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A DICHOTOMY FOR MEASURES OF MAXIMAL ENTROPY NEAR TIME-ONE MAPS OF TRANSITIVE ANOSOV FLOWS

JÉRÔME BUZZI, TODD FISHER, AND ALI TAHZIBI

ABSTRACT. We show that time-one maps of transitive Anosov flows of compact manifolds are accumulated by diffeomorphisms robustly satisfying the following dichotomy: either all of the measures of maximal entropy are non-hyperbolic, or there are exactly two ergodic measures of maximal entropy, one with a positive central exponent and the other with a negative central exponent.

We establish this dichotomy for certain partially hyperbolic diffeomorphisms isotopic to the identity whenever both of their strong foliations are minimal. Our proof builds on the approach developed by Margulis for Anosov flows where he constructs suitable families of measures on the dynamical foliations.

RÉSUMÉ. Nous montrons que l'application du temps 1 de tout flot d'Anosov transitif est dans l'adhérence des difféomorphismes présentant robustement la dichotomie suivante: ou bien aucune mesure maximisant l'entropie n'est hyperbolique, ou bien il existe exactement deux mesures ergodiques maximisant l'entropie, l'une ayant un exposant central strictement positif, l'autre strictement négatif.

Nous établissons cette dichotomie pour certains difféomorphismes partiellement hyperboliques isotopes à l'identité sous l'hypothèse de la minimalité de leurs feuilletages invariants forts. Notre preuve s'appuie sur l'approche développée par Margulis dans le cas des flots d'Anosov et la construction de familles de mesures convenables sur les différents feuilletages dynamiques.

1. INTRODUCTION

In his pioneering work [32], Margulis studied measures of maximal entropy of geodesic flows in order to count closed geodesics for manifolds with variable negative curvature. More precisely, he constructed a family of measures $\{m_x\}_{x \in M}$ such that for all $x \in M$ the measure m_x is carried by the unstable manifold at x , and for all $t \in \mathbb{R}$ we have

$$(\varphi^t)_* m_x = e^{-t \cdot h_{\text{top}}(\varphi)} m_{\varphi^t x}.$$

He then built an invariant probability measure which was observed to be a measure of maximal entropy and is now called the Bowen-Margulis measure. It was then proved to be the unique measure of maximal entropy. We refer to Ledrappier [30] for an introduction.

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In this paper, we will extend Margulis' construction to a class of partially hyperbolic maps and obtain a striking dichotomy.

Theorem 1.1. *If φ^t is a transitive Anosov flow on a compact manifold M , then there is an open set \mathcal{U} in $\text{Diff}^1(M)$ which contains φ^1 in its closure such that for any $f \in \mathcal{U} \cap \text{Diff}^2(M)$ we have the following dichotomy:*

- (1) *either all the measures of maximal entropy have zero central Lyapunov exponents, or*
- (2) *there are exactly two ergodic measures of maximal entropy where one has a positive central exponent and the other has a negative central exponent, and both measures are Bernoulli.*

Related results. These results are part of a larger program to understand properties of entropy beyond uniform hyperbolicity. In that classical setting, say for a transitive Anosov diffeomorphism, there is a unique measure of maximal entropy.¹ Beyond the hyperbolic setting, even though there are a number of significant results [35, 12, 13] there are still many fundamental open questions. Partially hyperbolic diffeomorphisms with one-dimensional center have been studied as the “next non-trivial class”. A MME always exists in this setting by entropy expansivity (see [15, 19, 31]). Its uniqueness is a more delicate question.

This uniqueness has been shown for certain systems that are *derived from Anosov*, a subclass introduced by Mañé, first for specific constructions, then in greater and greater generality [10, 16, 50, 20, 11].

The partially hyperbolic diffeomorphisms with a center foliation into circles form another subclass with a more subtle behavior. Assuming accessibility, [39] has established the following dichotomy:

- either the dynamics is isometric in the center direction and there exists a unique MME which is nonhyperbolic; or
- there are multiple hyperbolic MMEs.

Strategy of proof. We introduce a new subclass of partially hyperbolic diffeomorphisms with one-dimensional center which we call *flow-type*. They are isotopic to the identity and the fundamental examples are the perturbations of time-one maps of Anosov flows. Our main result is Theorem 3.10: the above dichotomy holds for partially hyperbolic flow-type diffeomorphisms whose strong foliations are both minimal.

The uniqueness of the MME for a given sign of the central exponent (say nonpositive) follows from a variant of Margulis' approach. Namely, we first build a family of measures on the center-unstable leaves which is invariant under stable holonomies and projectively invariant under the dynamics. Then we construct measures on unstable leaves, which we call Margulis u -conditionals. This is more difficult for maps than for flows.

We then use the entropy with respect to the unstable foliation as introduced by Ledrappier and Young [28] and an argument of Ledrappier [26] to show that, when its central exponent is nonpositive, a measure maximizes the entropy if and only if its disintegration along the unstable leaves is given by the Margulis conditionals.

¹Throughout this paper, we will abbreviate *ergodic Borel probability measure maximizing the entropy* to MME. Recall that a measure maximizes the entropy if and only if it is invariant and almost all its ergodic components are MMEs.

A Hopf argument shows that if there is a MME with negative central exponent, then any MME with nonpositive central exponent must coincide with it. The symmetry between positive and negative central exponents in the hyperbolic case follows from the one-dimensionality of the central leaves: we associate to any measure with, say negative central exponent, an isomorphic one with nonnegative central exponent.

A hyperbolic MME is isomorphic to a Bernoulli shift times a circular permutation, according to a general result by Ben Ovadia [36]. The triviality of the permutation follows by considering iterates. This concludes the proof of Theorem 3.10.

Finally, to prove Theorem 1.1 we establish Theorem 3.11, i.e., we find open sets of partially hyperbolic flow-type diffeomorphisms with both strong foliations minimal near any time-one map of a transitive Anosov flow. We first show that such time-one maps are robustly flow-type, partially hyperbolic diffeomorphisms. Then Bonatti and Díaz [6] provide a perturbation ensuring robust transitivity. Lastly, by a further perturbation following Bonatti, Díaz, and Urès [7] we ensure robust minimality of both strong foliations. Theorem 1.1 now follows from Theorem 3.10.

The use of Margulis conditionals. The construction of Margulis has given rise to a large body of work, mainly devoted to the estimation of the number of periodic orbits, sometimes beyond the uniformly hyperbolic setting [25]. We refer to Sharp's survey in [33], the long awaited publication of Margulis' thesis. The works of Hamenstädt [22] and Hasselblatt [23] that give a geometric description of the Margulis conditionals $\{m_x\}_{x \in M}$ are perhaps closer to our concerns. We also note Plante's point of view [38] (see also [45, 43]): measures invariant under stable holonomies can be seen as transverse measures. We have not pursued this point of view here.

While this work was being written, we learned that a different but related approach has been developed in [14]. This approach can deal with equilibrium measures (i.e., generalizations of measures of maximal entropy taking into account a weight function) but requires non-expansion along the center. Separately, Jiagang Yang has told us that he also has some related results.

Comments. First, we note that our result, Theorem 3.10, will be proved for a class of partially hyperbolic diffeomorphisms which we call *flow-type* (see Def. 3.1). This class turns out to coincide², at least in dimension 3, with the *discretized Anosov flows* introduced by Barthelmé, Fenley, Frankel, and Potrie in the classification of partially hyperbolic diffeomorphisms [5]. Work by Martinchich [34] is extending this to any dimension, and proves that the set of such diffeomorphisms is both open and closed among partially hyperbolic diffeomorphisms (making this property invariant by isotopy).

We point out that even if our result main's application is to perturbations of time-one maps of flows with very good properties, those perturbations are almost

²Discretized Anosov flows are partially hyperbolic diffeomorphisms that can be written as $x \mapsto \psi^{\tau(x)}(x)$ for some topological Anosov flow ψ and some positive continuous function τ . In dimension 3, [5, Proposition G.2] shows that discretized Anosov flows are, among the partially hyperbolic diffeomorphisms, exactly those which satisfy our Definition 3.1 with the additional condition that the underlying flow ψ is topological Anosov. However, this is automatic by [9, Theorem 2].

never time-one maps of a perturbed flow [37]. This well-known fact explains why some properties, such as stable ergodicity [21], have been extended from time-one maps to their perturbations only with great difficulties.

Second, we note that part of our results could be obtained from symbolic dynamics, using generalizations of ideas going back to the classical works of Sinai, Ruelle, and Bowen (see, e.g., [47, 8, 44]). More precisely, existence of at most one MME with, say, positive central exponent can be deduced from [12, Section 1.6] since, in the terminology of this work, our minimality assumption implies that there is a unique homoclinic class of measures with a given sign of the central exponent. However, the dichotomy does not seem to follow from this approach which is blind to nonhyperbolic measures.

Thirdly, one usually expects that results such as ours can be extended to $C^{1+\alpha}$ smoothness, for any $0 < \alpha < 1$, and generalized to equilibrium measures with respect to Hölder-continuous potentials (although uniqueness holds for generic potentials [44]).

Questions. Our techniques demand a very strong form of irreducibility: the minimality of both strong foliations. We ask:

Question 1. *In Theorem 3.10, can one replace minimality of both strong foliations by minimality of just one of those foliations or by robust transitivity of the diffeomorphism?*

This minimality is obtained in Theorem 1.1 by a C^1 perturbation assuming the topological transitivity of the Anosov flow. We believe that some non transitive Anosov flows could be accumulated by diffeomorphisms having more than two MMEs.

We expect both the hyperbolic and nonhyperbolic cases to occur with the hyperbolic case being generic. Indeed, in the conservative setting, there is a very strong rigidity result [4]. We think that some version of it may hold for MMEs in the dissipative setting.

