mathfrak{p}$.

It is proved in [2] that $\mathrm{ord}\mathbb{A}_{g,p}$ is a $(6g+3p-3)$-dimensional manifold. The proof reduces to the following facts.

(a) Each codimension 1 simplex lies in at most the closure of two $(6g+3p-3)$-dimensional simplices. See Remark 3.5.

(b) If σ'' is a simplex of $\mathrm{ord}\mathbb{A}_{g,p}$ of codimension at least 1 and σ, σ' are two $(6g+3p-3)$-dimensional simplices of $\mathrm{ord}\mathbb{A}_{g,p}$ such that σ'' is a face of both σ and σ', then there exists a sequence of $(6g+3p-3)$-dimensional simplices of $\mathrm{ord}\mathbb{A}_{g,p}$ such that the sequence starts with σ and ends with σ', and, we can move from one element of the sequence to the next element of the sequence through a codimension 1 face which has σ'' as a face.

From (a) and (b) above, by a general position argument, we have the following.

(i) γ is isotopic, relative to $\gamma(0) = \mathfrak{p}$ and $\gamma(1) = \psi \cdot \mathfrak{p}$, to a path lying in the interior of simplices of codimension at most one.

(ii) If γ lies in the interior of simplices of codimension at most one and γ bounds a disk $D \subseteq \mathrm{ord}\mathbb{A}_{g,p}$, then D is isotopic, relative to $\gamma = \partial D$, to a disk lying in the interior of simplices of codimension at most two.

By (i) above, we can suppose there exist a sequence of $(6g+3p-3)$-dimensional open simplices $\sigma(\mathfrak{p}_i) = \sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $1 \leq i \leq k$, and there exist edges $e_i \in E(\Gamma_i)$ and $f_{i+1} \in E(\Gamma_{i+1})$, for $1 \leq i \leq k-1$, such that $\sigma(\mathfrak{p}_i^{e_i}) = \sigma(\mathfrak{p}_{i+1}^{f_{i+1}}) \subseteq \mathrm{cl}(\sigma(\mathfrak{p}_i)) \cap \mathrm{cl}(\sigma(\mathfrak{p}_{i+1}))$ and

$$\gamma([0,1]) \subseteq \Big(\bigcup_{1\leq i\leq k} \sigma(\mathfrak{p}_i)\Big) \cup \Big(\bigcup_{1\leq i\leq k-1} \mathrm{cl}(\sigma(\mathfrak{p}_i^{e_i}))\Big). \tag{4}$$

Without lost of generality, we can suppose $\mathfrak{p}$ is the barycenter of $\sigma(\mathfrak{p}_1)$ and $\gamma([0,1])$ is the concatenation of segments from the barycenter of $\sigma(\mathfrak{p}_i)$ to the barycenter of $\sigma(\mathfrak{p}_i^{e_i})$ and from the barycenter of $\sigma(\mathfrak{p}_i^{e_i}) = \sigma(\mathfrak{p}_{i+1}^{f_{i+1}})$ to the barycenter of $\sigma(\mathfrak{p}_{i+1})$, for $1 \leq i \leq k-1$.

Since $\mathbb{P}_{g,p}$ is a fundamental domain for the action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$, for $1 \leq i \leq k$, there exists $\varphi_i \in \mathcal{AM}_{g,p,1}$ such that $\sigma(\mathfrak{p}_i) \subseteq \varphi_i \cdot \mathbb{P}_{g,p}$. Notice φ_i is uniquely determined and $\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i})$, $\varphi_{i+1}^{-1} \cdot \sigma(\mathfrak{p}_{i+1}^{f_{i+1}})$ are interior codimension one faces of $\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i)$, $\varphi_{i+1}^{-1} \cdot \sigma(\mathfrak{p}_{i+1}) \subseteq \mathbb{P}_{g,p}$, respectively. Let $\psi_i = \varphi_{i+1}^{-1}\varphi_i \in \mathcal{AM}_{g,p,1}$ for $1 \leq i \leq k-1$. Notice

$$\psi_i\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i}) = \varphi_{i+1}^{-1}\varphi_i\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i}) = \varphi_{i+1}^{-1} \cdot \sigma(\mathfrak{p}_{i+1}^{f_{i+1}}).$$

Hence, ψ_i sends the interior of the face $\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i})$ of $\mathbb{P}_{g,p}$ to the interior of the face $\varphi_{i+1}^{-1} \cdot \sigma(\mathfrak{p}_{i+1}^{f_{i+1}})$ of $\mathbb{P}_{g,p}$. Since $\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i})$, $\varphi_{i+1}^{-1} \cdot \sigma(\mathfrak{p}_{i+1}^{f_{i+1}})$ are codimension one faces of $\mathbb{P}_{g,p}$, we see ψ_i is represented by $\varphi_i^{-1} \cdot \sigma(\mathfrak{p}_i^{e_i})$. From $\sigma(\mathfrak{p}_1) \subseteq \mathbb{P}_{g,p}$, it follows φ_1 is the identity and $\psi_1^{-1}\psi_2^{-1} \cdots \psi_{k-1}^{-1} = (\varphi_1^{-1}\varphi_2) \cdot (\varphi_2^{-1}\varphi_3) \cdots (\varphi_{k-1}^{-1}\varphi_k) = \varphi_1^{-1}\varphi_k = \varphi_k$ sends $\mathfrak{p} \in \sigma(\mathfrak{p}_1)$ to $\varphi_k \cdot \mathfrak{p} \in \varphi_k \cdot \sigma(\mathfrak{p}_1) = \sigma(\mathfrak{p}_k)$. Since $\mathcal{AM}_{g,p,1}$ acts freely on $\mathrm{ord}\mathbb{A}_{g,p}$, we see $\varphi = \psi_1^{-1}\psi_2^{-1} \cdots \psi_{k-1}^{-1}$. We define a group homomorphism $\mathcal{AM}_{g,p,1} \to G$, $\varphi \mapsto x_1^{-1}x_2^{-1} \cdots x_{k-1}^{-1}$.

Suppose $\gamma : [0,1] \to \mathrm{ord}\mathbb{A}_{g,p}$ is a loop. To see $\mathcal{AM}_{g,p,1} \to G$ is well defined we have to prove $x_{k-1}x_{k-2} \cdots x_1$ is the identity. Since $\mathrm{ord}\mathbb{A}_{g,p}$ is simply connected, γ

bounds a disk D. By (i) and (ii) above, we can suppose $\gamma = \partial D$ lies in the union of simplices of codimension at most one, and, the interior of D lies in the union of open simplices of codimension at most two. Then D has a complex structure with 2-cells defined by $D \cap \sigma$, 1-cells defined by $D \cap \sigma'$ and 0-cells defined by $D \cap \sigma''$, where σ is a $(6g+3p-3)$-dimensional simplex, σ' is a codimension one simplex and σ'' is a codimension two simplex. Since γ is the boundary of D, by (i) above, all 0-cells are in the inetrior of D.

If D does not have any 0-cell, then the complex structure of D consist of arcs which do not intersect themselves and join points in $\gamma = \partial D$. Then, there exists an inner-most segment in γ which joins points $\sigma(\mathfrak{p}_i^{e_i}) \cap \gamma$ and $\sigma(\mathfrak{p}_{i+1}^{e_{i+1}}) \cap \gamma$. And there exists a subsequences of $(6g+3p-3)$-dimensional simplices $\sigma(\mathfrak{p}_{i-1})$, $\sigma(\mathfrak{p}_i)$, $\sigma(\mathfrak{p}_{i+1})$ such that $\sigma(\mathfrak{p}_{i-1}) = \sigma(\mathfrak{p}_{i+1})$. Hence, $\varphi_{i+1} = \varphi_{i-1}$ and $\psi_i = \varphi_{i+1}^{-1}\varphi_i = \varphi_{i-1}^{-1}\varphi_i = \psi_{i-1}^{-1}$. Thus, $x_{k-1}x_{k-2}\cdots x_1$ can be reduced by relations of the second set. Applying this relation to $x_{k-1}x_{k-2}\cdots x_1$ is equivalent to remove from D the 2-cell bounded by the arc of $\gamma = \partial D$ from $\sigma(\mathfrak{p}_i^{e_i}) \cap \gamma$ to $\sigma(\mathfrak{p}_{i+1}^{e_{i+1}}) \cap \gamma$ and the arc inside D from $\sigma(\mathfrak{p}_i^{e_i}) \cap \gamma$ to $\sigma(\mathfrak{p}_{i+1}^{e_{i+1}}) \cap \gamma$. The same argument can be applied to the new word.

If the complex structure of D has 0-cells, then there exisits a 1-cell which joins a 0-cell q_1 to a point $q_2 \in \gamma = \partial D$. Notice $q_2 = \sigma(\mathfrak{p}_{\mathfrak{i}}{}^{e_i}) \cap \gamma = \sigma(\mathfrak{p}_{i+1}^{f_{i+1}}) \cap \gamma$ and $q_1 = \sigma(\mathfrak{p}_{\mathfrak{i}}{}^{e_i,f}) \cap D$ for some $f \in E(\Gamma_i)$. The relation obtained by applying the procedure of the third set of relations to $\sigma(\mathfrak{p}_i)$ with pair of edges $\{e_i, f\}$ and collpasing e_i is a word $y_1y_2\cdots y_r$ such that $y_1 = x_i$. Then we can replace x_i in $x_{k-1}x_{k-2}\cdots x_1$ by $y_r^{-1}\cdots y_2^{-1}$. The new word $(x_{k-1}x_{k-2}\cdots x_{i+1})\cdot(y_r^{-1}\cdots y_2^{-1})\cdot(x_{i-1}\cdots x_1)$ represents a path which bounds a disk obtained from D by applying an homotopy supported in D. The homotopy moves γ beyond q_1 through the arc from q_2 to q_1. This new disk does not have q_1 in the interior. Hence, this new disk has a 0-cell fewer in the interior. The same argument can be applied to the new word. □

3.8 Examples. (a) For $(g,p,b) = (0,2,1)$, we have $\mathbb{P}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$ where

$$\begin{aligned} V(\Gamma) &= \{v_0, v_1, v_2\}, \\ E(\Gamma) &= \{e_1, e_2, e_3, e_4\}, \\ \mathrm{ord}(v_0^*) &= (e_1, e_2), \\ \mathrm{ord}(v_1^*) &= (\overline{e}_1, e_3, e_4), \\ \mathrm{ord}(v_2^*) &= (\overline{e}_2, \overline{e}_4, \overline{e}_3). \end{aligned}$$

We write

$$(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3).$$

Notice

$$w(\Gamma, v_0, \mathrm{ord}) = \{e_1e_3\overline{e}_2, [\overline{e}_1e_2\overline{e}_4], [\overline{e}_3e_4]\}.$$

By collapsing e_1 we have

$$(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}) = (e_3, e_4, e_2); (\overline{e}_2, \overline{e}_4, \overline{e}_3).$$

We can split $v_0^{e_1}$, where $\mathrm{ord}^{e_1}(v_0^{e_1}) = (e_3, e_4, e_2)$, in two different manners. Both of them give graphs without separating edges. Hence, $\sigma(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}, \phi^{e_1})$ is an interior face which represents a generator x_1 of $\mathcal{AM}_{0,2,1}$.

Similarly,

$$(\Gamma^{e_2}, v_0^{e_2}, \mathrm{ord}^{e_2}) = (e_1, \overline{e}_4, \overline{e}_3); (\overline{e}_1, e_3, e_4),$$

is an interior face which represents a generator x_2 of $\mathcal{AM}_{0,2,1}$.

By collapsing e_3 we have

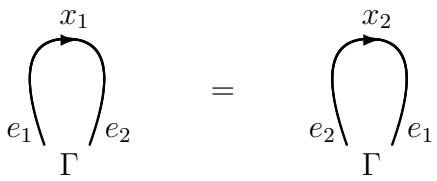
$$(\Gamma^{e_3}, v_0^{e_3}, \mathrm{ord}^{e_3}) = (e_1, e_2); (\overline{e}_1, \overline{e}_2, \overline{e}_4, e_4).$$

We can split the vertex corresponding to $(\overline{e}_1, \overline{e}_2, \overline{e}_4, e_4)$ in two manners, one case gives a graph with a separating edge. Hence, $\sigma(\Gamma^{e_3}, v_0^{e_3}, \mathrm{ord}^{e_3}, \phi^{e_3})$ is a boundary face. Similarly, $\sigma(\Gamma^{e_4}, v_0^{e_4}, \mathrm{ord}^{e_4}, \phi^{e_4})$ is a boundary face.

We see faces $\sigma(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}, \phi^{e_1})$ and $\sigma(\Gamma^{e_2}, v_0^{e_2}, \mathrm{ord}^{e_2}, \phi^{e_2})$ are in the same $\mathcal{AM}_{0,2,1}$-orbit by the isometry $h_{e_1,e_2} : (\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}) \to (\Gamma^{e_2}, v_0^{e_2}, \mathrm{ord}^{e_2})$ described by

$$h_{e_1,e_2} : \begin{cases} e_2 & \mapsto & \overline{e}_3, \\ e_3 & \mapsto & e_1, \\ e_4 & \mapsto & \overline{e}_4. \end{cases}$$

Hence, x_1, the generator represented by $\sigma(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}, \phi^{e_1})$, is the inverse of x_2, the generator represented by $\sigma(\Gamma^{e_2}, v_0^{e_2}, \mathrm{ord}^{e_2}, \phi^{e_2})$. We represent these generators and this relation:



The only possible relation comes from the pair of edges $\{e_1, e_2\}$. By splitting $v_0^{e_1}$ in $(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1})$ we have $(\Gamma, v_0, \mathrm{ord})$ and $(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_3, e_1); (\overline{e}_3, \overline{e}_2, \overline{e}_4), (\overline{e}_1, e_4, e_2)$. Then $(\Gamma, v_0, \mathrm{ord}, \phi)$ and $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ are in the same $\mathcal{AM}_{0,2,1}$-orbit with isometry $h : (\Gamma_2, v_{2,0}, \mathrm{ord}_2) \to (\Gamma, v_0, \mathrm{ord})$ such that

$$h : \begin{cases} e_1 & \mapsto & e_2 \\ \overline{e}_2 & \mapsto & e_3, \\ e_3 & \mapsto & e_1, \\ \overline{e}_4 & \mapsto & e_4. \end{cases}$$

We have to apply the same procedure to $(\Gamma, v_0, \mathrm{ord})$ with edges $h(e_2) = \overline{e}_3$, $h(e_1) = e_2$. Since $\sigma(\Gamma^{e_3}, v_0^{e_3}, \mathrm{ord}^{e_3}, \phi^{e_3})$ is a boundary face, we do not obtain a relation and $\mathcal{AM}_{0,2,1}$ is a free cyclic group generated by x_1. Notice the isometry $h : (\Gamma_2, v_{2,0}, \mathrm{ord}_2) \to (\Gamma, v_0, \mathrm{ord})$ can be deduced from the isometry $h_{e_1,e_2} : (\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1}) \to (\Gamma^{e_2}, v_0^{e_2}, \mathrm{ord}^{e_2})$ and $e_1 \mapsto e_2$.

(b) For $(g, p, b) = (1, 0, 1)$, we have $\mathbb{P}_{1,0} = \mathrm{cl}(\sigma(\Gamma', v_0', \mathrm{ord}', \phi'))$ where

$$(\Gamma', v_0', \mathrm{ord}') = (e_1, e_2); (\overline{e}_1, e_3, e_4)(\overline{e}_2, \overline{e}_3, \overline{e}_4).$$

And,

$$w(\Gamma', v_0', \mathrm{ord}') = \{e_1 e_3 \overline{e}_4 \overline{e}_1 e_2 \overline{e}_3 e_4 \overline{e}_2\}.$$

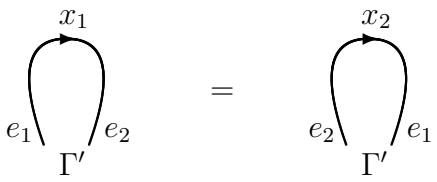
By collapsing e_1 and e_2 we have, respectively,

$$(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1}) = (e_3, e_4, e_2); (\overline{e}_2, \overline{e}_3, \overline{e}_4),$$
$$(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2}) = (e_1, \overline{e}_3, \overline{e}_4); (\overline{e}_1, e_3, e_4).$$

Notice $\sigma(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1}, \phi'^{e_1})$ and $\sigma(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2}, \phi'^{e_2})$ are in the same $\mathcal{AM}_{1,0,1}$-orbit. The isometry $h_{e_1,e_2} : (\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1}) \to (\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2})$ is described by

$$h_{e_1,e_2} : \begin{cases} e_2 & \mapsto & \overline{e}_4, \\ e_3 & \mapsto & e_1, \\ e_4 & \mapsto & \overline{e}_3. \end{cases}$$

Hence, both faces are interior and they represent generators of $\mathcal{AM}_{1,0,1}$, one inverse of the other. Let x_1 be the generator represented by $\sigma(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1}, \phi'^{e_1})$ and x_2 the generator represented by $\sigma(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2}, \phi'^{e_2})$. Then, $x_2 = x_1^{-1}$. We represent these generators and this relations:



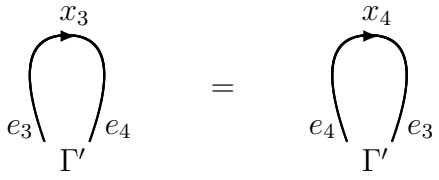
By collapsing e_3 and e_4 we have, respectively,

$$(\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3}) = (e_1, e_2); (\overline{e}_1, \overline{e}_4, \overline{e}_2, e_4),$$
$$(\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4}) = (e_1, e_2); (\overline{e}_1, e_3, \overline{e}_2, \overline{e}_3).$$

Notice $\sigma(\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3}, \phi'^{e_3})$ and $\sigma(\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4}, \phi'^{e_4})$ are in the same $\mathcal{AM}_{1,0,1}$-orbit. The isometry $h_{e_3,e_4} : (\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3}) \to (\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4})$ is described by

$$h_{e_3,e_4} : \begin{cases} e_1 & \mapsto & e_1, \\ e_2 & \mapsto & e_2, \\ e_4 & \mapsto & \overline{e}_3. \end{cases}$$

Hence, both faces are interior and they represent generators of $\mathcal{AM}_{1,0,1}$, one inverse of the other. Let x_3 be the generator represented by $\sigma(\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3}, \phi'^{e_3})$ and x_4 the generator represented by $\sigma(\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4}, \phi'^{e_4})$. Then, $x_4 = x_3^{-1}$. We represent these generators and this relations:

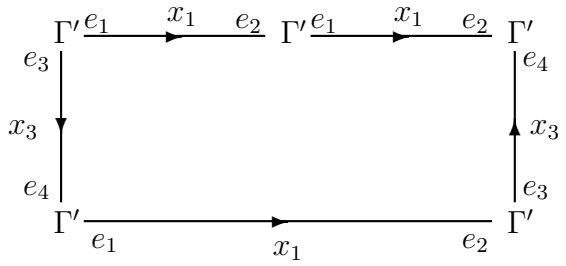


To deduce all relation, we have to consider the relations obtained from the pair of edges $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_3\}$ and $\{e_2, e_4\}$ of $(\Gamma', v_0', \mathrm{ord}')$. Notice the pair $\{e_3, e_4\}$ cannot be considered, since e_3 and e_4 are not the edges of a forest in $(\Gamma', v_0', \mathrm{ord}')$.

From the pair $\{e_1, e_2\}$ we have the following.

1. We collapse e_1 in $(\Gamma', v_0', \mathrm{ord}')$ and we obtain $(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1})$ which lies in the $\mathcal{AM}_{1,0,1}$-orbit of $(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2})$. Hence, we obtain the generator x_1 and the pair of edges $h_{e_1,e_2}(e_2) = \overline{e}_4$, $h_{e_1,e_2}(e_1) = e_2$.
2. We collapse e_4 in $(\Gamma', v_0', \mathrm{ord}')$ and we obtain $(\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4})$ which lies in the $\mathcal{AM}_{1,0,1}$-orbit of $(\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3})$. Hence, we obtain the generator $x_4 = x_3^{-1}$ and the pair of edges $h_{e_4,e_3}(e_2) = h^{-1}_{e_3,e_4}(e_2) = e_2$, $h_{e_4,e_3}(e_4) = h^{-1}_{e_3,e_4}(e_4) = e_3$.
3. We collapse e_2 in $(\Gamma', v_0', \mathrm{ord}')$ and we obtain $(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2})$ which lies in the $\mathcal{AM}_{1,0,1}$-orbit of $(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1})$. Hence, we obtain the generator $x_2 = x_1^{-1}$ and the pair of edges $h_{e_2,e_1}(e_3) = h^{-1}_{e_1,e_2}(e_3) = \overline{e}_4$, $h_{e_2,e_1}(e_2) = h^{-1}_{e_1,e_2}(e_2) = e_1$.
4. We collapse e_4 in $(\Gamma', v_0', \mathrm{ord}')$ and we obtain $(\Gamma'^{e_4}, v_0'^{e_4}, \mathrm{ord}'^{e_4})$ which lies in the $\mathcal{AM}_{1,0,1}$-orbit of $(\Gamma'^{e_3}, v_0'^{e_3}, \mathrm{ord}'^{e_3})$. Hence, we obtain the generator $x_4 = x_3^{-1}$ and the pair of edges $h_{e_4,e_3}(e_1) = h^{-1}_{e_3,e_4}(e_1) = e_1$, $h_{e_4,e_3}(e_4) = h^{-1}_{e_3,e_4}(e_4) = e_3$.
5. We collapse e_1 in $(\Gamma', v_0', \mathrm{ord}')$ and we obtain $(\Gamma'^{e_1}, v_0'^{e_1}, \mathrm{ord}'^{e_1})$ which lies in the $\mathcal{AM}_{1,0,1}$-orbit of $(\Gamma'^{e_2}, v_0'^{e_2}, \mathrm{ord}'^{e_2})$. Hence, we obtain the generator x_1 and the pair of edges $h_{e_1,e_2}(e_3) = e_1$, $h_{e_1,e_2}(e_1) = e_2$.

We are back to $(\Gamma', v_0', \mathrm{ord}')$ with pair of edges $\{e_1, e_2\}$ and we have to collapse e_1. Hence, we obtain the relation $x_1 x_3^{-1} x_1^{-1} x_3^{-1} x_1$. We represent this relation:



Since we have used the pairs of edges $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_2, e_3\}$ and $\{e_2, e_4\}$, there are no more relations and $\mathcal{AM}_{1,0,1}$ has the presentation

$$\langle x_1, x_3 \mid x_1 x_3^{-1} x_1^{-1} x_3^{-1} x_1 \rangle.$$

□

3.9 Remark. Let $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$ be a $(6g+3p-3)$-dimensional simplex of $\mathbb{P}_{g,p}$ and let e, $f \in E(\Gamma)$ be the edges of a forest in Γ. Suppose that applying the procedure in Theorem 3.7 to the pair of edges $\{e, f\}$ and collapsing e we obtain a relation r.

(a) If we apply the procedure in Theorem 3.7 to the pair of edges $\{e, f\}$ and collapsing f, then we obtain the word r^{-1}, the inverse of r.

(b) If e, f are the edges of a tree in Γ; then, after collapsing e and f in $(\Gamma, v_0, \mathrm{ord})$ we obtain a graph $(\Gamma^{e,f}, v_0^{e,f}, \mathrm{ord}^{e,f})$ with a vertex v such that either $|v^*| = 5$ if $v \neq v_0^{e,f}$, or, $|v^*| = 4$ if $v = v_0^{e,f}$. In both cases there are, at most, 5 simplices of dimension $(6g+3p-3)$ which have $\sigma(\mathfrak{p}^{e,f}) = \sigma(\Gamma^{e,f}, v_0^{e,f}, \mathrm{ord}^{e,f}, \phi^{e,f})$ as a face. Hence, r is a word of length five.

(c) If e, f are the edges of two non-connected trees in $(\Gamma, v_0, \mathrm{ord})$; then, after collapsing e and f in Γ we obtain a graph $(\Gamma^{e,f}, v_0^{e,f}, \mathrm{ord}^{e,f})$ with two vertices $u_1 \neq u_2$ such that either $|u_1^*| = |u_2^*| = 4$ if $u_1 \neq v_0^{e,f}$ and $u_2 \neq v_0^{e,f}$, or, $|u_1^*| = 3$, $|u_2^*| = 4$ if $u_1 = v_0^{e,f}$. In both cases there are, at most, 4 simplices of dimension $(6g+3p-3)$ which have $\sigma(\mathfrak{p}^{e,f}) = \sigma(\Gamma^{e,f}, v_0^{e,f}, \mathrm{ord}^{e,f}, \phi^{e,f})$ as a face. Hence, r is a word of length four.

□

4 The subcomplex $\mathbb{T}_{g,p}$ of $\mathrm{ord}\mathbb{A}_{g,p}$

We define a connected subcomplex, denoted $\mathbb{T}_{g,p}$, of $\mathrm{ord}\mathbb{A}_{g,p}$ which decomposes $\mathrm{ord}\mathbb{A}_{g,p}$.

4.1 Definition. Let $(\Gamma, v_0, \mathrm{ord})$ be a connected (g,p)-fat graph with a distinguished vertex. The *standard labelling* of $(\Gamma, v_0, \mathrm{ord})$ is a labelling of vertices and edges of Γ as follows. Recall $\mathrm{ord}(v_0^*)$ is a linear order on v_0^*. We label edges incident to v_0 according to the order on v_0^*, that is, $\mathrm{ord}(v_0^*) = (e_1, e_2, \ldots)$ or $\mathrm{ord}(v_0^*) = (e_1, \overline{e}_1, e_2, \ldots)$. Suppose we have labelled edges $e_1, e_2, \ldots, e_i \in E(\Gamma)$ and vertices $v_0, v_1, \ldots, v_j \in V(\Gamma)$. Let k, $1 \leq k \leq i$, be the least integer such that $\overline{e}_k \in u^*$ where $u \in V(\Gamma)$ has not been labelled by $v_1, v_2, \ldots, v_j$. Then we label u by v_{j+1}. Suppose $\mathrm{ord}(v_{j+1}^*) = (\overline{e}_k, f_1, f_2, \ldots, f_r)$. Notice some edges $f_1, f_2, \ldots f_r$ may have been labelled by some $\overline{e}_{k+1}, \overline{e}_{k+2}, \ldots, \overline{e}_i$. We label by $e_{i+1}, e_{i+2}, \ldots$ edges $f_1, f_2, \ldots, f_r$ which have not been labelled by $\overline{e}_{k+1}, \overline{e}_{k+2}, \ldots, \overline{e}_i$. Since $(\Gamma, v_0, \mathrm{ord})$ is connected, we label every vertex by $v_0, v_1, v_2, \ldots$ and every edge by $e_1, e_2, e_3, \ldots$. □

4.2 Example. Let $(\Gamma, v_0, \mathrm{ord})$ be a fat graph with a distinguished vertex where $V(\Gamma) = \{v_0, u_1, u_2, u_3\}$, $E(\Gamma) = \{f_1, f_2, \ldots, f_7\}$ and

$$\begin{aligned}(\Gamma, v_0, \mathrm{ord}) &= \mathrm{ord}(v_0^*); \mathrm{ord}(u_1^*), \mathrm{ord}(u_2^*), \mathrm{ord}(u_3^*) \\ &= (f_1, f_2); (f_3, \overline{f}_2, f_4), (f_5, f_6, \overline{f}_1, \overline{f}_4), (f_7, \overline{f}_5, \overline{f}_7, \overline{f}_3, \overline{f}_6).\end{aligned}$$

The standard labelling of $(\Gamma, v_0, \mathrm{ord})$ is obtained as follows. Since $\mathrm{ord}(v_0^*) = (f_1, f_2)$, f_1 is labelled e_1 and f_2 is labelled e_2. Hence, $\mathrm{ord}(v_0^*) = (e_1, e_2)$. Since $\overline{e}_1 = \overline{f}_1 \in u_2^*$, u_2 is labelled v_1 and $\mathrm{ord}(v_1^*) = (\overline{e}_1, \overline{f}_4, f_5, f_6)$. Then $\overline{f}_4$ is labelled e_3, f_5 is labelled e_4 and f_6 is labelled e_5. Hence, $\mathrm{ord}(v_1^*) = (\overline{e}_1, e_3, e_4, e_5)$. Since $\overline{e}_2 = \overline{f}_2 \in u_1^*$, u_1 is labelled v_2 and $\mathrm{ord}(v_2^*) = (\overline{e}_2, \overline{e}_3, f_3)$. Then f_3 is labelled e_6. Hence, $\mathrm{ord}(v_2^*) = (\overline{e}_2, \overline{e}_3, e_6)$. Since $\overline{e}_3 \in v_2^*$, $\overline{e}_4 = \overline{f}_5 \in u_3^*$, u_3 is labelled v_3 and $\mathrm{ord}(v_3^*) = (\overline{e}_4, \overline{f}_7, \overline{e}_6, \overline{e}_5, f_7)$. Then $\overline{f}_7$ is labelled e_7. Hence, $\mathrm{ord}(v_3^*) = (\overline{e}_4, e_7, \overline{e}_6, \overline{e}_5, \overline{e}_7)$. The standard labelling of $(\Gamma, v_0, \mathrm{ord})$ is $V(\Gamma) = \{v_0, v_1, v_2, v_3\}$, $E(\Gamma) = \{e_1, e_2, \ldots, e_7\}$ and

$$\begin{aligned}(\Gamma, v_0, \mathrm{ord}) &= \mathrm{ord}(v_0^*); \mathrm{ord}(v_1^*), \mathrm{ord}(v_2^*), \mathrm{ord}(v_3^*) \\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4, e_5), (\overline{e}_2, \overline{e}_3, e_6), (\overline{e}_4, e_7, \overline{e}_6, \overline{e}_5, \overline{e}_7).\end{aligned}$$

□

4.3 Remark. Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i), i = 1, 2$ be two points in $\mathrm{ord}\mathbb{A}_{g,p}$. Simplices $\sigma(\mathfrak{p}_1) = \sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ and $\sigma(\mathfrak{p}_2) = \sigma(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ are in the same $\mathcal{AM}_{g,p,1}$-orbit if and only if the standard labelling of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ equals to the standard labelling of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$. □

4.4 Notation. From now on, we suppose that all fat graph with a distinguished vertex $(\Gamma, v_0, \mathrm{ord})$ have the standard labelling. In particular, if $|v_0^*| = 2$ and $|V(\Gamma)| \geq 3$, then $\mathrm{ord}(v_0^*) = (e_1, e_2), \mathrm{ord}(v_1^*) = (\overline{e}_1, \ldots)$ and, either $\overline{e}_2 \in v_1^*$ or $\mathrm{ord}(v_2^*) = (\overline{e}_2, \ldots)$. □

4.5 Definition. Let $(\Gamma, v_0, \mathrm{ord})$ be a connected (g,p)-fat graph with a distinguished vertex such that $|v_0^*| = 2$. Suppose $(g,p) \neq (0,1)$. Since $(\Gamma, v_0, \mathrm{ord})$ has the standard labelling, $\mathrm{ord}(v_0^*) = (e_1, e_2)$ and $\mathrm{ord}(v_1^*) = (\overline{e}_1, \ldots)$. Since (Γ, v_0) is connected and $(g,p) \neq (0,1)$, we see $e_1 \neq \overline{e}_2$. We define the fat graph with a distinguished vertex $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ obtained from $(\Gamma, v_0, \mathrm{ord})$ by setting $v_0^{\downarrow} = v_1$, $V(\Gamma^{\downarrow}) = V(\Gamma) - \{v_0\}$ and $E(\Gamma^{\downarrow}) = E(\Gamma) - \{e_1, e_2\}$. We set $\mathrm{ord}^{\downarrow}(v) = \mathrm{ord}(v)$ for $v \in V(\Gamma^{\downarrow}) - \{v_0^{\downarrow}, u\}$ where $\overline{e}_2 \in u^*$. If $\overline{e}_2 \notin v_1^*$, then $u = v_2$ and we set $\mathrm{ord}^{\downarrow}(v_0^{\downarrow}) = (f_1^1, f_2^1, \ldots, f_r^1)$ and $\mathrm{ord}^{\downarrow}(v_2) = (f_1^2, f_2^2, \ldots, f_s^2)$ where $\mathrm{ord}(v_1) = (\overline{e}_1, f_1^1, f_2^1, \ldots, f_r^1)$ and $\mathrm{ord}(v_2) = (\overline{e}_2, f_1^2, f_2^2, \ldots, f_s^2)$. If $\overline{e}_2 \in v_1^*$, then $u = v_1$ and we set $\mathrm{ord}^{\downarrow}(v_0^{\downarrow}) = (f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_r)$ where $\mathrm{ord}(v_1) = (\overline{e}_1, f_1, f_2, \ldots, f_{k-1}, \overline{e}_2, f_{k+1}, \ldots, f_r)$

Notice $v_0^{\downarrow} = v_1, u \in E(\Gamma^{\downarrow})$ may have valency two. If $u \neq v_1$, then u is not the distinguished vertex of $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$. If $u \neq v_1$ and $|u^*| = 2$, then we set u as the midpoint of a new edge defined by concatenating the two edges incident to $u \in \Gamma^{\downarrow}$.

