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# UPPER BOUNDS FOR SUPERQUANTILES OF MARTINGALES

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## Abstract

Let  $(M_n)_n$  be a discrete martingale in  $L^p$  for  $p$  in  $]1, 2]$  or  $p = 3$ . In this note, we give upper bounds on the superquantiles of  $M_n$  and the quantiles and superquantiles of  $M_n^* = \max(M_0, M_1, \dots, M_n)$ .

## 1 Introduction

Throughout this note, we consider a nondecreasing filtration  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  and a real-valued martingale  $(M_n)_{n \in \mathbb{N}}$  adapted to this filtration. We use the notations  $X_n = M_n - M_{n-1}$  and  $M_n^* = \max(M_0, M_1, \dots, M_n)$  for any positive integer  $n$ .

The tail and tail-quantile functions of a real-valued random variable  $X$  are defined by

$$H_X(x) = \mathbb{P}(X > x) \text{ for } x \in \mathbb{R}, \quad Q_X(u) = \inf\{x \in \mathbb{R} : H_X(x) \leq u\} \text{ for } u \in ]0, 1]. \quad (1.1)$$

Recall that  $H_X$  is cadlag and nonincreasing and  $Q_X$  is the cadlag generalized inverse function of  $H_X$ . From the definition of  $Q_X$ , if  $U$  has the uniform law over  $[0, 1]$ , then  $Q_X(U)$  has the same law as  $X$ . The tail-quantile function  $Q_X$  is often called Value at Risk (VaR). The Conditional Value at Risk or superquantile  $\tilde{Q}_X$  of  $X$  is defined by

$$\tilde{Q}_X(u) = \int_0^1 Q_X(us) ds, \text{ for any } u \in ]0, 1]. \quad (1.2)$$

Since  $Q_X$  is nonincreasing,  $\tilde{Q}_X \geq Q_X$ . From a result which goes back to [1],

$$Q_{M_n^*}(u) \leq \tilde{Q}_{M_n}(u) \text{ for any } u \in ]0, 1]. \quad (1.3)$$

We also refer to [4] for a proof of this result. Consequently any upper bound on the superquantiles of  $M_n$  provides the same upper bound on the tail-quantiles of  $M_n^*$ . Furthermore (1.3) cannot be improved without additional conditions, as proved by [3]. These facts motivate this note.

Our approach to bound up  $\tilde{Q}_{M_n}$  is based on the  $p$ -risks  $Q_p(X, \cdot)$  introduced in [8]. Let  $z_+$  and  $z_-$  denote respectively the positive and the negative part of the real  $z$ . The  $p$ -risk  $Q_p(X, \cdot)$  of a real-valued random variable  $X$  is defined by

$$Q_p(X, u) = \inf\{-t + u^{-1/p} \|(X+t)_+\|_p : t \in \mathbb{R}\} \text{ for any } u \in ]0, 1]. \quad (1.4)$$

These  $p$ -risks are nondecreasing with respect to  $p$ . The main feature is that they are easier to bound up than the quantiles or superquantiles. Furthermore, in the case  $p = 1$ ,

$$Q_1(X, u) = \tilde{Q}_X(u) \text{ for any } u \in ]0, 1]. \quad (1.5)$$

Hence  $Q_1(X, \cdot)$  is exactly the superquantile of  $X$ . Therefrom

$$\tilde{Q}_X(u) \leq Q_p(X, u) \text{ for any } u \in ]0, 1] \text{ and any } p \geq 1. \quad (1.6)$$

We refer to [8] for more about the properties of the  $p$ -risks.