Question 2. *In the setting of Theorem 3.10, is the hyperbolic case open and dense? When the MME is nonhyperbolic, does this imply that the diffeomorphism is the time one map of a flow? does it at least exclude the existence of hyperbolic periodic points?*

Though we will identify the disintegrations of nonhyperbolic MMEs along *both* strong foliations, our analysis remains incomplete:

Question 3. *Consider a partially hyperbolic diffeomorphism f with flow-type and with minimality of both strong foliations. Assume that its MMEs are nonhyperbolic. Can their disintegration along the center be atomic like in the hyperbolic case? Can there be more than one nonhyperbolic MME? Are nonhyperbolic MMEs Bernoulli?*

Remark 1.2. *After the preprint version of this work was released, Crovisier and Poletti [17] have announced positive answers to Questions 2 and 3. They generalized Avila-Viana's invariance principle (see [27], [3] and [48]) to partial hyperbolicity with quasi-isometric center and applied it to the nonhyperbolic MMEs in our setting using the disintegration we establish here (see Proposition 5.5).*

We prove that the hyperbolic MMEs are Bernoulli, hence strongly mixing. One can try to establish some speed (see [51] for a related result).

Question 4. *If μ is a hyperbolic MME for a flow-type diffeomorphism f with minimality of both strong foliations, does it satisfy exponential decay of correlations for Hölder-continuous functions, i.e., for any Hölder-continuous functions $u, v : M \rightarrow \mathbb{R}$, does there exist a number $\kappa < 1$ such that:*

$$\int_M u \circ f^n \cdot v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)?$$

For Anosov flows, the topological entropy can obviously be changed by perturbations whereas it is locally constant for Anosov diffeomorphisms. What is the situation for the maps we consider?

Question 5. *Consider a flow-type diffeomorphism whose strong foliations are robustly both minimal. Is it true that the volume growth of each strong leaf is equal to the topological entropy? Does the topological entropy have a homological interpretation? Can an arbitrarily small perturbation make the topological entropy locally constant as a function of the diffeomorphism?*

Our approach avoids constructing the Margulis conditional on strong leaves that meet a compact leaf. We do not know if the construction could be extended to the whole manifold, e.g., by passing to some cover. Such continuity would imply uniqueness of these Margulis systems at least in the hyperbolic case. We therefore ask:

Question 6. *Does a flow-type diffeomorphism with both strong foliation robustly minimal admit continuous Margulis conditional on those?*

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2. BACKGROUND

In this section we review concepts of partial hyperbolicity, Lyapunov exponents, and disintegration of measures.

2.1. Partial hyperbolicity. For a diffeomorphism $f : M \rightarrow M$ of a compact manifold to itself recall the norm and conorm with respect to a subspace of $V \subset T_x M$ for some $x \in M$: $\|Df|V\| := \max\{\|Tf(v)\| : v \in V, \|v\| = 1\}$ and

$$\text{conorm}(Df|V) := \min\{\|Tf(v)\| : v \in V, \|v\| = 1\}.$$

A splitting $E \oplus F$ is dominated³ if it is nontrivial, invariant, and if there is some $N \geq 1$ such that, for all $x \in M$:

$$\|Df^N|E_x\| < \frac{1}{2} \text{conorm}(Df^N|F_x).$$

Definition 2.1. *A diffeomorphism is (strongly) partially hyperbolic if there is an invariant splitting of the tangent bundle: $TM = E^s \oplus E^c \oplus E^u$ such that $E^s \oplus (E^c \oplus E^u)$ and $(E^s \oplus E^c) \oplus E^u$ are dominated, E^s is uniformly contracted, and E^u is uniformly expanded.*

³This is sometimes called *pointwise* domination, see [2].

The stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are always uniquely integrable into stable and unstable foliations, respectively, denoted by \mathcal{F}^s and \mathcal{F}^u . The bundles E^c , $E^{cs} := E^s \oplus E^c$, and $E^{cu} := E^c \oplus E^u$ fail to be integrable for some strongly partially hyperbolic diffeomorphisms.

Definition 2.2. *A strongly partially hyperbolic diffeomorphism f is **dynamically coherent** if there exists invariant foliations \mathcal{F}^{cs} and \mathcal{F}^{cu} that are tangent to the E^{cs} and E^{cu} bundles respectively. In this case there is a center foliation \mathcal{F}^c given by $\mathcal{F}^c(x) = \mathcal{F}^{cs}(x) \cap \mathcal{F}^{cu}(x)$ for $x \in M$.*

We refer to [9] for various other definitions of dynamical coherence and their relationships.

For a dynamically coherent diffeomorphism f each leaf of \mathcal{F}^{cs} is subfoliated by the leaves of \mathcal{F}^c and the leaves of \mathcal{F}^s . A similar statement holds for the center-unstable foliation. Then for any points $p, q \in M$ where $q \in \mathcal{F}^s(p)$ there is a neighborhood U_p of p in the leaf $\mathcal{F}^c(p)$ and a homeomorphism $h_{p,q}^s : U_p \rightarrow \mathcal{F}^c(q)$ which slides points along the stable manifolds inside a foliation chart, i.e.,

$$h_{p,q}^s(x) \in \mathcal{F}_{\text{loc}}^s(x) \cap \mathcal{F}_{\text{loc}}^c(q).$$

The map $h_{p,q}^s$ is the *local stable holonomy map*. A holonomy map is any homeomorphism $h : A \rightarrow B$ which coincides locally with a composition of such local holonomy maps. Its *size* is $\Delta := \sup_{x \in A} d^s(p, h(p))$ where d^s is the distance given by the intrinsic metric of the stable manifolds. We then say that A and B are Δ -*equivalent along \mathcal{F}^s* or (s, Δ) -*equivalent* for short. These notions extends in the obvious way to the other dynamical foliations. Unstable holonomy maps are defined similarly.

2.2. Center Lyapunov exponents. For a strongly partially hyperbolic diffeomorphism $f : M \rightarrow M$ a real number χ is a *center Lyapunov exponent* at $x \in M$ if there exists a nonzero vector $v \in E_x^c$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n(v)\| = \chi.$$

If $\dim E^c = 1$, then the limit above only depends on x and exists m -almost everywhere for any f -invariant Borel probability measure m . For m an ergodic f -invariant Borel probability measure the limit takes on a single value for m -almost every $x \in M$.

2.3. Disintegration of a measure. Let X be a Polish space and μ be a finite Borel measure on X . Let \mathcal{P} be a partition of X into measurable sets. Let $\hat{\mu}$ be the induced measure on the σ -algebra generated by \mathcal{P} . A *system of conditional measures of μ* with respect to \mathcal{P} is a family $\{\mu_P\}_{P \in \mathcal{P}}$ of probability measures on X such that

- (1) $\mu_P(P) = 1$ for μ -almost every $P \in \mathcal{P}$, and
- (2) given any continuous function $\psi : X \rightarrow \mathbb{R}$, the function $P \mapsto \int \psi d\mu_P$ is integrable, and

$$\int_X \psi d\mu = \int_{\mathcal{P}} \left(\int \psi d\mu_P \right) d\hat{\mu}(P).$$

Rokhlin [40, 41] proved that if \mathcal{P} is a measurable partition, then the disintegration always exists and is essentially unique.

We will consider partitions given by foliations of a manifold. If a foliation has a positive measure set of noncompact leaves, then the result of Rokhlin does not immediately apply. However, one can extend the result of Rokhlin by disintegrating into *measures defined up to scaling* (see Avila, Viana, and Wilkinson [4]).

Let M be a manifold where $\dim(M) \geq 2$ and m be a locally finite measure on M . Let \mathcal{B} be a small foliation box. Then Rokhlin's result implies there is a disintegration $\{m_x^{\mathcal{B}} : x \in \mathcal{B}\}$ of the restriction of m to the foliation box into conditional probability measures along the local leaves of the foliation, i.e., the connected components $\mathcal{F}_{\mathcal{B}}(x)$ of $\mathcal{F}(x) \cap \mathcal{B}$ for $x \in \mathcal{B}$. From [4, Lemma 3.2] we know that if \mathcal{B} and \mathcal{B}' are foliation boxes and m -almost any $x \in \mathcal{B} \cap \mathcal{B}'$, then the restriction of $m_x^{\mathcal{B}}$ and $m_x^{\mathcal{B}'}$ coincide up to a constant factor.

We then know that for m -almost every $x \in M$ there is a projective measure m_x (i.e., defined up to some scaling possibly depending on x) such that $m_x(M \setminus \mathcal{F}(x)) = 0$. Furthermore, the function $x \mapsto m_x$ is constant along the leaves of \mathcal{F} , and the conditional probabilities $m_x^{\mathcal{B}}$ along the local leaves of any foliation box \mathcal{B} coincide almost everywhere with the normalized restriction of the m_x to the local leaves of \mathcal{B} .

Finally, we note that if the measure m is invariant and the one-dimensional foliation \mathcal{F} is fixed by some diffeomorphism (i.e., $f(\mathcal{F}(x)) = \mathcal{F}(x)$) without fixed points, one can replace the projective measures by true measures using the global normalization: $m_x([x, f(x)]_c) = 1$ for all $x \in M$ where $[x, f(x)]_c$ is the segment in the center manifold between the points x and $f(x)$.

2.4. Measurable systems of measures. We will work with families of measures carried by the leaves of the dynamical foliations up to a union of exceptional leaves.

Definition 2.3. *Given a foliation \mathcal{F} of some manifold M and some \mathcal{F} -saturated subset $M_1 \subset M$ a **measurable system of measures** on $\mathcal{F}|_{M_1}$ is a family $\{m_x\}_{x \in M_1}$ such that:*

- (i) for all $x \in M_1$, m_x is a Radon measure on $\mathcal{F}(x)$;
- (ii) for all $x, y \in M_1$, $m_x = m_y$ if $\mathcal{F}(x) = \mathcal{F}(y)$;
- (iii) M is covered by foliation charts B such that: $x \mapsto m_x(\phi|_{\mathcal{F}_B(x)})$ is measurable on M_1 for any $\phi \in C_c(B)$.

The Radon property (i) means that each m_x is a Borel measure and is finite on compact subsets of the leaf $\mathcal{F}(x)$ (here, and elsewhere, we consider the intrinsic topology on each leaf).