If $(\Gamma, v_0, \mathrm{ord})$ is a metric graph with total volume one, then $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ has a metric induced by the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$; and scaling the metric of $(\Gamma, v_0, \mathrm{ord})$ such that the total volume of $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is one. □

4.6 Remark. Notice $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ may have separating edges. □

4.7 Examples. (a) Let

$$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_5, \overline{e}_7).$$

Notice $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ has the standard labelling and

$$w(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = \{e_1 e_3 \overline{e}_6 \overline{e}_2, [\overline{e}_1 e_2 e_5 \overline{e}_7 \overline{e}_3 e_4 \overline{e}_5 e_6 e_7 \overline{e}_4]\}.$$

Since $\pi_1(\Gamma_1, v_{1,0})$ is a rank-3 free-group, we see $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ is a $(1,1)$-fat graph with a distinguished vertex.

Then

$$(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (e_3, e_4); (\overline{e}_3, f, e_7), (\overline{e}_4, \overline{f}, \overline{e}_7).$$

where edges e_5, e_6 have been concatenated to a unique edge f. We have $w(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = \{e_3 f\, \overline{e}_7 \overline{e}_3 e_4 \overline{f} e_7 \overline{e}_4\}$. Since $\pi_1(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow})$ is a rank-2 free-group, we see $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(1,0)$-fat graph with a distinguished vertex.

(b) Let

$$(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_6), (\overline{e}_4, \overline{e}_5, \overline{e}_7).$$

Notice $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ has the standard labelling and

$$w(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = \{e_1 e_3 e_7 \overline{e}_4 \overline{e}_1 e_2 e_5 \overline{e}_7 \overline{e}_6 \overline{e}_2, [\overline{e}_3 e_4 \overline{e}_5 e_6]\}.$$

Since $\pi_1(\Gamma_2, v_{2,0})$ is a rank-3 free group, we see $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ is a $(1,1)$-fat graph with a distinguished vertex.

Then

$$(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}) = (e_3, e_4); (\overline{e}_3, e_7, f), (\overline{e}_4, \overline{f}, \overline{e}_7)$$

where edges e_5, e_6 have been concatenated to a unique edge f. We have $w(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}) = \{e_3 e_7 \overline{e}_4, [\overline{e}_3 e_4 \overline{f}], [f\, \overline{e}_7]\}$. Since $\pi_1(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow})$ is a rank-2 free group, we see $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(0,2)$-fat graph with a distinguished vertex. □

The following lemma generalises these examples.

4.8 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in ord$\mathbb{A}_{g,p}$ such that $|v_0^*| = 2$. Suppose $w(\Gamma, v_0, ord) = \{w_0, [w_1], \ldots, [w_p]\}$. Then $w_0 = e_1 w_0' \overline{e}_2$ and $\overline{e}_1 e_2$ appears in $w(\Gamma, v_0, ord)$.*

(i) If $[w_i] = [\overline{e}_1 e_2 w_i']$ for some $1 \le i \le p$, then $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex.

(ii) If $w_0 = e_1 u_0' \overline{e}_1 e_2 v_0' \overline{e}_2$, then $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex.

Proof. Notice there exists an embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, which induces an injective group homomorphism $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to \pi_1(\Gamma, v_1)$. Then, $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ is a free group of rank $2g + p - 1 = n - 1$.

Since $(\Gamma, v_0, \mathrm{ord}, \phi)$ has the standard labelling, we have $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$ where $w_0 = e_1 w_0' \overline{e}_2$ and the subword $\overline{e}_1 e_2$ appears in $w(\Gamma, v_0, \mathrm{ord})$.

If $[w_i] = [\overline{e}_1 e_2 w_i']$ for some $1 \le i \le p$, then

$$w(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) = \{w_0' w_i', [w_1], [w_2], \ldots, [w_{i-1}], [w_{i+1}], \ldots, [w_p]\}.$$

If $w_0 = e_1 w_0' \overline{e}_2 = e_1 u_0' \overline{e}_1 e_2 v_0' \overline{e}_2$, then

$$w(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) = \{u_0', [v_0'], [w_1], [w_2], \ldots, [w_p]\}.$$

□

4.9 Definition. Let $\mathbb{T}_{g,p}$ be the set of all points $(\Gamma, v_0, \mathrm{ord}, \phi)$ in ord$\mathbb{A}_{g,p}$ such that $|v_0^*| \ge 3$. □

4.10 Remark. In [2, Remark 4.20 (d)], it is proved that ord$\mathbb{A}_{g,p}$ is connected by proving that $\mathbb{T}_{g,p}$ is connected. □

4.11 Remark. If a point $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ in ord$\mathbb{A}_{g,p}$ lies in $\mathbb{T}_{g,p}$, then the open simplex $\sigma(\mathfrak{p})$ lies in $\mathbb{T}_{g,p}$. Hence, $\mathbb{T}_{g,p}$ and (ord$\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$) have a simplex structure. In particular, we have the following.

If $\sigma(\mathfrak{p})$ is a $(6g+3p-3)$-dimensional simplex, then $|v_0^*| = 2$ and $\sigma(\mathfrak{p}) \subseteq \mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$.

If $\sigma(\mathfrak{p}) \subseteq \mathrm{ord}\mathbb{A}_{g,p}$ is a simplex such that $|v_0^*| = 2$, then a face σ' of $\sigma(\mathfrak{p})$ is missing in $\sigma(\mathfrak{p}) \cap (\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p})$ if and only if σ' is obtained from $\sigma(\mathfrak{p})$ by collapsing a forest of Γ which contains either e_1 or e_2. □

4.12 Definition. Let $\mathfrak{p}$ be a point in $\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$.

We denote by $\mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}}$ the connected component of $\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$ which contains $\mathfrak{p}$.

We define

$$\mathcal{AM}_{g,p,1}^{\mathfrak{p}} = \{\varphi \in \mathcal{AM}_{g,p,1} \mid \varphi \cdot \mathfrak{p} \in \mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}}\}.$$

□

4.13 Remarks. Let $\mathfrak{p}$ be a point in $\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$.

(a) $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ is a subgroup of $\mathcal{AM}_{g,p,1}$ which acts on $\mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}}$.

(b) By Remark 4.11, $\mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}}$ has a complex structure.

(c) Let $\varphi \in \mathcal{AM}_{g,p,1}$. Then $\mathcal{AM}_{g,p,1}^{\varphi\cdot\mathfrak{p}} = \varphi \cdot \mathcal{AM}_{g,p,1}^{\mathfrak{p}} \cdot \varphi^{-1}$.

(d) Suppose $g \geq 1$, $p \geq 1$. Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, $i = 1, 2$, be points in $\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$ such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. Then $\mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}_1} \cap \mathrm{ord}\mathbb{A}_{g,p}^{\mathfrak{p}_2} = \emptyset$.

□

5 The reduced ordered Auter space

We define a subcomplex of $\mathrm{ord}\mathbb{A}_{g,p}$, called the *reduced ordered Auter space*. This subspace, denoted $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$, is a homotopy retract of $\mathrm{ord}\mathbb{A}_{g,p}$. Since $\mathrm{ord}\mathbb{A}_{g,p}$ is simply connected, $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ is simply connected. The action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$ restricts to an action on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. On the other hand, $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ is a $(6g+3p-3)$-dimensional manifold, and, Poincaré's Polyhedron theorem can be applied to the action of $\mathcal{AM}_{g,p,1}$ on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$.

5.1 Definition. Let $(\Gamma, v_0, \mathrm{ord})$ be a connected (g,p)-fat graph with a distinguished vertex such that $|v_0^*| = 2$ and $|v^*| = 3$ for every $v \in V(\Gamma) - \{v_0\}$.

For $2g + p = 1$, we say that $(\Gamma, v_0, \mathrm{ord})$ is a *strongly non-separating graph* if there are not separating edges in $(\Gamma, v_0, \mathrm{ord})$.

For $2g + p \geq 2$, we say that $(\Gamma, v_0, \mathrm{ord})$ is a *strongly non-separating graph* if there are not separating edges in $(\Gamma, v_0, \mathrm{ord})$ and $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a strongly non-separating graph. □

Recall a point $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \mathrm{ord}\mathbb{A}_{g,p}$ defines a simplex $\sigma(\mathfrak{p})$ of dimension $6g+3p-3$ if and only if $|v_0^*| = 2$ and $|v^*| = 3$ for every $v \in V(\Gamma) - \{v_0\}$.

5.2 Definition. We define the *reduced ordered Auter space*, denoted $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$, as the closure of $(6g+3p-3)$-dimensional simplices $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi) \subseteq \mathrm{ord}\mathbb{A}_{g,p}$ such that $(\Gamma, v_0, \mathrm{ord})$ is a strongly non-separating graph. □

5.3 Lemma. *Let (Γ, v_0, ord) be a fat graph with a distinguished vertex. Suppose there are not separating edges in (Γ, v_0, ord). Let $u \in V(\Gamma)$ such that $|u^*| \geq 4$ if $u \neq v_0$, or, $|u^*| \geq 3$ if $u = v_0$. Then u can be split such that (Γ^u, v_0^u, ord^u) has no separating edges.*

Proof. Suppose $|u| = r$ and $\mathrm{ord}(u) = (e_1, e_2, \ldots, e_r)$. We split u according to $1 \leq k_1 < k_2 \leq r$, $(k_1, k_2) \neq (1, r)$. See Definition 2.7. If $(\Gamma^u, v_0^u, \mathrm{ord}^u)$ has separating edges, then, since $(k_1, k_2) \neq (1, r)$, we can split u according to either $1 \leq k_1 - 1 < k_2 - 1 \leq r$ or $1 \leq k_1 + 1 < k_2 + 1 \leq r$ and the new splitting gives a graph with a distinguished vertex without separating edges. □

5.4 Lemma. *$\widehat{ord\mathbb{A}}_{g,p}$ is a homotopy retract of $ord\mathbb{A}_{g,p}$.*

Proof. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \mathrm{ord}\mathbb{A}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a $(6g+3p-3)$-dimensional simplex, and $\mathfrak{p} \notin \widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. Suppose $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ has separating edges. The embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$ identifies every edge of $\Gamma^{\downarrow}$, but one, with an edge of Γ, and one edge of $\Gamma^{\downarrow}$ with the concatenation of two edges of Γ. Since $(\Gamma, v_0, \mathrm{ord})$ does not have separating edges, collapsing separating edges of $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ can be extend to collapsing the corresponding edges of $(\Gamma, v_0, \mathrm{ord})$. This defines a homotopy retraction which sends $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ to $\mathfrak{p}' = (\Gamma', v_0', \mathrm{ord}', \phi')$. By Lemma 5.3, $\mathfrak{p}'$ lies in the closure of a $(6g+3p-3)$-dimensional simplex $\sigma(\mathfrak{p}_1) = \sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ does not have separating edges.

If $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ does not have separating edges, after a finite number of times applying Definition 4.5, we obtain a graph with separating edges and the same argument can be applied. The homotopy retraction can be extend to $\mathrm{ord}\mathbb{A}_{g,p}$. □

5.5 Remark. The action of $\mathcal{AM}_{g,p,1}$ on $\mathrm{ord}\mathbb{A}_{g,p}$ described in Definition 2.11 gives an action of $\mathcal{AM}_{g,p,1}$ on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. □

5.6 Lemma. *$\widehat{ord\mathbb{A}}_{g,p}$ is a manifold of dimension $6g+3p-3$ with boundary of dimension $6g+3p-4$.*

Proof. It is proved in [2, Lemma 4.15] that $\mathrm{ord}\mathbb{A}_{g,p}$ is a $(6g+3p-3)$-dimensional manifold. The proof reduces to the following facts.

(a) Each codimension 1 simplex lies, at most, in the closure of two $(6g+3p-3)$-dimensional simplices. See Remark 3.5.

(b) If σ'' is a simplex of $\mathrm{ord}\mathbb{A}_{g,p}$ of codimension at least 1 and σ, σ' are two $(6g+3p-3)$-dimensional simplices of $\mathrm{ord}\mathbb{A}_{g,p}$ which have σ'' is a face, then there exists a sequence of $(6g+3p-3)$-dimensional simplices of $\mathrm{ord}\mathbb{A}_{g,p}$ such that the sequence starts with σ and ends with σ', and, we can move from one element of the sequence to the next element of the sequence through a codimension 1 face which has σ'' as a face.

Since $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ is a subcomplex of $\mathrm{ord}\mathbb{A}_{g,p}$, condition (a) above holds for $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$.

The proof of condition (b) above is the same for $\mathrm{ord}\mathbb{A}_{g,p}$ and $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. □

5.7 Remark. By Lemma 5.4, $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ is simply connected. By Lemma 5.6, $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ is a manifold. Hence, Poincaré's Polyhedron theorem can be applied to the action of $\mathcal{AM}_{g,p,1}$ on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. □

5.8 Definition. We define

$$\widehat{\mathbb{P}}_{g,p} = \widehat{\mathrm{ord}\mathbb{A}}_{g,p} \cap \mathbb{P}_{g,p},$$
$$\widehat{\mathbb{T}}_{g,p} = \widehat{\mathrm{ord}\mathbb{A}}_{g,p} \cap \mathbb{T}_{g,p}.$$

For $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$, we define

$$\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} = \widehat{\mathrm{ord}\mathbb{A}}_{g,p} \cap \mathrm{ord}\mathbb{A}^{\mathfrak{p}}_{g,p}.$$

□

6 A decomposition for $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$

6.1 Definitions. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$.

Recall $\mathrm{ord}(v_0) = (e_1, e_2)$. We say that edges e_1, $e_2 \in E(\Gamma)$ are *g-edges of level* $n = 2g + p$.

We denote by $\frac{1}{2}\text{-}\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ the subspace of $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ consisting of points such that the length of each g-edge of level n, e_1 and e_2, is $\frac{1}{4}$. □

6.2 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Then $\frac{1}{2}\text{-}\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is a homotopy retract of $\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$.*

Proof. We define an homotopy

$$[0,1] \times [0,1] \times \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} \to \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p},$$
$$(s, t, (\Gamma', v_0', \mathrm{ord}', \phi')) \mapsto (\Gamma'', v_0', \mathrm{ord}', \phi').$$

such that $(\Gamma'', v_0', \mathrm{ord}', \phi')$ equals $(\Gamma', v_0', \mathrm{ord}', \phi')$ excepts for the length of edges as follows. If we denote the length of an edge e of Γ by $\ell_\Gamma(e)$, then $\ell_{\Gamma''}(e_1) = \frac{s}{4} + (1-s)\ell_{\Gamma'}(e_1)$, $\ell_{\Gamma''}(e_2) = \frac{t}{4} + (1-t)\ell_{\Gamma'}(e_2)$ and $\ell_{\Gamma''}(f)$, where $f \in E(\Gamma'') - \{e_1, e_2\} = E(\Gamma') - \{e_1, e_2\}$, equals $\ell_{\Gamma'}(f)$ scaled by a factor such that the sum of lengths of all edges in $E(\Gamma'')$ is 1. □

6.3 Definition. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $\overline{e}_2 \in u^*$, $u \in V(\Gamma)$ and $|u^*| = 3$. We say that edges in $u^* - \{\overline{e}_2\}$ are *h-edges of level* $n = 2g + p$. □

6.4 Example. By Example 3.8, $\widehat{\mathbb{P}}_{0,2} = \mathbb{P}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$ where $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$ and $\widehat{\mathbb{P}}_{1,0} = \mathbb{P}_{1,0} = \mathrm{cl}(\sigma(\Gamma', v_0', \mathrm{ord}', \phi'))$ where $(\Gamma', v_0', \mathrm{ord}') = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4)$. In both cases, e_1 and e_2 are g-edges of level 2, and, e_3 and e_4 are h-edges of level 2. □

6.5 Remark. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $\overline{e}_2 \in u^*$, $u \in V(\Gamma)$ and $|u^*| = 3$. Notice the embedding $(\Gamma^\downarrow, v_0^\downarrow, \mathrm{ord}^\downarrow) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^\downarrow \mapsto v_1$, identifies every edge of Γ which is neither a g-edge nor an h-edge of level n with an edge of $\Gamma^\downarrow$, and, identifies the concatenation of both h-edges of level n with an edge of $\Gamma^\downarrow$. □

6.6 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\frac{1}{2}$-$\widehat{ord\mathbb{A}}_{g,p}$. Suppose $\sigma(\mathfrak{p})$ is a simplex of dimension $3n - 3 = 6g + 3p - 3$. Let h, q be integers such that $(\Gamma^\downarrow, v_0^\downarrow, ord^\downarrow)$ is a (h,q)-fat graph with a distinguished vertex. Let $\phi^\downarrow : \pi_1(\Gamma^\downarrow, v_0^\downarrow) \to F_{h,q,1}$ be an isomorphism such that $\mathfrak{p}^\downarrow = (\Gamma^\downarrow, v_0^\downarrow, ord^\downarrow, \phi^\downarrow)$ is a point in $\widehat{ord\mathbb{A}}_{h,q}$.*

Then, a path in $\widehat{ord\mathbb{A}}_{h,q}$ lying in simplices of codimension at most one and starting at $\mathfrak{p}^\downarrow$ can be lifted to a unique path, up to h-edges of level n and length of h-edges of level n, starting at $\mathfrak{p}$ and lying in simplices of $\frac{1}{2}$-$\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ of codimension at most one.

Proof. By Lemma 4.8, $(\Gamma^\downarrow, v_0^\downarrow, \mathrm{ord}^\downarrow)$ is a (h,q)-fat graph where either $(h,q) = (g-1, p+1)$ or $(h,q) = (g, p-1)$. By definition of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$, there exists a point $\mathfrak{p}^\downarrow = (\Gamma^\downarrow, v_0^\downarrow, \mathrm{ord}^\downarrow, \phi^\downarrow)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$. By Remark 6.5, a path starting at $\mathfrak{p}^\downarrow$ in the open simplex $\sigma(\mathfrak{p}^\downarrow)$ can be lifted to a path starting at $\mathfrak{p}$ inside the open simplex $\sigma(\mathfrak{p})$ such that the length of g-edges of level n of Γ is constant. Notice such a lift is unique up to the length of h-edges of level n of Γ.

A path in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ which lies in simplices of codimension at most one is a sequence of collapsing edges and splitting the vertex of valence four, if it is not the distinguished vertex, or the vertex of valence three, if it is the distinguished vertex. By Remark 6.5, such a path in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ can be lifted to $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ if the edge of $\Gamma^\downarrow$ which is collapsed is not identified with the concatenation of both h-edges of level n of $(\Gamma, v_0, \mathrm{ord})$.

Collapsing the edge of $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ which is identified with the concatenation of both h-edges of level n of Γ can be lifted by previously collapsing one of the two h-edges of level n and splitting the vertex of valence four. After this sequence of collapsing and splitting, the edge of $\mathfrak{p}^{\downarrow}$ which is collapsed is not identified with the concatenation of both h-edge of level n of $\mathfrak{p}$.

Notice the length of g-edges of level n can be left constant and the lifted path lies in codimension at most one simplices. Hence, the lifted path lies in $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$. □

6.7 Proposition. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\widehat{ord\mathbb{A}}_{g,p}$. Suppose $\sigma(\mathfrak{p})$ is a simplex of dimension $3n-3 = 6g+3p-3$. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a (h,q)-fat graph with a distinguished vertex. Let $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ be an isomorphism such that $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$ is a point in $\widehat{ord\mathbb{A}}_{h,q}$.*

Then, there exists an onto and continuous map

$$\mu : \widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p} \to \widehat{ord\mathbb{A}}_{h,q},$$
$$\mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}.$$

such that μ is constant on paths where any two points in the path have different length of h-edges of level n.

Proof. By definition of the metric on $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ and Remark 6.5, the assignment $\mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}$, extends to a continuous map between open simplices $\sigma(\mathfrak{p}) \to \sigma(\mathfrak{p}^{\downarrow})$. Notice this map is constant on points which only differ in the length of h-edges of level n. By Remark 4.11 and Remark 6.5, $\sigma(\mathfrak{p}) \to \sigma(\mathfrak{p}^{\downarrow})$ extends to faces of $\sigma(\mathfrak{p})$ obtained without collapsing none of the g-edges of level n. Hence, $\sigma(\mathfrak{p}) \to \sigma(\mathfrak{p}^{\downarrow})$ extends to faces of $\sigma(\mathfrak{p})$ lying in $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$. By Remark 6.5, the continuous map $\sigma(\mathfrak{p}) \to \sigma(\mathfrak{p}^{\downarrow})$ extends to a continuous map from simplices where $\sigma(\mathfrak{p})$ is a face. Since $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is path-connected, we have defined μ. We have to see that μ is well-defined.

Let $\gamma : [0,1] \to \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ be a path such that $\gamma(0) = \mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$. Recall γ is a sequence of collapsing edges and splitting vertices. Let $\gamma(1) = (\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$. Then $\phi_1^{-1} \cdot \phi : \pi_1(\Gamma, v_0) \to \pi_1(\Gamma_1, v_{1,0})$ is an isomorphism obtained by the process of collapsing the edges and splitting the vertices described in γ. Consider the path $\gamma^{\downarrow} = \mu\gamma : [0,1] \to \widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ and $\gamma^{\downarrow}(1) = (\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}, \phi_1^{\downarrow})$. Notice $(\phi_1^{\downarrow})^{-1} \cdot \phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to \pi_1(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow})$ is the restriction of $\phi_1^{-1} \cdot \phi$ to $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ under the inclusion induced by $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$. Suppose $\gamma(1) = \gamma(0) = \mathfrak{p}$. To see μ is well defined, we have to see $\gamma^{\downarrow}(1) = \gamma^{\downarrow}(0) = \mathfrak{p}^{\downarrow}$. Since $\gamma(1) = \mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ and $\gamma^{\downarrow}(1) = (\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}, \phi_1^{\downarrow})$, we see $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$. And $(\phi_1^{\downarrow})^{-1} \cdot \phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ is the identity since it is the restriction to $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ of the identity of $\pi_1(\Gamma, v_0)$.

We can suppose both g-edges of level n of Γ have length $\frac{1}{4}$. Since $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ is path-connected by paths lying in codimension at most one simplices of $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$, by Lemma 6.6, such a path in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ starting at $\mathfrak{p}^{\downarrow}$ lifts to a path in $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ staring at $\mathfrak{p}$. Then μ maps the end point of the lifted path starting at $\mathfrak{p}$ to the end point of the path starting at $\mathfrak{p}^{\downarrow}$. □

6.8 Remark. Notice $\mu : \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} \to \widehat{\mathrm{ord}\mathbb{A}}_{h,q}$, $\mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}$ defined in Proposition 6.7 depends on the choice of the isomorphism $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ which defines $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$. □

6.9 Corollary. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Then $\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is simply connected.*

Proof. Let γ be a closed path in $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$. Then $\mu\gamma$ is a closed path in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$. Since $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ is simply connected, $\mu\gamma$ bounds a disk. This disk can be lifted, by Lemma 6.6, to a disk in $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ with boundary γ. □

6.10 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose the length of each g-edge is $\frac{1}{4}$. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a (h,q)-fat graph with a distinguished vertex. Let $\mathfrak{p}' = (\Gamma', v_0', ord', \phi')$ be a point in $\widehat{ord\mathbb{A}}_{h,q}$ such that $\sigma(\mathfrak{p}')$ is a simplex of dimension $3(n-1)-3 = 6h+3q-3$.*

Then a connected component of the anti-image of $\mathfrak{p}'$ under μ restricted to $\frac{1}{2}$-$\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is obtained by changing the length of h-edges of level n, collapsing h-edges of level n and splitting the vertex of valence four, and, scaling the length of every edge but g-edges and h-edges of level n such that the total volume is one.

(i) If $(h,q) = (g, p-1)$, then the anti-image of $\mathfrak{p}'$ under μ restricted to $\frac{1}{2}$-$\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is homeomorphic to $[0,1]$.

(ii) If $(h,q) = (g-1, p+1)$, then the anti-image of $\mathfrak{p}'$ under μ restricted to $\frac{1}{2}$-$\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ is homeomorphic to the disjoint union of $(p+1)$ copies of $\mathbb{R}$.

Proof. Recall $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ has dimension $3n-3 = 6g+3p-3$, $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ has dimension $3n-5 = 6g+3p-5$ and $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ has dimension $3(n-1)-3 = 6g+3p-6$. Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 1, 2$, be a point in $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ such that $\sigma(\mathfrak{p}_i)$ is a simplex of dimension $3n-3 = 6g+3p-3$ and $\mu(\mathfrak{p}_i) = \mathfrak{p}'$, for $i = 1, 2$. Then $(\Gamma_i^{\downarrow}, v_{i,0}^{\downarrow}, \mathrm{ord}_i^{\downarrow}) = (\Gamma', v_0', \mathrm{ord}')$, for $i = 1, 2$. Notice $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ and $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ are obtained from $(\Gamma', v_0', \mathrm{ord}')$ by choosing an oriented edge which corresponds to the concatenation of both h-edges of level n. Hence, h-edges of level n of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ and $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ either are the same, possibly with different lengths, or, are different edges. Since μ is constant on paths where two

points in the path only differ in the length of h-edges, if both points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are in the same component of the anti-image under μ, then there is a sequence of collapsing an h-edge and splitting the valence four vertex which transforms $\mathfrak{p}_1 = (\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ into $\mathfrak{p}_2 = (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$.

Suppose $w(\Gamma', v_0', \mathrm{ord}') = \{w_0', [w_1'], [w_2'], \ldots, [w_q']\}$.

If $(h, q) = (g, p-1)$, then the chosen oriented edge appears in w_0'. Hence, the anti-image is parametrized by $[0, 1]$.

If $(h, q) = (g-1, p+1)$, then the chosen oriented edge appears in $[w_i']$, for $1 \leq i \leq p+1$. Hence, the anti-image is parametrized by $(p+1)$ copies of $\mathbb{R}$. $\square$

6.11 Proposition. *Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$, $i = 1, 2$, be points such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, ord_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, ord_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. Then*

$$\widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p} = \bigcup_{i=1,2} \Big(\cup_{\varphi_i} (\varphi_i \cdot \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_i}) \Big),$$

where φ_i ranges over $\mathcal{AM}_{g,p,1}/\mathcal{AM}_{g,p,1}^{\mathfrak{p}_i}$, $i = 1, 2$; and every pair of sets on the right-hand side are disjoint.

Proof. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Since $|v_0^*| = 2$, by Lemma 4.8, $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is either a $(g, p-1)$-fat graph with a distinguished vertex or a $(g-1, p+1)$-fat graph with a distinguished vertex.

Suppose $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. Let $\phi_2^{\downarrow} : \pi_1(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}) \to F_{g-1,p+1,1}$ be an isomorphism such that $\mathfrak{p}_2^{\downarrow} = (\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}, \phi_2^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1}$. By Proposition 6.7, there exists a continuous map

$$\mu : \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2} \to \widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1},\ \mathfrak{p}_2 \mapsto \mathfrak{p}_2^{\downarrow}.$$

Let $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{g-1,p+1,1}$ be an isomorphism such that $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1}$. Let $\gamma^{\downarrow}$ be a path in $\widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1}$ from $\mathfrak{p}_2^{\downarrow}$ to $\mathfrak{p}^{\downarrow}$ such that each simplex intersected by $\gamma^{\downarrow}$ has codimension at most one. By Lemma 6.6, $\gamma^{\downarrow}$ lifts to a path γ in $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ from $\mathfrak{p}_2$ to $\mathfrak{p}' = (\Gamma', v_0', \mathrm{ord}', \phi')$. Then $\mu(\mathfrak{p}') = \mathfrak{p}^{\downarrow}$. By Lemma 6.10, there exists $\mathfrak{p}'' = (\Gamma'', v_0'', \mathrm{ord}'', \phi'') \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ such that $\mu(\mathfrak{p}'') = \mu(\mathfrak{p}') = \mathfrak{p}^{\downarrow}$ and $(\Gamma'', v_0'', \mathrm{ord}'') = (\Gamma, v_0, \mathrm{ord})$. Hence, $\mathfrak{p}$ and $\mathfrak{p}''$ are in the same $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$-orbit and $\mathfrak{p}'' \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$.

If $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex, the same argument can be applied with $\mathfrak{p}_2$ replaced by $\mathfrak{p}_1$.