In order to bound up  $\tilde{Q}_{M_n^*}$ , we will introduce supersuperquantiles. Let  $U$  be a random variable with uniform law over  $[0, 1]$ . For a real-valued random variable  $X$ , the supersuperquantile  $Q_{1,1}(X, \cdot)$  of  $X$  is defined by

$$Q_{1,1}(X, u) = \tilde{Q}_{\tilde{Q}_X(U)}(u) \text{ for any } u \in ]0, 1]. \quad (1.7)$$

Then, from (1.3),

$$\tilde{Q}_{M_n^*}(u) \leq Q_{1,1}(M_n, u) \text{ for any } u \in ]0, 1], \quad (1.8)$$

so that any upper bound on the supersuperquantile of  $M_n$  yields the same upper bound on  $\tilde{Q}_{M_n^*}$ . Therefore the upper bounds on  $\tilde{Q}_{M_n^*}$  will be derived from the inequality below, proved in Section 2: for any  $p > 1$  and any  $u$  in  $]0, 1]$ ,

$$Q_{1,1}(X, u) \leq Q_p(X, (\Pi(q))^{1-p}u) \text{ where } q = p/(p-1), \quad \Pi(q) = \int_0^\infty t^q e^{-t} dt. \quad (1.9)$$

According to the above inequalities, it is enough to bound up the  $p$ -risks of  $M_n$ . For martingales in  $L^p$  for some  $p$  in  $[1, 2]$ , these upper bounds will be derived from one-sided von Bahr-Esseen type inequalities stated in Section 3. In the case of martingales in  $L^2$  satisfying an additional condition of order 3, these upper bounds will be derived from Inequality (1.11) below. For random variables  $Y$  and  $Z$  such that  $\mathbb{E}(Y_+^p) < \infty$  and  $\mathbb{E}(Z_+^p) < \infty$ , let

$$D_p^+(Y, Z) = \sup \{ \mathbb{E}((Z+t)_+^p - (Y+t)_+^p) : t \in \mathbb{R} \}. \quad (1.10)$$

Then, from the definition (1.4) of the  $p$ -risks, it is immediate that, for any  $u$  in  $]0, 1]$ ,

$$Q_p(Z, u) \leq \inf \{ -t + u^{-1/p} (\mathbb{E}(Y+t)_+^p + D_p^+(Y, Z))^{1/p} : t \in \mathbb{R} \} \leq Q_p(Y, u) + u^{-1/p} (D_p^+(Y, Z))^{1/p}. \quad (1.11)$$

This inequality will be used in Section 4 to provide upper bounds on the superquantiles of martingales under additional assumptions on the conditional variances of the increments and the moments of order 3 of their positive parts.

## 2 Comparison inequalities for risks

In this section we prove the comparison inequality (1.9) and we give applications of this inequality to upper bounds on the superquantiles of  $M_n^*$ . We now state the main results of this section.

**Proposition 2.1.** *Let  $p > 1$  and  $X$  be an integrable real-valued random variable such that  $\mathbb{E}(X_+^p) < \infty$ . Then  $Q_{1,1}(X, u) \leq Q_p(X, (\Pi(q))^{1-p} u)$  for any  $u$  in  $]0, 1]$ , where  $q = p/(p-1)$  and  $\Pi(q) = \int_0^\infty t^q e^{-t} dt$ .*

From Proposition 2.1 and (1.8), we immediately get the result below.

**Corollary 2.1.** *Let  $(M_n)_n$  be a martingale such that  $\mathbb{E}(M_{n+}^p) < \infty$  for some  $p > 1$ . Set  $M_n^* = \max(M_0, M_1, \dots, M_n)$ . Then  $Q_1(M_n^*, u) \leq Q_p(M_n, (\Pi(q))^{1-p} u)$  for any  $u$  in  $]0, 1]$ .*

**Proof of Proposition 2.1.** Let  $V$  is a random variable with uniform law over  $[0, 1]$ . Integrating  $\tilde{Q}_X$ , we get from (1.2) that

$$Q_{1,1}(X, u) = \mathbb{E}(Q_X(uV) \log(1/V)), \quad (2.1)$$

where  $\log$  denotes the Neper logarithm. Next, since  $\mathbb{E} \log(1/V) = 1$ ,

$$Q_{1,1}(X, u) = -t + \mathbb{E}(\log(1/V)(Q_X(uV) + t)) \leq -t + \mathbb{E}(\log(1/V)(Q_X(uV) + t)_+). \quad (2.2)$$