If $\{\mu_x\}_{x \in M}$ is the disintegration of some probability measure μ along a foliation \mathcal{F} as defined in the previous definition and if $\{m_x\}_{x \in M_1}$ is a measurable system of measures on $\mathcal{F}|_{M_1}$, we will say that they *coincide* if $\mu(M_1) = 1$ and for μ -a.e. $x \in M_1$, μ_x and m_x are proportional.

Definition 2.4. *Assume that \mathcal{F} is a foliation which is invariant under some diffeomorphism $f : M \rightarrow M$, i.e., for all $x \in M$, $f(\mathcal{F}(x)) = \mathcal{F}(f(x))$. Let $M_1 \subset M$ be \mathcal{F} -saturated. A measurable system of measures $\{m_x\}_{x \in M_1}$ on $\mathcal{F}|_{M_1}$ is **dilated** if there is some number $D > 0$ such that for all $x \in M_1 \cap f^{-1}(M_1)$:*

$$(1) \quad f_* m_x = D^{-1} m_{f(x)}.$$

D is called the **dilation factor**. We call the family $\{m_x\}_{x \in M_1}$ a **Margulis system** on \mathcal{F} and the measures m_x the **Margulis \mathcal{F} -conditionals**.

Our construction (following Margulis) relies on properties of the holonomy between foliations defined as follows:

Definition 2.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be foliations which are invariant under some diffeomorphism $f \in \text{Diff}^1(M)$. Let M_1 be an \mathcal{F}_1 -saturated subset of M . Assume that $\{m_x\}_{x \in M_1}$ is a Margulis system of measures on $\mathcal{F}_1|_{M_1}$ and that \mathcal{F}_2 is transverse to \mathcal{F}_1 . The system $\{m_x\}_{x \in M_1}$ is **invariant**, respectively **quasi-invariant**, along \mathcal{F}_2 if, for all \mathcal{F}_2 -holonomies $h : U \rightarrow V$ with U, V contained in \mathcal{F}_1 -leaves included in M_1 :

$$(2) \quad h_*(m_x|U) = m_{h(x)}|V \text{ for any } x \in U,$$

respectively:

$$(3) \quad h_*(m_x|U) \text{ and } m_{h(x)}|V \text{ are equivalent for any } x \in U.$$

Remark 2.6. The quasi-invariance in (3) can be characterized by the absolute continuity of the holonomies along \mathcal{F}_2 with respect to a class of transversal measures defined by the Margulis system on \mathcal{F}_1 .

Remark 2.7. A system of measures along a foliation \mathcal{F}_1 which is invariant along \mathcal{F}_2 can be seen as a \mathcal{F}_2 -transverse measure in the sense of Plante [38], Ruelle [43], and Ruelle-Sullivan [45].

Though an arbitrary measurable system of measures along the strong unstable foliation does not need to correspond to the disintegration of any invariant probability measure, those we construct in this paper will (see Proposition 5.1).

3. MAIN RESULTS

This section collects our main results. Our techniques deal with the following type of diffeomorphisms. For convenience, we fix some Riemannian structure on the compact manifold M .

Definition 3.1. A diffeomorphism $f : M \rightarrow M$ is **flow-type** if:

- (I) PARTIAL HYPERBOLICITY: f is strongly partially hyperbolic with splitting $TM = E^s \oplus E^c \oplus E^u$ and $\dim E^c = 1$;
- (II) DYNAMICAL COHERENCE: there are invariant foliations \mathcal{F}^{cs} and \mathcal{F}^{cu} tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$;

Let \mathcal{F}^c be the foliation whose leaves are the connected components of the intersections $\mathcal{F}^{cs}(x) \cap \mathcal{F}^{cu}(x)$, $x \in M$.

- (III) CENTER LEAVES: The center foliation \mathcal{F}^c is oriented and has at least one compact leaf.

Let $F : \mathbb{R} \times M \rightarrow M$ be the continuous flow along \mathcal{F}^c with unit positive speed.

- (IV) FLOW LIKE DYNAMICS: there is a continuous $\tau : M \rightarrow \mathbb{R}$ such that, for all $x \in M$, $f(x) = F(x, \tau(x))$ and $\tau(x) > 0$.

Remarks 3.2. Let us note the following:

- A diffeomorphism is flow-type if and only if its inverse is.
- A flow obviously satisfies the flow like condition (IV). The converse is far from true (see the comments in the introduction).
- The assumption of a compact leaf in condition (III) might seem artificial. It is only used to prove that the dilation of the unstable Margulis system is $D_u > 1$

(Lemma 4.7) and there are many other sufficient conditions verified near the time-one map of an Anosov flow such as the uniform volume expansion/contraction of the center-unstable/center-stable leaves. However discretized Anosov flows have compact leaves (see [5, Appendix G]).

Following Margulis, we build special measures on strong stable and strong unstable leaves. More precisely, let $M^u := M \setminus C^u$ where C^u is the union of the unstable leaves that intersect some compact center leaf. Define M^s and C^s likewise. Recall the notion of equivalence along some foliation from Sec. 2.1.

Theorem 3.3. *Let f be a flow-type C^2 diffeomorphism with minimal stable foliation on a compact manifold M . Then there is a measurable system of measures $\{m_x^u\}_{x \in M^u}$ such that:*

- (1) each m_x^u is an atomless, fully-supported Radon measure on $\mathcal{F}^u(x)$ with respect to the intrinsic topology;
- (2) $f_* m_x^u = D_u^{-1} m_{f(x)}^u$ for some constant $D_u > 1$ (independent of x);
- (3) the system $\{m_x^u\}_{x \in M^u}$ is cs-quasi-invariant;
- (4) M^u is dense with full measure for any ergodic measure not supported on a closed leaf.

In fact a stronger result than Item (3) holds: For any $\delta > 0$ there exists $C(\delta) > 1$ such that if A_1 and A_2 inside u -leaves are δ -equivalent along \mathcal{F}^{cs} , then

$$(4) \quad \frac{1}{C(\delta)} < \frac{m^u(A_1)}{m^u(A_2)} < C(\delta).$$

Item (2) says that $(m_x^u)_{x \in M^u}$ is a Margulis system on $\mathcal{F}^u|_{M^u}$ with dilation D_u (see Def. 2.4). We call $\{m_x^u\}_{x \in M^u}$ the *unstable Margulis system*. Considering f^{-1} , we define similarly the measures m_x^s , called the *unstable Margulis conditionals*.

Addendum 3.4. *In the setting of the previous theorem, assuming minimality of both stable and unstable foliations, the dilation factor D_u of the unstable Margulis system satisfies: $D_u = \exp h_{\text{top}}(f)$.*

The theorem will be proved in Section 4 and the addendum in Section 5.

Remarks 3.5.

(1) The above theorem and addendum, applied to f^{-1} , define a stable Margulis system $\{m_x^s\}_{x \in M^s}$ with dilation factor $D_s = D_u^{-1} = \exp(-h_{\text{top}}(f)) < 1$.

(2) The C^2 smoothness assumption is only used by Theorem 4.3 to establish absolute continuity of the s -holonomy but it is probably enough to assume $C^{1+\alpha}$ smoothness. We do not know about the C^1 smooth case.

(3) The strong Margulis systems are unique up to a set negligible for the MMEs: e.g., for any two unstable Margulis systems $\{m_x^u\}$ and $\{\tilde{m}_x^u\}$ satisfying items (1) and (2) of Theorem 3.3, the uniqueness of the disintegration of a probability measure yields $\mu(\{x : m_x^u \neq \tilde{m}_x^u\}) = 0$ for any MME μ with $\lambda^c(\mu) \leq 0$.

Using tools from Ledrappier and Young [28], we prove the following result in Section 5.

Theorem 3.6. *Let f be a C^2 diffeomorphism that is flow-type and minimal stable and unstable foliations on a compact manifold M . Let μ be an ergodic MME.*

If $\lambda^c(\mu) \leq 0$, then the disintegration of μ along \mathcal{F}^u is given by the unstable Margulis system $\{m_x^u\}_{x \in M^u}$ from Theorem 3.3. In particular, the measure μ has full support.

The above applied to f^{-1} shows that an ergodic MME with $\lambda^c(\mu) \geq 0$ has disintegration along \mathcal{F}^s given by $\{m_x^s\}_{x \in M^s}$. In particular, any MME has full support.

Remark 3.7. *The above theorem gives more information in the non-hyperbolic case. Indeed, if μ is an ergodic measure of maximal entropy with $\lambda^c(\mu) = 0$, then the disintegrations, along both strong foliations \mathcal{F}^u and \mathcal{F}^s , are given by the corresponding Margulis systems from Theorem 3.3.*

The dichotomy will follow from two results about hyperbolic measures of maximal entropy. The first is a uniqueness result, based on the Hopf argument.

Proposition 3.8. *Let f be a flow-type C^2 diffeomorphism and minimal stable and unstable strong foliations on a compact manifold M . Let μ be some ergodic MME. If μ is hyperbolic, say $\lambda^c(\mu) < 0$, then there is no other ergodic MME ν with $\lambda^c(\nu) \leq 0$.*

The second result is a symmetry argument, using the one-dimensional center leaves. It builds so-called twin measures (see [39, 18]).

Proposition 3.9. *Let f be a C^1 diffeomorphism of a compact manifold M . Let \mathcal{F}^c be an orientable one-dimensional foliation (with continuously varying C^1 leaves). Assume that, for all $x \in M$, f maps $\mathcal{F}^c(x)$ to itself in an orientation-preserving way. Let $\mu \in \mathbb{P}_{\text{erg}}(f)$ satisfy:*

- (1) *its Lyapunov exponent along \mathcal{F}^c is $\lambda_c(\mu) < 0$;*
- (2) *for μ -a.e. $x \in M$, the following set is relatively compact in the intrinsic topology of $\mathcal{F}^c(x)$:*

$$\mathcal{W}^c(x) := \{y \in \mathcal{F}^c(x) : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^n x, f^n y) < 0\};$$

- (3) *and for μ -a.e. $x \in M$, the leaf $\mathcal{F}^c(x)$ is noncompact and contains no fixed point.*

Then there is another invariant probability measure ν which is isomorphic to μ and with exponent $\lambda_c(\nu) \geq 0$.