Every pair of sets on the right-hand side of the equality in the statement are disjoint by definition and Remark 4.13(d). $\square$

6.12 Remark. Recall $\widehat{\mathbb{P}}_{g,p}$ is a fundamental domain for the action of $\mathcal{AM}_{g,p,1}$ on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$, $i = 1, 2$, be points such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. By Proposition 6.11, we can choose $\widehat{\mathbb{P}}_{g,p}$ such that

$$\widehat{\mathbb{P}}_{g,p} - \widehat{\mathbb{T}}_{g,p} \subseteq \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} \cup \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}.$$

Since sets on the right-hand side above are disjoint, we have a decomposition of $\widehat{\mathbb{P}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ in two disjoint sets: $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1}$ and $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2} \cap \widehat{\mathbb{P}}_{g,p}$. Notice $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathbb{T}}_{g,p}$ lies in the closure of $(6g+3p-3)$-dimensional simplices of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} \cap \widehat{\mathbb{P}}_{g,p}$ and $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2} \cap \widehat{\mathbb{P}}_{g,p}$.

Recall $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_i}$, $i = 1, 2$, is the subgroup of $\mathcal{AM}_{g,p,1}$ consisting of automorphisms $\varphi \in \mathcal{AM}_{g,p,1}$ such that $\varphi \cdot \mathfrak{p}_i \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_i}$, $i = 1, 2$. Hence, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_i}$, $i = 1, 2$, is the subgroup of $\mathcal{AM}_{g,p,1}$ which leaves $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_i}$, $i = 1, 2$, invariant. By Theorem 3.7, a presentation for $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_i}, i = 1, 2$, can be deduced from $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_i}$, $i = 1, 2$. □

The following Proposition identifies $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$.

6.13 Proposition. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi)$ be a point in $\widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a simplex of dimension $3n - 3 = 6g + 3p - 3$.*

(i) If $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex, then $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ is isomorphic to $\mathcal{AM}_{g,p-1,1}$.

(ii) If $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex, then $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ is isomorphic to $\mathcal{AM}_{g-1,p,2}$.

Proof. Suppose $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2], \ldots, [w_p]\}$. We identify $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ with its image under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$. Then $e_1\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1$ is a free factor of $\pi_1(\Gamma, v_0)$ and

$$\pi_1(\Gamma, v_0) = (e_1\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1) * \langle e_1\gamma_{v_2}\overline{e}_2\rangle,$$

where γ_{v_2} is a path in $\Gamma^{\downarrow} \subseteq \Gamma$ from $v_0^{\downarrow} = v_1$ to v_2 where $\overline{e}_2 \in v_2^*$.

Let $\varphi \in \mathcal{AM}_{g,p,1}^{\mathfrak{p}}$. Recall $\phi : \pi_1(\Gamma, v_0) \to F_{g,p,1}$ is an isomorphism. Notice $\phi^{-1}\varphi\phi = \varphi^{\phi}$ is an automorphism of $\pi_1(\Gamma, v_0)$ which preserves w_0 and the set of conjugacy classes $\{[w_1], [w_2], \ldots, [w_p]\}$. We will define an automorphism of $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ with the appropriated properties in both cases.

Since $\mathcal{AM}_{g,p,1}$ acts freely on the simply connected space $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$, we see φ is determined by a path $\gamma : [0, 1] \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}} \subseteq \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ from $\gamma(0) = \mathfrak{p}$ to

$\gamma(1) = \varphi \cdot \mathfrak{p} = \varphi \cdot (\Gamma, v_0, \mathrm{ord}, \phi) = (\Gamma, v_0, \mathrm{ord}, \varphi \circ \phi)$. We can suppose $\gamma([0,1])$ lies in the union of simplices of codimension at most 1. Hence, there exists a sequence of $(6g+3p-3)$-dimensional open simplices $\sigma(\mathfrak{p}_1), \sigma(\mathfrak{p}_2), \ldots, \sigma(\mathfrak{p}_k)$ such that $\sigma(\mathfrak{p}) = \sigma(\mathfrak{p}_1)$, $\varphi \cdot \sigma(\mathfrak{p}) = \sigma(\mathfrak{p}_k)$ and

$$\gamma([0,1]) \subseteq \Big(\bigcup_{1 \le i \le k} \sigma(\mathfrak{p}_i) \Big) \cup \Big(\bigcup_{1 \le i \le k-1} \Big(\mathrm{cl}(\sigma(\mathfrak{p}_i)) \cap \mathrm{cl}(\sigma(\mathfrak{p}_{i+1})) \Big) \Big),$$

where $\mathrm{cl}(\sigma(\mathfrak{p}_i)) \cap \mathrm{cl}(\sigma(\mathfrak{p}_{i+1}))$ contains a codimension 1 face of $\sigma(\mathfrak{p}_i)$ and $\sigma(\mathfrak{p}_{i+1})$. Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$ for $1 \le i \le k$. Then $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$. Since the codimension one face in $\mathrm{cl}(\sigma(\mathfrak{p}_1)) \cap \mathrm{cl}(\sigma(\mathfrak{p}_2))$ is obtained without collapsing none of the two g-edges of level n, moving from $\sigma(\mathfrak{p}_1)$ to $\sigma(\mathfrak{p}_2)$ through the codimension one face in $\mathrm{cl}(\sigma(\mathfrak{p}_1)) \cap \mathrm{cl}(\sigma(\mathfrak{p}_2))$ sends the subgraph $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ to the subgraph $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$, sends $v_{1,0}$ to $v_{2,0}$ and sends $\mathrm{ord}_1(v_{1,0}^*)$ to $\mathrm{ord}_2(v_{2,0}^*)$. This process can be repeated until $\sigma(\mathfrak{p}_k) = \sigma(\Gamma_k, v_{k,0}, \mathrm{ord}_k, \phi_k) = \sigma(\Gamma, v_0, \mathrm{ord}, \varphi \circ \phi) = \sigma(\varphi \cdot (\Gamma, v_0, \mathrm{ord}, \phi)) = \varphi \cdot \sigma(\mathfrak{p})$. Hence, the subgroup $e_1\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1 = e_1\pi_1(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow})\overline{e}_1 = e_1\pi_1(\Gamma_k^{\downarrow}, v_{k,0}^{\downarrow})\overline{e}_1$ is invariant under φ^{ϕ} and the image of $e_1\gamma_{v_2}\overline{e}_2$ under φ^{ϕ} lies in the right coset $e_1\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1 \cdot (e_1\gamma_{v_2}\overline{e}_2)$ of $(e_1\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1)\backslash\pi_1(\Gamma, v_0)$. We define an automorphism $\rho(\varphi)$ of $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$

$$\rho(\varphi) : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}), \quad w \mapsto \overline{e}_1\varphi^{\phi}(e_1 w \overline{e}_1)e_1.$$

Let $u \in \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ such that $\varphi^{\phi}(e_1\gamma_{v_2}\overline{e}_2) = e_1 u \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2$. Recall $w_0 = e_1 w_0' \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2$ and $\overline{e}_1 e_2$ appears in $w(\Gamma, v_0, \mathrm{ord})$.

If $\overline{e}_1 e_2$ appears in $[w_i]$ for some $1 \le i \le p$; then, by Lemma 4.8, $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex, and, $w_0 = e_1 w_0' \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2$, $[w_i] = [e_2\overline{\gamma}_{v_2}\overline{e}_1 \cdot e_1 w_i' \overline{e}_1]$ for some $1 \le i \le p$ where w_0', $w_i' \in \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$ and $[w_1]$, $[w_2], \ldots [w_{i-1}], [w_{i+1}], \ldots, [w_p]$ are conjugacy classes in $\pi_1(\Gamma^{\downarrow}, v_0^{\downarrow})$. Then

$$\begin{aligned}
\varphi^{\phi}(w_0) &= \varphi^{\phi}(e_1 w_0' \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2) \\
&= \varphi^{\phi}(e_1 w_0' \overline{e}_1) \cdot \varphi^{\phi}(e_1\gamma_{v_2}\overline{e}_2) \\
&= \varphi^{\phi}(e_1 w_0' \overline{e}_1) \cdot e_1 u \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2, \\
[\varphi^{\phi}(w_i)] &= [\varphi^{\phi}(e_2\overline{\gamma}_{v_2}\overline{e}_1 \cdot e_1 w_i' \overline{e}_1)] \\
&= [\varphi^{\phi}(e_2\overline{\gamma}_{v_2}\overline{e}_1) \cdot \varphi^{\phi}(e_1 w_i' \overline{e}_1)] \\
&= [e_2\overline{\gamma}_{v_2}\overline{e}_1 \cdot e_1\overline{u}\ \overline{e}_1 \cdot \varphi^{\phi}(e_1 w_i' \overline{e}_1)].
\end{aligned}$$

Since these equalities hold in the free group $\pi_1(\Gamma, v_0) = (e_1\pi(\Gamma^{\downarrow}, v_0^{\downarrow})\overline{e}_1) * \langle e_1\gamma_{v_2}\overline{e}_2 \rangle$ and $\varphi^{\phi}(w_0) = w_0 = e_1 w_0' \overline{e}_1 \cdot e_1\gamma_{v_2}\overline{e}_2$, $[\varphi^{\phi}(w_i)] = [w_i] = [e_2\overline{\gamma}_{v_2}\overline{e}_1 \cdot e_1 w_i' \overline{e}_1]$, we see

$$\begin{aligned}
\varphi^{\phi}(e_1 w_0' \overline{e}_1) &= e_1 w_0' \overline{e}_1 \cdot e_1\overline{u}\ \overline{e}_1 \\
&= e_1 w_0' \overline{u}\ \overline{e}_1; \\
\varphi^{\phi}(e_1 w_i' \overline{e}_1) &= e_1 u \overline{e}_1 \cdot e_1 w_i' \overline{e}_1 \\
&= e_1 u w_i' \overline{e}_1.
\end{aligned}$$

Hence,

$$\begin{aligned}\varphi^{\phi}(e_1w_0'w_i'\overline{e}_1) &= \varphi^{\phi}(e_1w_0'\overline{e}_1\cdot e_1w_i'\overline{e}_1)\\ &= \varphi^{\phi}(e_1w_0'\overline{e}_1)\cdot\varphi^{\phi}(e_1w_i'\overline{e}_1)\\ &= e_1w_0'\overline{u}\ \overline{e}_1\cdot e_1uw_i'\overline{e}_1\\ &= e_1w_0'w_i'\overline{e}_1.\end{aligned}$$

And,

$$\begin{aligned}\rho(\varphi)(w_0'w_i') &= w_0'w_i',\\ \{[\rho(\varphi)(w_1)],\dots,[\rho(\varphi)(w_{i-1})],&\\ [\rho(\varphi)(w_{i+1})],\dots,[\rho(\varphi)(w_p)]\} &= \{[w_1],\dots,[w_{i-1}],[w_{i+1}],\dots,[w_p]\}.\end{aligned}$$

We fix $\phi^{\downarrow}:\pi_1(\Gamma^{\downarrow},v_0^{\downarrow})\to F_{g,p-1,1}$, independent of φ, such that $(\Gamma^{\downarrow},v_0^{\downarrow},\mathrm{ord}^{\downarrow},\phi^{\downarrow})\in \mathrm{ord}\mathbb{A}_{g,p-1}$ and we have $\phi^{\downarrow}\rho(\varphi)(\phi^{\downarrow})^{-1}\in\mathcal{AM}_{g,p-1,1}$.

If $\overline{e}_1e_2$ appears in w_0; then, by Lemma 4.8, $(\Gamma^{\downarrow},v_0^{\downarrow},\mathrm{ord}^{\downarrow})$ is a $(g-1,p+1)$-fat graph with a distinguished vertex, and,

$$w_0=e_1w_0'\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2=e_1u_0'\overline{e}_1\cdot e_2\overline{\gamma}_{v_2}\overline{e}_1\cdot e_1v_0'\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2\in\pi_1(\Gamma,v_0),$$

where $u_0',\,v_0'\in\pi_1(\Gamma^{\downarrow},v_0^{\downarrow})$ and $[w_1]$, $[w_2],\dots,[w_p]$ are conjugacy classes in $\pi_1(\Gamma^{\downarrow},v_0^{\downarrow})$. Then

$$\begin{aligned}\varphi^{\phi}(w_0) &= \varphi^{\phi}(e_1u_0'\overline{e}_1\cdot e_2\overline{\gamma}_{v_2}\overline{e}_1\cdot e_1v_0'\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2)\\ &= \varphi^{\phi}(e_1u_0'\overline{e}_1)\cdot\varphi^{\phi}(e_2\overline{\gamma}_{v_2}\overline{e}_1)\cdot\varphi^{\phi}(e_1v_0'\overline{e}_1)\cdot\varphi^{\phi}(e_1\gamma_{v_2}\overline{e}_2)\\ &= \varphi^{\phi}(e_1u_0'\overline{e}_1)\cdot e_2\overline{\gamma}_{v_2}\overline{e}_1\cdot e_1\overline{u}\ \overline{e}_1\cdot\varphi^{\phi}(e_1v_0'\overline{e}_1)\cdot e_1u\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2.\end{aligned}$$

Since this equality holds in the free group $\pi_1(\Gamma,v_0)=(e_1\pi(\Gamma^{\downarrow},v_0^{\downarrow})\overline{e}_1)*\langle e_1\gamma_{v_2}\overline{e}_2\rangle$ and $\varphi^{\phi}(w_0)=w_0=e_1u_0'\overline{e}_1\cdot e_2\overline{\gamma}_{v_2}\overline{e}_1\cdot e_1v_0'\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2$, we see

$$\begin{aligned}\varphi^{\phi}(e_1u_0'\overline{e}_1) &= e_1u_0'\overline{e}_1,\\ \varphi^{\phi}(e_1v_0'\overline{e}_1) &= e_1u\overline{e}_1\cdot e_1v_0'\overline{e}_1\cdot e_1\overline{u}\ \overline{e}_1\\ &= e_1uv_0'\overline{u}\ \overline{e}_1,\\ \varphi^{\phi}((e_1v_0'\overline{e}_1)^{e_1\gamma_{v_2}\overline{e}_2}) &= (e_1uv_0'\overline{u}\ \overline{e}_1)^{e_1u\overline{e}_1\cdot e_1\gamma_{v_2}\overline{e}_2}\\ &= (e_1v_0'\overline{e}_1)^{e_1\gamma_{v_2}\overline{e}_2}.\end{aligned}$$

Hence,

$$\begin{aligned}\rho(\varphi)(u_0') &= u_0',\\ [\rho(\varphi)(v_0')] &= [v_0'],\\ \rho(\varphi)((v_0')^{\gamma_{v_2}\overline{e}_2e_1}) &= (v_0')^{\gamma_{v_2}\overline{e}_2e_1},\\ \{[\rho(\varphi)(w_1)],[\rho(\varphi)(w_2)],\dots,[\rho(\varphi)(w_p)]\} &= \{[w_1],[w_2],\dots,[w_p]\}.\end{aligned}$$

We fix $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{g-1,p+1,1}$, independent of φ, such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow}) \in \mathrm{ord}\mathbb{A}_{g-1,p+1}$. We see $\phi^{\downarrow}\rho(\varphi)(\phi^{\downarrow})^{-1} \in \mathcal{AM}_{g-1,p,2}$ by considering that u_0', v_0' represent boundary components and $\gamma_{v_2}\overline{e}_2 e_1$ represents an arc from the boundary component represented by u_0' to the boundary component represented by v_0'. $\square$

The following remarks gives an inductive description of $\widehat{\mathbb{P}}_{g,p}$.

6.14 Remark. Let $\mathfrak{p}_1 = (\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ be a point such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex.

Let $\phi_1^{\downarrow} : \pi_1(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}) \to F_{g,p-1,1}$ be an isomorphism such that $\mathfrak{p}_1^{\downarrow} = (\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}, \phi_1^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p-1}$. Recall there is a continuous map

$$\mu : \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p-1},\ \mathfrak{p}_1 \mapsto \mathfrak{p}_1^{\downarrow}$$

By Lemma 6.10, the anti-image of a $(6g+3(p-1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g,p-1}$ under μ restricted to $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1}$ is parametrized by $[0,1]$. This restricted anti-image is described by changing the length of h-edges of level n and collapsing h-edges of level n. We choose $\widehat{\mathbb{P}}_{g,p}$ such that the anti-image of each $(6g+3(p-1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g,p-1}$ is contained in $\widehat{\mathbb{P}}_{g,p}$. Hence, every codimension one face of $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1}$ corresponding to collapse an h-edge of level n is interior in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1}$. Since μ is onto, we choose $\widehat{\mathbb{P}}_{g,p}$ such that $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1}$ maps onto $\widehat{\mathbb{P}}_{g,p-1}$ under μ. $\square$

6.15 Example. We compute $\widehat{\mathbb{P}}_{0,3}$ following Remark 6.14.

By Definition 5.8 and Example 3.8(a), $\widehat{\mathbb{P}}_{0,2} = \mathbb{P}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$, where $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$. We have to compute a connected anti-image of $(\Gamma, v_0, \mathrm{ord}, \phi)$ by collapsing h-edges of level 3. Since $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2]\}$, $w_0 = e_1 e_3 \overline{e}_2$, $[w_1] = [\overline{e}_1 e_2 \overline{e}_4]$, $[w_2] = [\overline{e}_3 e_4]$, we choose an oriented edge in $w_0 = e_1 e_3 \overline{e}_2$ to be identified with h-edges of level 3.

If we choose e_1 in $w_0 = e_1 e_3 \overline{e}_2$, we have

$$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_3), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6).$$

Notice $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (\Gamma, v_0, \mathrm{ord})$ and $e_1 \in E(\Gamma)$ is identified with the concatenation of e_5, $\overline{e}_3 \in E(\Gamma_1)$, which are the h-edges of level 3 of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. Collapsing $\overline{e}_3$ gives a boundary face of $\sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$. Hence, collapsing $\overline{e}_3$ does not give a new simplex in the anti-image of $(\Gamma, v_0, \mathrm{ord}, \phi)$ under μ. Collapsing e_5 gives an interior face of $\sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$. Then, collapsing e_5 gives a new simplex in the anti-image of $(\Gamma, v_0, \mathrm{ord}, \phi)$ under μ as follows.

$$\begin{aligned}
(\Gamma_1^{e_5}, v_{1,0}^{e_5}, \mathrm{ord}_1^{e_5}) &= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_7, \overline{e}_6, \overline{e}_3), (\overline{e}_4, e_6, e_7), \\
(\Gamma_2, v_{2,0}, \mathrm{ord}_2) &= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_7, f), (\overline{f}, \overline{e}_6, \overline{e}_3), (\overline{e}_4, e_6, e_7) \\
&= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_7, \overline{e}_5),
\end{aligned}$$

where last notation of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ is standard labelling. Notice e_5 and $\overline{e}_3$ are h-egdes of level 3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. Hence, there exist a generator corresponding to collapse e_5 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. We denote this generator by $h_{3,1}$ since it is represented by collapsing an h-edge of level 3 lying in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$.

Collapsing e_6 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ we go back to $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ and collapsing e_5 gives an interior face of $\sigma(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ as follows.

$$\begin{aligned}(\Gamma_2^{e_5}, v_{2,0}^{e_5}, \mathrm{ord}_2^{e_5}) &= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_7, e_6), (\overline{e}_3, \overline{e}_6, e_7)\\ (\Gamma_3, v_{3,0}, \mathrm{ord}_3) &= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, f), (\overline{f}, \overline{e}_7, e_6), (\overline{e}_3, \overline{e}_6, e_7)\\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, e_5), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6),\end{aligned}$$

where last notation of $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$ is standard labelling. Notice e_5 and e_6 are h-egdes of level 3 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$. Hence, there exist a generator corresponding to collapse e_5 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$. We denote this generator by $h_{3,2}$ since it is represented by collapsing an h-edge of level 3 lying in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$.

Collapsing e_5 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$ we go back to $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ and collapsing $\overline{e}_4$ gives a boundary face of $\sigma(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$. Notice $\overline{e}_4$ and e_5 are h-egdes of level 3 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$.

We represent these generators:

$$\Gamma_1 \xrightarrow{e_5 \quad h_{3,1} \quad e_6} \Gamma_2 \xrightarrow{e_5 \quad h_{3,2} \quad e_5} \Gamma_3$$

We have $\widehat{\mathbb{P}}_{0,3} \cap \widehat{\mathrm{ord}\mathbb{A}}_{0,3}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \cup_{i=1}^3 \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i))$ and

$$\widehat{\mathbb{P}}_{0,3} = \cup_{i=1}^3 \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)).$$

□

6.16 Remark. Let $\mathfrak{p}_2 = (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ be a point such that $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex.

Let $\phi_2^{\downarrow} : \pi_1(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}) \to F_{g-1,p+1,1}$ be an isomorphism such that $\mathfrak{p}_2^{\downarrow} = (\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}, \phi_2^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1}$. Recall there is a continuous map

$$\mu : \widehat{\mathrm{ord}\mathbb{A}}_{(g,p)}^{\mathfrak{p}_2} \to \widehat{\mathrm{ord}\mathbb{A}}_{g-1,p+1}, \ \mathfrak{p}_2 \mapsto \mathfrak{p}_2^{\downarrow}.$$

By Lemma 6.10, the anti-image of a $(6(g-1)+3(p+1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g-1,p+1}$ under μ restricted to $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ is parametrized by $p+1$ copies of $\mathbb{R}$. A connected component of this anti-image restricted to $\frac{1}{2}$-$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ is described by changing the length of h-edges of level n and collapsing h-edges of level n. We choose $\widehat{\mathbb{P}}_{g,p}$ such that the anti-image of each $(6(g-1)+3(p+1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g-1,p+1}$ has $p+1$ connected components contained in

$\widehat{\mathbb{P}}_{g,p}$. Hence, for each simplex of dimension $3(n-1)-3 = 6(g-1)+3(p+1)-3$ of $\widehat{\mathbb{P}}_{g-1,p+1}$, every codimension one face of $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_2}_{g,p}$ corresponding to collapse an h-edge of level n is interior in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_2}_{g,p}$, but one for every connected component. Since μ is onto, we choose $\widehat{\mathbb{P}}_{g,p}$ such that $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_2}_{g,p}$ has $p+1$ connected components which map onto $\widehat{\mathbb{P}}_{g-1,p+1}$ under μ. We explain how to connect these $p+1$ connected components in next section. □

6.17 Example. We compute $\widehat{\mathbb{P}}_{1,1}$.

By Definition 5.8, and, Example 3.8(a) and Example 3.8(b), $\widehat{\mathbb{P}}_{0,2} = \mathbb{P}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$ and $\widehat{\mathbb{P}}_{1,0} = \mathbb{P}_{1,0} = \mathrm{cl}(\sigma(\Gamma', v_0', \mathrm{ord}', \phi'))$ where $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$ and $(\Gamma', v_0', \mathrm{ord}') = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4)$. We have to compute connected anti-images of $(\Gamma, v_0, \mathrm{ord}, \phi)$ and $(\Gamma', v_0', \mathrm{ord}', \phi')$ by collapsing h-edges of level 3.

We follow Remark 6.14. Since $w(\Gamma', v_0', \mathrm{ord}') = \{w_0'\}$, $w_0' = e_1 e_3 \overline{e}_4 \overline{e}_1 e_2 \overline{e}_3 e_4 \overline{e}_2$, we choose the (oriented) edge e_1 which appears in w_0' to be the image of h-edges of level 3; and we have

$$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_3), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7).$$

Notice $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}^{\downarrow}) = (\Gamma', v_0', \mathrm{ord}')$ and $e_1 \in E(\Gamma')$ is identified with the concatenation of edges $e_5, \overline{e}_3 \in E(\Gamma_1)$ which are h-edges of level 3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. Collapsing $\overline{e}_3$ in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ we have a boundary face. Collapsing e_5 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ we have

$$(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_5, \overline{e}_7).$$

Collapsing e_5 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ we have

$$(\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_5), (\overline{e}_4, \overline{e}_7, \overline{e}_6).$$

Collapsing e_5 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$ we have

$$(\Gamma_4, v_{4,0}, \mathrm{ord}_4) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, e_5), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7).$$

Collapsing $\overline{e}_3$ in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4)$ we have

$$(\Gamma_5, v_{5,0}, \mathrm{ord}_5) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_4), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7).$$

Collapsing e_5 in $(\Gamma_5, v_{5,0}, \mathrm{ord}_5)$ we have

$$(\Gamma_6, v_{6,0}, \mathrm{ord}_6) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_5, e_7), (\overline{e}_4, \overline{e}_6, \overline{e}_7).$$

Collapsing e_5 in $(\Gamma_6, v_{6,0}, \mathrm{ord}_6)$ we have

$$(\Gamma_7, v_{7,0}, \mathrm{ord}_7) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_6), (\overline{e}_4, \overline{e}_7, \overline{e}_5).$$

Collapsing e_5 in $(\Gamma_7, v_{7,0}, \mathrm{ord}_7)$ we have

$$(\Gamma_8, v_{8,0}, \mathrm{ord}_8) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, e_5), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7).$$

Collapsing $\overline{e}_4$ in $(\Gamma_8, v_{8,0}, \mathrm{ord}_8)$ we have a boundary face. We represent these generators:

$$\Gamma_1 \xrightarrow[e_5]{h_{3,1}} \xrightarrow[e_6]{} \Gamma_2 \xrightarrow[e_5]{h_{3,2}} \xrightarrow[e_6]{} \Gamma_3 \xrightarrow[e_5]{h_{3,3}} \xrightarrow[e_5]{} \Gamma_4 \xrightarrow[e_3]{h_{3,4}} \xrightarrow[e_4]{} \Gamma_5 \xrightarrow[e_5]{h_{3,5}} \xrightarrow[e_6]{} \Gamma_6 \xrightarrow[e_5]{h_{3,6}} \xrightarrow[e_6]{} \Gamma_7 \xrightarrow[e_5]{h_{3,7}} \xrightarrow[e_5]{} \Gamma_8$$

There is one connected anti-image of $(\Gamma', v_0', \mathrm{ord}', \phi')$ under μ. We have $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \cup_{i=1}^{8} \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i))$.

We follow Remark 6.16. Since $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], [w_2]\}$, $w_0 = e_1 e_3 \overline{e}_2$, $[w_1] = [\overline{e}_1 e_2 \overline{e}_4]$, $[w_2] = [\overline{e}_3 e_4]$, the anti-image of $(\Gamma, v_0, \mathrm{ord}, \phi)$ has one connected components for each cyclic word. We choose the oriented edge $\overline{e}_1$ which appears in $[w_1] = [\overline{e}_1 e_2 \overline{e}_4]$ to be the anti-image of h-edges of level 3; and we have

$$(\Gamma_9, v_{9,0}, \mathrm{ord}_9) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, e_5), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6).$$

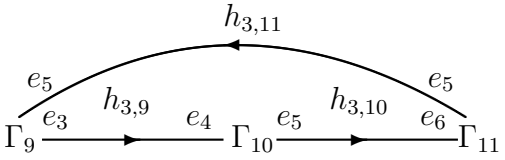
Collapsing $\overline{e}_3$ in $(\Gamma_9, v_{9,0}, \mathrm{ord}_9)$ we have

$$(\Gamma_{10}, v_{10,0}, \mathrm{ord}_{10}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_4), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6).$$

Collapsing e_5 in $(\Gamma_{10}, v_{10,0}, \mathrm{ord}_{10})$ we have

$$(\Gamma_{11}, v_{11,0}, \mathrm{ord}_{11}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_5), (\overline{e}_4, \overline{e}_6, \overline{e}_7).$$

Collapsing e_5 in $(\Gamma_{11}, v_{11,0}, \mathrm{ord}_{11})$ we go back to $(\Gamma_9, v_{9,0}, \mathrm{ord}_9)$. Hence, we have computed a connected component of the anti-image of $\sigma(\Gamma, v_0, \mathrm{ord}, \phi)$. We represent these generators:



Notice one of the three codimension one faces is not interior in $\widehat{\mathbb{P}}_{1,1}$.

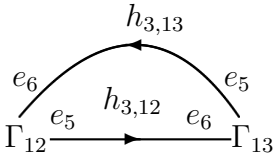
To compute the other connected component, we choose $\overline{e}_3$ in $[w_2] = [\overline{e}_3 e_4]$ to be the anti-image of h-edges of level 3; and we have

$$(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_5, e_7), (\overline{e}_4, \overline{e}_7, \overline{e}_6).$$

Collapsing e_5 in $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12})$ we have

$$(\Gamma_{13}, v_{13,0}, \mathrm{ord}_{13}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_6), (\overline{e}_4, \overline{e}_5, \overline{e}_7).$$

Collapsing e_5 in $(\Gamma_{13}, v_{13,0}, \mathrm{ord}_{13})$ we go back to $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12})$. Hence, we have computed another connected component of the anti-image of $\sigma(\Gamma, v_0, \mathrm{ord}, \phi)$. We represent these generators:



Notice one of the two codimension one faces is not interior in $\widehat{\mathbb{P}}_{1,1}$.

These two connected anti-images of $(\Gamma, v_0, \mathrm{ord}, \phi)$ under μ need to be connected through a codimension one face. We do this in next section. We deduce $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)} = \cup_{i=9}^{13} \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i))$.

To define $\widehat{\mathbb{P}}_{1,1}$, it remains to declare which codimension one face is interior in $\widehat{\mathbb{P}}_{1,1}$ in order to connected the connected components $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$ and $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)}$. This interior face of $\widehat{\mathbb{P}}_{1,1}$ corresponds to collapse a g-edge of level 3. Then

$$\widehat{\mathbb{P}}_{1,1} = \cup_{i=1}^{13} \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi)).$$

□

7 An inductive argument

For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ such that $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. We give explicit descriptions of the isomorphisms $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1} \simeq \mathcal{AM}_{g,p-1,1}$ and $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2} \simeq \mathcal{AM}_{g-1,p,2}$ in Proposition 6.13.

Notice for $p = 0$ such a $\mathfrak{p}_1$ does not exist, and, for $g = 0$ such a $\mathfrak{p}_2$ does not exist. To avoid distinguished cases, we will consider points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ as above with the convention that if the point does not exist, there is no consideration.

7.1 Definition. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ such that $|u^*| = 3$ where $\overline{e}_2 \in u^*$, $u \in V(\Gamma)$. Then $\mathrm{ord}(v_0^*) = (e_1, e_2)$ and $\mathrm{ord}(u^*) = (\overline{e}_2, f_1, f_2)$, for some f_1, $f_2 \in E(\Gamma) \cup \overline{E}(\Gamma)$. We say that e_1 and f_1 are *left* edges, and, e_2 and f_2 are *right* edges. □

7.2 Example. By Example 3.8, $\widehat{\mathbb{P}}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$ where $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$. Thus, e_1 and $\overline{e}_4$ are left edges, and, e_2 and $\overline{e}_3$ are right edges of Γ.

Similarly, by Example 3.8, $\widehat{\mathbb{P}}_{1,0} = \mathrm{cl}(\sigma(\Gamma', v_0', \mathrm{ord}', \phi'))$ where $(\Gamma', v_0', \mathrm{ord}') = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4)$. Thus, e_1 and $\overline{e}_3$ are left edges, and, e_2 and $\overline{e}_4$ are right edges of Γ'. □

The following extends the definition above.