Now, applying the Hölder inequality, with exponents  $q = p/(p-1)$  and  $p$ ,

$$\mathbb{E}(\log(1/V)(Q_X(uV) + t)_+) \leq \|\log(1/V)\|_q \|(Q_X(uV) + t)_+\|_p.$$

Since  $\log(1/V)$  has the law  $\mathcal{E}(1)$ ,  $\|\log(1/V)\|_q = (\Pi(q))^{1/q}$  and, setting  $w = uv$ ,

$$\int_0^1 (Q_X(uv) + t)_+^p dv = u^{-1} \int_0^u (Q_X(w) + t)_+^p dw \leq u^{-1} \int_0^1 (Q_X(w) + t)_+^p dw.$$

Hence

$$\mathbb{E}(\log(1/V)(Q_X(uV) + t)_+) \leq (\Pi(q))^{1/q} u^{-1/p} \|(X + t)_+\|_p. \quad (2.3)$$

Combining (2.2) and (2.3), we now get that, for any real  $t$ ,

$$Q_{1,1}(X, u) \leq -t + ((\Pi(q))^{1-p} u)^{-1/p} \|(X + t)_+\|_p, \quad (2.4)$$

which implies Proposition 2.1.  $\diamond$

**Remark 2.1.** *From (1.4),  $Q_p(M_n, u) \leq u^{-1/p} \|M_{n+}\|_p$ . Hence, if  $M_0 = 0$ , Corollary 2.1 applied with  $u = 1$  implies the known inequality  $\|M_n^*\|_1 \leq (\Pi(q))^{1/q} \|M_{n+}\|_p$ . The constant  $(\Pi(q))^{1/q}$  in this inequality is sharp, which proves that our constant is also sharp. We refer to Theorem 7.8 in [7] for more about this.*

We now discuss Corollary 2.1. If the martingale  $(M_n)_n$  is conditionally symmetric, then, by the Lévy symmetrization inequality,  $H_{M_n^*}(x) \leq 2H_{M_n}(x)$  for any real  $x$ , which implies that  $Q_p(M_n^*, u) \leq Q_p(M_n, u/2)$  for  $p \geq 1$  and  $u$  in  $]0, 1]$ . Therefrom, for conditionally symmetric martingales,

$$Q_1(M_n^*, u) \leq Q_p(M_n, u/2) \text{ for any } p \geq 1. \quad (2.5)$$

If  $p = 2$ , Corollary 2.1 also yields  $Q_1(M_n^*, u) \leq Q_2(M_n, u/2)$ . Recall now that  $\Pi(q) = \mathbb{E}(\tau^q)$ , if  $\tau$  is a random variable with law  $\mathcal{E}(1)$ . Thus, if  $p > 2$ , then  $1 < q < 2$  and  $\Pi(q) = \mathbb{E}(\tau^q) < (\mathbb{E}\tau)^{2-q} (\mathbb{E}\tau^2)^{q-1} = 2^{q-1}$ , which implies that  $(\Pi(q))^{1-p} > 1/2$ , since  $(q-1)(1-p) = -1$ . Consequently, for  $p > 2$  Corollary 2.1 is more efficient than (2.5). For example, if  $p = 3$ ,  $Q_1(M_n^*, u) \leq Q_3(M_n, 16u/(9\pi))$  by Corollary 2.1, and  $16/(9\pi) = 0.565... > 1/2$ .

### 3 Martingales in $L^p$ for $p$ in $]1, 2]$

In this section,  $p$  is any real in  $]1, 2]$  and  $(M_n)_n$  is a martingale in  $L^p$ . Our aim is to obtain upper bounds on the risks of  $M_n$  and  $M_n^*$ . From (1.4), these upper bounds can be derived from upper bounds on the moments of order  $p$  of  $(M_n + t)_+$ . At the present time, moment inequalities with sharp constants are only available for the absolute value of  $M_n$ . More precisely, by Proposition 1.8 in [9],

$$\mathbb{E}(|M_n|^p) \leq \mathbb{E}(|M_0|^p) + K_p \mathbb{E}(|X_1|^p + \cdots + |X_n|^p), \quad \text{where } K_p = \sup_{x \in [0,1]} (px^{p-1} + (1-x)^p - x^p). \quad (3.1)$$