Finally, we state the abstract version of our main result:

Theorem 3.10. *For any flow-type C^2 diffeomorphism f with minimal stable and unstable foliations on a compact manifold M , we have the following dichotomy:*

- (1) *either all the measures of maximal entropy have zero central Lyapunov exponents, or*
- (2) *there are exactly two ergodic measures of maximal entropy where one has a positive central exponent and one has a negative central exponent, and both are Bernoulli.*

The next theorem shows that there is an abundance of diffeomorphisms satisfying the above assumptions. It follows from properties of perturbations of time-one maps of transitive Anosov flows established in [6] and [7], as discussed in Section 6.

Theorem 3.11. *If ϕ^t is a transitive Anosov flow on a compact manifold M , then for all $T \neq 0$ there exists a C^1 open set \mathcal{U} in $\text{Diff}^1(M)$ such that ϕ^T belongs to the C^1 -closure of \mathcal{U} and every $f \in \mathcal{U} \cap \text{Diff}^2(M)$ is flow-type with both stable and unstable foliations minimal.*

4. BUILDING MARGULIS SYSTEMS OF MEASURES

In this section, we consider flow-type diffeomorphisms whose strong stable foliation is minimal. To begin with, we follow Margulis' construction of a system of measures on the cu -leaves that are invariant under stable holonomies. We then deduce from this a system of u -conditionals that are quasi-invariant under center-stable holonomies. This will prove Theorem 3.3.

4.1. **The cu -conditionals.** We follow Margulis' construction.

Proposition 4.1. *Let $f \in \text{Diff}^2(M)$ on a compact manifold M with a dominated splitting $E^s \oplus E^{cu}$ with E^s uniformly contracted. Assume that:*

- (1) *there are foliations \mathcal{F}^{cu} and \mathcal{F}^s which are tangent to, respectively E^{cu} and E^s ;*
- (2) *\mathcal{F}^s is minimal.*

Then there is a Margulis system $\{m_x^{cu}\}_{x \in M}$ on \mathcal{F}^{cu} which is invariant under \mathcal{F}^s -holonomy and such that each m_x^{cu} is atomless, Radon, and fully supported on $\mathcal{F}^{cu}(x)$.

If some cu -leaf contains an invariant topological circle, then the dilation constant of this Margulis system is $D_u > 1$.

We introduce some convenient definitions. Let $\sigma \in \{c, cu, cs, s, u\}$. We denote by λ^σ the intrinsic Riemannian volume on each σ -leaf: for any subset E of a σ -leaf, $\lambda^\sigma(E)$ is its volume with respect to the Riemannian structure on the leaf. We denote by d_σ the distance defined on each leaf by the induced Riemannian structure and defining the intrinsic topology.

The σ -balls are $B^\sigma(x, r) := \{y \in \mathcal{F}^\sigma(x) : d_\sigma(y, x) < r\}$. For a subset A of such a leaf, we set $B^\sigma(A, r) := \bigcup_{x \in A} B^\sigma(x, r)$. A σ -subset is a d_σ -bounded subset of a σ -leaf. A σ -test function is a nonnegative function $\psi : M \rightarrow \mathbb{R}$ such that $\{x \in M : \psi(x) \neq 0\}$ is a σ -subset and the restriction of ψ to

$$\text{supp}(\psi) := \overline{\{x \in M : \psi(x) \neq 0\}}$$

is continuous. We write $\psi > 0$ if $\psi \geq 0$ and $\{\psi > 0\}$ has non-empty interior in the intrinsic topology. We denote by \mathcal{T}^σ the collection of all σ -test functions.

We recalled the notions of holonomy, holonomy size, and holonomy equivalence for sets in Sec. 2.1. Two functions ψ, ϕ are ϵ -equivalent along \mathcal{F}^σ if their supports are equivalent through a σ -holonomy h with size at most ϵ and satisfying $\phi = \psi \circ h$. Two submanifolds N', N'' are t -transverse if they are transverse and if for every $x \in N' \cap N''$, the angle between any two nonzero vectors in $T_x N'$ and $T_x N''$ is at least t .

Following Margulis, we consider functionals $\lambda : \mathcal{T}^{cu} \rightarrow \mathbb{R}$. Note that λ^{cu} defines one such functional. The map f acts on them by:

$$\forall \psi \in \mathcal{T}^{cu} \quad f(\lambda)(\psi) := \lambda(\psi \circ f^{-1}).$$

A key class of such functionals are $\ell_n := f^n(\lambda^{cu})$ for any $n \in \mathbb{N}$. That is, for any $\phi \in \mathcal{T}^{cu}$,

$$\ell_n(\phi) := \int \phi \circ f^{-n} d\lambda^{cu}.$$

To normalize, we fix some $\phi_1 \in \mathcal{T}^{cu}$ with $\phi_1 > 0$. Considering the topology of pointwise convergence (i.e., working in $\mathbb{R}^{\mathcal{T}^{cu}}$ with the product topology), let \mathbb{L} be the closure of the following set:

$$\mathbb{L}_1 := \left\{ \lambda = \sum_{i=1}^n c_i \ell_{t_i} : n \in \mathbb{N}^*, t_1, \dots, t_n \in \mathbb{N}, c_1, \dots, c_n > 0 \text{ with } \lambda(\phi_1) = 1 \right\}.$$

We will use the following covering numbers. For K , a cu -subset, and $\rho > 0$, we denote by $r^{cu}(K, \rho)$ the smallest integer $k \geq 0$ such that there are $x_1, \dots, x_k \in K$ with $K \subset \bigcup_{i=1}^k B^{cu}(x_i, \rho)$.

We build the cu -system as a functional.

Proposition 4.2. *Let $f \in \text{Diff}^2(M)$ be partially hyperbolic with a splitting $TM = E^s \oplus E^{cu}$ and an invariant foliation \mathcal{F}^{cu} tangent to E^{cu} . Assume that the stable foliation is minimal. Then there exist $\Lambda \in \mathbb{L}$ and $D_u > 0$ such that*

$$f(\Lambda) = D_u \cdot \Lambda$$

and, for some positive numbers C, R , for any $\phi \in \mathcal{T}^{cu}$:

- (a) $\Lambda(\phi) \leq Cr^{cu}(\text{supp } \phi, R) \|\phi\|_\infty$;
- (b) if $\phi > 0$, then $\Lambda(\phi) > 0$; and
- (c) if $\psi \in \mathcal{T}^{cu}$ is s -equivalent to ϕ , then $\Lambda(\phi) = \Lambda(\psi)$.

To prove this, we will show that \mathbb{L} is a convex and compact set and then apply the Schauder-Tychonoff fixed point Theorem to a normalized action of f on \mathbb{L} . We note that \mathbb{L} is not a Banach space, but this is not required to apply the Schauder-Tychonoff fixed point Theorem.

We will relate the iterations of different cu -test functions by using the invariance under s -holonomy and especially the following theorem (see, e.g., [2, Theorem C]).

Theorem 4.3. *Let f be a C^2 diffeomorphism on a compact Riemannian manifold M . Assume that there is a dominated splitting $E^s \oplus E^{cu}$ with E^s uniformly contracting. Fix $t > 0$ and let A_1, A_2 be two submanifolds t -transverse to E^s and s -equivalent through $h : A_1 \rightarrow A_2$. Then h is absolutely continuous.*

More precisely, writing λ_1, λ_2 for the Riemannian volume on A_1, A_2 , the measures $h_*\lambda_1$ and λ_2 are equivalent and there are constants $C < \infty$ and $\alpha > 0$ depending only on f, E^s , and t such that, letting ϵ be the size of the holonomy h :

$$(5) \quad \left| \frac{dh_*\lambda_1}{d\lambda_2} - 1 \right| \leq C^{1+\epsilon} \epsilon^\alpha.$$

Note that [2, Theorem C] gives the usual (stronger) bound by $C_0\epsilon^\alpha$ but only for $0 \leq \epsilon \leq \epsilon_0$. The above bound for all $\epsilon \geq 0$ is obtained by a routine computation by decomposing a holonomy of some large size $\epsilon \geq \epsilon_0$ into a product of $\lceil \epsilon/\epsilon_0 \rceil$ holonomies of size at most ϵ_0 .

In the rest of Subsection 4.1, we assume that $f \in \text{Diff}^2(M)$ is partially hyperbolic with a splitting $TM = E^s \oplus E^{cu}$ and an invariant foliation \mathcal{F}^{cu} tangent to E^{cu} and a minimal strong stable foliation \mathcal{F}^s .

We need two additional lemmas. The first one will give a uniform bound on the volume growth in the center unstable leaves.

Lemma 4.4. *For any open, non-empty cu -subset A , there are constants $C_A < \infty$ and $r_A > 0$ such that,*

$$(6) \quad \forall x \in M \forall n \geq 0 \lambda^{cu}(f^n B^{cu}(x, r_A)) \leq C_A \lambda^{cu}(f^n A).$$

Proof. Fix $r_1 > 0$ so small that $A_1 := A \setminus B^{cu}(\partial A, r_1)$ is not empty. The minimality implies that any x is s -equivalent to some point in A_1 . For every $x \in M$, the continuity of the foliation \mathcal{F}^s and its transversality to A yield $r_x > 0$ and $R_x < \infty$ such that, for any $y \in B(x, r_x)$, $B^{cu}(y, r_x)$ is (R_x, s) -equivalent to a subset of A . The compactness of M allows to choose $r_A > 0$ and $R_A < \infty$ independent of $x \in M$.