7.3 Definition. Suppose $n \geq 3$. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a simplex of dimension $3n - 3 = 6g + 3p - 3$. Notice, in $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$, there are left and right g-edges, and, left and right h-edges of level $n - 1$.

A *left (resp. right) g-edge of level $n-1$* of $(\Gamma, v_0, \mathrm{ord})$ is an edge of Γ which is identified under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with a left (resp. right) g-edge of level $n-1$ of $\Gamma^{\downarrow}$.

A *left (resp. right) h-edge of level $n-1$* of $(\Gamma, v_0, \mathrm{ord})$ is an edge of Γ which is identified under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with a left (resp. right) h-edge of level $n-1$ of $\Gamma^{\downarrow}$.

For $n \geq 4$, we define left and right g-edges, and, left and right h-edges of level k of Γ, where $2 \leq k \leq n-2$, as edges which are identified under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with left and right g-edges, and, left and right h-edges of level k of $\Gamma^{\downarrow}$, respectively. □

7.4 Example. By Example 6.15, $\widehat{\mathbb{P}}_{0,3} = \cup_{i=1}^{3} \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i))$ where
$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_3), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6)$,
$(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (f, e_4); (\overline{e}_4, e_6, e_7), (\overline{f}, \overline{e}_7, \overline{e}_6)$, and,

Γ_1	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	$\overline{e}_3$
level 2		e_4	e_6	e_7

$(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_7, \overline{e}_5)$,
$(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}) = (e_3, e_4); (\overline{e}_3, f, e_7), (\overline{e}_4, \overline{e}_7, \overline{f})$, and,

Γ_2	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	e_6
level 2	e_3	e_4	$\overline{e}_7$	

$(\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, e_5), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6)$,
$(\Gamma_3^{\downarrow}, v_{3,0}^{\downarrow}, \mathrm{ord}_3^{\downarrow}) = (e_3, f); (\overline{e}_3, e_6, e_7), (\overline{f}, \overline{e}_7, \overline{e}_6)$, and,

Γ_3	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	$\overline{e}_4$	e_5
level 2	e_3		$\overline{e}_7$	$\overline{e}_6$

□

7.5 Example. By Example 6.17, $\widehat{\mathbb{P}}_{1,1} = \cup_{i=1}^{13} \mathrm{cl}(\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i))$ where
$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_3), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7)$,
$(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (f, e_4); (\overline{e}_4, e_6, e_7), (\overline{f}, \overline{e}_6, \overline{e}_7)$, and,

Γ_1	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	$\overline{e}_3$
level 2		e_4	e_6	e_7

$(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_5, \overline{e}_7)$,
$(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}) = (e_3, e_4); (\overline{e}_3, f, e_7), (\overline{e}_4, \overline{f}, \overline{e}_7)$, and,

Γ_2	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	e_6
level 2	e_3	e_4		$\overline{e}_7$

$(\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, \overline{e}_5), (\overline{e}_4, \overline{e}_7, \overline{e}_6)$,
$(\Gamma_3^{\downarrow}, v_{3,0}^{\downarrow}, \mathrm{ord}_3^{\downarrow}) = (e_3, e_4); (\overline{e}_3, e_7, f), (\overline{e}_4, \overline{e}_7, \overline{f})$, and,

Γ_3	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	e_6
level 2	e_3	e_4	$\overline{e}_7$	

$(\Gamma_4, v_{4,0}, \mathrm{ord}_4) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, e_5), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7)$,
$(\Gamma_4^{\downarrow}, v_{4,0}^{\downarrow}, \mathrm{ord}_4^{\downarrow}) = (f, e_4); (\overline{e}_4, e_6, e_7), (\overline{f}, \overline{e}_6, \overline{e}_7)$, and,

Γ_4	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	$\overline{e}_3$	e_5
level 2		e_4	e_6	e_7

$(\Gamma_5, v_{5,0}, \mathrm{ord}_5) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_4), (\overline{e}_3, e_6, e_7), (\overline{e}_5, \overline{e}_6, \overline{e}_7)$,
$(\Gamma_5^{\downarrow}, v_{5,0}^{\downarrow}, \mathrm{ord}_5^{\downarrow}) = (e_3, f); (\overline{e}_3, e_6, e_7), (\overline{f}, \overline{e}_6, \overline{e}_7)$, and,

Γ_5	g-edge		h-edge	
	left	right	left	right
level 3	e_1	e_2	e_5	$\overline{e}_4$
level 2	e_3		$\overline{e}_6$	$\overline{e}_7$

The remaining cases are similar. □

7.1 Generators

We give a description of generators of $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}$ where $\mathfrak{p}$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. These are generators of $\mathcal{AM}_{g,p,1}$ but generators corresponding to collapse g-edges of level n.

7.6 Remark. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. Suppose $\sigma(\mathfrak{p})$ is a simplex of dimension $3n-3 = 6g+3p-3$. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a (h,q)-fat graph with a distinguished vertex. Let f be an edge of Γ of level less than n, and, let $f^{\downarrow}$ be the edge of $\Gamma^{\downarrow}$ identified with f under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$. Then the following holds.

(a) f is a left (resp. right) edge of level $k < n$ if, and only if, $f^{\downarrow}$ is a left (resp. right) edge of level k.

(b) f is a g-edge (resp. h-edge) of level $k < n$ if, and only if, $f^{\downarrow}$ is a g-edge (resp. h-edge) of level k.

□

7.7 Lemma. *Let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i)$, $i = 1, 2$, be points in $\widehat{ord\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p}_i)$, $i = 1, 2$, are simplices of dimension $3n-3 = 6g+3p-3$. Let f_i be edges of Γ_i, $i = 1, 2$, such that $\sigma(\mathfrak{p}_1^{f_1}) = \sigma(\mathfrak{p}_2^{f_2})$.*

Then, for $2 \leq k \leq n$, f_1 is a left g-edge (resp. h-edge) of level k if, and only if, f_2 is a right g-edge (resp. h-edge) of level k.

Proof. If f_1 is a left g-edge of level n, then $|(v_{1,0}^{f_1})^*| = 3$. Since $\mathfrak{p}_1^{f_1} = (\Gamma_1^{f_1}, v_{1,0}^{f_1}, \mathrm{ord}_1^{f_1}, \phi_1^{f_1}) = (\Gamma_2^{f_2}, v_{2,0}^{f_2}, \mathrm{ord}_2^{f_2}, \phi_2^{f_2}) = \mathfrak{p}_1^{f_2}$, it follows $|(v_{2,0}^{f_2})^*| = 3$. Then, f_2 is a g-edge of level n. And, f_1 is a left edge if and only if f_2 is a right edge.

If f_1 is a left h-edge of level n, then f_1 is incident to $v_{1,2}$, where $\overline{e}_2 \in v_{1,2}^*$, and $|(v_{1,2}^{f_1})^*| = 4$. Since $\mathfrak{p}_1^{f_1} = (\Gamma_1^{f_1}, v_{1,0}^{f_1}, \mathrm{ord}_1^{f_1}, \phi_1^{f_1}) = (\Gamma_2^{f_2}, v_{2,0}^{f_2}, \mathrm{ord}_2^{f_2}, \phi_2^{f_2}) = \mathfrak{p}_2^{f_2}$, we see $|(v_{2,2}^{f_2})^*| = 4$, where $\overline{e}_2 \in v_{2,2}^*$, and f_2 is an h-edge of level n. Then, f_1 is a left edge if and only if f_2 is a right edge.

Let k_i be an integer such that $f_i \in E(\Gamma_i)$ has level $k_i, i = 1, 2$. Suppose $k_1 \geq k_2$. We have proved that if $k_1 = n$, then the result holds. Suppose $k_1 < n$. Let h, q be integers such that $(\Gamma_i^{\downarrow}, v_{i,0}^{\downarrow}, \mathrm{ord}_i^{\downarrow}), i = 1, 2$, are (h,q)-fat graphs. We choose $\phi_1^{\downarrow} : \pi_1(\Gamma_1, v_{1,0}) \to F_{h,q,1}$ such that $\mathfrak{p}_1^{\downarrow} = (\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}, \phi_1^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$. Let $\phi_2^{\downarrow} : \pi_2(\Gamma_2, v_{2,0}) \to F_{h,q,1}$ such that $\mu(\mathfrak{p}_2) = (\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}, \phi_2^{\downarrow}) = \mathfrak{p}_2^{\downarrow}$ where $\mu : \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} \to \widehat{\mathrm{ord}\mathbb{A}}_{h,q}$, $\mathfrak{p}_1 \mapsto \mathfrak{p}_1^{\downarrow}$. By induction hypothesis, the result holds for $f_i^{\downarrow}$ and $\mathfrak{p}_i^{\downarrow} = (\Gamma_i^{\downarrow}, v_{i,0}^{\downarrow}, \mathrm{ord}_i^{\downarrow}, \phi_i^{\downarrow}), i = 1, 2$, where $f_i^{\downarrow}$ is identified with f_i under the embedding $(\Gamma_i^{\downarrow}, v_{i,0}^{\downarrow}, \mathrm{ord}_i^{\downarrow}) \hookrightarrow (\Gamma_i, v_{i,0}, \mathrm{ord}_i)$, $v_{i,0}^{\downarrow} \mapsto v_{i,1}$, $i = 1, 2$. Then, conclusions hold for f_i and $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i), i = 1, 2$. □

Since generators of $\mathcal{AM}_{g,p,1}$ correspond to collapse edges, we have the following definition.

7.8 Definition. A *g-generator (resp. an h-generator) of level k* of $\mathcal{AM}_{g,p,1}$ is a generator which corresponds to collapse a left g-edge (resp. h-edge) of level k. □

7.9 Remark. By Lemma 7.7, the inverse of a g-generator (resp. an h-generator) of level k corresponds to collapse a right g-edge (resp. h-generator) of level k. □

7.10 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi) \in \widehat{ord\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a simplex of dimension $3n-3 = 6g+3p-3$. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ is a (h,q)-fat graph with a distinguished vertex, and, let $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ be an isomorphism such that $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$ is a point in $\widehat{ord\mathbb{A}}_{h,q}$.*

Let f be a g-edge (resp. an h-edge) of Γ of level $k < n$, and, let $f^{\downarrow}$ be the edge of $\Gamma^{\downarrow}$ identified with f under the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}) \hookrightarrow (\Gamma, v_0, ord)$, $v_0^{\downarrow} \mapsto v_1$. Then there exists a g-generator (resp. an h-generator) of level k of $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}$ corresponding to collapse f in (Γ, v_0, ord, ϕ) if and only if there exists a g-generator (resp. an h-generator) of level k of $\mathcal{AM}_{h,q,1}$ corresponding to collapse $f^{\downarrow}$ in $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$.

Proof. Since level of f is $k < n$, there exists $f^{\downarrow} \in E(\Gamma^{\downarrow})$ as in the statement. By Proposition 6.7, there exists a continuous map which is onto

$$\mu : \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} \to \widehat{\mathrm{ord}\mathbb{A}}_{h,q},\ \mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}.$$

If there exists a generator corresponding to collapse f in $(\Gamma, v_0, \mathrm{ord}, \phi)$, then there exist a simplex $\sigma(\mathfrak{p}') = \sigma(\Gamma', v_0', \mathrm{ord}', \phi')$ of dimension $3n-3 = 6g+3p-3$ which has the codimension one face $\sigma(\mathfrak{p}^f) = \sigma(\Gamma^f, v_0^f, \mathrm{ord}^f, \phi^f)$. Hence, $\mu(\mathfrak{p}')$ is a simplex of dimension $3(n-1)-3$ which has the codimension one face obtained by collapsing $f^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$.

Since μ is onto, if there exists a generator corresponding to collapse $f^{\downarrow}$ in $\mathfrak{p}^{\downarrow}$, then a similar argument can be applied. □

7.11 Definition. With the hypothesis of the Corollary above, we say the generator of $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}$ corresponding to collapse f in $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi)$ is a *lift* of the generator of $\mathcal{AM}_{h,q,1}$ corresponding to collapse $f^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$. □

7.12 Example. We compute generators of $\mathcal{AM}_{0,3,1}$ of level less than 3. By Lemma 7.10 and Example 6.15, we have to lift generators of $\mathcal{AM}_{0,2,1}$. By Example 3.8(a), $\mathcal{AM}_{0,2,1} = \langle x_1 \mid \rangle$, where x_1 corresponds to collapse e_1. Recall $\widehat{\mathbb{P}}_{0,2} = \mathrm{cl}(\sigma(\Gamma, v_0, \mathrm{ord}, \phi))$, where $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$. By Example 6.4 and Definition 7.1, e_1 is a left g-edge of level 2. Then, we denote by g_2 the generator x_1 of $\mathcal{AM}_{0,2,1}$. Hence, $\mathcal{AM}_{0,2,1} = \langle g_2 \mid \rangle$.

To lift $g_2 \in \mathcal{AM}_{0,2,1}$, we follow Example 7.4. Notice there is no left g-edge of level 2 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, e_3 is the left g-edge of level 2 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ and e_3 is the left g-edge of level 2 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$. By Lemma 7.10, $\mathcal{AM}_{0,3,1}$ has two g-generators of level 2 corresponding to collapse e_3 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$

and e_3 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$. To identify these generators we collapse left and right g-edges of level 2.

Collapsing left g-edges of level 2 we have the following.

$$\begin{aligned}(\Gamma_2^{e_3}, v_{2,0}^{e_3}, \mathrm{ord}_2^{e_3}) &= (e_1, e_2); (\overline{e}_1, \overline{e}_6, e_7, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_4, \overline{e}_7, \overline{e}_5)\\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4, e_5), (\overline{e}_2, e_6, \overline{e}_3), (\overline{e}_4, \overline{e}_6, \overline{e}_5)\\ (\Gamma_3^{e_3}, v_{3,0}^{e_3}, \mathrm{ord}_3^{e_3}) &= (e_1, e_2); (\overline{e}_1, e_6, e_7, e_4), (\overline{e}_2, \overline{e}_4, e_5), (\overline{e}_5, \overline{e}_7, \overline{e}_6)\\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4, e_5), (\overline{e}_2, \overline{e}_5, e_6), (\overline{e}_3, \overline{e}_6, \overline{e}_4)\end{aligned}$$

Collapsing right g-edges of level 2 we have the following.

$$\begin{aligned}(\Gamma_1^{e_4}, v_{1,0}^{e_4}, \mathrm{ord}_1^{e_4}) &= (e_1, e_2); (\overline{e}_1, e_3, e_6, e_7), (\overline{e}_2, e_5, \overline{e}_3), (\overline{e}_5, \overline{e}_7, \overline{e}_6)\\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4, e_5), (\overline{e}_2, e_6, \overline{e}_3), (\overline{e}_4, \overline{e}_6, \overline{e}_5)\\ (\Gamma_2^{e_4}, v_{2,0}^{e_4}, \mathrm{ord}_2^{e_4}) &= (e_1, e_2); (\overline{e}_1, e_3, \overline{e}_7, \overline{e}_5), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7)\\ &= (e_1, e_2); (\overline{e}_1, e_3, e_4, e_5), (\overline{e}_2, \overline{e}_5, e_6), (\overline{e}_3, \overline{e}_6, \overline{e}_4)\end{aligned}$$

Since $(\Gamma_2^{e_3}, v_{2,0}^{e_3}, \mathrm{ord}_2^{e_3}) = (\Gamma_1^{e_4}, v_{1,0}^{e_4}, \mathrm{ord}_1^{e_4})$ and e_3 is a left edge of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$, we define the generator $g_{2,2}$ which corresponds to collapse e_3 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$.

Since $(\Gamma_3^{e_3}, v_{3,0}^{e_3}, \mathrm{ord}_3^{e_3}) = (\Gamma_2^{e_4}, v_{2,0}^{e_4}, \mathrm{ord}_2^{e_4})$ and e_3 is a left edge of $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$, we define the generator $g_{2,3}$ which corresponds to collapse e_3 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$. We represent these generators:

$$\Gamma_1 \xleftarrow[g_{2,2}]{e_4 \quad e_3} \Gamma_2 \xleftarrow[g_{2,3}]{e_4 \quad e_3} \Gamma_3$$

We denote these generators by a g_2 since they are lifts of the g-generator g_2 in $\mathcal{AM}_{0,2,1}$, and a second subindex which denotes the fat graph where the left edge which needs to be collapsed lies. Hence, $g_{2,2}$ is obtained by collapsing the left g-edge of level 2 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ and $g_{2,3}$ is obtained by collapsing the left g-edge of level 2 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$. □

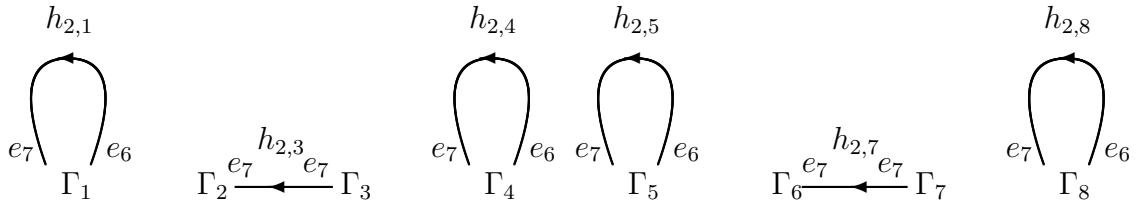
7.13 Remark. We complete Remark 6.16. Each of the $p+1$ connected components in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ which map onto $\widehat{\mathbb{P}}_{g-1,p+1}$ under μ corresponds to a cyclic word in $w(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ as follows. Each connected component in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ contains points $(\Gamma, v_0, \mathrm{ord}, \phi)$ such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) = (\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$. By the embedding $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_{2,0}^{\downarrow} \mapsto v_1$, there is a cyclic word in $w(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ which does not appears in $w(\Gamma, v_0, \mathrm{ord})$. Recall $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph and $(\Gamma, v_0, \mathrm{ord})$ is a (g,p)-fat graph. There exists a g-generator of level $k < n$ in $\mathcal{AM}_{g-1,p+1,1}$ whose corresponding left g-edge is incident to this cyclic word. Then, the lift of this g-generator of level $k < n$ in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ connects the connected component containing $\mathfrak{p}$ with another connected component.

□

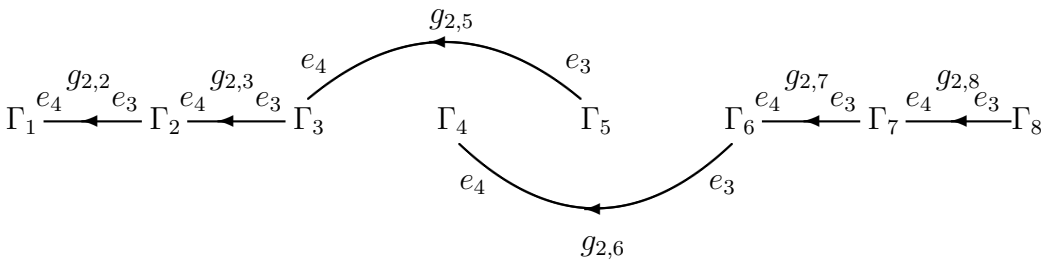
7.14 Example. We compute generators of $\mathcal{AM}_{1,1,1}$ of level less than 3. By Lemma 7.10 and Example 6.17, we have to lift generators of $\mathcal{AM}_{1,0,1}$ and $\mathcal{AM}_{0,2,1}$.

By Example 3.8(b), $\mathcal{AM}_{1,0,1} = \langle x_1, x_3 \mid x_1 x_3^{-1} x_1^{-1} x_3^{-1} x_1 \rangle$, where x_1 corresponds to collapse e_1, and, x_3 corresponds to collapse e_3. Recall $\widehat{\mathbb{P}}_{1,0} = \mathrm{cl}(\sigma(\Gamma', v_0', \mathrm{ord}', \phi'))$, where $(\Gamma', v_0', \mathrm{ord}') = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, \overline{e}_4)$. By Example 6.4 and Definition 7.1, e_3 is a left h-edge of level 2, and, e_1 is a left g-edge of level 2. Then, we denote by h_2 the generator x_3 of $\mathcal{AM}_{1,0,1}$, and, by g_2 the generator x_1 of $\mathcal{AM}_{1,0,1}$. Hence, $\mathcal{AM}_{1,0,1} = \langle h_2,\ g_2 \mid g_2 h_2^{-1} g_2^{-1} h_2^{-1} g_2 \rangle$.

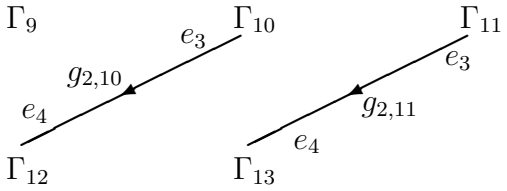
To lift $h_2, g_2 \in \mathcal{AM}_{1,0,1}$, we have to complete Example 7.5. If we complete Example 7.5, we will see e_6 is the left h-edge of level 2 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, there is no left h-edge of level 2 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$, $\overline{e}_7$ is the left h-edge of level 2 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$, e_6 is the left h-edge of level 2 in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4)$, $\overline{e}_6$ is the left h-edge of level 2 in $(\Gamma_5, v_{5,0}, \mathrm{ord}_5, \phi_5)$, there is no left h-edge of level 2 in $(\Gamma_6, v_{6,0}, \mathrm{ord}_6, \phi_6)$, $\overline{e}_7$ is the left h-edge of level 2 in $(\Gamma_7, v_{7,0}, \mathrm{ord}_7, \phi_7)$, and, $\overline{e}_6$ is the left h-edge of level 2 in $(\Gamma_8, v_{8,0}, \mathrm{ord}_8, \phi_8)$. By Lemma 7.10, $\mathcal{AM}_{1,1,1}$ has six h-generators of level 2 corresponding to collapse these left h-edges of level 2. To identify these generators, we can collapse left and right h-edges of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 1, \ldots, 8$. We denote by $h_{2,1}$ the generator corresponding to collapse e_6 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, $h_{2,3}$ the generator corresponding to collapse $\overline{e}_7$ in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$, $h_{2,4}$ the generator corresponding to collapse e_6 in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4)$, $h_{2,5}$ the generator corresponding to collapse $\overline{e}_6$ in $(\Gamma_5, v_{5,0}, \mathrm{ord}_5, \phi_5)$, $h_{2,7}$ the generator corresponding to collapse $\overline{e}_7$ in $(\Gamma_7, v_{7,0}, \mathrm{ord}_7, \phi_7)$, and, $h_{2,8}$ the generator corresponding to collapse $\overline{e}_6$ in $(\Gamma_8, v_{8,0}, \mathrm{ord}_8, \phi_8)$. We represent these generators:

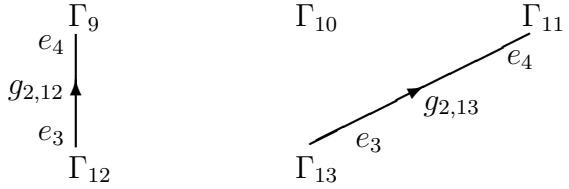


On the other hand, there are neither left g-edge of level 2 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ nor in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4)$, and, e_3 is the left g-edge of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 2, 3, 5, 6, 7, 8$. By Lemma 7.10, $\mathcal{AM}_{1,1,1}$ has six g-generators of level 2 corresponding to collapse these left g-edges of level 2. To identify this generators, we can collapse left and right g-edges of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 1, \ldots, 8$. We denote by $g_{2,i}$ the generator corresponding to collapse left g-edges of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 2, 3, 5, 6, 7, 8$. We represent these generators:



As in the previous example, $\mathcal{AM}_{0,2,1} = \langle g_2 \mid \rangle$ where g_2 corresponds to collapse e_1 in $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_4, \overline{e}_3)$. To lift $g_2 \in \mathcal{AM}_{0,2,1}$, we follow Example 7.5. Notice there is no left g-egde of level 2 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ and e_3 is the left g-edge of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 10, \ldots, 13$. By Lemma 7.10, $\mathcal{AM}_{1,1,1}$ has four g-generators of level 2 corresponding to collapse e_3 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 10, \ldots, 13$. To identify these generators, we can collapse left and right g-edges of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 9, \ldots, 13$. We denote by $g_{2,i}$ the generator corresponding to collapse e_3 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 10, \ldots, 13$. We represent these generators:





By Example 6.17, the first component of $\widehat{\mathbb{P}}_{0,2}$ contains $\Gamma_9, \Gamma_{10}, \Gamma_{11}$ and the second component contains Γ_{12}, Γ_{13}. These two components are connected through codimension one faces obtained by collapsing g-edges of level 2. □

7.2 Relations

We give a description of relations of $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}$ where $\mathfrak{p}$ is a point in $\mathrm{ord}\mathbb{A}_{g,p} - \mathbb{T}_{g,p}$. These are relations of $\mathcal{AM}_{g,p,1}$ which do not involve g-generators of level n.

Notice h-generators of level n are described in Lemma 6.10. The following lemma describes relations involving generators of level $k < n$ and h-generators

of level n.

7.15 Lemma. *Let $\mathfrak{p} = (\Gamma, v_0, ord, \phi) \in \widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i)$ be a point in $\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}}$ such that $\sigma(\mathfrak{p}_i)$ is a simplex of dimension $3n - 3 = 6g + 3p - 3$. For $i = 1, 2$, let f_i be a left g-edge (resp. h-edge) of Γ_i of level $k < n$ such that there exists a g-generator (resp. an h-generator) corresponding to collapse f_i in $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i)$.*

If there exists a path γ from $\mathfrak{p}_1$ to $\mathfrak{p}_2$ such that γ lies in simplices of codimension at most one where every codimension one simplex intersected by γ is obtained by collapsing an h-edge of level n, then the generator corresponding to collapse f_1 in $\mathfrak{p}_1 = (\Gamma_1, v_{1,0}, ord_1, \phi_1)$ is equal to the generator corresponding to collapse f_2 in $\mathfrak{p}_2 = (\Gamma_2, v_{2,0}, ord_2, \phi_2)$, up to h-generators of level n.

Proof. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a (h, q)-fat graph with a distinguished vertex and let $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ be an isomorphism such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$.

By Proposition 6.7, there exists a continuous map

$$\mu : \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}} \to \widehat{\mathrm{ord}\mathbb{A}}_{h,q},\ \mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}.$$

Since every codimension one simplex intersected by γ is obtained by collapsing an h-edge of level n, we can suppose μ is constant on γ. Then

$$\mu(\mathfrak{p}_1) = \mu(\mathfrak{p}_2)$$

and γ is homotopic to a path γ^f from $\mathfrak{p}^{f_1} = (\Gamma_1^{f_1}, v_{1,0}^{f_1}, \mathrm{ord}_1^{f_1}, \phi_1^{f_1})$ to $\mathfrak{p}^{f_2} = (\Gamma_2^{f_2}, v_{2,0}^{f_2}, \mathrm{ord}_2^{f_2}, \phi_2^{f_2})$ such that μ is constant on γ^f. Then γ^f lies in simplices of codimension 1 or 2. The graph of each codimension 1 simplex which contains a segment of γ^f has a vertex of valence 4. This vertex can be split in two manners. One manner gives a $(3n-3)$-dimensional simplex which contains a segment of γ. Since there exists generators corresponding to collapse f_i in $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, $i = 1, 2$, there exists simplices $\sigma(\mathfrak{p}_i') = \sigma(\Gamma_i', v_{i,0}', \mathrm{ord}_i', \phi_i')$, $i = 1, 2$, such that $\sigma(\Gamma_i^{f_i}, v_{i,0}^{f_i}, \mathrm{ord}_i^{f_i}, \phi_i^{f_i})$ is a common face of both $\sigma(\mathfrak{p}_i)$ and $\sigma(\mathfrak{p}'_i)$, $i = 1, 2$. From $\mu(\mathfrak{p}_1^{f_1}) = \mu(\mathfrak{p}_2^{f_2})$ we see $\mu(\mathfrak{p}'_1) = \mu(\mathfrak{p}'_2)$. By Lemma 7.10, the other manner to split the valence 4 vertex gives a $(3n-3)$-dimensional simplex of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$ which maps to $\sigma(\mu(\mathfrak{p}'_1))$ under μ. Then γ^f is homotopic to a path γ' from $\mathfrak{p}'_1$ to $\mathfrak{p}'_2$ lying in simplices of codimension at most one where every codimension one simplex is obtained by collapsing an h-edge of level n. For $i = 1, 2$, let β_i be a path from $\mathfrak{p}_i$ to $\mathfrak{p}'_i$ through the codimension one face $\sigma(\mathfrak{p}_i^{f_i})$. Hence, there exists a disc in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{(\Gamma, v_0, \mathrm{ord}, \phi)}$ bounded by β_1, γ', $\overline{\beta}_2$ and $\overline{\gamma}$. □

7.16 Remark. Let $\mathfrak{p}_1 = (\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex.

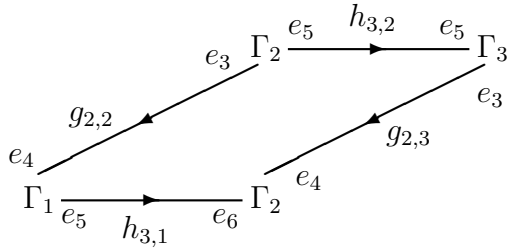
Generators of $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1}$ are h-generators of level at most n and g-generators of level at most $n-1$.

By Remark 6.14, all h-edges of level n of $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1}$ are declared to be the identity. See the first set of relations in Theorem 3.7 and Lemma 6.10.

By Lemma 7.10 and Definition 7.11, every generator of level at most $n-1$ of $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1}$ is a lift of a generator of $\mathcal{AM}_{g,p-1,1}$.