As shown in [9], the constant  $K_p$  is sharp. The constant  $K_p$  is decreasing with respect to  $p$ ,  $K_2 = 1$  and  $\lim_{p \searrow 1} K_p = 2$ . However, for conditionally symmetric martingales, it is known since a long time that the constant in the above inequality is equal to 1 for any  $p$  in  $]1, 2]$ . So it seems clear that the constants in the one-sided case are smaller than  $K_p$ . Below we give a new inequality.

**Theorem 3.1.** *Let  $p$  be any real in  $]1, 2]$  and  $(M_n)_n$  be a martingale in  $L^p$ . Then*

$$\mathbb{E}(M_{n+}^p) \leq \mathbb{E}(M_{0+}^p) + \Delta_p, \quad \text{with } \Delta_p = \mathbb{E}(X_{1+}^p + \cdots + X_{n+}^p) + (p-1)^{p-1} \mathbb{E}(X_{1-}^p + \cdots + X_{n-}^p). \quad (3.2)$$

Before proving Theorem 3.1, we give an application to risks.

**Corollary 3.1.** *Let  $p$  be any real in  $]1, 2]$  and  $(M_n)_n$  be a martingale in  $L^p$  such that  $M_0 = 0$ . Set  $q = p/(p-1)$ . Then  $Q_p(M_n, u) \leq \Delta_p^{1/p} (u^{1-q} - 1)^{1/q}$  and  $Q_1(M_n^*, u) \leq \Delta_p^{1/p} (\Pi(q) u^{1-q} - 1)^{1/q}$  for any  $u$  in  $]0, 1]$ .*

**Remark 3.1.** *If  $p = 2$ ,  $q = 2$  and  $\Pi(q) = 2$ . Then we get from Corollary 3.1 that*

$$Q_2(M_n, u) \leq \sqrt{\mathbb{E}(M_n^2)(1/u - 1)}, \quad Q_1(M_n^*, u) \leq \sqrt{\mathbb{E}(M_n^2)(2/u - 1)}. \quad (3.3)$$

The first inequality is a version of an inequality of Tchebichef [12], often called Cantelli's inequality. For  $p < 2$ ,  $(p-1)^{p-1} < 1$ . In that case the results are new.

**Proof of Corollary 3.1.** We start by the first inequality. Let  $u$  be any real in  $]0, 1]$ . From Theorem 3.1 applied to  $(t + M_n)_n$ , we get  $Q_p(M_n, u) \leq -t + u^{-1/p} (t^p + \Delta_p)^{1/p}$ . Now the function  $f : t \mapsto -t + u^{-1/p} (\Delta_p + t^p)^{1/p}$  has a unique minimum at point  $t = t_u = \Delta_p^{1/p} (u^{1-q} - 1)^{-1/p}$  and  $f(t_u) = \Delta_p^{1/p} (u^{1-q} - 1)^{1/q}$ , which completes the proof of the first inequality in the case  $u < 1$ . Since  $Q_p(M_n, \cdot)$  is nonincreasing, the case  $u = 1$  follows by taking the limit as  $u \uparrow 1$ . The second part follows from the first part, Corollary 2.1 and the fact that  $(1-p)(1-q) = 1$ .  $\diamond$

**Proof of Theorem 3.1.** Theorem 3.1 follows immediately from the Lemma below by induction on  $n$ .  $\diamond$

**Lemma 3.1.** *Let  $Z$  and  $X$  be real-valued random variables in  $L^p$  for some  $p$  in  $]1, 2]$ . If  $\mathbb{E}(X \mid Z) = 0$ , then  $\mathbb{E}((Z + X)_+^p) \leq \mathbb{E}(Z_+^p) + \mathbb{E}(X_+^p) + (p-1)^{p-1} \mathbb{E}(X_-^p)$ .*