Since \mathcal{F}^s is contracted, there are numbers $C < \infty$ and $\kappa < 1$ such that, for all $n \geq 0$ and all $x \in M$, the set $f^n(B^{cu}(x, r_A))$ is $(C\kappa^n R_A, s)$ -equivalent to a subset of $f^n(A)$. We conclude by Theorem 4.3. \square

Corollary 4.5. *For any $\phi \in \mathcal{T}^{cu}$ with $\phi > 0$, there are numbers $C(\phi) < \infty$ and $R(\phi) > 0$ such that for any $\psi \in \mathcal{T}^{cu}$ and any $n \geq 0$:*

$$\int \psi \circ f^{-n} d\lambda^{cu} \leq C(\phi) r^{cu}(\text{supp } \psi, R(\phi)) \|\psi\|_\infty \int \phi \circ f^{-n} d\lambda^{cu}.$$

Proof. The left hand side is bounded by $\|\psi\|_\infty \cdot \lambda^{cu}(f^n(\text{supp } \psi))$. Fix some $0 < t < \sup \phi$. The previous lemma with $A := \{\phi > t\}$ yields $r_A > 0$ and $C_A < \infty$. Since $\text{supp}(\phi)$ is compact in its cu -leaf $\mathcal{F}^c(A)$, there are $x_1, \dots, x_N \in \mathcal{F}^c(A)$ with $N = r^{cu}(\text{supp } \psi, r_A)$ such that $\text{supp}(\psi) \subset \bigcup_{i=1}^N B^{cu}(x_i, r_A)$. Now

$$\lambda^{cu}(f^n(B^{cu}(x_i, r_A))) \leq C_A \lambda^{cu}(f^n A) \leq C_A \lambda^{cu}(\phi \circ f^{-n})/t.$$

Summing over the cover of $\text{supp } \psi$, the claim follows with $C(\phi) := C_A/t$ and $R(\phi) := r_A$. \square

The next lemma establishes approximate holonomy invariance.

Lemma 4.6. *There are numbers $C < \infty$ and $0 < \kappa < 1$ with the following properties. Let $\psi, \phi \in \mathcal{T}^{cu}$ be (s, Δ) -equivalent for some $\Delta < \infty$.*

First, if $\psi > 0$, then, for all $n \geq 0$:

$$(7) \quad \forall \lambda \in \mathbb{L} \quad |\lambda(\psi \circ f^{-n}) - \lambda(\phi \circ f^{-n})| \leq C^{1+\Delta\kappa^n} \kappa^{\alpha n} \Delta^\alpha \lambda(\psi \circ f^{-n}).$$

Second, for any $\phi_1 \in \mathcal{T}^{cu}$ with $\phi_1 > 0$, there are numbers $C(\phi_1)$ and $R(\phi_1) > 0$ such that, for any $n \geq 0$ we have

$$(8) \quad |\ell_n(\psi) - \ell_n(\phi)| \leq C(\phi_1)^{1+\Delta\kappa^n} r^{cu}(\text{supp}(\psi), R(\phi_1)) \|\psi\|_\infty \kappa^{\alpha n} \Delta^\alpha \ell_n(\phi_1).$$

Proof. Since ϕ and ψ are Δ -equivalent, there is $h : \text{supp}(\psi) \rightarrow \text{supp}(\phi)$ with size at most Δ such that $\psi = \phi \circ h$. Since \mathcal{F}^s is uniformly contracted, $\phi \circ f^{-n}$ and $\psi \circ f^{-n}$ are $C\kappa^n \Delta$ -equivalent through $h_n := f^n \circ h \circ f^{-n}$ for some $C < \infty$ and

$\kappa < 1$. Theorem 4.3 yields (replacing C by a larger constant each time needs arise):

$$\begin{aligned}
|\ell_n(\phi) - \ell_n(\psi)| &= \left| \int \phi \circ f^{-n} d\lambda_{f^n y}^{cu} - \int \psi \circ f^{-n} d\lambda_{f^n x}^{cu} \right| \\
&= \left| \int \left(\phi \circ f^{-n} \circ h_n \frac{d(h_n)_* \lambda_{f^n y}^{cu}}{d\lambda_{f^n x}^{cu}} - \psi \circ f^{-n} \right) d\lambda_{f^n x}^{cu} \right| \\
&= \left| \int \psi \circ f^{-n} \left(\frac{d(h_n)_* \lambda_{f^n y}^{cu}}{d\lambda_{f^n x}^{cu}} - 1 \right) d\lambda_{f^n x}^{cu} \right| \\
&\leq C^{1+C\Delta\kappa^n} \Delta^\alpha \kappa^{\alpha n} \int |\psi \circ f^{-n}| d\lambda_{f^n x}^{cu} \\
&\leq C^{1+\Delta\kappa^n} \Delta^\alpha \kappa^{\alpha n} \ell_n(|\psi|).
\end{aligned}$$

Let $\lambda \in \mathbb{L}_1$ so that $\lambda = \sum_{i=1}^I c_i \ell_{t_i}$. Using the identity $\ell_t(\phi \circ f^{-n}) = \ell_n(\phi \circ f^{-t})$, we get: $\lambda(\phi \circ f^{-n}) = \sum_{i=1}^I c_i \ell_n(\phi \circ f^{-t_i})$. Hence, if $\psi > 0$,

$$|\lambda(\phi \circ f^{-n}) - \lambda(\psi \circ f^{-n})| \leq C^{1+\Delta\kappa^n} \Delta^\alpha \kappa^{\alpha n} \lambda(\psi \circ f^{-n}),$$

proving eq. (7) for all $\lambda \in \mathbb{L}$ by continuity.

Let $\phi_1 \in \mathcal{T}^{cu}$ with $\phi_1 > 0$. Applying Corollary 4.5, we obtain $C(\phi_1)$ and $R(\phi_1) > 0$ such that eq. (8) holds. \square

Proof of Proposition 4.2. We check the claims (a), (b), and (c) in turn.

Step 1: *Claim (a):* $\exists R > 0 \forall \lambda \in \mathbb{L} \lambda(\psi) \leq Cr^{cu}(\text{supp } \psi, R) \|\psi\|_\infty$.

Corollary 4.5 applied to the test function $\phi = \phi_1$ (used in the normalization) yields numbers $C < \infty$ and $R > 0$ such that for any $\psi \in \mathcal{T}^{cu}$:

$$\forall n \geq 0 \ell_n(\psi) \leq Cr^{cu}(\text{supp } \psi, R) \|\psi\|_\infty \ell_n(\phi_1).$$

Therefore, for any $\lambda \in \mathbb{L}_1$:

$$\begin{aligned}
(9) \quad \lambda(\psi) &= \sum_{i=1}^I c_i \ell_{t_i}(\psi) \leq \sum_{i=1}^I c_i Cr^{cu}(\text{supp } \psi, R) \|\psi\|_\infty \ell_{t_i}(\phi_1) \\
&= Cr^{cu}(\text{supp } \psi, R) \|\psi\|_\infty \lambda(\phi_1).
\end{aligned}$$

Since $\lambda(\phi_1) = 1$, this proves the claim, first for $\lambda \in \mathbb{L}_1$, then for $\mathbb{L} = \overline{\mathbb{L}_1}$ by continuity. Step 1 is done.

Note that eq. (9) implies that \mathbb{L}_1 is a subset of the compact set

$$\prod_{\psi \in \mathcal{T}^{cu}} [0, C \cdot r^{cu}(\text{supp } \psi, R) \|\psi\|].$$

Hence its closure \mathbb{L} is compact.

Step 2: *Claim (b):* $\forall \psi \in \mathcal{T}^{cu}$ if $\psi > 0$ there is $C(\psi) > 0$ s.t. $\forall \lambda \in \mathbb{L}, \lambda(\psi) \geq C(\psi)$.

We now assume that $\psi > 0$ so we can apply Corollary 4.5 exchanging ψ and ϕ_1 . We get new numbers C' and R' , depending on ψ . Setting $C_1 := C' r^{cu}(\text{supp } \phi_1, R')$, we have, for all $n \geq 0$, $\ell_n(\phi_1) \leq C_1 \ell_n(\psi)$. That is, $\ell_n(\psi) \geq (1/C_1) \ell_n(\phi_1)$ so that, for any $\lambda \in \mathbb{L}$:

$$(10) \quad \lambda(\psi) \geq \frac{1}{C_1} \lambda(\phi_1) = \frac{1}{C_1},$$

concluding Step 2 with the lower bound $1/C_1(\psi)$.

Step 3: *Existence of $\Lambda \in \mathbb{L}$ with $f(\Lambda) = D \cdot \Lambda$*

We now build the functional Λ as a fixed point of the map

$$\bar{f} : \mathbb{L} \rightarrow \mathbb{L}, \quad \lambda \mapsto \frac{\lambda(\cdot \circ f^{-1})}{\lambda(\phi_1 \circ f^{-1})}.$$

We claim that \bar{f} is well-defined and continuous from \mathbb{L} to $\mathbb{R}^{\mathcal{T}^{cu}}$. Indeed, the map $\lambda \mapsto \lambda(\phi_1 \circ f^{-1})$ from \mathbb{L} to \mathbb{R} is well-defined since $\phi_1 \circ f^{-1} \in \mathcal{T}^{cu}$, is obviously continuous, and is positive by Step 2. Moreover, $\lambda(\cdot \circ f^{-1}) : \mathcal{T}^{cu} \rightarrow \mathbb{R}$ is well-defined and $\lambda \mapsto \lambda(\cdot \circ f^{-1})$ is continuous from \mathbb{L} to $\mathbb{R}^{\mathcal{T}^{cu}}$. The claim is proved.

Finally, it is obvious that $\bar{f}(\mathbb{L}_1) = \mathbb{L}_1$, hence $\bar{f} : \mathbb{L} \rightarrow \mathbb{L}$ is a well-defined continuous map. Since \mathbb{L} is a convex, compact subset of the locally convex linear space $\mathbb{R}^{\mathcal{T}^{cu}}$, the Schauder-Tychonoff Theorem applies and yields $\Lambda \in \mathbb{L}$ with $\bar{f}(\Lambda) = \Lambda$.

In other words, $f(\Lambda) = D \cdot \Lambda$ with $D = \Lambda(\phi_1 \circ f^{-1}) > 0$.