By Remark 6.14, all lifts of a generators of $\mathcal{AM}_{g,p-1,1}$ are connected through codimension one simplices corresponding to collapse h-edges of level n. Then, by Lemma 7.15, the set of generators of $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1}$ is reduced to the set of generators of $\mathcal{AM}_{g,p-1,1}$. □

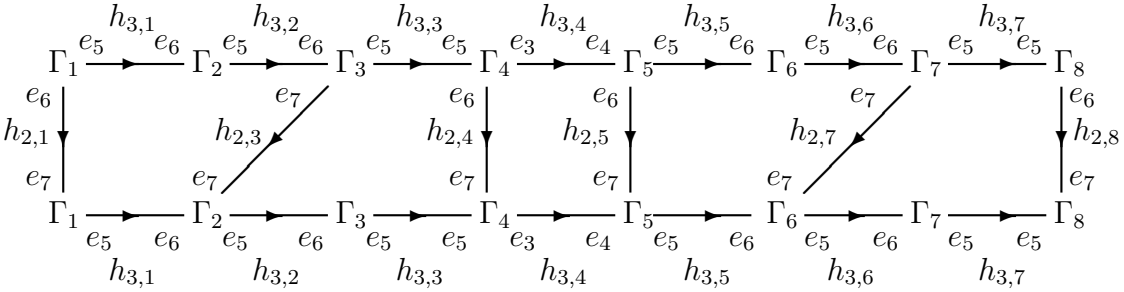
7.17 Example. By Example 7.12, $\mathcal{AM}_{0,3,1}$ has two lift of $g_2 \in \mathcal{AM}_{0,2,1}$ denoted $g_{2,2}$ and $g_{2,3}$. Recall $g_{2,2}$ and $g_{2,3}$ correspond to collapse the left g-edge of level 2 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ and $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$, respectively. By Example 6.15, $\sigma(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ and $\sigma(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$ are connected through a codimension one face obtained by collapsing an h-edge of level 3. Then, by Lemma 7.15, $g_{2,2}$ and $g_{2,3}$ are equal, up to h-generators of level 3. We apply Theorem 3.7 to $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$, the pair of edges $\{e_3, e_5\}$ and collapsing e_3. We obtain the relation $g_{2,2}\, h_{3,1}\, g_{2,3}^{-1}\, h_{3,2}^{-1}$. We represent this relation:



By Remark 6.14, in $\mathcal{AM}_{0,3,1}$, we have $h_{3,1} = h_{3,2} = 1$. Then, relation $g_{2,2}\, h_{3,1}\, g_{2,3}^{-1}\, h_{3,2}^{-1}$ reduces to $g_{2,2} = g_{2,3}$. Hence, $\mathcal{AM}_{0,3,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$ is generated by $g_{2,2}$. Since $\mathcal{AM}_{0,2,1}$ is generated by g_2, this agrees with Remark 7.16. □

7.18 Example. By Example 7.14, $\mathcal{AM}_{1,1,1}$ has six lifts of h_2 in $\mathcal{AM}_{1,0,1}$ denoted $h_{2,i}$ for $i = 1, 3, 4, 5, 7, 8$. Recall $h_{2,i}$ corresponds to collapse the left h-edge of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 1, 3, 4, 5, 7, 8$.

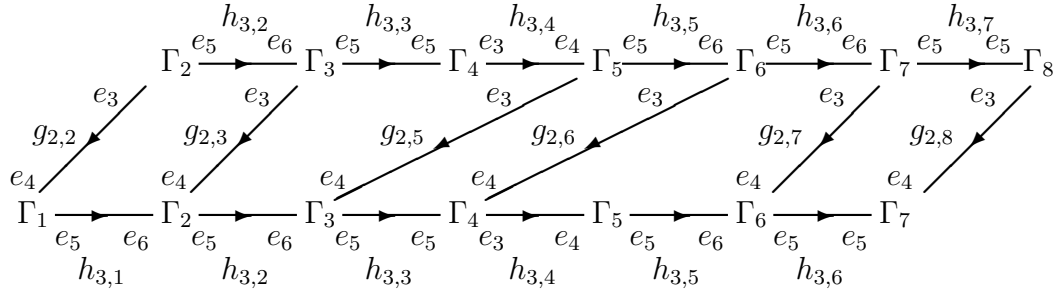
By Example 6.17, for $i = 1, \ldots, 7$, $\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$ and $\sigma(\Gamma_{i+1}, v_{i+1,0}, \mathrm{ord}_{i+1}, \phi_{i+1})$ are connected through a codimension one face obtained by collapsing an h-edge of level 3. We represent the relations involving h-generators of level 2 and 3 that we obtain by Theorem 3.7.



By Remark 6.16, in $\mathcal{AM}_{1,1,1}$, we have $h_{3,i} = 1$, $i = 1, \ldots, 7$. Then, the obtained relations reduce to $h_{2,1} = h_{2,3} = h_{2,4} = h_{2,5} = h_{2,7} = h_{2,8}$.

By Example 7.14, $\mathcal{AM}_{1,1,1}$ has six lifts of g_2 in $\mathcal{AM}_{1,0,1}$ denoted $g_{2,i}$ for $i = 2, 3, 5, 6, 7, 8$. Recall these generators correspond to collapse the left g-edge of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 2, 3, 5, 6, 7, 8$.

By Example 6.17, for $i = 1, \ldots, 7$, $\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$ and $\sigma(\Gamma_{i+1}, v_{i+1,0}, \mathrm{ord}_{i+1}, \phi_{i+1})$ are connected through a codimension one face obtained by collapsing an h-edge of level 3. We represent the relations involving h-generators of level 3 and g-generators of level 2 that we obtain by Theorem 3.7.



By Remark 6.16, in $\mathcal{AM}_{1,1,1}$, we have $h_{3,i} = 1$, $i = 1, \ldots, 8$. Then, the obtained relations reduce to $g_{2,2} = g_{2,3} = g_{2,5} = g_{2,6} = g_{2,7} = g_{2,8}$.

Hence, $\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$ is generated by $h_{2,1}$ and $g_{2,2}$. Since $\mathcal{AM}_{0,2,1}$ is generated by h_2 and g_2, this agrees with Remark 7.16. □

7.19 Remark. Let $\mathfrak{p}_2 = (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ be a point in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex.

Generators of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$ are h-generators of level at most n and g-generators of level at most $n-1$. By Remark 6.16, for each $(3(n-1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g-1,p+1}$, there are $(p+1)$ components connected through codimension one faces obtained by collapsing h-edges of level n.

For each of these components, by Remark 6.16, all h-generators of level n of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$ are declared to be the identity, but one. See the first set of relations in Theorem 3.7 and Lemma 6.10.

By Lemma 7.10 and Definition 7.11, every generator of level at most $n-1$ of $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1}$ is a lift of a generator of $\mathcal{AM}_{g-1,p+1,1}$.

Hence, by Lemma 7.15, the set of generators of $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1}$ is reduced to a set of $p+1$ h-generators of level n for each $(3(n-1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g-1,p+1}$ (one generator for every connected component), and, $(p+1)$ lifts of each generator of $\mathcal{AM}_{g-1,p+1,1}$ (one lift for every connected component).

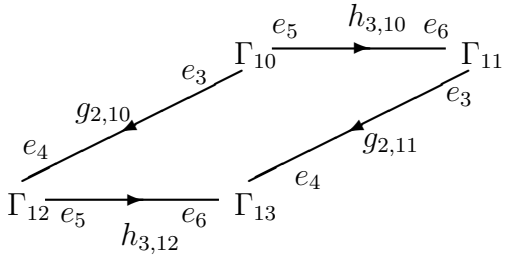
By Remark 7.13, there are p g-generators of level $k<n$ which are declareted to be the identity. Recall that the corresponding edge of these g-generators must connect two connected components. □

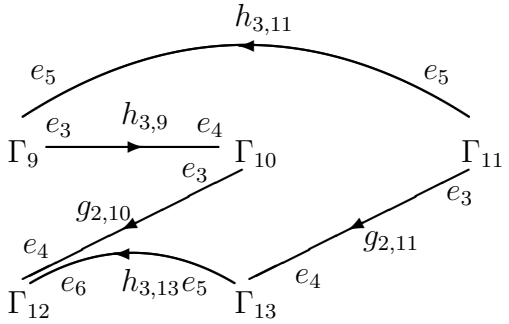
7.20 Example. By Example 6.17, $(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)$, $(\Gamma_{10}, v_{10,0}, \mathrm{ord}_{10}, \phi_{10})$ and $(\Gamma_{11}, v_{11,0}, \mathrm{ord}_{11}, \phi_{11})$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,1}$ are connected by two paths contained in simplices of codimension at most one where every codimension one simplex is obtained by collapsing an h-edge of level 3. Similarly, $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12}, \phi_{12})$ and $(\Gamma_{13}, v_{13,0}, \mathrm{ord}_{13}, \phi_{13})$ are connected by two paths contained in simplices of codimension at most one where every codimension one simplex is obtained by collapsing an h-edge of level 3.

By Remark 6.16, in $\mathcal{AM}_{1,1,1}$, we have $h_{3,10} = h_{3,11} = 1$ and $h_{3,13} = 1$. Hence, $h_{3,9}$ and $h_{3,12}$ are the h-generators of level 3 which are not declared to the identity for the components containing $(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)$ and $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12}, \phi_{12})$, respectively.

By Example 7.14, $\mathcal{AM}_{1,1,1}$ has four lifts of g_2 in $\mathcal{AM}_{0,2,1}$ denoted $g_{2,i}$ for $i = 10, \ldots, 13$. Recall $g_{2,i}$ corresponds to collapse the left g-edge of level 2 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$, for $i = 10, \ldots, 13$. Notice $g_{2,10}$ and $g_{2,11}$ are the g-generators of level 2 for the component containing $(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)$; $g_{2,12}$ and $g_{2,13}$ are the g-generators of level 2 for the component containing $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12}, \phi_{12})$.

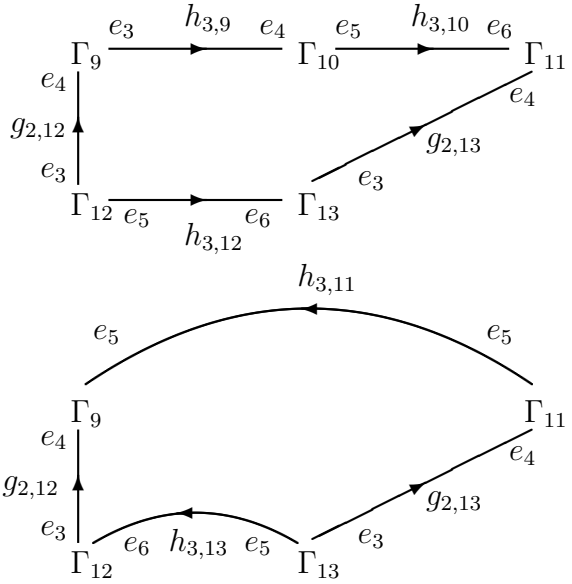
By Theorem 3.7 we obtain relations involving $g_{2,10}$ and $g_{2,11}$: $g_{2,10}\, h_{3,12}\, g_{2,11}^{-1}\, h_{3,10}^{-1}$ and $g_{2,10}\, h_{3,13}^{-1}\, g_{2,11}^{-1}\, h_{3,11}\, h_{3,9}$. We represent these relations.





Since, in $\mathcal{AM}_{1,1,1}$, we have $h_{3,10} = h_{3,11} = 1$ and $h_{3,13} = 1$, these relations reduce to $g_{2,10}h_{3,12} = g_{2,11}$ and $h_{3,9}g_{2,10} = g_{2,11}$.

By Theorem 3.7 we obtain relations involving $g_{2,12}$ and $g_{2,13}$: $g_{2,12}^{-1}\, h_{3,12}\, g_{2,13}\, h_{3,10}^{-1}\, h_{3,9}^{-1}$ and $g_{2,12}^{-1}\, h_{3,13}^{-1}\, g_{2,13}\, h_{3,11}$. We represent these relations.



Since, $h_{3,10} = h_{3,11} = 1$, $h_{3,13} = 1$ hold in $\mathcal{AM}_{1,1,1}$, these relations reduce to $h_{3,12}g_{2,13} = g_{2,12}h_{3,9}$ and $g_{2,13} = g_{2,12}$.

Hence, $\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$ is generated by two h-generators of leve 3: $h_{3,9}$, $h_{3,12}$; and, two g-generators of level 2: $g_{2,10}$, $g_{2,12}$. Notice g-generators of level 2 connect the two connected components. And one of the g-generators of level 2 needs to declareted the identity to connect the two connected components. Since $\mathcal{AM}_{0,2,1}$ is generated by a g-generator of level 2, this agrees with Remark 7.19. If we declare $g_{2,12} = 1$, then $h_{3,9} = h_{3,12}$ and $g_{2,10}h_{3,9} = h_{3,9}g_{2,10}$. □

7.21 Lemma. *Let* $\mathfrak{p} = (\Gamma, v_0, ord, \phi) \in \widehat{ord\mathbb{A}}_{g,p}$ *such that* $\sigma(\mathfrak{p})$ *is a simplex of dimension* $3n - 3 = 6g + 3p - 3$. *Let* $f_i \in E(\Gamma)$, $i = 1, 2$, *such that there*

exists a codimention two face $\sigma(\mathfrak{p}^{f_1,f_2}) = \sigma(\Gamma^{f_1,f_2}, v_0^{f_1,f_2}, ord^{f_1,f_2}, \phi^{f_1,f_2})$ *in* $\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ *obtained by collapsing* f_1 *and* f_2. *Let* h, q *be integers such that* $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ *is a* (h,q)*-fat graph with a distinguished vertex, and, let* $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ *be an isomorphism such that* $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$ *is a point in* $\widehat{ord\mathbb{A}}_{h,q}$. *Let*

$$\mu : \widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p} \to \widehat{ord\mathbb{A}}_{h,q},\ \mathfrak{p} \mapsto \mathfrak{p}^{\downarrow}.$$

be the map described in Proposition 6.7.

(i) If either f_1 *or* f_2 *is an* h*-edges of level* n*, then* $\mu(\sigma(\mathfrak{p}^{f_1,f_2}))$ *is a codimention one face of* $\widehat{ord\mathbb{A}}_{h,q}$.

(ii) If neither f_1 *nor* f_2 *is an* h*-edge of level* n*, then* $\mu(\sigma(\mathfrak{p}^{f_1,f_2}))$ *is a codimention two face of* $\widehat{ord\mathbb{A}}_{h,q}$.

Proof. Notice neither f_1 nor f_2 is a g-edge of level n.

(i) Suppose f_1 is an h-edge of level n.

If f_2 is an h-edge of level n, then there exists an edge $f^{\downarrow}$ in $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ identified under $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with the concatenation f_1 and $\overline{f}_2$ in $(\Gamma, v_0, \mathrm{ord})$. Hence, μ sends the codimension two face σ^{f_1,f_2} to the codimension one face of $\sigma(\mathfrak{p}^{\downarrow}) = \sigma(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ obtained by collapsing $f^{\downarrow}$.

If f_2 is not an h-edge of level n, then there exists an edge $f_2^{\downarrow}$ in $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ identified under $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with f_2 in $(\Gamma, v_0, \mathrm{ord})$. Hence, μ sends the codimension two face σ^{f_1,f_2} to the codimension one face of $\sigma(\mathfrak{p}^{\downarrow}) = \sigma(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ obtained by collapsing $f_2^{\downarrow}$.

(ii) If neither f_1 nor f_2 is an h-edge of level n, then there exist edges $f_1^{\downarrow}, f_2^{\downarrow}$ in $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ identified under $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$, with f_1, f_2 in $(\Gamma, v_0, \mathrm{ord})$, respectively. Hence, μ sends the codimension two face σ^{f_1,f_2} to the codimension two face of $\sigma(\mathfrak{p}^{\downarrow}) = \sigma(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ obtained by collapsing $f_1^{\downarrow}$ and $f_2^{\downarrow}$. □

7.22 Corollary. *Let* $\mathfrak{p} = (\Gamma, v_0, ord, \phi) \in \widehat{ord\mathbb{A}}_{g,p}$ *such that* $\sigma(\mathfrak{p})$ *is a simplex of dimension* $3n - 3 = 6g + 3p - 3$. *Let* h, q *be integers such that* $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ *is a* (h,q)*-fat graph with a distinguished vertex, and, let* $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ *be an isomorphism such that* $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$ *is a point in* $\widehat{ord\mathbb{A}}_{h,q}$.

For $i = 1, 2$, *let* $f_i \in E(\Gamma)$ *of level at most* $n-1$, *and, let* $f_i^{\downarrow}$ *be identified with* f_i *under* $(\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}) \hookrightarrow (\Gamma, v_0, ord)$, $v_0^{\downarrow} \mapsto v_1$.

(i) There exists the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2}) = \sigma(\Gamma^{f_1,f_2}, v_0^{f_1,f_2}, ord^{f_1,f_2}, \phi^{f_1,f_2})$ *in* $\widehat{ord\mathbb{A}}^{\mathfrak{p}}_{g,p}$ *obtained by collapsing* f_1 *and* f_2 *if, and only if, there exists the codimension two face in* $\widehat{ord\mathbb{A}}_{h,q}$ *obtained by collapsing* $f_1^{\downarrow}$ *and* $f_2^{\downarrow}$ *in* $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$.

(ii) There exists a relation corresponding to the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2}) = \sigma(\Gamma^{f_1,f_2}, v_0^{f_1,f_2}, ord^{f_1,f_2}, \phi^{f_1,f_2})$ *in* $\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}}$ *obtained by collapsing* f_1 *and* f_2 *if, and only if, there exists a relation corresponding to the codimension two face in* $\widehat{ord\mathbb{A}}_{h,q}$ *obtained by collapsing* $f_1^{\downarrow}$ *and* $f_2^{\downarrow}$ *in* $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow}, \phi^{\downarrow})$.

Proof. (i) If there exists the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2})$ of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$, by Lemma 7.21(ii), there exists the codimension two face corresponding to collapse $f_1^{\downarrow}$ and $f_2^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$.

If there exists the codimension two face corresponding to collapse $f_1^{\downarrow}$ and $f_2^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$, then $f_1^{\downarrow}$ and $f_2^{\downarrow}$ can be collapsed in $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$. Hence, f_1 and f_2 can be collapsed in $(\Gamma, v_0, \mathrm{ord})$ and there exists the codimension two face σ^{f_1,f_2} of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$.

(ii) Follows from (i) above and Lemma 7.10. □

7.23 Definition. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a simplex of dimension $3n-3 = 6g+3p-3$. Let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a (h,q)-fat graph with a distinguished vertex, and, let $\phi^{\downarrow} : \pi_1(\Gamma^{\downarrow}, v_0^{\downarrow}) \to F_{h,q,1}$ be an isomorphism such that $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$.

For $i = 1, 2$, let $f_i \in E(\Gamma)$, such that there exists a codimention two face $\sigma(\mathfrak{p}^{f_1,f_2}) = \sigma(\Gamma^{f_1,f_2}, v_0^{f_1,f_2}, \mathrm{ord}^{f_1,f_2}, \phi^{f_1,f_2})$ in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$ obtained by collapsing f_1 and f_2. Suppose f_i, $i = 1, 2$, is an edge of level at most $n-1$, and, let $f_i^{\downarrow}$ be identified with f_i under $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$.

We say that the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2})$ of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$ is a *lift* of the codimension two face in $\widehat{\mathrm{ord}\mathbb{A}}_{h,q}$ obtained by collapsing $f_1^{\downarrow}$ and $f_2^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$.

If there exists a relation r in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ deduced from the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2})$, then we say that the relation r of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ is a *lift* of the relation of $\mathcal{AM}_{h,q,1}^{\mathfrak{p}}$ deduced from the codimension two face obtained by collapsing $f_1^{\downarrow}$ and $f_2^{\downarrow}$ in $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}, \phi^{\downarrow})$. □

7.24 Corollary. *Let* $\mathfrak{p} = (\Gamma, v_0, ord, \phi) \in \widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$.

(i) Every relation in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ *obtained by Theorem 3.7 which involves* h*-generators of level* n *is deduced from Lemma 7.15.*

(ii) Every relation in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$ *obtained by Theorem 3.7 which does not involve* h*-generators of level* n *is a lift of a relation in* $\mathcal{AM}_{h,q,1}$*, where* $\mathfrak{p}^{\downarrow} = (\Gamma^{\downarrow}, v_0^{\downarrow}, ord^{\downarrow})$ *is a* (h,q)*-fat graph with a distinguished vertex.*

Proof. (i) Follows from Lemma 7.21(i).

(ii) Follows from Lemma 7.21(ii) and Definition 7.23. □

7.25 Remark. Let $\mathfrak{p}_1 = (\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $\mathfrak{p}_1^{\downarrow} = (\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex

By Proposition 6.13, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1} \simeq \mathcal{AM}_{g,p-1,1}$. Results on this section give an explicit description of this isomorphism as follows. By Remark 7.16, there is a bijection between generators of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1}$ and generators of $\mathcal{AM}_{g,p-1,1}$. By Corollary 7.24, relations not involving h-generators of level n in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1}$ are lifts of relations in $\mathcal{AM}_{g,p-1,1}$. □

7.26 Example. Recall $\mathcal{AM}_{0,2,1} = \langle g_2 \mid \rangle$. Following Example 7.17, by Corollary 7.24, $\mathcal{AM}_{0,3,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \langle g_{2,2} \mid \rangle \simeq \mathcal{AM}_{0,2,1}$. □

7.27 Example. Recall $\mathcal{AM}_{1,0,1} = \langle h_2,\, g_2 \mid g_2 h_2^{-1} g_2^{-1} h_2^{-1} g_2 \rangle$. The relation in $\mathcal{AM}_{1,0,1}$ lifts to the following relations in $\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$: $g_{2,2} h_{2,1}^{-1} g_{2,2}^{-1} h_{2,3}^{-1} g_{2,3}$; $g_{2,3} h_{2,3}^{-1} g_{2,5}^{-1} h_{2,5}^{-1} g_{2,5}$; $g_{2,6} h_{2,4}^{-1} g_{2,6}^{-1} h_{2,7}^{-1} g_{2,7}$ and $g_{2,7} h_{2,7}^{-1} g_{2,8}^{-1} h_{2,8}^{-1} g_{2,8}$. Following Example 7.18, these relations reduce to the same relation. By Corollary 7.24, $\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \langle h_{2,1},\, g_{2,2} \mid g_{2,2} h_{2,1}^{-1} g_{2,2}^{-1} h_{2,1}^{-1} g_{2,2} \rangle \simeq \mathcal{AM}_{1,0,1}$. □

7.28 Remark. Let $\mathfrak{p}_2 = (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $\mathfrak{p}_2^{\downarrow} = (\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex.

By Proposition 6.13, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2} \simeq \mathcal{AM}_{g-1,p,2}$. Results on this section give an explicit description of this isomorphism as follows. By Remark 7.16, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$ is generated by $(p+1)$ copies of the generators of $\mathcal{AM}_{g,p-1,1}$ and $(p+1)$ h-generators of level n for each $(3(n-1)-3)$-dimensional simplex of $\widehat{\mathbb{P}}_{g-1,p+1}$. On the other hand, relations not involving h-generators of level n in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$ are lifts of relations in $\mathcal{AM}_{g-1,p+1,1}$. Recall $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2}$ has $p+1$ connected components which map onto $\widehat{\mathbb{P}}_{g-1,p+1}$ under μ. Each of these connected components correspond to a cyclic word in $w(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$. See 6.16 and Remark 7.13. There are p g-generators of level $k < n$ which are declareted to be the identity. □

7.29 Example. Recall $\mathcal{AM}_{0,2,1} = \langle g_2 \mid \rangle$. By Example 7.20 and by Corollary 7.24, $\mathcal{AM}_{1,1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)} = \langle h_{3,9}, g_{2,10} \mid h_{3,9} g_{2,10} = g_{2,10} h_{3,9} \rangle$. □

8 g-generators of level n

We describe a refinement of the decomposition of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ which gives g-generators of level n and some relations involving g-generators of level n. We deduce $\mathcal{AM}_{g,p,1}$ is a quotient of the fundamental group of a graph of groups. We define a second refinement of the decomposition of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ which gives the missing relations.

8.1 Definition. For $i = 1, 2$, let $\widehat{\mathbb{T}}_{g,p}^{(i)}$ be the set of points $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$ such that $|u^*| \geq 4$ where $\overline{e}_i \in u^*$, $u \in V(\Gamma)$. □

8.2 Remark. For $i = 1, 2$, if $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(i)}_{g,p})$, then $|v_0^*| = 2$ and $|v_i^*| = 3$. □

8.3 Definition. For $i = 1, 2$, let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(i)}_{g,p})$. We denote by $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(i)}_{g,p}$ the connected component of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(i)}_{g,p})$ which contains $\mathfrak{p}$.

We define

$$\mathcal{AM}^{\mathfrak{p},(i)}_{g,p,1} = \{\varphi \in \mathcal{AM}_{g,p,1} \mid \varphi \cdot \mathfrak{p} \in \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(i)}_{g,p}\}.$$

□

8.4 Remark. For $i = 1, 2$, let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(i)}_{g,p})$. Notice $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(i)}_{g,p}$ can be defined as the connected component of $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} - \widehat{\mathbb{T}}^{(i)}_{g,p}$ which contains $\mathfrak{p}$.

Since $\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(i)}_{g,p} \subset \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$, we see $\mathcal{AM}^{\mathfrak{p},(i)}_{g,p,1}$ is a subgroup of $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}$. □

8.5 Remark. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(1)}_{g,p})$, and, let h, q be integers such that $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow})$ is a (h, q)-fat graph with a distinguished vertex. Since $\mathfrak{p} \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(1)}_{g,p})$, we see $|v_0^{\downarrow}| = 2$. Hence, we can consider the (k, r)-fat graph with a distinguished vertex $(\Gamma^{\downarrow\downarrow}, v_0^{\downarrow\downarrow}, \mathrm{ord}^{\downarrow\downarrow})$ where k, r are integers such that either $(k, r) = (h, q-1)$ or $(k, r) = (h-1, q+1)$.

We consider the composition of embeddings

$$\begin{array}{ccccc} (\Gamma^{\downarrow\downarrow}, v_0^{\downarrow\downarrow}, \mathrm{ord}^{\downarrow\downarrow}) & \hookrightarrow & (\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) & \hookrightarrow & (\Gamma, v_0, \mathrm{ord}) \\ v_0^{\downarrow\downarrow} & \mapsto & v_1^{\downarrow} & & \\ & & v_0^{\downarrow} & \mapsto & v_1 \end{array}$$

Notice $e_1, e_2, e_3, e_4 \in E(\Gamma)$ do not lie in the image of this composition.

We can obtain results for the composition above analogous to the results obtained for the embedding $(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) \hookrightarrow (\Gamma, v_0, \mathrm{ord})$, $v_0^{\downarrow} \mapsto v_1$.

Let $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], \ldots, [w_p]\}$. Since

$$(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \ldots), \ldots$$

w_0 starts with $e_1 e_3$ and ends with $\overline{e}_2$, and, $\overline{e}_4 \overline{e}_1 e_2$ and $\overline{e}_3 e_4$ appear in $w(\Gamma, v_0, \mathrm{ord})$. Lemma 4.8 can be extended as follows.

(i) If $w_0 = e_1 e_3 w_0' \overline{e}_2, [w_i] = [\overline{e}_4 \overline{e}_1 e_2 w_i']$ for some $1 \leq i \leq p$, then $(h, q) = (g, p-1)$ and exactly one of the cases holds.

  (a) If $w_0' = u_0' \overline{e}_3 e_4 v_0'$, then $(k, r) = (h-1, q+1) = (g-1, p)$. Hence, g and p satisfy $g \geq 1$ and $p \geq 1$.

  (b) If $w_i' = u_i' \overline{e}_3 e_4 v_i'$, then $(k, r) = (h-1, q+1) = (g-1, p)$. Hence, g and p satisfy $g \geq 1$ and $p \geq 1$.

(c) If $[w_j] = [\overline{e}_3 e_4 v_j]$ for some $1 \leq j \leq p, j \neq i$, then $(k,r) = (h, q-1) = (g, p-2)$. Hence, p satisfies $p \geq 2$.

Notice $w(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) = \{w_0^{\downarrow}, [w_1^{\downarrow}], \ldots, [w_{i-1}^{\downarrow}], [w_{i+1}^{\downarrow}], \ldots, [w_p^{\downarrow}]\}$ where $w_0^{\downarrow} = e_3 w_0' w_i' \overline{e}_4$ and, $[w_j^{\downarrow}] = [w_j]$ for $1 \leq j \leq p$, $j \neq i$. The three cases above are deduced by considering where $\overline{e}_3 e_4$ appears.

(ii) If $w_0 = e_1 e_3 w_0' \overline{e}_4 \overline{e}_1 e_2 w_0'' \overline{e}_2$, then $(h,q) = (g-1, p+1)$ and exactly one of the cases holds.

(d) If $w_0' = u_0' \overline{e}_3 e_4 v_0'$, then $(k,r) = (h-1, q+1) = (g-2, p+2)$. Hence, g satisfies $g \geq 2$.

(e) If $w_0'' = u_0'' \overline{e}_3 e_4 v_0''$, then $(k,r) = (h, q-1) = (g-1, p)$. Hence, g satisfies $g \geq 1$.

(f) If $[w_i] = [\overline{e}_3 e_4 u_i]$ for some $1 \leq i \leq p$, then $(k,r) = (h, q-1) = (g-1, p)$. Hence, g and p satisfy $g \geq 1$ and $p \geq 1$.

Notice $w(\Gamma^{\downarrow}, v_0^{\downarrow}, \mathrm{ord}^{\downarrow}) = \{w_0^{\downarrow}, [w_1^{\downarrow}], \ldots, [w_p^{\downarrow}], [w_{p+1}^{\downarrow}]\}$ where $w_0^{\downarrow} = e_3 w_0' \overline{e}_4$, $[w_{p+1}^{\downarrow}] = [w_0'']$ and $[w_j^{\downarrow}] = [w_j]$ for $1 \leq j \leq p$. The three cases above are deduced by considering where $\overline{e}_3 e_4$ appears.

Lemma 6.6 can be extended as follows. Let $\phi^{\downarrow\downarrow} : \pi_1(\Gamma^{\downarrow\downarrow}, v_0^{\downarrow\downarrow}) \to F_{k,r,1}$ be an isomorphism such that $\mathfrak{p}^{\downarrow\downarrow} = (\Gamma^{\downarrow\downarrow}, v_0^{\downarrow\downarrow}, \mathrm{ord}^{\downarrow\downarrow}, \phi^{\downarrow\downarrow})$ is a point in $\widehat{\mathrm{ord}\mathbb{A}}_{k,r}$. If $\sigma(\mathfrak{p})$ is a $(6g+3p-3)$-dimensional simplex, then a path in $\widehat{\mathrm{ord}\mathbb{A}}_{k,r}$ lying in simplices of codimentions at most one and starting at $\mathfrak{p}^{\downarrow\downarrow}$ can be lifted to a path in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1)}$ lying in simplices of codimension at most one and starting at $\mathfrak{p}$.

Proposition 6.7 can be extended as follows. We can define a continuous map

$$\mu^{(1)} : \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1)} \to \widehat{\mathrm{ord}\mathbb{A}}_{k,r},\ \mathfrak{p} \mapsto \mathfrak{p}^{\downarrow\downarrow}.$$

We see $\widehat{\mathrm{ord}\mathbb{A}}_{(g,p)}^{\mathfrak{p},(1)}$ is simply connected. See Corollary 6.9.

Since $\widehat{\mathrm{ord}\mathbb{A}}_{(g,p)}^{\mathfrak{p},(1)}$ has a complex structure, a presentation for $\mathcal{AM}_{g,p,1}^{\mathfrak{p},(1)}$ can be obtained by Theorem 3.7 applied to the action of $\mathcal{AM}_{g,p,1}^{\mathfrak{p},(1)}$ on $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1)}$.