**Proof of Lemma 3.1.** Define the function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\varphi(z, x) = (z + x)_+^p - z_+^p - p z_+^{p-1} x. \quad (3.4)$$

From the assumption  $\mathbb{E}(X \mid Z) = 0$ ,  $\mathbb{E}((Z + X)_+^p) - \mathbb{E}(Z_+^p) = \mathbb{E}(\varphi(Z, X))$ . Consequently Lemma 3.1 follows immediately from the upper bound

$$\varphi(z, x) \leq x_+^p + (p-1)^{p-1} x_-^p \quad \text{for any } (x, z) \in \mathbb{R} \times \mathbb{R}. \quad (3.5)$$

We now prove (3.5). If  $z \leq 0$ , then  $\varphi(z, x) = (z + x)_+^p \leq x_+^p$ , which proves (3.5) for  $z \leq 0$ .

If  $z \geq 0$ , let the function  $\eta_x$  be defined by  $\eta_x(z) = \varphi(z, x)$ . The function  $\eta_x$  is continuous on  $[0, \infty[$ , differentiable on  $]0, \infty[$ , and  $\eta'_x(z) = p((z + x)_+^{p-1} - z^{p-1} - (p-1)z^{p-2}x)$  for  $z > 0$ . If  $z \geq x_-$ ,  $z + x \geq x_+ + x \geq 0$ , which implies that  $(z + x)_+^{p-1} = (z + x)^{p-1}$ . Then the concavity of  $t \mapsto t^{p-1}$  ensures that  $\eta'_x(z) \leq 0$ . It follows that  $\eta_x$  is nonincreasing on  $[x_-, \infty[$ . If  $x \geq 0$ , then  $x_- = 0$  and  $\eta_x(z) \leq \eta_x(0) = x_+^p$ , which proves (3.5) for  $z \geq 0$  and  $x \geq 0$ .

Finally, if  $z \geq 0$  and  $x < 0$ ,  $z + x \leq 0$  for  $z$  in  $[0, x_-]$ . Thus  $\eta'_x(z) = p z^{p-2} (-z + (p-1)x_-)$  for  $z$  in  $]0, x_-]$ . Since  $\eta_x$  is nonincreasing on  $[x_-, \infty[$ , it follows that  $\eta_x$  has a unique maximum at point  $z = (p-1)x_-$  and, subsequently,

$$\eta_x(z) \leq \eta_x((p-1)x_-) = (-(p-1)^p + p(p-1)^{p-1})x_-^p = (p-1)^{p-1}x_-^p, \quad (3.6)$$

which proves (3.5) for  $z \geq 0$  and  $x < 0$ , therefore completing the proof of (3.5).  $\diamond$

**Numerical comparisons.** To conclude this section, we compare the upper bounds given by Corollary 3.1 with the inequality below, derived from (3.1) and Theorem 4.1 in [11]:

$$Q_1(M_n, u) \leq \Sigma_p^{1/p} u^{-1/p} (1 + (1 - u)^{1-p} u^{p-1})^{-1/p}, \quad \text{with } \Sigma_p = K_p \mathbb{E}(|X_1|^p + \dots + |X_n|^p). \quad (3.7)$$

For the numerical comparisons we assume that  $\sum_{k=1}^n \mathbb{E}(X_{k+}^p) = \sum_{k=1}^n \mathbb{E}(X_{k-}^p) = 1$ . Then Corollary 3.1 yields

$$Q_1(M_n, u) \leq (1 + (p-1)^{p-1})^{1/p} u^{-1/p} (1 - u^{q-1})^{1/q}, \quad \text{with } q = p/(p-1), \quad (3.8)$$

and  $\Sigma_p = 2K_p$  in (3.7). The table below gives values of the upper bounds (3.7) and (3.8) for  $p = 3/2$ , in which case  $2K_p = 2(1 + 1/\sqrt{2})^{1/2}$  and  $1 + (p-1)^{p-1} = 1 + 1/\sqrt{2}$ . Here (3.8) provides better bounds for  $u \leq 0.25$  and  $u \geq 0.9922$ .