Step 4: *Claim (c): s -holonomy invariance of Λ*

Let $\phi, \psi \in \mathcal{T}^{cu}$. Observe that positive (respectively negative) parts of ϕ and ψ must then be s -equivalent. Thus we can assume $\phi, \psi > 0$. Assume that ψ and ϕ are s -equivalent. By compactness of their support, they are (s, Δ) -equivalent for some $\Delta > 0$ and therefore $\phi \circ f^{-n}$ and $\psi \circ f^{-n}$ are $(s, C\kappa^n \Delta)$ -equivalent with $0 < \kappa < 1$. Using the dilation by $D \neq 0$ and eq. (7) from Lemma 4.6 (approximate holonomy invariance), we get that, for any $\epsilon > 0$, for large enough $n \geq 0$:

$$\begin{aligned} |\Lambda(\phi) - \Lambda(\psi)| &= D^{-n} |\Lambda(\phi \circ f^{-n}) - \Lambda(\psi \circ f^{-n})| \\ &\leq D^{-n} \epsilon \cdot \Lambda(\psi \circ f^{-n}) = \epsilon \cdot \Lambda(\psi) \end{aligned}$$

As $\epsilon > 0$ is arbitrarily small this implies $\Lambda(\phi) = \Lambda(\psi)$ and concludes the step.

Proposition 4.2 is established. \square

We deduce a Margulis system of cu -measures from the functional Λ .

Proof of Proposition 4.1. Proposition 4.2 yields a functional Λ on \mathcal{T}^{cu} , which contains $C_c(\mathcal{F}^{cu}(x))$ for all $x \in M$. We note that, for each $x \in M$, $\mathcal{F}^{cu}(x)$ is a locally compact metric space and $\Lambda|_{C_c(\mathcal{F}^{cu}(x))}$ is linear and positive (because this holds for all $\lambda \in \mathbb{L}_1$ and extends by continuity to \mathbb{L}). Hence, Riesz's Representation Theorem [42, 2.14] gives a Radon measure m_x on $\mathcal{F}^{cu}(x)$, for each $x \in M$, characterized by:

$$\forall \phi \in C_c(\mathcal{F}^{cu}(x)), \quad m_x(\phi) = \Lambda(\phi).$$

The full support, and s -invariance of each m_x follows from properties (b) and (c) from Proposition 4.2.

We prove that each m_x^{cu} is atomless using the holonomy invariance. Indeed, assume by contradiction that there is $y \in \mathcal{F}^{cu}(x)$ with $m_x^{cu}(\{y\}) > 0$. Consider the stable leaf $\mathcal{F}^s(y)$. By assumption it is dense in M , hence by transversality in $\mathcal{F}^{cu}(x)$. By s -invariance, m_x^{cu} must have a dense set of atoms $z \in \mathcal{F}^{cu}(x)$, all of which have measure $m_x^{cu}(\{z\}) > 0$. This contradicts the finiteness of m_x^{cu} on compact sets.

We finally deduce the continuity from the holonomy invariance. As \mathcal{F}^{cu} and \mathcal{F}^s are transverse, for any $x_0 \in M$, there is a neighborhood B of x_0 and a continuous map $h_0 : B \times \mathcal{F}_B^{cu}(x_0) \rightarrow B$ with $\{h_0(x, y)\} = \mathcal{F}_{\text{loc}}^{cu}(x) \cap \mathcal{F}_{\text{loc}}^s(y)$ (in particular,

$h_0(x_0, y) = y$). Let $\phi \in C(M)$. By holonomy invariance, $m_x^{cu}(\phi|\mathcal{F}_B^{cu}(x)) = m_{x_0}^{cu}(\phi \circ h_0(x, \cdot))$. Hence,

$$|m_x^{cu}(\phi|\mathcal{F}_B^{cu}(x)) - m_{x_0}^{cu}(\phi|\mathcal{F}_B^{cu}(x_0))| \leq \int_{\mathcal{F}_B^{cu}(x_0)} |\phi \circ h_0(x, \cdot) - \phi| dm_{x_0}^{cu}$$

which converges to 0 as x goes to x_0 . This is the continuity property.

The assertion about the dilation factor is $D_u > 1$ follows from the next Lemma. \square

Lemma 4.7. *Let $f \in \text{Diff}^2(M)$ have a dominated splitting $TM = E^s \oplus E^{cu}$ with tangent foliations \mathcal{F}^s and \mathcal{F}^{cu} . Assume that \mathcal{F}^s is minimal and that there is a Margulis system $\{m_x^{cu}\}_{x \in M}$ carried by \mathcal{F}^{cu} .*

If there is a invariant circle γ contained in some cu -leaf, then the dilation of the Margulis system on \mathcal{F}^{cu} satisfies: $D_u > 1$.

Proof. Let γ be the invariant circle and let $\mathcal{F}^{cu}(x)$ be its cu -leaf endowed with its intrinsic topology. Since γ is a topological attractor for the restriction of f^{-1} to the invariant set $\mathcal{F}^{cu}(x)$, there is a relatively compact neighborhood U of γ in $\mathcal{F}^{cu}(x)$ such that $U \setminus f^{-1}(U)$ has non empty interior. As m_x^{cu} has full support in $\mathcal{F}^{cu}(x)$, it follows that:

$$D_u^{-1} m_x^{cu}(U) = m_x^{cu}(f^{-1}U) < m_x^{cu}(U),$$

proving $D_u > 1$. \square

4.2. Building the u -conditionals. We complete the proof of Theorem 3.3.

We start with the previously built Margulis cu -system $\{m_x^{cu}\}_{x \in M}$ and define the family of measures $\{m_x^u\}_{x \in M^u}$ by extending subsets of u -leaves to subsets of cu -leaves along the center foliation. For flows, Margulis used the formula

$$m_x^u(A) = m_x^{cu} \left(\bigcup_{0 \leq t < t_0} \phi^t(A) \right).$$

Here $t_0 > 0$ is chosen small so that disjoint subsets of the same u -leaf correspond to disjoint subsets of the cu -leaf. This ensures that m_x^u inherits the σ -additivity of m_x^{cu} .

In our setting there is no flow commuting with the dynamics and we have to proceed differently. To keep the equivariance, we will replace $(\phi^t(x))_{0 \leq t < t_0}$ by the c -segment $I_c(x)$ “between x and $f(x)$ ”. Such center curves cannot be assumed to be arbitrarily short, so even restricting x to a small subset of a single u -leaf, the curves $I_c(x)$ might intersect and destroy the additivity property. It turns out that this problem only occurs on \mathcal{C}^u , i.e., the union of the cu -leaves containing a compact center leaf. This is why we will build u -measures m_x^u only for $x \in M \setminus \mathcal{C}^u$.⁴

Preparations. We recall or establish some useful properties of the dynamical foliations and of the cu -system built in Section 4. In particular, it allows us to disregard the closed center leaves.

Remark 4.8. *There can be at most countably many compact center leaves, since each one is normally hyperbolic (see [24, Theorem 4.1(b)]).*

⁴Alternatively, one could consider a covering of M where all center leaves are noncompact. Note also that this problem could be altogether avoided by restricting to perturbation of time t maps for small enough t , instead of taking $t = 1$.

Lemma 4.9. *Let $f \in \text{Diff}(M)$ be a flow-type diffeomorphism. If a center leaf meets some unstable leaf in more than one point, then it is contained in the cu-leaf of a compact center leaf.*

Proof. To simplify notations a bit, we prove the symmetric statement involving stable and center stable manifolds.

Let x, y be two distinct points with $y \in \mathcal{F}^s(x) \cap \mathcal{F}^c(x)$ and $x \leq y$ along the center leaf (recall that f being flow-type, it maps each center leaf to itself preserving some orientation). Let L be the center segment $[x, y]_c$. We are going to show that $f^n(L)$ converges to a closed center leaf γ as a compact subset with respect to the intrinsic distance in $\mathcal{F}^{cu}(x)$. The existence of γ in $\mathcal{F}^{cu}(x)$ will prove the lemma.

There are two cases, depending on the position of y with respect to $f(x)$. In the first case, we have:

$$(11) \quad x \leq f(x) \leq y \leq f(y)$$

hence the decompositions (the unions are disjoint up to one point):

$$\begin{aligned} L &= [x, y]_c = [x, f(x)]_c \cup [f(x), y]_c \\ f(L) &= [f(x), f(y)]_c = [f(x), y]_c \cup [y, f(y)]_c. \end{aligned}$$

For large enough $k \geq 0$, the points $f^k(x)$ and $f^k(y)$ are arbitrarily close while the length of $[f^k(x), f(f^k(x))]_c$ is bounded independently of k . We can assume that this holds already for $k = 0$. Thus there is a s -holonomy $h : [x, f(x)]_c \rightarrow [y, f(y)]_c$ such that $h(x) = y$.

Now define the map $\psi : L \rightarrow f(L)$ by $\psi|[x, f(x)]_c = h$ and $\psi|[f(x), y]_c = \text{Id}$. Observe that it is a bijection such that $\psi(x) \in \mathcal{F}_{\text{loc}}^s(x)$ and that there is a finite bound $\delta := \sup_{z \in L} d(z, \psi(z))$ on the Hausdorff distance $d_H(L, f(L))$.

This implies that, for any $n \geq 0$,

$$d_H(f^n(L), f^{n+1}(L)) \leq \sup_{z \in L} d(f^n(z), f^n(\psi(z)))$$

decays exponentially fast since \mathcal{F}^s is uniformly contracted. It follows that

$$\sum_{n \geq 0} d_H(f^n(L), f^{n+1}(L)) < \infty$$

and so $f^n(L)$ converges to some nonempty compact γ in the complete space of all nonempty compact subsets of $\mathcal{F}^{cu}(x)$. Obviously γ is a closed center leaf. This proves the lemma in the case (11) holds. In the other case,

$$x \leq y < f(x) \leq f(y).$$

As above, perhaps after replacing x, y by some forward iterates, we find a s -holonomy $h : [x, f(x)]_c \rightarrow [y, f(y)]_c$ such that $h(x) = y$. By transversality of \mathcal{F}^c and \mathcal{F}^s , the length of $[z, h(z)]_c$ is lower bounded for $z \in M$. Hence $h^N(x) < f(x) \leq h^{N+1}(x)$ for some $N \geq 1$, and one has:

$$\begin{aligned} L &= [x, h^{-N}(f(x))]_c \cup [h^{-N}(f(x)), y]_c \\ f(L) &= [f(x), h^N(y)]_c \cup [h^N(y), f(y)]_c \end{aligned}$$

and the following maps are s -holonomies:

$$\begin{aligned} h^N &: [h^{-N}(f(x)), y]_c \longrightarrow [f(x), h^N(y)]_c \\ h^{N+1} &: [x, h^{-N}(f(x))]_c \longrightarrow [h^N(y), f(y)]_c \end{aligned}$$

(since $h(f(x)) = f(h(x)) = f(y)$).