Proposition 6.11 can be extended as follows. If $(h,q) = (g, p-1)$, let $\mathfrak{p}_a = (\Gamma_a, v_{a,0}, \mathrm{ord}_a, \phi_a)$, $\mathfrak{p}_b = (\Gamma_b, v_{b,0}, \mathrm{ord}_b, \phi_b)$, $\mathfrak{p}_c = (\Gamma_c, v_{c,0}, \mathrm{ord}_c, \phi_c) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}} - \widehat{\mathbb{T}}_{g,p}^{(1)}$ such that $(\Gamma_a, v_{a,0}, \mathrm{ord}_a)$, $(\Gamma_b, v_{b,0}, \mathrm{ord}_b)$ and $(\Gamma_c, v_{c,0}, \mathrm{ord}_c)$ satisfy conditions (ia), (ib) and (ic) above, respectively. Then

$$\begin{aligned}\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}} - \widehat{\mathbb{T}}_{g,p}^{(1)} =& \Big(\cup_{\varphi_a} (\varphi_a \cdot \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_a,(1)}) \Big) \cup \Big(\cup_{\varphi_b} (\varphi_b \cdot \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_b,(1)}) \Big) \\ & \cup \Big(\cup_{\varphi_c} (\varphi_c \cdot \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_c,(1)}) \Big),\end{aligned}$$

where φ_a ranges over $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}/\mathcal{AM}^{\mathfrak{p}_a,(1)}_{g,p,1}$, φ_b ranges over $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}/\mathcal{AM}^{\mathfrak{p}_b,(1)}_{g,p,1}$ and φ_c ranges over $\mathcal{AM}^{\mathfrak{p}}_{g,p,1}/\mathcal{AM}^{\mathfrak{p}_c,(1)}_{g,p,1}$; and every pair of sets on the right-hand side are disjoint.

A similar result holds if $(h,q) = (g-1,p+1)$ and $\mathfrak{p}_a, \mathfrak{p}_b$ and $\mathfrak{p}_c$ are replaced by $\mathfrak{p}_d = (\Gamma_d, v_{d,0}, \mathrm{ord}_d, \phi_d)$, $\mathfrak{p}_e = (\Gamma_e, v_{e,0}, \mathrm{ord}_e, \phi_e)$, $\mathfrak{p}_f = (\Gamma_f, v_{f,0}, \mathrm{ord}_f, \phi_f) \in \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} - \widehat{\mathbb{T}}^{(1)}_{g,p}$ such that $(\Gamma_d, v_{d,0}, \mathrm{ord}_d)$, $(\Gamma_e, v_{e,0}, \mathrm{ord}_e)$ and $(\Gamma_f, v_{f,0}, \mathrm{ord}_f)$ satisfy conditions (iid), (iie) and (iif) above, respectively.

We can choose $\widehat{\mathbb{P}}_{g,p}$ such that there are at most three connected components in $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} - \widehat{\mathbb{T}}^{(1)}_{g,p})$, one for every case above. Notice $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p} - \widehat{\mathbb{T}}^{(1)}_{g,p})$ can be computed from $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}}_{g,p}$ by removing codimension one faces corresponding to collapse edges e_3 and e_4.

Proposition 6.13 can be extended as follows, where $\mathfrak{p}_a, \mathfrak{p}_b, \mathfrak{p}_c, \mathfrak{p}_d, \mathfrak{p}_e$ and $\mathfrak{p}_f$ are as above.

(iii) If $(h,q) = (g, p-1)$, then

- (a) $\mathcal{AM}^{\mathfrak{p}_a,(1)}_{g,p,1} \simeq \mathcal{AM}_{g-1,p-1,2}$.
- (b) $\mathcal{AM}^{\mathfrak{p}_b,(1)}_{g,p,1} \simeq \mathcal{AM}_{g-1,p-1,2}$
- (c) $\mathcal{AM}^{\mathfrak{p}_c,(1)}_{g,p,1} \simeq \mathcal{AM}_{g,p-2,1}$.

(iv) If $(h,q) = (g-1, p)$, then

- (d) $\mathcal{AM}^{\mathfrak{p}_d,(1)}_{g,p,1} \simeq \mathcal{AM}_{g-2,p,3}$.
- (e) $\mathcal{AM}^{\mathfrak{p}_e,(1)}_{g,p,1} \simeq \mathcal{AM}_{g-1,p,1}$
- (f) $\mathcal{AM}^{\mathfrak{p}_f,(1)}_{g,p,1} \simeq \mathcal{AM}_{g-1,p-1,2}$.

□

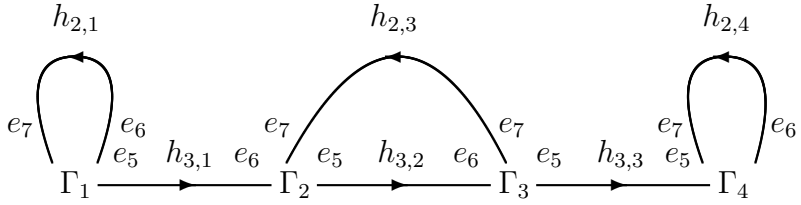
8.6 Example. We apply Remark 8.5 to $(g,p) = (0,3)$. The case $(h,q) = (g-1,p+1)$ is empty and the case $(h,q) = (g,p-1)$ does not satisfies neither (ia) nor (ib) in Remark 8.5. Following Example 6.15 and Example 7.12, $\widehat{\mathbb{P}}_{0,3} \cap (\widehat{\mathrm{ord}\mathbb{A}}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}_{3,0} - \widehat{\mathbb{T}}^{(1)}_{0,3})$ has one component corresponding to (ic) in Remark 8.5. This component is computed from $\widehat{\mathbb{P}}_{0,3} \cap \widehat{\mathrm{ord}\mathbb{A}}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}_{3,0}$ by removing codimension one faces obtained by collapsing edges e_3 and e_4. We obtain the following representation of $\widehat{\mathbb{P}}_{0,3} \cap \widehat{\mathrm{ord}\mathbb{A}}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1),(1)}_{3,0}$.

$$\Gamma_1 \xrightarrow{e_5 \quad h_{3,1} \quad e_6} \Gamma_2 \xrightarrow{e_5 \quad h_{3,2} \quad e_5} \Gamma_3$$

By (iiic) in Remark 8.5, $\mathcal{AM}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1),(1)}_{0,3,1} \simeq \mathcal{AM}_{0,1,1}$ is the trivial group. □

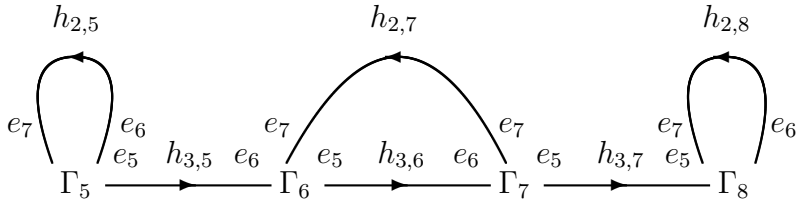
8.7 Example. We apply Remark 8.5 to $(g,p) = (1,1)$. The case $(h,q) = (g,p-1)$ does not satisfies (ic) in Remark 8.5. Following Example 6.17 and Example 7.13, $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)} - \widehat{\mathbb{T}}_{1,1}^{(1)})$ has two components: $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)}$ corresponding to (ib), and, $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\phi_5),(1)}$ corresponding to (ia). These components are computed from $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$ by removing codimension one faces obtained by collapsing edges e_3 and e_4.

$\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)}$ has the following representation.



By (iiib) in Remark 8.5, $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)} \simeq \mathcal{AM}_{0,0,2}$ is a cyclic infinite group generated by $h_{2,1}$.

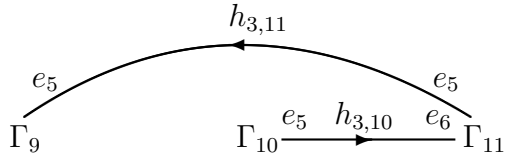
And, $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\phi_5),(1)}$ has the following representation.



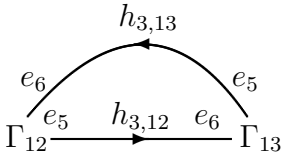
By (iiia) in Remark 8.5, $\mathcal{AM}_{1,1,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\phi_5),(1)} \simeq \mathcal{AM}_{0,0,2}$ is a cyclic infinite group generated by $h_{2,5}$.

The case $(h,q) = (g-1,p+1)$ does not satisfies condition (iid) in Remark 8.5. Following Example 6.17 and Example 7.13, $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)} - \widehat{\mathbb{T}}_{1,1}^{(1)})$ has two components: $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9),(1)}$ corresponding to (iie), and, $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\phi_{12}),(1)}$ corresponding to (iif). These components are computed from $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)}$ by removing codimension one faces obtained by collapsing edges e_3 and e_4.

$\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9),(1)}$ has the following representation.



By (ive) in Remark 8.5, $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9),(1)} \simeq \mathcal{AM}_{0,1,1}$ is the trivial group. And, $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\phi_{12}),(1)}$ has the following representation.



By (ivf) in Remark 8.5, $\mathcal{AM}_{1,1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\phi_{12}),(1)} \simeq \mathcal{AM}_{0,0,2}$ is a cyclic infinite group generated by $h_{3,12}h_{3,13}$.

□

8.8 Remark. For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(i)})$. Suppose

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}). \tag{5}$$

Then edges of $(\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1})$ are identified with edges of $(\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2})$ through the codimension one face (5). In particular, the following identifications hold.

$$\begin{aligned} E(\Gamma_1) - \{e_1\} &\leftrightarrow E(\Gamma_2) - \{e_2\} \\ e_2 &\leftrightarrow \text{right } h\text{-edge of level } n, \\ e_3 &\leftrightarrow e_1, \\ e_4 &\leftrightarrow \text{left } h\text{-edge of level } n. \end{aligned}$$

And, edges of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ but e_1, e_2, e_3, e_4 are identified with edges of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ but e_1, e_2, left h-edge of level n, right h-edge of level n. □

8.9 Lemma. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(i)})$ such that*

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, ord_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}).$$

Then, there exist continuous maps

$$\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)} \to \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)} \quad \text{and} \quad \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)} \to \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$$

which are homotopy inverse. Hence, $\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$ and $\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$ are homotopy equivalent.

Proof. We define $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)} \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$. Then $|v_0^*| = 2$ and $|v_1^*| = 3$ hold in $(\Gamma, v_0, \mathrm{ord})$. We define the homotopy which reduces the length of e_1 to 0 and increases the length of e_3 in the same proportion. Since, in $(\Gamma, v_0, \mathrm{ord})$, $|v_0^*| = 2$ and $|v_1^*| = 3$, the distinguished vertex, $v_0^{e_1}$, of $(\Gamma^{e_1}, v_0^{e_1}, \mathrm{ord}^{e_1})$ has valency 3. Then $v_0^{e_1}$ can be split in a unique manner to produce points $\mathfrak{p}' = (\Gamma', v_0', \mathrm{ord}', \phi') \notin \sigma(\mathfrak{p})$. We define a second homotopy which splits $v_0^{e_1}$ such that the length of e_2 in $(\Gamma', v_0', \mathrm{ord}')$ increases from 0 to the length of e_1 in $(\Gamma, v_0, \mathrm{ord})$, and the length of e_1 in $(\Gamma', v_0', \mathrm{ord}')$ decreases from sum of the lengths of e_1 and e_3 in $(\Gamma, v_0, \mathrm{ord})$ to the length of e_3 in $(\Gamma, v_0, \mathrm{ord})$. If $(\Gamma', v_0', \mathrm{ord}')$ is not a strongly non-separating graph, then $\mathfrak{p}'$ does not lie in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$. In this case, to obtain points in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$ we have to collapse edges in $(\Gamma', v_0', \mathrm{ord}')$ to obtain a strongly non-separating graph.

The inverse map $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)} \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$ is defined similarly, by firstly collapsing e_2 in $\mathfrak{p} \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$.

It is clear that these maps are homotopy inverse. □

8.10 Corollary. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(i)})$ such that*

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, ord_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}).$$

Then, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1,(1)}$ and $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$ are isomorphic.

Proof. Follows from Proposition 8.9 and Definition 8.3. □

8.11 Proposition. *For $i = 1, 2, 3, 4$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p}_i)$ is a simplex of dimension $6g + 3p - 3 = 3n - 3$. Suppose $\mathfrak{p}_1, \mathfrak{p}_3 \in \widehat{\mathbb{P}}_{g,p}$,*

$$\widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)} = \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_3,(1)},$$

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, ord_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}),$$

and,

$$\sigma(\mathfrak{p}_3^{e_1}) = \sigma(\Gamma_3^{e_1}, v_{3,0}^{e_1}, ord_3^{e_1}, \phi_3^{e_1}) = \sigma(\Gamma_4^{e_2}, v_{4,0}^{e_2}, ord_4^{e_2}, \phi_4^{e_2}) = \sigma(\mathfrak{p}_4^{e_2}).$$

Let $g_{n,1}$ be the generator of $\mathcal{AM}_{g,p,1}$ corresponding to collapse e_1 in $\mathfrak{p}_1$, and, let $g_{n,3}$ be the generator of $\mathcal{AM}_{g,p,1}$ corresponding to collapse e_1 in $\mathfrak{p}_3$. Then $g_{n,3} = g_{n,1}\, w$ for some $w \in \mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)} \subseteq \mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$.

Proof. Let γ_1 be a path from $\mathfrak{p}_1$ to $\mathfrak{p}_3$ contained in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$. The map

$$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)} \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$$

in Lemma 8.9 gives a path γ_2 homotopic to γ_1 from $\mathfrak{p}_2$ to $\mathfrak{p}_4$. Let β_1 be the path from $\mathfrak{p}_1$ to $\mathfrak{p}_2$ through the codimension one face $\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\mathfrak{p}_2^{e_2})$. And, let β_3 be the path from $\mathfrak{p}_3$ to $\mathfrak{p}_4$ through the codimension one face $\sigma(\mathfrak{p}_3^{e_1}) = \sigma(\mathfrak{p}_4^{e_2})$. Then, $\gamma_1\beta_3\overline{\gamma}_2\overline{\beta}_1$ bounds a disc in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. Notice β_1 and β_3 correspond to the generators $g_{n,1}$ and $g_{n,3}$, respectively. We can suppose γ_1 and γ_2 lie in simplices of codimension at most 1. Since γ_1 is contained in $\widehat{\mathbb{P}}_{g,p}$, its corresponding word is the identity. Since γ_2 is contained in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$, the word w corresponding to γ_2 lies in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$. Hence, the relation $g_{n,3}w^{-1}g_{n,1}^{-1}$ holds in $\mathcal{AM}_{g,p,1}$. □

8.12 Remark. For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)})$. Suppose $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. By Proposition 6.11 and Proposition 8.11, $\mathcal{AM}_{g,p,1}$ is generated by generators of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_i}$ for $i = 1, 2$ and a g-generators of level n for each connected component of $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} - \widehat{\mathbb{T}}_{g,p}^{(1)})$ and $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2} - \widehat{\mathbb{T}}_{g,p}^{(1)})$. □

8.13 Example. Recall $\widehat{\mathbb{P}}_{0,3}$ has three simplices of dimension $6g + 3p - 3 = 6$ described in Example 6.15. To compute g-generators of level 3 in $\mathcal{AM}_{0,3,1}$, we collapse edges e_1 and e_2 for each of these simplices.

$$\begin{aligned}
(\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1}) &= (e_3, e_4, e_2); (\overline{e}_3, \overline{e}_2, e_5), (\overline{e}_4, e_6, e_7), (\overline{e}_5, \overline{e}_7, \overline{e}_6),\\
&= (e_1, e_2, e_3); (\overline{e}_1, \overline{e}_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_4, \overline{e}_6, \overline{e}_5).\\
(\Gamma_2^{e_1}, v_{2,0}^{e_1}, \mathrm{ord}_2^{e_1}) &= (e_3, e_4, e_2); (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_7, \overline{e}_5), (\overline{e}_2, e_5, e_6),\\
&= (e_1, e_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, \overline{e}_5, e_6), (\overline{e}_3, \overline{e}_6, \overline{e}_4).\\
(\Gamma_3^{e_1}, v_{3,0}^{e_1}, \mathrm{ord}_3^{e_1}) &= (e_3, e_4, e_2); (\overline{e}_3, e_6, e_7), (\overline{e}_4, e_5, \overline{e}_2), (\overline{e}_6, \overline{e}_5, \overline{e}_7),\\
&= (e_1, e_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, e_6, \overline{e}_3), (\overline{e}_4, \overline{e}_6, \overline{e}_5).
\end{aligned}$$

And,

$$\begin{aligned}
(\Gamma_1^{e_2}, v_{1,0}^{e_2}, \mathrm{ord}_1^{e_2}) &= (e_1, e_5, \overline{e}_3); (\overline{e}_1, e_3, e_4), (\overline{e}_5, \overline{e}_7, \overline{e}_6), (\overline{e}_4, e_6, e_7),\\
&= (e_1, e_2, e_3); (\overline{e}_1, \overline{e}_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_4, \overline{e}_6, \overline{e}_5).\\
(\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2}) &= (e_1, \overline{e}_5, e_6); (\overline{e}_1, e_3, e_4), (\overline{e}_5, \overline{e}_4, \overline{e}_7), (\overline{e}_3, \overline{e}_6, e_7),\\
&= (e_1, e_2, e_3); (\overline{e}_1, e_4, e_5), (\overline{e}_2, \overline{e}_5, e_6), (\overline{e}_3, \overline{e}_6, \overline{e}_4).\\
(\Gamma_3^{e_2}, v_{3,0}^{e_2}, \mathrm{ord}_3^{e_2}) &= (e_1, e_4, e_5); (\overline{e}_1, e_3, e_4), (\overline{e}_5, \overline{e}_7, \overline{e}_6), (\overline{e}_3, e_6, e_7),\\
&= (e_1, e_2, e_3); (\overline{e}_1, e_4, \overline{e}_2), (\overline{e}_3, e_5, e_6), (\overline{e}_4, \overline{e}_6, \overline{e}_5).
\end{aligned}$$

Since $(\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1}) = (\Gamma_1^{e_2}, v_{1,0}^{e_2}, \mathrm{ord}_1^{e_2})$ and $(\Gamma_2^{e_1}, v_{2,0}^{e_1}, \mathrm{ord}_2^{e_1}) = (\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2})$, there are two g-generators of level 3 denoted $g_{3,1}$ and $g_{3,2}$, respectively. By Example 8.6, $\widehat{\mathbb{P}}_{0,3} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{0,3}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} - \widehat{\mathbb{T}}_{0,3}^{(1)})$ has one component and $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathbb{P}}_{0,3} \cap \widehat{\mathrm{ord}\mathbb{A}}_{0,3}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1),(1)}$ can be joined by a path through

the codimension one face $(\Gamma_1^{e_5}, v_{1,0}^{e_5}, \mathrm{ord}_1^{e_5}, \phi_1^{e_5}) = (\Gamma_2^{e_6}, v_{2,0}^{e_6}, \mathrm{ord}_2^{e_6}, \phi_2^{e_6})$ contained in $\widehat{\mathbb{P}}_{0,3} \cap \widehat{\mathrm{ord}\mathbb{A}}_{0,3}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1),(1)}$. By Proposition 8.11, $g_{3,1} = g_{3,2}w$ for some $w \in \mathcal{AM}_{0,3,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$. If we apply Theorem 3.7 to $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$, the pair of edges $\{e_1, e_5\}$ and collapsing e_1; we have the relation $g_{3,1}g_{2,2}^{-1}g_{3,2}^{-1}h_{3,1}^{-1}$. Since in $\mathcal{AM}_{0,3,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$ there is the relation $h_{3,1} = 1$, we see $g_{3,2} = g_{3,1}g_{2,2}^{-1}$. Hence, $g_{3,2} \in g_{3,1}\mathcal{AM}_{0,3,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$. □

8.14 Example. Recall $\widehat{\mathbb{P}}_{1,1}$ has thirteen simplices of dimension $6g + 3p - 3 = 6$ described in Example 6.17. To compute g-generators of level 3 in $\mathcal{AM}_{1,1,1}$, we collapse edges e_1 and e_2. We obtain the following ten g-generators of level 3:

$$\begin{aligned}
g_{3,1} \text{ since } (\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1}, \phi_1^{e_1}) &= (\Gamma_1^{e_2}, v_{1,0}^{e_2}, \mathrm{ord}_1^{e_2}, \phi_1^{e_2}),\\
g_{3,2} \text{ since } (\Gamma_2^{e_1}, v_{2,0}^{e_1}, \mathrm{ord}_2^{e_1}, \phi_2^{e_1}) &= (\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2}, \phi_2^{e_2}),\\
g_{3,3} \text{ since } (\Gamma_3^{e_1}, v_{3,0}^{e_1}, \mathrm{ord}_3^{e_1}, \phi_3^{e_1}) &= (\Gamma_3^{e_2}, v_{3,0}^{e_2}, \mathrm{ord}_3^{e_2}, \phi_3^{e_2}),\\
g_{3,4} \text{ since } (\Gamma_4^{e_1}, v_{4,0}^{e_1}, \mathrm{ord}_4^{e_1}, \phi_4^{e_1}) &= (\Gamma_5^{e_2}, v_{5,0}^{e_2}, \mathrm{ord}_5^{e_2}, \phi_5^{e_2}),\\
g_{3,6} \text{ since } (\Gamma_6^{e_1}, v_{6,0}^{e_1}, \mathrm{ord}_6^{e_1}, \phi_6^{e_1}) &= (\Gamma_{13}^{e_2}, v_{13,0}^{e_2}, \mathrm{ord}_{13}^{e_2}, \phi_{13}^{e_2}),\\
g_{3,7} \text{ since } (\Gamma_7^{e_1}, v_{7,0}^{e_1}, \mathrm{ord}_7^{e_1}, \phi_7^{e_1}) &= (\Gamma_{11}^{e_2}, v_{11,0}^{e_2}, \mathrm{ord}_{11}^{e_2}, \phi_{11}^{e_2}),\\
g_{3,9} \text{ since } (\Gamma_9^{e_1}, v_{9,0}^{e_1}, \mathrm{ord}_9^{e_1}, \phi_9^{e_1}) &= (\Gamma_{10}^{e_2}, v_{10,0}^{e_2}, \mathrm{ord}_{10}^{e_2}, \phi_{10}^{e_2}),\\
g_{3,11} \text{ since } (\Gamma_{11}^{e_1}, v_{11,0}^{e_1}, \mathrm{ord}_{11}^{e_1}, \phi_{11}^{e_1}) &= (\Gamma_{12}^{e_2}, v_{12,0}^{e_2}, \mathrm{ord}_{12}^{e_2}, \phi_{12}^{e_2}),\\
g_{3,12} \text{ since } (\Gamma_{12}^{e_1}, v_{12,0}^{e_1}, \mathrm{ord}_{12}^{e_1}, \phi_{12}^{e_1}) &= (\Gamma_7^{e_2}, v_{7,0}^{e_2}, \mathrm{ord}_7^{e_2}, \phi_7^{e_2}),\\
g_{3,13} \text{ since } (\Gamma_{13}^{e_1}, v_{13,0}^{e_1}, \mathrm{ord}_{13}^{e_1}, \phi_{13}^{e_1}) &= (\Gamma_6^{e_2}, v_{6,0}^{e_2}, \mathrm{ord}_6^{e_2}, \phi_6^{e_2}).
\end{aligned}$$

By Example 8.7, $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} - \widehat{\mathbb{T}}_{1,1}^{(1)})$ has two component: $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1),(1)}$ and $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_5, v_{5,0}, \mathrm{ord}_5, \phi_5),(1)}$. And, $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)} - \widehat{\mathbb{T}}_{1,1}^{(1)})$ has two component: $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9),(1)}$ and $\widehat{\mathbb{P}}_{1,1} \cap \widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12}, \phi_{12}),(1)}$. By Proposition 8.11, $g_{3,2}, g_{3,3}, g_{3,4} \in g_{3,1}\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$, $g_{3,7} \in g_{3,6}\mathcal{AM}_{1,1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)}$, $g_{3,11} \in g_{3,9}\mathcal{AM}_{1,1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)}$, $g_{3,13} \in g_{3,12}\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)}$. By Theorem 3.7, it can be computed: $g_{3,1}g_{2,2}^{-1} = h_{3,1}g_{3,2}$, $g_{3,2}h_{2,3}^{-1} = h_{3,2}g_{3,3}$, $g_{3,3}g_{2,5}^{-1} = h_{3,3}g_{3,4}$, $g_{3,6}g_{2,13} = h_{3,6}g_{3,7}$, $g_{3,9}g_{2,10} = h_{3,11}g_{3,11}$ and $g_{3,12}h_{2,7} = h_{3,13}^{-1}g_{3,13}$. Recall $h_{3,1} = h_{3,2} = h_{3,3} = h_{3,6} = h_{3,11} = h_{3,13} = 1$.

□

8.15 Lemma. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(i)})$ such that*

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, ord_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}).$$

For $i = 1, 2$, let h_i, q_i be integers such that $(\Gamma_i^{\downarrow}, v_{i,0}^{\downarrow}, ord_i^{\downarrow})$ is a (h_i, q_i)-fat graph with a distinguished vertex, and, $w(\Gamma_i, v_{i,0}, ord_i) = \{w_{i,0}, [w_{i,1}], \ldots, [w_{i,p}]\}$.

(i) If $(h_1, q_1) = (g, p-1)$, then $w_{1,0} = e_1 e_3 w'_{1,0} \overline{e}_2$, $[w_{1,j}] = [\overline{e}_4 \overline{e}_1 e_2 w'_{1,j}]$ for some $1 \leq j \leq p$ and exactly one of the cases holds.

(a) If $w'_{1,0} = u'_{1,0} \overline{e}_3 e_4 v'_{1,0}$, then $(h_2, q_2) = (g-1, p+1)$.

(b) If $w'_{1,j} = u'_{1,j} \overline{e}_3 e_4 v'_{1,j}$, then $(h_2, q_2) = (g, p-1)$.

(c) If $[w_{1,k}] = [\overline{e}_3 e_4 v_{1,k}]$ for some $1 \leq k \leq p, k \neq j$, then $(h_2, q_2) = (g, p-1)$.

(ii) If $(h_1, q_1) = (g-1, p+1)$, then $w_{1,0} = e_1 e_3 w'_{1,0} \overline{e}_4 \overline{e}_1 e_2 w''_{1,0} \overline{e}_2$ and exactly one of the cases holds.

(d) If $w'_{1,0} = u'_{1,0} \overline{e}_3 e_4 v'_{1,0}$, then $(h_2, q_2) = (g-1, p+1)$.

(e) If $w''_{1,0} = u''_{1,0} \overline{e}_3 e_4 v''_{1,0}$, then $(h_2, q_2) = (g-1, p+1)$.

(f) If $[w_{1,j}] = [\overline{e}_3 e_4 v_{1,j}]$ for some $1 \leq j \leq p$, then $(h_2, q_2) = (g, p-1)$.

Proof. Notice $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$ can be obtained from $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ by removing the distinguished vertex $v_{1,0}$, declaring the concatenation of edges $\overline{e}_1$ and e_2 a new edge denoted f, divinding e_3 in two new edges denoted f_1 and f_2 and declaring the new vertex inside e_3 the new distinguished vertex $v_{2,0}$.

(i) If $(h_1, q_1) = (g, p-1)$, then $w_{2,0} = f_1 w'_{2,0} \overline{f}\, \overline{f}_2$, $[w_{2,j}] = [\overline{e}_4 f w'_{2,j}]$ for some $1 \leq j \leq p$ and exactly one of the cases holds.

(a) If $w'_{1,0} = u'_{1,0} \overline{e}_3 e_4 v'_{1,0}$, then $w'_{2,0} = u'_{2,0} f_2 \overline{f}_1 e_4 v'_{2,0}$.

(b) If $w'_{1,j} = u'_{1,j} \overline{e}_3 e_4 v'_{1,j}$, then $w'_{2,j} = u'_{2,j} f_2 \overline{f}_1 e_4 v'_{2,j}$.

(c) If $[w_{1,k}] = [\overline{e}_3 e_4 v_{1,k}]$ for some $1 \leq k \leq p, j \neq k$, then $[w_{2,k}] = [f_2 \overline{f}_1 e_4 v_{2,k}]$.

(ii) If $(h_1, q_1) = (g-1, p+1)$, then $w_{2,0} = f_1 w'_{2,0} \overline{e}_4 f w''_{2,0} \overline{f}\, \overline{f}_2$ and exactly one of the cases holds.

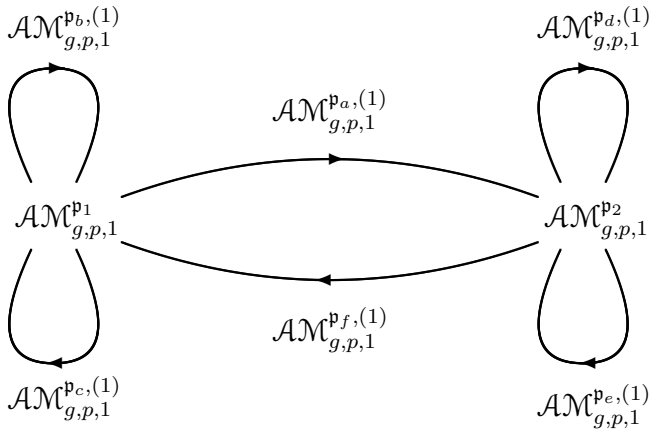
(d) If $w'_{1,0} = u'_{1,0} \overline{e}_3 e_4 v'_{1,0}$, then $w'_{1,0} = u'_{2,0} f_2 \overline{f}_1 e_4 v'_{2,0}$.

(e) If $w''_{1,0} = u''_{1,0} \overline{e}_3 e_4 v''_{1,0}$, then $w''_{2,0} = u''_{2,0} f_2 \overline{f}_1 e_4 v''_{2,0}$.

(f) If $[w_{1,j}] = [\overline{e}_3 e_4 v_{1,j}]$ for some $1 \leq j \leq p$, then $[w_{2,j}] = [f_2 \overline{f}_1 e_4 v_{2,j}]$.