Value of $u$	0.999	0.990	0.900	0.75	0.50	0.25	0.10	0.010	0.001
(3.7)	0.186	<b>0.387</b>	<b>0.808</b>	<b>1.18</b>	<b>1.90</b>	3.53	7.27	38.34	185.8
(3.8)	<b>0.180</b>	0.390	0.881	1.31	2.06	<b>3.52</b>	<b>6.61</b>	<b>30.77</b>	<b>142.8</b>

## 4 The case $p = 3$

In this section,  $(M_n)_n$  is a martingale in  $L^2$  such that  $M_0 = 0$ . We assume that, for some sequence  $(\sigma_k)_{k>0}$  of nonrandom positive reals,

$$\mathbb{E}(X_{k+}^3) < \infty \quad \text{and} \quad \mathbb{E}(X_k^2 | \mathcal{F}_{k-1}) \leq \sigma_k^2 \quad \text{almost surely, for any positive } k. \quad (4.1)$$

Although the above condition on the conditional variances is very strong, it is sometimes fulfilled. In particular, the second part of (4.1) holds for martingale decompositions associated to dynamical systems or suprema of empirical processes. We refer to [2], Inequality (4.9), page 861, for dynamical systems and to [6] for empirical processes. The main result of this section is the upper bound below on  $\mathbb{E}((M_n + t)_+^3)$ .

**Theorem 4.1.** *Let  $Y$  be a random variable with law  $N(0, 1)$  and  $(M_n)_n$  be a martingale such that  $M_0 = 0$ , satisfying (4.1). Set  $V_n = \sigma_1^2 + \dots + \sigma_n^2$ . Then  $\mathbb{E}((M_n + t)_+^3) \leq \mathbb{E}((Y\sqrt{V_n} + t)_+^3) + \sum_{k=1}^n \mathbb{E}(X_{k+}^3)$  for any real  $t$ .*

**Remark 4.1.** *From Theorem 4.1 with  $t = 0$ ,  $\mathbb{E}(M_{n+}^3) \leq V_n^{3/2} \mathbb{E}(Y_+^3) + \sum_{k=1}^n \mathbb{E}(X_{k+}^3)$ , which is a one-sided version of the Rosenthal inequality, with the optimal constants. We refer to [10] and the references therein for more about the constants in the Rosenthal inequalities.*

**Proof of Theorem 4.1.** Let  $(Y_k)_{k>0}$  be a sequence of independent random variables with law  $N(0, 1)$ , independent of the sequence  $(M_n)_n$ . Define the random variables  $T_k^n$  and the reals  $D_k^n$  for  $k$  in  $[1, n]$  by

$$T_k^n = M_{k-1} + (\sigma_{k+1}Y_{k+1} + \dots + \sigma_n Y_n), \quad D_k^n = \mathbb{E}((T_k^n + X_k)_+^3 - (T_k^n + \sigma_k Y_k)_+^3), \quad (4.2)$$

with the convention that  $T_n^n = M_{n-1}$ . Then

$$\mathbb{E}((M_n + t)_+^3 - (Y\sqrt{V_n} + t)_+^3) = D_1^n + \dots + D_n^n. \quad (4.3)$$

Now the function  $\varphi$  defined by  $\varphi(x) = x_+^3$  for  $x$  in  $\mathbb{R}$  is two times continuously differentiable and  $\varphi'(x) = 3x_+^2$ ,  $\varphi''(x) = 6x_+$ . Hence, applying the Taylor integral formula at order 2 to the function  $\varphi$  at point  $T_k^n$ ,

$$D_k^n = 3\mathbb{E}((T_{k+}^n)^2(X_k - \sigma_k Y_k)) + 3\mathbb{E}(T_{k+}^n(X_k^2 - \sigma_k^2 Y_k^2)) + 6 \int_0^1 (1-t) R_{k,n}(t) dt, \quad (4.4)$$

$$\text{with } R_{k,n}(t) = \mathbb{E}(((T_k^n + tX_k)_+ - T_{k+}^n)X_k^2 - ((T_k^n + t\sigma_k Y_k)_+ - T_{k+}^n)\sigma_k^2 Y_k^2). \quad (4.5)$$