As before, we conclude that $f^n(L)$ converges to a compact center leaf in $\mathcal{F}^{cu}(x)$. \square

Center-unstable leaves have a simple structure:

Lemma 4.10. *Let $f \in \text{Diff}^1(M)$ be a flow-type diffeomorphism. For any $x \in M$,*

$$\mathcal{F}^{cu}(x) = \bigcup_{y \in \mathcal{F}^c(x)} \mathcal{F}^u(y).$$

In particular, $\mathcal{C}^u = \bigcup_{\gamma} \mathcal{F}^u(\gamma)$ where γ ranges over the countably many compact center leaves.

This is [46, Prop. 2.9]. Note that the statement there assume that f is a perturbation of some Anosov flow, but inspection of the proof shows that flow-type is sufficient. Now we can show that the exceptional sets \mathcal{C}^u and \mathcal{C}^s are negligible.

Lemma 4.11. *Let f be a diffeomorphism on a compact manifold with flow-type. Then the following holds:*

- (1) *For any $\mu \in \mathbb{P}_{\text{erg}}(f)$ not carried by a compact center leaf, $\mu(\mathcal{C}^s) = \mu(\mathcal{C}^u) = 0$.*
- (2) *If f admits a Margulis system of cu -measures $\{m_x^{cu}\}_{x \in M}$ with dilation factor $D_u \neq 1$ for any $x \in M$, any compact center leaf has zero m_x^{cu} -measure and more generally $m_x^{cu}(\mathcal{C}^s) = 0$.*

We will deduce from item (2) the unstable Margulis conditionals are atomless.

Proof. We prove item (1) by contradiction. Let $\mu \in \mathbb{P}_{\text{erg}}(f)$ be such that $\mu(\mathcal{C}^s) > 0$. The previous lemma and σ -additivity imply that μ gives positive measure to $\mathcal{F}^s(\gamma)$ for some compact center leaf γ . By ergodicity, we have $\mu(\mathcal{F}^s(\gamma)) = 1$. Iterating forward, this implies that $\mu(\gamma) = 1$. This contradicts the assumption and proves $\mu(\mathcal{C}^s) = 0$. Iterating backwards yield $\mu(\mathcal{C}^u) = 0$.

We turn to item (2). First let γ be a compact center leaf. Since it is fixed, $m_x^{cu}(\gamma) = D_u \cdot m_x^{cu}(\gamma)$. As $D_u \neq 1$ and $m_x^{cu}(\gamma) < \infty$ (by the Radon property), we obtain $m_x^{cu}(\gamma) = 0$, the first claim.

Now let us prove that $m_x^{cu}(\mathcal{F}^s(\gamma)) = 0$ for all arbitrary compact center leaves γ . Fix such a leaf γ . Since $\gamma \subset \mathcal{F}^s(\gamma) \cap \mathcal{F}^u(\gamma)$, we can find a countable cover

$$\{y \in \mathcal{F}^{cu}(x) : y \in \mathcal{F}^s(\gamma)\} \subset \bigcup_{i \geq 1} h_i^s(\gamma \cap U_i)$$

where $h_i^s : U_i \rightarrow \mathcal{F}^{cu}(x)$, $i \geq 1$, are s -holonomies with $U_i \subset \mathcal{F}^{cu}(\gamma)$. Now, by s -invariance of the cu -system and the fact that $m_x^{cu}(\gamma) = 0$, each term on the right hand side has zero m_x^{cu} -measure. This shows that $m_x^{cu}(\mathcal{F}^s(\gamma)) = 0$

The claim $m_x^{cu}(\mathcal{C}^s) = 0$ follows by σ -additivity and the previous lemma. \square

Proof of Theorem 3.3 . We begin by defining the measures m_x^u along the unstable foliation. For each $x \in M^u$ and Borel subset $A \subset \mathcal{F}^u(x)$ we let

$$m_x^u(A) = m_x^{cu}(\hat{A}) \text{ with } \hat{A} := \bigcup_{y \in A} [x, f(x)]_c.$$

This is well-defined since \hat{A} is measurable when A is. Consider a u -leaf $\mathcal{F}^u(x)$ for some $x \in M^u$. Let A_1, A_2, \dots be pairwise disjoint measurable sets. By Lemma 4.9, the extensions $\hat{A}_1, \hat{A}_2, \dots$ are also pairwise disjoint. Hence,

$$m_x^u\left(\bigsqcup_n A_n\right) = m_x^{cu}\left(\widehat{\bigsqcup_n A_n}\right) = m_x^{cu}\left(\bigsqcup_n \hat{A}_n\right) = \sum_n m_x^u(A_n),$$

proving the σ -additivity. Obviously, $m_x^u(\emptyset) = 0$. Thus m_x^u is a measure.

\mathcal{C}^u is a countable union of submanifolds with positive codimension. Thus the Baire theorem shows that $M^u := M \setminus \mathcal{C}^u$ is dense. Lemma 4.11 shows that \mathcal{C}^u has zero measure for any ergodic measure not concentrated on a single compact leaf. Item (4) is proven.

Observe that relative compactness, respectively nonempty interior, hold for \hat{A} if it holds for A yielding that m_x^u is Radon and fully supported. Item (2) of Lemma 4.11 implies that any center leaf is m_x^{cu} -negligible for all $x \in M$, hence each m_x^u is atomless. Item (1) follows.

We prove that $\{m_x^u\}_{x \in M \setminus \mathcal{C}^u}$ is a Margulis system. Observe that $\{m_x^{cu}\}_{x \in M}$ is a Margulis system with dilation $D_u > 1$ by Proposition 4.1. To determine the dilation of $\{m_x^u\}$, note that $\widehat{f(A)} = f(\hat{A})$, since $f([x, f(x)]_c) = [f(x), f^2(x)]_c$ so that:

$$m_x^u(f(A)) = m_x^{cu}(\widehat{f(A)}) = m_x^{cu}(f(\hat{A})) = D_u m_x^{cu}(\hat{A}) = D_u m_x^u(A),$$

Item (2) is proved, except for the measurability which will be established at the end of this section.

We turn to item (3), the quasi-invariance under cs -holonomies. We show the stronger claim in eq. (4) (recall that a holonomy is by definition a composition of local holonomies). Since E^c and E^s are transverse, we see that:

Lemma 4.12. *Let $x, y \in M$ satisfy $\mathcal{F}^{cs}(x) = \mathcal{F}^{cs}(y)$ with $d_{cs}(x, y)$ small. Then there exists a local s -holonomy $h^s : U \rightarrow V$ with U a neighborhood of $[x, f(x)]_c$ s.t.*

$$h^s([x, f(x)]_c) \subset (f^{-1}(y), f^2(y))_c.$$

Let us prove the cs -quasi-invariance. Since this is a local property, we can assume that $A \subset \mathcal{F}_\delta^u(x)$ for $x \in M^u$ with $\delta > 0$ small so that the cs -holonomy extends to $h^{cs}(A) = B$. By construction and the Margulis property,

$$m_y^{cu}(f^{-1}(\hat{B}) \cup \hat{B} \cup f(\hat{B})) = (D_u^{-1} + 1 + D_u)m_y^u(B).$$

By the previous lemma, $h^s(\hat{A}) \subset f^{-1}(\hat{B}) \cup \hat{B} \cup f(\hat{B})$, so the s -invariance of the cu -conditionals yields:

$$m_x^u(A) = m_x^{cu}(\hat{A}) = m_y^{cu}(h^s(\hat{A})) \leq (D_u^{-1} + 1 + D_u)m_y^u(B).$$

We have shown $m_x^u \ll (h^{cs})_*^{-1}(m_y^u)$ with Radon-Nikodym derivative bounded by $D_u^{-1} + 1 + D_u$. Exchanging the roles of x and y we obtain the equivalence of the measures.

This finishes the proof of item (3) and of Theorem 3.3 except for the measurability which will follow from the next lemma. \square

Recall that $M^u = M \setminus \mathcal{C}^u$ is a measurable subset. Fix a local product cube C around $p \in M^u$. Using the local product structure, C may be identified as $\mathcal{F}_{\text{loc}}^{cs}(p) \times \mathcal{F}_{\text{loc}}^u(p)$.

Lemma 4.13. *For any Borel measurable, bounded function $\phi : M \rightarrow \mathbb{R}$, the map*

$$x \in \mathcal{F}_{loc}^{cs}(p) \cap M^u \mapsto m_x^u(1_{\mathcal{F}_{loc}^u(x)} \cdot \phi)$$

is measurable.

Proof. Clearly, it is enough to show that for any Borel measurable subset $K \subset C$ the map

$$x \in \mathcal{F}_{loc}^{cs}(p) \cap M^u \mapsto m_x^u(K_x)$$

is measurable where $K_x = K \cap \mathcal{F}_{loc}^u(x)$. We begin by the special case when K is a compact subset. As \mathcal{F}_{loc}^u varies continuously, and K is compact, we conclude that $x \mapsto K_x$ is upper semicontinuous. Now we proceed to show that the map $x \mapsto m_x^u(K_x)$ defined on $\mathcal{F}_{loc}^{cs}(p) \cap \mathcal{C}^u$ is upper semi continuous and in particular measurable.