□

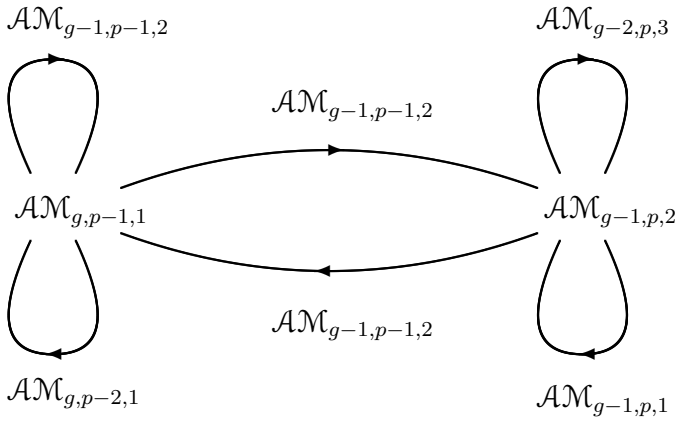
8.16 Theorem. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p} - \widehat{\mathbb{T}}_{g,p}$. Suppose $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, ord_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, ord_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. Then $\mathcal{AM}_{g,p,1}$ is a quotient of the fundamental group of the graph of groups*



where $\mathfrak{p}_a = (\Gamma_a, v_{a,0}, ord_a, \phi_a), \mathfrak{p}_b = (\Gamma_b, v_{b,0}, ord_b, \phi_b), \mathfrak{p}_c = (\Gamma_c, v_{c,0}, ord_c, \phi_c) \in \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_1} - \widehat{\mathbb{T}}_{g,p}^{(1)}$ *and* $\mathfrak{p}_d = (\Gamma_d, v_{d,0}, ord_d, \phi_d), \mathfrak{p}_e = (\Gamma_e, v_{e,0}, ord_e, \phi_e), \mathfrak{p}_f = (\Gamma_f, v_{f,0}, ord_f, \phi_f) \in \widehat{ord\mathbb{A}}_{g,p}^{\mathfrak{p}_2} - \widehat{\mathbb{T}}_{g,p}^{(1)}$ *satisfy conditions* (ia), (ib), (ic), (iid), (iie) *and* (iif) *in Remark 8.5, respectively. For every edge, there exists a monomorphism which goes from the subgroup that labels the edge to the subgroup that labels the vertex where the edge points.*

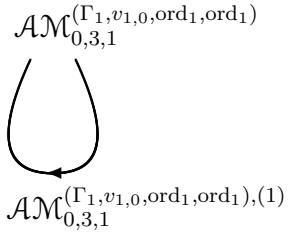
Proof. By definition, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1}$ and $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$ are subgroups of $\mathcal{AM}_{g,p,1}$. By Remark 8.12, $\mathcal{AM}_{g,p,1}$ is generated by these two subgroups and a g-generator of level n for each connected component of $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1} - \widehat{\mathbb{T}}_{g,p}^{(1)})$ and $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2} - \widehat{\mathbb{T}}_{g,p}^{(1)})$. By Remark 8.5, these connected components are $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_a,(1)}$, $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_b,(1)}$ and $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_c,(1)}$; and, $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_d,(1)}$, $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_e,(1)}$ and $\widehat{\text{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_f,(1)}$. Hence, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_a,(1)}$, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_b,(1)}$ and $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_c,(1)}$ are subgroups of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1}$; and, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_d,(1)}$, $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_e,(1)}$ and $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_f,(1)}$ are subgroups of $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2}$. By Corollary 8.10, the g-generator for each of these connected component is a monomorphism from the corresponding subgroup to the subgroup given by Lemma 8.15. □

8.17 Corollary. $\mathcal{AM}_{g,p,1}$ *is a quotien of the fundamental group of the graph of groups*



Proof. Follows from Theorem 8.16, Proposition 6.13 and Remark 8.5. □

8.18 Example. Following Example 8.6, by Theorem 8.16, $\mathcal{AM}_{0,3,1}$ is a quotient of the fundamental group of the graph of groups



where $\mathcal{AM}_{0,3,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1)} \simeq \mathcal{AM}_{0,2,1}$. Since $\mathcal{AM}_{0,3,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1),(1)} \simeq \mathcal{AM}_{0,1,1}$ is the trivial group, the monomorphism corresponding to the edge is trivial. □

We can give an explicit description of the monomorphism of the edges in Theorem 8.16.

8.19 Proposition. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{ord\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p}_i)$ is a simplex of dimension $6g + 3p - 3 = 3n - 3$. Suppose*

$$\sigma(\mathfrak{p}_1^{e_1}) = \sigma(\Gamma_1^{e_1}, v_{1,0}^{e_1}, ord_1^{e_1}, \phi_1^{e_1}) = \sigma(\Gamma_2^{e_2}, v_{2,0}^{e_2}, ord_2^{e_2}, \phi_2^{e_2}) = \sigma(\mathfrak{p}_2^{e_2}).$$

Let $\varphi \in \mathcal{AM}_{g,p,1}^{\mathfrak{p}_1,(1)}$, and, let $\psi \in \mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$ be the image of φ under the isomorphism $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1,(1)} \simeq \mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$ described in Corollary 8.10. Let $w_{\varphi,1}$ be a word representing φ in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_1,(1)}$, let $w_{\psi,2}$ be a word representing ψ in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$ and let $g_{n,1}$ be the generator of $\mathcal{AM}_{g,p,1}$ corresponding to collapse e_1 in $\mathfrak{p}_1$. Then $g_{n,1}^{-1}\, w_{\varphi,1}\, g_{n,1} = w_{\psi,2}$.

Proof. Since $\varphi \in \mathcal{AM}_{g,p,1}^{\mathfrak{p}_1,(1)}$, we see $\varphi \cdot \mathfrak{p}_1 \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$. Since $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$ is connected, there exists a path $\gamma_1 \subseteq \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)}$ from $\mathfrak{p}_1$ to $\varphi \cdot \mathfrak{p}_1$ which represents φ.

The map

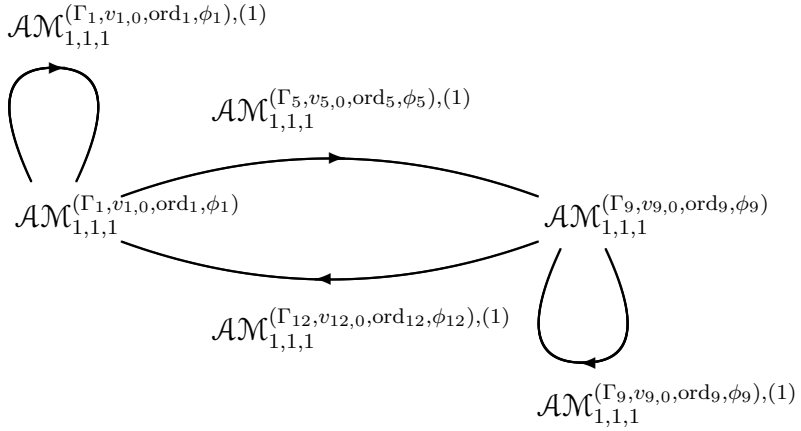
$$\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_1,(1)} \to \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$$

in Lemma 8.9 gives a path γ_2 homotopic to γ_1 from $\mathfrak{p}_2$ to $\mathfrak{p}'_2 = (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi'_2)$. Hence, $\mathfrak{p}'_2 \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$ and $\psi = \phi'_2 \circ \phi_2^{-1}$ lies in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}_2,(2)}$. Notice $\mathfrak{p}'_2 = \psi \cdot \mathfrak{p}_2$.

We can suppose γ_1 lies in simplices of codimension at most 1. Then $w_{\varphi,1}$ is obtaiend from γ_1. And $w_{\psi,2}$ is obtained from a path contained in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}_2,(2)}$ homotopic to γ_2 and lying in simplices of codimension at most 1.

Let β be a path from $\mathfrak{p}_1$ to $\mathfrak{p}_2$ which goes through the codimension one simplex corresponding to collapse e_1 in $\mathfrak{p}_1$. Notice $\varphi \cdot \beta$ is a path from $\psi \cdot \mathfrak{p}_1$ to $\psi \cdot \mathfrak{p}_2$ which goes through the codimension one simplex corresponding to collapse e_1 in $\psi \cdot \mathfrak{p}_1$. Then, $\overline{\beta}\gamma_1(\varphi \cdot \beta)\overline{\gamma}_2$ bounds a disc in $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}$. Hence, the relation $g_{n,1}^{-1} w_{\varphi,1} g_{n,1} w_{\psi,2}^{-1}$ holds in $\mathcal{AM}_{g,p,1}$. □

8.20 Example. Following Example 8.7, by Theorem 8.16, $\mathcal{AM}_{1,1,1}$ is a quotient of the fundamental group of the graph of groups



where $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1)} \simeq \mathcal{AM}_{1,0,1}$ and $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\mathrm{ord}_9)} \simeq \mathcal{AM}_{0,1,2}$.

Since $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\mathrm{ord}_9),(1)} \simeq \mathcal{AM}_{0,1,1}$ is the trivial group, the monomorphism corresponding to this edge is trivial. To describe the monomorphisms of the other edges, we apply Proposition 8.19 to g-generators of level 3 given in Example 8.14 and subgroups given in Example 8.7. Notice $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1),(1)} \simeq \mathcal{AM}_{0,0,2}$, $\mathcal{AM}_{1,1,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\mathrm{ord}_5),(1)} \simeq \mathcal{AM}_{0,0,2}$ and $\mathcal{AM}_{1,1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\mathrm{ord}_{12}),(1)} \simeq \mathcal{AM}_{0,0,2}$ are cyclic infinite groups.

The monomorphism for the edge labelled $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1),(1)}$ is described by $g_{3,1}$ such that $g_{3,1}^{-1}\, h_{2,1}\, g_{3,1} = h_{2,1}$.

The monomorphism for the edge labelled $\mathcal{AM}_{1,1,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\mathrm{ord}_5),(1)}$ is described by $g_{3,6}$ such that $g_{3,6}^{-1}\, h_{3,6} h_{2,7}\, g_{3,6} = g_{2,13} g_{2,11}$.

The monomorphism for the edge labelled $\mathcal{AM}_{1,1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\mathrm{ord}_{12}),(1)}$ is described by $g_{3,12}$ such that $g_{3,12}^{-1}\, h_{3,12} h_{3,13}\, g_{3,12} = g_{2,7} h_{2,7}^{-1}$.

By Example 7.27, $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1)} = \langle h_{2,1},\, g_{2,2} \mid g_{2,2}h_{2,1}^{-1}g_{2,2}^{-1}h_{2,1}^{-1}g_{2,2}\rangle$.

By Example 7.29, $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)} = \langle h_{3,9}, g_{2,10} \mid h_{3,9}g_{2,10} = g_{2,10}h_{3,9}\rangle$.

By Example 7.18, $h_{3,6}h_{2,7} = h_{2,1}$ and $g_{2,7}h_{2,7}^{-1} = g_{2,2}h_{2,1}^{-1}$. By Example 7.20, $g_{2,13}g_{2,11}^{-1} = h_{3,9}g_{2,10}$ and $h_{3,12}h_{3,13} = h_{3,9}$. Hence, the monomorphisms of the edges are:

$$\begin{aligned} g_{3,1}^{-1}\, h_{2,1}\, g_{3,1} &= h_{2,1} \\ g_{3,6}^{-1}\, h_{2,1}\, g_{3,6} &= h_{3,9}g_{2,10}, \\ g_{3,12}^{-1}\, h_{3,9}\, g_{3,12} &= g_{2,2}h_{2,1}^{-1}. \end{aligned}$$

□

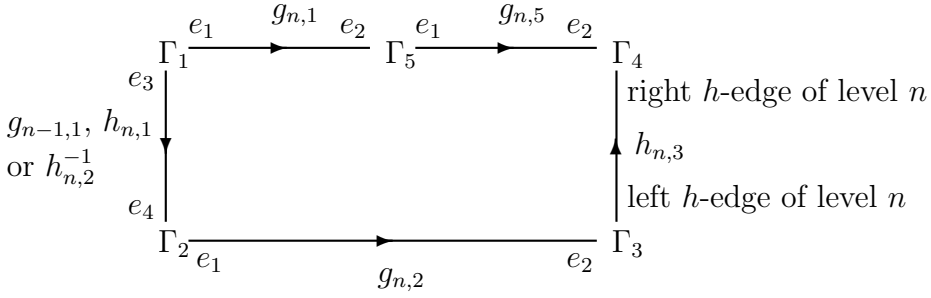
8.21 Remark. To obtain a presentation for $\mathcal{AM}_{g,p,1}$ from Theorem 8.16, we need to add relations from Theorem 3.7 corresponding to some codimension two faces of simplices of dimension $6g+3p-3 = 3n-3$ of $\widehat{\mathbb{P}}_{g,p}$.

Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathbb{P}}_{g,p}$ such that $\sigma(\mathfrak{p})$ is a simplex of dimension $6g+3p-3 = 3n-3$. Let $f_1, f_2 \in E(\Gamma)$ such that there exists a relations corresponding to the codimension two face $\sigma(\mathfrak{p}^{f_1,f_2}) = (\Gamma^{f_1,f_2}, v_0^{f_1,f_2}, \mathrm{ord}^{f_1,f_2}, \phi^{f_1,f_2})$ obtained by collapsing f_1 and f_2. To simplify cases, we suppose that in the standard labelling of $(\Gamma, v_0, \mathrm{ord})$, f_1 has been labelled before f_2. For example, if $f_1 = e_2$ then $f_2 \neq e_1$, or, if $f_1 = e_3$ then $f_2 \neq e_1, e_2$.

If $f_1 \neq e_1, e_2$, then $\sigma(\mathfrak{p}^{f_1,f_2}) \subseteq \widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p}}$ and the relation deduced form $\sigma(\mathfrak{p}^{f_1,f_2})$ appears in $\mathcal{AM}_{g,p,1}^{\mathfrak{p}}$, a vertex group in the graph of groups in Theorem 8.16.

If $f_1 = e_1$ and $f_2 \notin \{e_2, e_3, e_4\}$, or, $f_1 = e_2$ and $f_2 \notin \{$left h-edge of level n, right h-edge of level $n\}$; then the relation deduced form $\sigma(\mathfrak{p}^{f_1,f_2})$ appears either in Proposition 8.11 or in Proposition 8.19. In both cases, the relation deduced form $\sigma(\mathfrak{p}^{f_1,f_2})$ appears in the monomorphism of the edges in the graph of groups in Theorem 8.16.

We have to consider the cases where either $f_1 = e_1$ and $f_2 \in \{e_2, e_3, e_4\}$, or, $f_1 = e_2$ and $f_2 \in \{$left h-edge of level n, right h-edge of level $n\}$. The relation deduced from these codimension two faces can be computed by Remark 8.8 and represented as follows.



Notice this relation is similar to relation of length 5 in Example 3.8(b).

If the left side of the rectangle above corresponds to the generator $h_{n,1}$, then $\overline{e}_3 \in v_2^*$ and $(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, \overline{e}_3, e_5), \ldots$. It follows, $(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, \overline{e}_2, f), (\overline{f}, e_5, e_4), \ldots$. Hence, $(\Gamma_1^{e_3}, v_{1,0}^{e_3}, \mathrm{ord}_1^{e_3}, \phi_1^{e_3})$ is a boundary face and there is no such a relation.

If the left side of the rectangle above corresponds to the generator $h_{n,2}^{-1}$ and $n \geq 3$, then $\overline{e}_3 \in v_2^*$ and $(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, \overline{e}_3), \ldots$. It follows, $(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_5, f_1), (\overline{f}_1, \overline{e}_2, e_4), \ldots$ and $(\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_5, f_2); (\overline{f}_2, f_1, e_2), (\overline{f}_1, \overline{e}_2, e_4), \ldots$. Since $n \geq 3$, $(\Gamma_3^{\downarrow}, v_{3,0}^{\downarrow}, \mathrm{ord}_3^{\downarrow})$ has a separating edge, $(\Gamma_2^{e_1}, v_{2,0}^{e_1}, \mathrm{ord}_2^{e_1}, \phi_2^{e_1})$ is a boundary face and there is no such a relation.

Hence, for $n \geq 3$, the relations that need to be added are

$$g_{n,1}g_{n,5} = g_{n-1,1}g_{n,2}h_{n,3}.$$

We will see that no more than 24 of these relations need to be added. □

8.22 Definition. We denote by $\widehat{\mathbb{T}}_{g,p}^{(1),(3)}$ the set of point $\mathfrak{p} = (\Gamma, v_0\mathrm{ord}, \phi) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)})$ such that $|u^*| \geq 4$ where $\overline{e}_3 \in u^*$, $u \in V(\Gamma)$. □

8.23 Definition. Let $\mathfrak{p} \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)} \cup \widehat{\mathbb{T}}_{g,p}^{(1),(3)})$. We denote by $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1),(3)}$ the connected component of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)} \cup \widehat{\mathbb{T}}_{g,p}^{(1),(3)})$ which contains $\mathfrak{p}$. □

8.24 Remark. Let $\mathfrak{p} \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)} \cup \widehat{\mathbb{T}}_{g,p}^{(1),(3)})$.

Notice $|v_0^*| = 2$ and $|v_1^*| = 3$. If $\overline{e}_3 \in v_2^*$, then $|v_2^*| = 3$. If $\overline{e}_3 \in v_3^*$, then $|v_3^*| = 3$. In both cases, $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1),(3)}$ can be defined as the connected component of $\widehat{\mathrm{ord}\mathbb{A}}_{g,p}^{\mathfrak{p},(1)} - \widehat{\mathbb{T}}_{g,p}^{(1),(3)}$ which contains $\mathfrak{p}$. □

8.25 Remark. Let $\mathfrak{p} = (\Gamma, v_0, \mathrm{ord}, \phi) \in \widehat{\mathbb{P}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)} \cup \widehat{\mathbb{T}}_{g,p}^{(1),(3)})$. Suppose there exists a relation in $\mathcal{AM}_{g,p,1}$ deduced from the codimension two face $\sigma(\mathfrak{p}^{e_1,e_3}) = \sigma(\Gamma^{e_1,e_3}, v_0^{e_1,e_3}, \mathrm{ord}^{e_1,e_3}, \phi^{e_1,e_3})$. Hence, $(\Gamma, v_0, \mathrm{ord}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), \ldots$ and $\mathrm{ord}(v_3^*) = (\overline{e}_3, f_1, f_2)$. Let $w(\Gamma, v_0, \mathrm{ord}) = \{w_0, [w_1], \ldots, [w_p]\}$. Then $w_0 = e_1e_3f_1w_0'\overline{e}_2$ and $\overline{e}_4\overline{e}_1e_2$, $\overline{f}_2\overline{e}_3e_4$, $\overline{f}_1f_2$ appear in $w(\Gamma, v_0, \mathrm{ord})$. It is show in Lemma 4.8 that there are at most two cases where $\overline{e}_1e_2$ can appear. It is shown in Remark 8.5 that for each of these two cases, there are at most three cases where $\overline{e}_3e_4$ can appear. Similarly, for each of these six cases there are at most four cases where $\overline{f}_3f_4$ can appear. For example, in Remark 8.5 (ia) we have $w_0 = e_1e_3f_1w_0'\overline{e}_2$, $[w_i] = [\overline{e}_4\overline{e}_1e_2w_i']$ and $w_0' = u_0'\overline{f}_2\overline{e}_3e_4v_0'$, under the condition $g \geq 1$ and $p \geq 1$. Then there are four cases according to whether $\overline{f}_1f_2$ appears in u_0', v_0', w_i' or, $[w_j]$ for $j \neq i$. Notice $\overline{f}_1f_2$ can appear in u_0' under the condition $g \geq 2$ and $\overline{f}_1f_2$ can appear in $[w_j]$, for $j \neq i$, under the condition $p \geq 2$.

In general, $\widehat{\mathbb{P}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}_{g,p}^{(1)} \cup \widehat{\mathbb{T}}_{g,p}^{(1),(3)})$ has, at most, 24 components with relations corresponding to collapse e_1 and e_3.

In particular, for $g = 0$, $\widehat{\mathbb{P}}_{0,p} - (\widehat{\mathbb{T}}_{0,p} \cup \widehat{\mathbb{T}}_{0,p}^{(1)} \cup \widehat{\mathbb{T}}_{0,p}^{(1),(3)})$ has one component with relations corresponding to collapse e_1 and e_3.

For $g = 2$ and $p = 0$, $\widehat{\mathbb{P}}_{2,0} - (\widehat{\mathbb{T}}_{2,0} \cup \widehat{\mathbb{T}}^{(1)}_{2,0} \cup \widehat{\mathbb{T}}^{(1),(3)}_{2,0})$ has five components with relations corresponding to collapse e_1 and e_3. And for $g \geq 3$ and $p = 0$, $\widehat{\mathbb{P}}_{g,0} - (\widehat{\mathbb{T}}_{g,0} \cup \widehat{\mathbb{T}}^{(1)}_{g,0} \cup \widehat{\mathbb{T}}^{(1),(3)}_{g,0})$ has six components with relations corresponding to collapse e_1 and e_3.

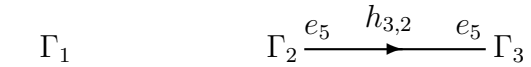
Notice $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(1),(3)}_{g,p}$ can be computed from $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p},(1)}_{g,p}$ by removing codimesion one face corresponding to collapse edges f_1 and f_2. Recall $\mathrm{ord}(v_3^*) = (\overline{e}_3, f_1, f_2)$. □

The next Lemma expresses that all relations deduced from Theorem 3.7 corresponding to collapse edges e_1 and e_3 in a connected component of $\widehat{\mathbb{P}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(1)}_{g,p} \cup \widehat{\mathbb{T}}^{(1),(3)}_{g,p})$ are equal in the fundamental group of the graph of groups in Theorem 8.16.

8.26 Lemma. *For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, ord_i, \phi_i) \in \widehat{\mathbb{P}}_{g,p} - (\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(1)}_{g,p} \cup \widehat{\mathbb{T}}^{(1),(3)}_{g,p})$ such that $\widehat{ord\mathbb{A}}^{\mathfrak{p}_1,(1),(3)}_{g,p} = \widehat{ord\mathbb{A}}^{\mathfrak{p}_2,(1),(3)}_{g,p}$. If there exists the relation r_i in $\mathcal{AM}_{g,p,1}$ corresponding to collapse e_1 and e_3 in $(\Gamma_i, v_{i,0}, ord_i)$, for $i = 1, 2$, then $r_1 = r_2$ in the fundamental group of the graph of groups in Theroem 8.16.*

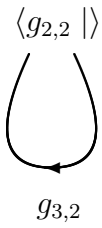
Proof. For $i = 1, 2$, the relation r_i is deduced from the codimension two face $\sigma(\mathfrak{p}_i^{e_1,e_3})$ of $\sigma(\mathfrak{p}_i)$ obtained by collapsing e_1 and e_3 in $(\Gamma_i, v_{i,0}, \mathrm{ord}_i)$ where $\mathfrak{p}_i^{e_1,e_3} = (\Gamma_i^{e_1,e_3}, v_{i,0}^{e_1,e_3}, \mathrm{ord}_i^{e_1,e_3}, \phi_i^{e_1,e_3})$ and $|v_{i,0}^{e_1,e_3}| = 4$. Let γ be a path from $\mathfrak{p}_1$ to $\mathfrak{p}_2$ contained in $\widehat{\mathbb{P}}_{g,p} \cap \widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_1,(1),(3)}_{g,p}$. Notice γ is a sequence of collapsing edges and splitting vertices such that if γ intersects $\sigma(\mathfrak{p}) = \sigma(\Gamma, v_0, \mathrm{ord}, \phi)$, then $\sigma(\mathfrak{p})$ does not intersect $\widehat{\mathbb{T}}_{g,p} \cup \widehat{\mathbb{T}}^{(1)}_{g,p} \cup \widehat{\mathbb{T}}^{(1),(3)}_{g,p}$. Hence, $|v_0^*| = 2$ and $|v_1^*| = |v_3^*| = 3$ in $(\Gamma, v_0, \mathrm{ord}, \phi)$; and, the subtree with edge e_1, e_2, e_3, e_4 and f_1, f_2 is left invariant through the path γ. We conclude γ is homotopic to a path γ^{e_1,e_3} from $\sigma(\mathfrak{p}_1^{e_1,e_3})$ to $\sigma(\mathfrak{p}_2^{e_1,e_3})$ such that every simplex $\sigma(\mathfrak{p}^{e_1,e_3}) = \sigma(\Gamma^{e_1,e_3}, v_0^{e_1,e_3}, \mathrm{ord}^{e_1,e_3}, \phi^{e_1,e_3})$ intersected by γ^{e_1,e_3} satisfies $|(v_0^{e_1,e_3})^*| = 4$. Since $\gamma \subseteq \widehat{\mathbb{P}}_{g,p}$, the word in $\mathcal{AM}^{\mathfrak{p}_1,(1)}_{g,p,1}$ described by γ is the identity and $r_1 = r_2$ in the fundamental group of the graph of groups in Theorem 8.16. □

8.27 Example. We compute the connected components of $\widehat{\mathbb{P}}_{0,3} - (\widehat{\mathbb{T}}_{3,0} \cup \widehat{\mathbb{T}}^{(1)}_{3,0} \cup \widehat{\mathbb{T}}^{(1),(3)}_{3,0})$ by removing codimension one faces from $\widehat{\mathbb{P}}_{0,3} - (\widehat{\mathbb{T}}_{3,0} \cup \widehat{\mathbb{T}}^{(1)}_{3,0})$. See Example 8.6. From Example 6.15, we see that the codimension one faces that need to be removed correspond to collapse $\overline{e}_2$ and e_5 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$, $\overline{e}_6$ and e_5 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$, and, e_6 and e_7 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$. Hence, $\widehat{\mathbb{P}}_{0,3} - (\widehat{\mathbb{T}}_{3,0} \cup \widehat{\mathbb{T}}^{(1)}_{3,0} \cup \widehat{\mathbb{T}}^{(1),(3)}_{3,0})$ has the following connected components.



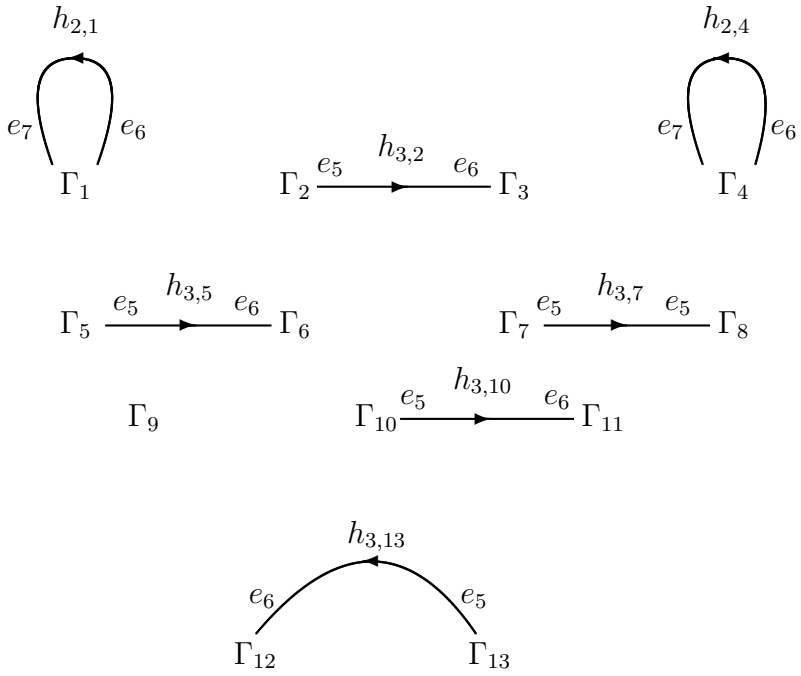
Since $\overline{e}_3 \in v_2^*$ in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$, there is no relation corresponding to collapse e_1 and e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. The relation $g_{3,2}g_{3,2} = g_{2,2}g_{3,1}h_{3,1}$ is deduced by collapsing e_1 an e_3 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$.

By Example 7.26, $\mathcal{AM}_{0,3,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\mathrm{ord}_1)} = \langle g_{2,2} \mid \rangle$. By Example 8.18, $\mathcal{AM}_{0,3,1}$ is a quotient of the fundamental group of the graph of groups



where the edge group of $g_{3,2}$ is the trivial group. By Example 8.13, $g_{3,1} = g_{3,2}g_{2,2}$. The relation $g_{3,2}g_{3,2} = g_{2,2}g_{3,2}g_{2,2}$ needs to be added in the fundamental group of the graph of groups above. □

8.28 Example. We compute the connected components of $\widehat{\mathbb{P}}_{1,1} - (\widehat{\mathbb{T}}_{1,1} \cup \widehat{\mathbb{T}}_{1,1}^{(1)} \cup \widehat{\mathbb{T}}_{1,1}^{(1),(3)})$ by removing codimension one faces from $\widehat{\mathbb{P}}_{1,1} - (\widehat{\mathbb{T}}_{1,1} \cup \widehat{\mathbb{T}}_{1,1}^{(1)})$. See Example 8.7. From Example 6.17, we see which codimension one faces need to be removed. We conclude that $\widehat{\mathbb{P}}_{1,1} - (\widehat{\mathbb{T}}_{1,1} \cup \widehat{\mathbb{T}}_{1,1}^{(1)} \cup \widehat{\mathbb{T}}_{1,1}^{(1),(3)})$ has eight connected components represented as follows.



Since $\overline{e}_3 \in v_2^*$ in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$, $(\Gamma_4, v_{4,0}, \mathrm{ord}_4)$ and $(\Gamma_9, v_{9,0}, \mathrm{ord}_9)$, there is no relation corresponding to collapse e_1 and e_3 in these cases.

Collapsing e_1 and e_3 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$, we have the relation $g_{3,2}g_{3,2} = g_{2,2}g_{3,1}h_{3,1}$;

Collapsing e_1 and e_3 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$, we have the relation $g_{3,3}g_{3,3} = g_{2,3}g_{3,2}h_{3,2}$;

Collapsing e_1 and e_3 in $(\Gamma_6, v_{6,0}, \mathrm{ord}_6)$, we have the relation $g_{3,6}g_{3,13} = g_{2,6}g_{3,4}h_{3,5}$;

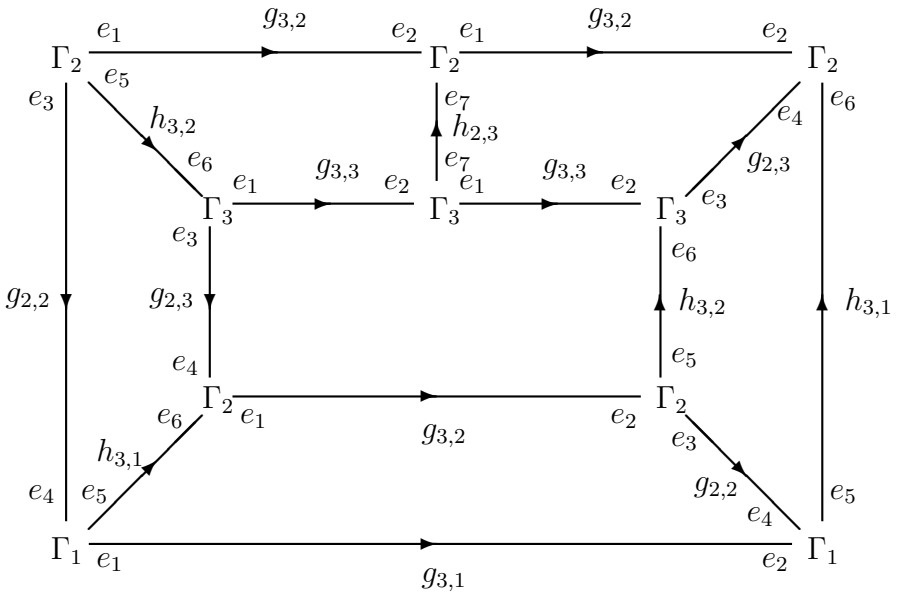
Collapsing e_1 and e_3 in $(\Gamma_7, v_{7,0}, \mathrm{ord}_7)$, we have the relation $g_{3,7}g_{3,11} = g_{2,7}g_{3,6}h_{3,13}$;

Collapsing e_1 and e_3 in $(\Gamma_{11}, v_{11,0}, \mathrm{ord}_{11})$, we have the relation $g_{3,11}g_{3,12} = g_{2,11}g_{3,13}h_{3,6}$;

Collapsing e_1 and e_3 in $(\Gamma_{12}, v_{12,0}, \mathrm{ord}_{12})$, we have the relation $g_{3,12}g_{3,7} = g_{2,12}g_{3,9}h_{3,10}$;

Collapsing e_1 and e_3 in $(\Gamma_{13}, v_{13,0}, \mathrm{ord}_{13})$, we have the relations $g_{3,13}g_{3,6} = g_{2,13}g_{3,11}h_{3,12}$.