From the martingale assumption, the first term on right hand in (4.4) is equal to 0. Next

$$\mathbb{E}(T_{k+}^n(X_k^2 - \sigma_k^2 Y_k^2)) = \mathbb{E}(T_{k+}^n(\mathbb{E}(X_k^2 | \mathcal{F}_{k-1}) - \sigma_k^2)) \leq 0,$$

since  $T_{k+}^n \geq 0$  and  $\mathbb{E}(X_k^2 | \mathcal{F}_{k-1}) - \sigma_k^2 \leq 0$  almost surely.

From the above inequalities, the two first terms in (4.4) are nonpositive. It remains to bound up the integral term in (4.4). First  $(T_{k,n} + tX_k)_+ - T_{k,n} \leq tX_k^+$  for any  $t$  in  $[0, 1]$ , which implies that

$$\mathbb{E}(((T_k^n + tX_k)_+ - T_{k+}^n)X_k^2) \leq t\mathbb{E}(X_{k+}^3). \quad (4.6)$$

And second the normal law is symmetric, whence

$$\mathbb{E}(((T_k^n + t\sigma_k Y_k)_+ - T_{k+}^n)Y_k^2) = \frac{1}{2} \mathbb{E}(((T_k^n + t\sigma_k Y_k)_+ + (T_k^n - t\sigma_k Y_k)_+ - 2T_{k+}^n)Y_k^2).$$

Since the function  $x \mapsto x_+$  is convex,  $(T_k^n + t\sigma_k Y_k)_+ + (T_k^n - t\sigma_k Y_k)_+ - 2T_{k+}^n \geq 0$ . It follows that

$$\mathbb{E}(((T_k^n + t\sigma_k Y_k)_+ - T_{k+}^n)Y_k^2) \geq 0. \quad (4.7)$$

Now (4.5), (4.6) and (4.7) imply that  $R_{k,n}(t) \leq t\mathbb{E}(X_{k+}^3)$ . Finally, putting this inequality in (4.4) and integrating, we get that  $D_k^n \leq \mathbb{E}(X_{k+}^3)$ , which, by (4.3), implies Theorem 4.1.  $\diamond$

From Theorem 4.1, (1.11) and Corollary 2.1, we immediately get the following asymptotically subGaussian upper bounds on the superquantiles of  $M_n$  and  $M_n^*$ .

**Corollary 4.1.** *Let  $Y$  be a random variable with law  $N(0, 1)$  and  $(M_n)_n$  be a martingale such that  $M_0 = 0$ , satisfying (4.1). Set  $V_n = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$ . Then, for any  $p$  in  $[1, 3]$  and any  $u$  in  $]0, 1]$*

$$Q_p(M_n, u) \leq \inf_{t \in \mathbb{R}} \left\{ -t + u^{-\frac{1}{3}} \left( \mathbb{E}((Y\sqrt{V_n} + t)_+^3) + \sum_{k=1}^n \mathbb{E}(Y_{k+}^3) \right)^{\frac{1}{3}} \right\} \leq \sqrt{V_n} Q_3(Y, u) + u^{-\frac{1}{3}} \left( \sum_{k=1}^n \mathbb{E}(Y_{k+}^3) \right)^{\frac{1}{3}}, \quad (a)$$

$$Q_1(M_n^*, u) \leq \inf \left\{ -t + \left( \frac{9\pi}{16u} \right)^{1/3} \left( \mathbb{E}((Y\sqrt{V_n} + t)_+^3) + \sum_{k=1}^n \mathbb{E}(Y_{k+}^3) \right)^{1/3} : t \in \mathbb{R} \right\}. \quad (b)$$

**Numerical comparisons.** To conclude this section, we compare Corollary 4.1(a) with upper bounds derived from moment inequalities or estimates of the Kantorovich distance in the central limit theorem. From Theorem 4.1 applied with  $t = 0$  and the Hölder inequality,

$$Q_1(M_n, u) \leq u^{-1/3} (\sqrt{2/\pi} + \mathbb{E}(X_{1+}^3 + \dots + X_{n+}^3))^{1/3}. \quad (4.8)$$