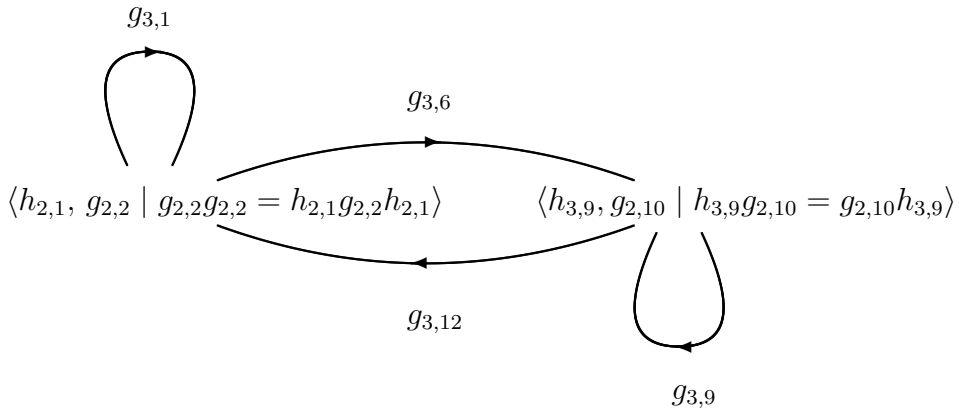
By Lemma 8.26, relations $g_{3,2}g_{3,2} = g_{2,2}g_{3,1}h_{3,1}$ and $g_{3,3}g_{3,3} = g_{2,3}g_{3,2}h_{3,2}$ are equal in the fundamental group of the graph of groups in Example 8.20. We represent this equality in the following picture where rectangles represent the relations, and, quadrilaterals are relations in the fundamental group of the graph of groups in Example 8.20.



Similarly, relations $g_{3,12}g_{3,7} = g_{2,12}g_{3,9}h_{3,10}$ and $g_{3,13}g_{3,6} = g_{2,13}g_{3,11}h_{3,12}$ are equal in the fundamental group of the graph of groups in Example 8.20.

By Example 7.27, $\mathcal{AM}_{1,1,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \langle h_{2,1},\, g_{2,2} \mid g_{2,2}g_{2,2} = h_{2,1}g_{2,2}h_{2,1} \rangle$. By Example 7.29, $\mathcal{AM}_{1,1,1}^{(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)} = \langle h_{3,9}, g_{2,10} \mid h_{3,9}g_{2,10} = g_{2,10}h_{3,9} \rangle$. By

Example 8.20, $\mathcal{AM}_{1,1,1}$ is a quotient of the fundamental group of the graph of groups



where the monomorphisms of the edges are given in Example 8.20.

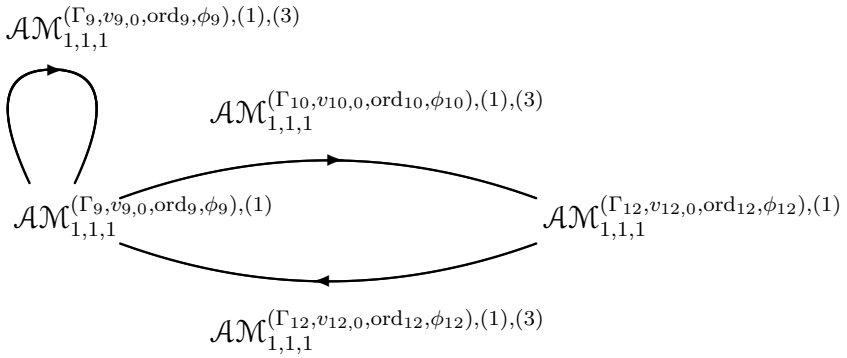
There are five relations corresponding to collapse e_1 and e_3 that need to be added in the fundamental group of the graph of groups above. From Example 7.27, Example 7.29 and Example 8.14, we can write these relations in the fundamental group of the graph of groups:

$$\begin{aligned}
g_{3,2}g_{3,2} = g_{2,2}g_{3,1}h_{3,1} &\Rightarrow g_{3,1}g_{2,2}^{-1}g_{3,1}g_{2,2}^{-1} = g_{2,2}g_{3,1},\\
g_{3,6}g_{3,13} = g_{2,6}g_{3,4}h_{3,5} &\Rightarrow g_{3,6}g_{3,12}h_{2,1} = g_{2,2}g_{3,1}g_{2,2}^{-1}h_{2,1}^{-1}g_{2,2}^{-1},\\
g_{3,7}g_{3,11} = g_{2,7}g_{3,6}h_{3,13} &\Rightarrow g_{3,6}g_{3,9}g_{2,10} = g_{2,2}g_{3,6},\\
g_{3,11}g_{3,12} = g_{2,11}g_{3,13}h_{3,6} &\Rightarrow g_{3,9}g_{2,10}g_{3,12} = h_{3,9}g_{2,10}g_{3,12}h_{2,1},\\
g_{3,12}g_{3,7} = g_{2,12}g_{3,9}h_{3,10} &\Rightarrow g_{3,12}g_{3,6} = g_{3,9}.
\end{aligned}$$

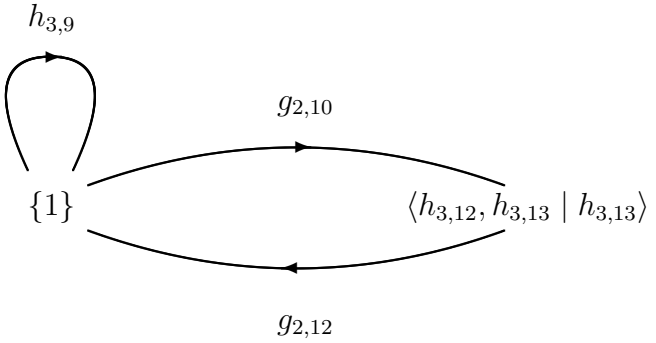
Recall, $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)} \simeq \mathcal{AM}_{1,0,1}$ by Proposition 6.13. Instead of a direct computation for the presentation for $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$ as in Example 7.27, we can compute a presentation for $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$ by lifting a presentation for $\mathcal{AM}_{1,0,1}$. This lifting is obtained by declaring all h-generators of level 3 to be the identity. Then, all h-generators and g-generators of level 2 are identified and the relation in the presentation for $\mathcal{AM}_{1,0,1}$ is lifted to a relation in $\mathcal{AM}_{1,1,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$.

Recall, $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)} \simeq \mathcal{AM}_{0,1,2}$ by Proposition 6.13. Instead of a direct computation for the presentation for $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)}$ as in Example 7.29, we can compute a presentation for $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)}$ by Theorem 8.16 applied one level deeper. Hence, we have a graph of groups with vertices corresponding to connected components of $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)} - \widehat{\mathbb{T}}_{1,1}^{(1)})$ and edges corresponding to connected components of $\widehat{\mathbb{P}}_{1,1} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)} - (\widehat{\mathbb{T}}_{1,1}^{(1)} \cup \widehat{\mathbb{T}}_{1,1}^{(1),(3)}))$. See

Examples 8.7 and the beginning of this Example, respectively. We have the following graph of groups.



where $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\mathrm{ord}_9),(1)} \simeq \mathcal{AM}_{0,1,1}$ is the trivial group, $\mathcal{AM}_{1,1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\mathrm{ord}_{12}),(1)} \simeq \mathcal{AM}_{0,0,2}$ is a cyclic infinite group, and, $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\mathrm{ord}_9),(1),(3)} \simeq \mathcal{AM}_{0,0,1}$, $\mathcal{AM}_{1,1,1}^{(\Gamma_{10},v_{10,0},\mathrm{ord}_{10},\mathrm{ord}_{10}),(1),(3)} \simeq \mathcal{AM}_{0,0,1}$, $\mathcal{AM}_{1,1,1}^{(\Gamma_{12},v_{12,0},\mathrm{ord}_{12},\mathrm{ord}_{12}),(1),(3)} \simeq \mathcal{AM}_{0,0,1}$ are the trivial group. Hence, we have the following graph of groups.



where the monomorphisms of the edges are trivial since the edge groups are trivial. The relations that need to be added are $h_{3,9}g_{2,10} = h_{3,11}^{-1}g_{2,11}h_{3,13}$ and $g_{2,12}h_{3,9} = h_{3,12}g_{2,13}h_{3,10}^{-1}$. Since $h_{3,10}g_{2,11} = g_{2,10}h_{3,12}$, $h_{3,13}g_{2,12} = g_{2,13}h_{3,11}$ and $h_{3,10}, h_{3,11} \in \mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9),(1)} = \{1\}$, we have a presentation for $\mathcal{AM}_{1,1,1}^{(\Gamma_9,v_{9,0},\mathrm{ord}_9,\phi_9)}$. □

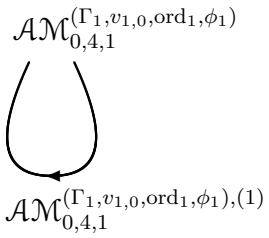
9 Examples

For $i = 1, 2$, let $\mathfrak{p}_i = (\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i) \in \widehat{\mathrm{ord}\mathbb{A}}_{g,p}$ such that $\sigma(\mathfrak{p}_i)$ is a simplex of dimension $3n-3 = 6g+3p-3$. Suppose $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(g, p-1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$ is a $(g-1, p+1)$-fat graph with a distinguished vertex. A presentation for $\mathcal{AM}_{g,p,1}$ can be deduced by Theorem 8.16 by computing the monomorphisms of the edges and the relations

that need to be added.

Last two paragraphs in Example 8.28 can be generalised as follows. Recall $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1} \simeq \mathcal{AM}_{g,p-1,1}$ and $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1} \simeq \mathcal{AM}_{g-1,p-1,2}$ are the vertices of the graph of groups for $\mathcal{AM}_{g,p,1}$. A presentation for $\mathcal{AM}_{g,p-1,1}$ can be lifted to $\mathcal{AM}^{\mathfrak{p}_1}_{g,p,1}$. A presentation for $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1}$ can be computed by applying Theorem 8.16 one level deeper. In particular, connected components in $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_2}_{g,p} - \widehat{\mathbb{T}}^{(1)}_{g,p})$ correspond to vertices in the graph of groups for $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1}$, and, connected components in $\widehat{\mathbb{P}}_{g,p} \cap (\widehat{\mathrm{ord}\mathbb{A}}^{\mathfrak{p}_2}_{g,p} - (\widehat{\mathbb{T}}^{(1)}_{g,p} \cup \widehat{\mathbb{T}}^{(1),(3)}_{g,p}))$ correspond to edges in the graph of groups for $\mathcal{AM}^{\mathfrak{p}_2}_{g,p,1}$.

9.1 Example. By Theorem 8.16, $\mathcal{AM}_{0,4,1}$ is a quotient of the fundamental group of the following graph of groups.



where $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$. We can suppose $\sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ is a simplex of dimension $6g + 3p - 3 = 9$. By Proposition 6.13, $\mathcal{AM}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}_{0,4,1} \simeq \mathcal{AM}_{0,3,1}$. By results in Section 7, these isomorphisms are realised by declaring all h-generators of level 4 to be the identity and lifting relations from $\mathcal{AM}_{0,3,1}$ to $\mathcal{AM}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}_{0,4,1}$. Then, lifts of a generator in $\mathcal{AM}_{0,3,1}$ are identified in $\mathcal{AM}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}_{0,4,1}$. By Remark 8.5, $\mathcal{AM}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)}_{0,4,1} \simeq \mathcal{AM}_{0,2,1}$.

We choose $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$ such that there exist generators corresponding to collapse e_1, e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ and $\overline{e}_3 \notin v_2^*$.

We choose $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{0,4}$ to be a lift of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{0,3}$. Hence,

$$(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, \overline{e}_7, \overline{e}_5).$$

If e_6 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,3}$ is lifted to the concatenation of both h-edges of level 4, we have

$$(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2);(\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, e_8), (\overline{e}_4, e_9, \overline{e}_5), (\overline{e}_6, e_{10}, \overline{e}_7), (\overline{e}_8, \overline{e}_{10}, \overline{e}_9).$$

Then $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow}) = (e_3, e_4); (\overline{e}_3, e_7, e_8), (\overline{e}_4, e_9, f), (\overline{e}_7, \overline{f}, e_{10}), (\overline{e}_8, \overline{e}_{10}, \overline{e}_9),$

$(\Gamma_1^{\downdownarrows}, v_{1,0}^{\downdownarrows}, \mathrm{ord}_1^{\downdownarrows}) = (e_7, e_8); (\overline{e}_7, g, e_{10}), (\overline{e}_8, \overline{e}_{10}, \overline{g})$, and,

Γ_1	g-edge		h-edge	
	left	right	left	right
level 4	e_1	e_2	e_5	e_6
level 3	e_3	e_4	e_9	
level 2	e_7	e_8	$\overline{e}_{10}$	

A lift of a g-generator of level 3 in $\mathcal{AM}_{0,3,1}$ is obtained by collapsing e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. We have $(\Gamma_1^{e_3}, v_{1,0}^{e_3}, \mathrm{ord}_1^{e_3}) = (\Gamma_2^{e_4}, v_{2,0}^{e_4}, \mathrm{ord}_2^{e_4})$ where

$$(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, e_8, e_9), \\ (\overline{e}_5, \overline{e}_9, e_{10}), (\overline{e}_7, \overline{e}_{10}, \overline{e}_8).$$

We denote by $g_{3,2,1}$ the generator of $\mathcal{AM}_{0,4,1}$ corresponding to collapse e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. Since e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$ is a lift of e_1 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,3}$, $g_{3,2,1}$ is a lift of the generator $g_{3,2}$ in $\mathcal{AM}_{0,3,1}$.

A lift of a g-generator of level 2 in $\mathcal{AM}_{0,3,1}$ is obtained by collapsing e_7 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. We have $(\Gamma_1^{e_7}, v_{1,0}^{e_7}, \mathrm{ord}_1^{e_7}) = (\Gamma_3^{e_7}, v_{3,0}^{e_7}, \mathrm{ord}_3^{e_7})$ where

$$(\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_6, e_7), (\overline{e}_4, e_8, \overline{e}_5), \\ (\overline{e}_7, e_9, e_{10}), (\overline{e}_8, \overline{e}_{10}, \overline{e}_9).$$

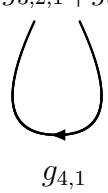
We denote by $g_{2,2,1}$ the generator of $\mathcal{AM}_{0,4,1}$ corresponding to collapse e_7 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$. Since e_7 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$ is a left g-edge of level 2, $g_{2,2,1}$ is a lift of the generator $g_{2,2}$ in $\mathcal{AM}_{0,3,1}$.

A presentations for $\mathcal{AM}_{0,3,1}$ is given in Example 8.27. By lifting that presentation, we deduce $\mathcal{AM}_{0,4,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} = \langle g_{2,2,1}, g_{3,2,1} \mid g_{3,2,1}g_{3,2,1} = g_{2,2,1}g_{3,2,1}g_{2,2,1} \rangle$ and $\mathcal{AM}_{0,4,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1), (1)} = \langle g_{2,2,1} \mid \rangle$.

Notice $(\Gamma_1^{e_1}, v_{1,0}^{e_1}, \mathrm{ord}_1^{e_1}) = (\Gamma_4^{e_2}, v_{4,0}^{e_2}, \mathrm{ord}_4^{e_2})$ where

$$(\Gamma_4, v_{4,0}, \mathrm{ord}_4) = (e_1, e_2); (\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, e_8), (\overline{e}_4, \overline{e}_8, e_9), \\ (\overline{e}_5, \overline{e}_9, e_{10}), (\overline{e}_6, \overline{e}_{10}, \overline{e}_7).$$

Hence, we have a g-generator of level 4, denoted $g_{4,1}$, obtained by collapsing e_1 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$. Then, $\mathcal{AM}_{0,4,1}$ is a quotient of the fundamental group of the following graph of groups



where the edge group is $\langle g_{2,2,1} \mid \rangle$.

It remains to describe the monomorphism of the edge in the graph of groups above and to find the relation that needs to be added.

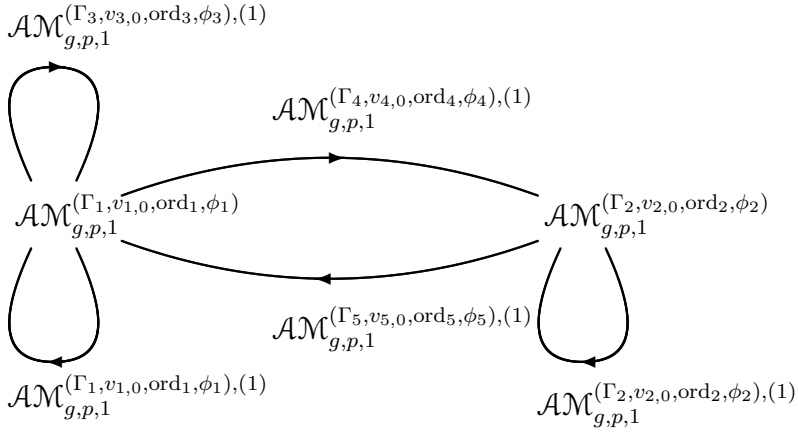
Notice $(\Gamma_3^{e_8}, v_{3,0}^{e_8}, \mathrm{ord}_3^{e_8}) = (\Gamma_2^{e_9}, v_{2,0}^{e_9}, \mathrm{ord}_2^{e_9})$, e_8 is the left h-edge of level 3 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$. Notice $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{0,4}$ is a lift of $(\Gamma_1, v_{1,0}, \mathrm{ord}_1)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{0,3}$. Hence, there exists a generator $h_{3,1,3}$ in $\mathcal{AM}_{0,4,1}$ corresponding to collapse e_8 in $(\Gamma_3, v_{3,0}, \mathrm{ord}_3)$.

Notice $(\Gamma_2^{e_5}, v_{2,0}^{e_5}, \mathrm{ord}_2^{e_5}) = (\Gamma_1^{e_6}, v_{1,0}^{e_6}, \mathrm{ord}_1^{e_6})$ and e_5 is the left h-edge of level 4 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$. Hence, there exists a generator $h_{4,2}$ in $\mathcal{AM}_{0,4,1}$ corresponding to collapse e_5 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2)$.

Then, $\mathcal{AM}_{0,4,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1), (1)}$ contains the automorphism $g_{2,2,1}h_{3,1,3}h_{4,2}$. It is easy to see that the monomorphism of the edge in the graph of groups is described by $g_{4,1}^{-1}(g_{2,2,1}h_{3,1,3}h_{4,2})g_{4,1} = h_{3,2,1}^{-1}g_{3,2,1}g_{2,3,4}^{-1}$. By collapsing e_7 and e_9 in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4)$, we have the relation $g_{2,3,4}h_{3,1,3}^{-1} = h_{3,2,1}^{-1}g_{2,2,1}$. Hence, the monomorphism of the edge group is described by $g_{4,1}^{-1}g_{2,2,1}g_{4,1} = g_{3,2,1}g_{2,2,1}^{-1}$.

The relation that needs to be added is deduced by collapsing e_1 and e_3 in $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$. Since $(\Gamma_4^{e_1}, v_{4,0}^{e_1}, \mathrm{ord}_4^{e_1}) = (\Gamma_1^{e_2}, v_{1,0}^{e_2}, \mathrm{ord}_1^{e_2})$ and $(\Gamma_2^{e_1}, v_{2,0}^{e_1}, \mathrm{ord}_2^{e_1}) = (\Gamma_2^{e_2}, v_{2,0}^{e_2}, \mathrm{ord}_2^{e_2})$, we have generators $g_{4,4}$ and $g_{4,2}$, respectively. And, the relation that needs to be added is $g_{4,1}g_{4,4} = g_{3,2,1}g_{4,2}h_{4,2}$. Collapsing e_1 and e_9 in $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$, we obtain the relation $h_{3,2,1}g_{4,4} = g_{4,1}h_{3,2,1}$. Collapsing e_1 and e_5 in $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{0,4}$, we obtain the relation $h_{4,2}g_{4,1} = g_{4,2}g_{2,3,4}^{-1}$. And, the relation that needs to be added is $g_{4,1}g_{4,1} = g_{3,2,1}g_{4,1}g_{2,2,1}$. □

9.2 Example. By Theorem 8.16, $\mathcal{AM}_{1,2,1}$ is a quotient of the fundamental group of the following graph of groups.



where $(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i) \in \widehat{\mathrm{ord}\mathbb{A}}_{1,2}$, for $i = 1, \ldots, 5$. We can suppose $\sigma(\Gamma_i, v_{i,0}, \mathrm{ord}_i, \phi_i)$ is a simplex of dimension $6g + 3p - 3 = 9$, for $i = 1, \ldots, 5$. By Proposition 6.13, $\mathcal{AM}_{1,2,1}^{(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)} \simeq \mathcal{AM}_{1,1,1}$ and $\mathcal{AM}_{1,2,1}^{(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)} \simeq \mathcal{AM}_{0,2,2}$. Notice $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(1,1)$-fat graph with a distinguished vertex and $(\Gamma_2^{\downarrow}, v_{2,0}^{\downarrow}, \mathrm{ord}_2^{\downarrow})$

is a $(0,3)$-fat graph with a distinguished vertex.

A presentation for $\mathcal{AM}_{1,1,1}$ can be lifted to $\mathcal{AM}_{1,2,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$. By Remark 8.5, $\mathcal{AM}_{1,2,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)} \simeq \mathcal{AM}_{1,0,1}$, $\mathcal{AM}_{1,2,1}^{(\Gamma_3,v_{3,0},\mathrm{ord}_3,\phi_3),(1)} \simeq \mathcal{AM}_{0,1,2}$ and $\mathcal{AM}_{1,2,1}^{(\Gamma_4,v_{4,0},\mathrm{ord}_4,\phi_4),(1)} \simeq \mathcal{AM}_{0,1,2}$. Notice $(\Gamma_1^{\downarrow\downarrow}, v_{1,0}^{\downarrow\downarrow}, \mathrm{ord}_1^{\downarrow\downarrow})$ is a $(1,0)$-fat graph with a distinguished vertex, and, $(\Gamma_3^{\downarrow\downarrow}, v_{3,0}^{\downarrow\downarrow}, \mathrm{ord}_3^{\downarrow\downarrow})$ and $(\Gamma_4^{\downarrow\downarrow}, v_{4,0}^{\downarrow\downarrow}, \mathrm{ord}_4^{\downarrow\downarrow})$ are $(0,2)$-fat graphs with a distinguished vertex. We can take

$$\begin{aligned}(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1,e_2);&(\overline{e}_1,e_3,e_4),(\overline{e}_2,e_5,e_6),(\overline{e}_3,e_7,e_8),(\overline{e}_4,e_9,\overline{e}_5),\\ &(\overline{e}_6,e_{10},\overline{e}_7),(\overline{e}_8,\overline{e}_9,\overline{e}_{10})\\ (\Gamma_3, v_{3,0}, \mathrm{ord}_3) = (e_1,e_2);&(\overline{e}_1,e_3,e_4),(\overline{e}_2,e_5,e_6),(\overline{e}_3,e_7,\overline{e}_5),(\overline{e}_4,\overline{e}_7,e_8),\\ &(\overline{e}_6,e_9,e_{10}),(\overline{e}_8,\overline{e}_{10},\overline{e}_9)\\ (\Gamma_4, v_{4,0}, \mathrm{ord}_4) = (e_1,e_2);&(\overline{e}_1,e_3,e_4),(\overline{e}_2,e_5,e_6),(\overline{e}_3,e_7,e_8),(\overline{e}_4,\overline{e}_7,e_9),\\ &(\overline{e}_5,\overline{e}_9,e_{10}),(\overline{e}_6,\overline{e}_{10},\overline{e}_8)\end{aligned}$$

Notice $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,2}$ is a lift of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,1}$; and, $(\Gamma_3, v_{3,0}, \mathrm{ord}_3, \phi_3)$ and $(\Gamma_4, v_{4,0}, \mathrm{ord}_4, \phi_4)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,2}$ are lifts of $(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,1}$.

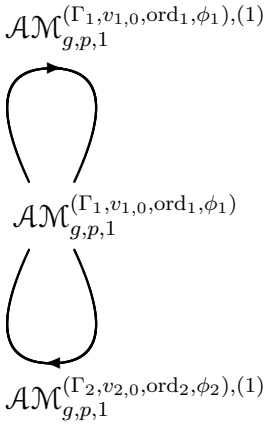
A presentation for $\mathcal{AM}_{1,2,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2)}$ can be obtained from a graph of groups with vertices $\mathcal{AM}_{1,2,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2),(1)}$ and $\mathcal{AM}_{1,2,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\phi_5),(1)}$, and, one edge for every connected component of $\widehat{\mathbb{P}}_{1,2} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{1,2}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2)} - (\widehat{\mathbb{T}}_{1,2}^{(1)} \cup \widehat{\mathbb{T}}_{1,2}^{(1),(3)}))$. By Remark 8.5, $\mathcal{AM}_{1,2,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2),(1)} \simeq \mathcal{AM}_{0,2,1}$ and $\mathcal{AM}_{1,2,1}^{(\Gamma_5,v_{5,0},\mathrm{ord}_5,\phi_5),(1)} \simeq \mathcal{AM}_{0,1,2}$. The subgroups of the edges in the graph of groups of $\mathcal{AM}_{1,2,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2)}$ are isomorphic to either $\mathcal{AM}_{0,1,1}$ or $\mathcal{AM}_{0,0,2}$. We can take

$$\begin{aligned}(\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1,e_2);&(\overline{e}_1,e_3,e_4),(\overline{e}_2,e_5,e_6),(\overline{e}_3,e_7,\overline{e}_5),(\overline{e}_4,e_8,e_9),\\ &(\overline{e}_6,e_{10},\overline{e}_8),(\overline{e}_7,\overline{e}_9,\overline{e}_{10})\\ (\Gamma_5, v_{5,0}, \mathrm{ord}_5) = (e_1,e_2);&(\overline{e}_1,e_3,e_4),(\overline{e}_2,e_5,e_6),(\overline{e}_3,e_7,e_8),(\overline{e}_4,e_9,\overline{e}_6),\\ &(\overline{e}_5,e_{10},\overline{e}_7),(\overline{e}_8,\overline{e}_{10},\overline{e}_9)\end{aligned}$$

Notice $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ and $(\Gamma_5, v_{5,0}, \mathrm{ord}_5, \phi_5)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,2}$ are lifts of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{0,3}$.

It remains to describe the monomorphisms of the edges in the graph of groups for $\mathcal{AM}_{1,2,1}$ and to find the eleven relations that need to be added. □

9.3 Example. By Theorem 8.16, $\mathcal{AM}_{2,0,1}$ is a quotient of the fundamental group of the following graph of groups.



where $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1), (\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2) \in \widehat{\mathrm{ord}\mathbb{A}}_{2,0}$. We can suppose $\sigma(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ and $\sigma(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ are simplices of dimension $6g + 3p - 3 = 9$. By Proposition 6.13, $\mathcal{AM}_{2,0,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)} \simeq \mathcal{AM}_{1,0,2}$. Notice $(\Gamma_1^{\downarrow}, v_{1,0}^{\downarrow}, \mathrm{ord}_1^{\downarrow})$ is a $(1,1)$-fat graph with a distinguished vertex.

A presentation for $\mathcal{AM}_{2,0,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$ can be obtained from a graph of groups with vertices $\mathcal{AM}_{2,0,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)}$ and $\mathcal{AM}_{2,0,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2),(1)}$, and, one edge for every connected component of $\widehat{\mathbb{P}}_{2,0} \cap (\widehat{\mathrm{ord}\mathbb{A}}_{2,0}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)} - (\widehat{\mathbb{T}}_{2,0}^{(1)} \cup \widehat{\mathbb{T}}_{2,0}^{(1),(3)}))$. By Remark 8.5, $\mathcal{AM}_{2,0,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1),(1)} \simeq \mathcal{AM}_{1,0,1}$ and $\mathcal{AM}_{2,0,1}^{(\Gamma_2,v_{2,0},\mathrm{ord}_2,\phi_2),(1)} \simeq \mathcal{AM}_{0,0,3}$. The subgroups of the edges in the graph of groups of $\mathcal{AM}_{2,0,1}^{(\Gamma_1,v_{1,0},\mathrm{ord}_1,\phi_1)}$ are isomorphic to $\mathcal{AM}_{0,0,2}$. We can take

$$\begin{aligned}(\Gamma_1, v_{1,0}, \mathrm{ord}_1) = (e_1, e_2);&(\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, \overline{e}_5, e_7), (\overline{e}_4, e_8, e_9),\\ &(\overline{e}_6, \overline{e}_9, e_{10}), (\overline{e}_7, \overline{e}_8, \overline{e}_{10})\\ (\Gamma_2, v_{2,0}, \mathrm{ord}_2) = (e_1, e_2);&(\overline{e}_1, e_3, e_4), (\overline{e}_2, e_5, e_6), (\overline{e}_3, e_7, e_8), (\overline{e}_4, \overline{e}_7, e_9),\\ &(\overline{e}_5, e_{10}, \overline{e}_8), (\overline{e}_6, \overline{e}_9, \overline{e}_{10})\end{aligned}$$

Notice $(\Gamma_1, v_{1,0}, \mathrm{ord}_1, \phi_1)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{2,0}$ is a lift of $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,1}$ and $(\Gamma_2, v_{2,0}, \mathrm{ord}_2, \phi_2)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{2,0}$ is a lift of $(\Gamma_9, v_{9,0}, \mathrm{ord}_9, \phi_9)$ in $\widehat{\mathrm{ord}\mathbb{A}}_{1,1}$.

It remains to describe the monomorphisms of the edges in the graph of groups for $\mathcal{AM}_{2,0,1}$ and to find the five relations that need to be added. □

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