Such a result is called weak  $L^3$  concentration inequality in [2]. If furthermore the increments  $X_k$  are independent and in  $L^3$ , then, by Theorem 1.1 in [5], for any 1-Lipschitz function  $f$ ,

$$\mathbb{E}(f(M_n) - f(Y\sqrt{V_n})) \leq \mathbb{E}(|X_1|^3 + \dots + |X_n|^3), \quad \text{with } V_n = \text{Var} M_n. \quad (4.9)$$

Now, since  $\mathbb{E}(M_n) = \mathbb{E}(Y) = 0$ , by (4.9) and the elementary equality  $x_+ = (x + |x|)/2$ ,

$$\mathbb{E}((M_n + t)_+ - (Y\sqrt{V_n} + t)_+) \leq \frac{1}{2} \mathbb{E}(|X_1|^3 + \dots + |X_n|^3)$$

for any real  $t$ . Hence, by (1.11) applied with  $p = 1$ , for any  $u$  in  $]0, 1]$ ,

$$Q_1(M_n, u) \leq \sqrt{V_n} Q_1(Y, u) + (2u)^{-1} \mathbb{E}(|X_1|^3 + \dots + |X_n|^3). \quad (4.10)$$

The table below gives numerical values for the upper bounds of (4.10), Corollary 4.1(a), their respective limits  $Q_1(Y, u)$  and  $Q_3(Y, u)$  (as the Liapounov ratio tends to 0) and (4.8) in the case  $V_n = 1$  and  $L_3^+ := \mathbb{E}(X_{1+}^3 + \dots + X_{n+}^3) = \mathbb{E}(X_{1-}^3 + \dots + X_{n-}^3) := L_3^-$ , for  $L_3^+ = 10^{-m}$ ,  $m = 1, 2, 3$  and  $u = 2^{k-2} 10^{-k}$ ,  $k = 0, 1, 2$ . For sake of completeness, the values of the usual subGaussian bound  $\sqrt{2|\log u|}$  (which is larger than  $Q_p(Y, u)$ , as shown in [8]) are also included. One can observe that the convergence to the limit is much faster in Corollary 4.1(a) than in (4.10). As a by-product, Corollary 4.1(a) still provides better bounds for  $u \leq 1/20$  if  $L_3^+ = 10^{-2}$ , which is in the range of normal approximation, since the Liapounov ratio  $L_3 := L_3^+ + L_3^-$  is equal to  $2 \cdot 10^{-2}$ . For all the values of the Liapounov ratio in the table, Inequality (4.8) is of poor quality for  $u = 1/20$  and very poor quality for  $u = 10^{-2}$ , which shows that moment inequalities are not a suitable tool to achieve efficient concentration inequalities if the Liapounov ratio is small.

Value of $(k, m)$	(0,1)	(1, 1)	(2,1)	(0,2)	(1,2)	(2,2)	(0,3)	(1,3)	(2,3)
Value of $Q_1(Y, u)$	1.27	2.06	2.67	1.27	2.06	2.67	1.27	2.06	2.67
Inequality (4.10)	1.67	4.06	12.67	<b>1.31</b>	<b>2.26</b>	3.67	<b>1.274</b>	<b>2.08</b>	<b>2.77</b>
Inequality (4.8)	1.53	2.62	4.48	1.478	2.53	4.32	1.473	2.52	4.31
Corollary 4.1(a)	<b>1.50</b>	<b>2.48</b>	<b>3.67</b>	1.467	<b>2.26</b>	<b>3.01</b>	1.463	2.23	2.84
Value of $Q_3(Y, u)$	1.462	2.22	2.81	1.462	2.22	2.81	1.462	2.22	2.81
Value of $\sqrt{2 \log u }$	1.665	2.447	3.035	1.665	2.447	3.035	1.665	2.447	3.035

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