As m_x^{cu} is a locally finite measure, given any $\epsilon > 0$ there exists $\epsilon_0 > 0$ such that $m^{cu}(U_{\epsilon_0}(\hat{K}_x)) \leq m^{cu}(\hat{K}_x) + \epsilon$ where $U_{\epsilon_0}(\hat{K}_x)$ is the ϵ_0 -neighborhood of K_x inside the cu -leaf. By construction $m_x^u(K_x) = m^{cu}(\hat{K}_x)$. If y is close enough to x we have

$$h_{y,x}^s(\hat{K}_y) \subset U_{\epsilon_0}(\hat{K}_x).$$

Indeed, there exists $R > 0$ such that for any such $y \in C$, $\hat{K}_y \subset \mathcal{F}^{cu}(y, R)$. Now, if y is close enough to x the restriction of stable holonomy $h_{y,x}^s$ to $\mathcal{F}^{cu}(y, R)$ is close to the identity. So we obtain that:

$$\begin{aligned} m^u(K_y) &= m^{cu}(\hat{K}_y) = m^{cu}(h_{y,x}^s(\hat{K}_y)) \leq m^{cu}(U_{\epsilon_0}(\hat{K}_x)) \\ &\leq m^{cu}(\hat{K}_x) + \epsilon = m^u(K_x) + \epsilon. \end{aligned}$$

and this proves the upper semicontinuity in the case K is closed.

Let \mathcal{K} be the collection of all subsets $K \subset C$ such that $x \mapsto m_x^u(K_x)$ is measurable. Note that this is a Dynkin system, i.e.,

- (1) If $A, B \in \mathcal{K}$ and $A \subset B$ then $B \setminus A \in \mathcal{K}$,
- (2) If $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots \in \mathcal{K}$ then $\bigcup A_i \in \mathcal{K}$
- (3) $C \in \mathcal{K}$.

Since \mathcal{K} contains the closed subsets of C (which form a π -system), Dynkin π - λ -theorem implies that it contains the σ -algebra generated by the closed subsets, i.e., all the Borel subsets of C . \square

5. DICHOTOMY FOR FLOW-TYPE DIFFEOMORPHISMS

This section is devoted to the proof of our main result on the MMEs of flow-type diffeomorphisms (Theorem 3.10).

We first build an invariant probability measure $\mu^{cu \otimes s}$ from the cu - and s -Margulis conditionals. We deduce that dilation factors of these conditionals are inverse of each other: $D_s D_u = 1$. We then show that any MME with nonnegative central exponent has u -disintegrations given by the u -Margulis conditionals by using the entropy along the unstable foliation as introduced by Ledrappier and Young [28] and a classical convexity argument of Ledrappier (see [30] for a pedagogical exposition).

A Hopf argument now gives the uniqueness of the MME with nonpositive center exponent. We finish the proof of the dichotomy by building twin measures and considering f^{-1} instead of f . We conclude this section by showing some additional properties: uniqueness of the Margulis systems, properties on the sign of the central exponent of $\lambda_c(\mu^{cu \otimes u})$, and the Bernoulli property of hyperbolic MMEs.

5.1. Quasi-product measures. Given a diffeomorphism $f \in \text{Diff}(M)$ that is flow-type and minimal strong foliations, we build an invariant probability measure from the Margulis systems $\{m_x^{cu}\}_{x \in M}$ and $\{m_x^s\}_{x \in M \setminus \mathcal{C}^s}$ as those provided by Proposition 4.1 and Theorem 3.3 applied to the inverse f^{-1} .

Proposition 5.1. *Let $f \in \text{Diff}^1(M)$ be flow-type with minimal strong foliations. Assume that there are Margulis systems:*

- $\{m_x^{cu}\}_{x \in M}$ on \mathcal{F}^{cu} with dilation D^u which is s -invariant;
- $\{m_x^s\}_{x \in M \setminus \mathcal{C}^s}$ on \mathcal{F}^s with dilation D^s which is cu -quasi-invariant;
- \mathcal{C}^s satisfies item (2) of Lemma 4.11.

Then there is an invariant Borel probability measure $\mu^{cu \otimes s}$ such that for each $y \in M$, there is a neighborhood in which its conditional measures along \mathcal{F}^s are given by $\{m_x^s\}_{x \in M \setminus \mathcal{C}^s}$ with m_y^{cu} as quotient measure. Moreover, $D_u = D_s^{-1}$.

Proof. We first define local measures $\{m_p\}_{p \in M}$. For any $p \in M$ we define a small foliated box (for both \mathcal{F}^s and \mathcal{F}^{cu}) neighbourhood U_p containing p as follows: Let $\mathcal{F}_{loc}^s(p), \mathcal{F}_{loc}^{cu}(p)$ be small neighborhoods of p in $\mathcal{F}^s(p), \mathcal{F}^{cu}(p)$ and U_p be the homeomorphic image of $\mathcal{F}_{loc}^s(p) \times \mathcal{F}_{loc}^{cu}(p)$ by the local product map $(x, y) \mapsto \mathcal{F}_{loc}^{cu}(x) \cap \mathcal{F}_{loc}^s(y)$.

For any $z \in U_p$ by $\mathcal{F}^s(z, U_p)$ we mean the plaque of \mathcal{F}^s passing through z inside U_p . Similarly we define $\mathcal{F}^{cu}(z, U_p)$.

One can define two projections $\pi_p^{cu} : U_p \rightarrow \mathcal{F}_{loc}^{cu}(p)$ and $\pi_p^s : U_p \rightarrow \mathcal{F}_{loc}^s(p)$ in a natural way. By definition, for any $y \in \mathcal{F}^{cu}(p, U_p), \pi_p^s$ sends $\mathcal{F}^s(y, U_p)$ homeomorphically to $\mathcal{F}^s(p, U_p)$. This is a local cu -holonomy map.

We are going to define a measure m_p on U_p by integrating the measures m_y^s with respect to the measure m_p^{cu} . We define

$$m_p := \int_{\mathcal{F}^{cu}(p, U_p)} m_y^s dm_p^{cu}(y)$$

and justify this definition as follows: Fix ϕ a continuous function with $\text{supp } \phi \subset U_p$. Let $\alpha_\phi^p : \mathcal{F}^{cu}(p, U_p) \rightarrow \mathbb{R}$ be defined by

$$\alpha_\phi^p(y) := \begin{cases} m_y^s(1_{\mathcal{F}^s(y, U_p)} \cdot \phi) & \text{if } y \notin \mathcal{C}^s, \\ 0 & \text{otherwise.} \end{cases}$$

By item (2) of Lemma 4.11, the set $\mathcal{C}^s \cap \mathcal{F}^{cu}(p, U_p)$ has zero m_p^{cu} -measure. By Lemma 4.13, α_ϕ^p is defined and measurable. Observe that

$$m_y^s(\mathcal{F}^s(y, U_p)) \leq C m_p^s(\mathcal{F}^s(p, U_p))$$

for some constant C given by item (3) of Theorem 3.3 and $m_p^s(\mathcal{F}^s(p, U_p))$ is finite by the Radon property of m_p^s . Therefore the measurable function α_ϕ^p is bounded hence integrable for the Radon measure m_p^{cu} . As announced, we get a well-defined, finite, positive Borel measures for every $p \in M$ by setting:

$$(12) \quad m_p(\phi) := \int_{\mathcal{F}^{cu}(p, U_p)} \alpha_\phi^p(y) dm_p^{cu}(y).$$

Claim 5.2. *The previously defined measures $m_p, p \in M$, with support contained in some neighborhood U_p satisfy the following compatibility condition:*

$$(13) \quad \text{If } p, q \in M \text{ and } \phi \in C(U_p \cap U_q), \text{ then } m_p(\phi) = m_q(\phi).$$

To prove this claim, note that, since U_p and U_q are small (and may be assumed to intersect) there exists a well defined s -holonomy $h : \pi_p^{cu}(U_p \cap U_q) \rightarrow \pi_q^{cu}(U_p \cap U_q)$.

Since $\phi \in C(U_p \cap U_q)$, for any $y \in \pi_p^{cu}(U_p \cap U_q)$ it follows that

$$\alpha_\phi^p(y) = \alpha_\phi^q(h(y)).$$

Since $\{m_x^{cu}\}_{x \in M}$ is invariant under s -holonomy;

$$\begin{aligned} m_p(\phi) &= \int_{\pi_p^{cu}(U_p \cap U_q)} \alpha_\phi^p(y) dm_p^{cu}(y) = \int_{\pi_p^{cu}(U_p \cap U_q)} \alpha_\phi^q(h(y)) dm_p^{cu}(y) \\ &= \int_{\pi_q^{cu}(U_p \cap U_q)} \alpha_\phi^q(z) dm_q^{cu}(z) = m_q(\phi), \end{aligned}$$

proving the claim.

We now define a finite Borel measure m on M by picking a partition of unity $1 = \chi_1 + \dots + \chi_r$ subordinated to a finite cover U_1, \dots, U_r determined by points p_1, \dots, p_r and setting:

$$m(\phi) = m_{p_1}(\phi\chi_1) + \dots + m_{p_r}(\phi\chi_r).$$

Finally, we set $\mu^{cu \otimes s} = m(M)^{-1} \cdot m$. Observe it is a Borel probability measure on M . It is locally finite, hence finite on the compact set M .

Observe that m (and $\mu^{cu \otimes s}$) does not depend on the choice of the partition of unity. Indeed, if $1 = \chi'_1 + \dots + \chi'_{r'}$ is another partition of unity subordinated to some finite cover defined by points $p'_1, \dots, p'_{r'}$, then, for every $\phi \in C(M)$,

$$\begin{aligned} m_{p_1}(\phi\chi_1) + \dots + m_{p_r}(\phi\chi_r) &= \sum_{i=1}^r \sum_{j=1}^{r'} m_{p_i}(\phi\chi_i\chi'_j) = \sum_{i=1}^r \sum_{j=1}^{r'} m_{p'_j}(\phi\chi_i\chi'_j) \\ &= m_{p'_1}(\phi\chi_1) + \dots + m_{p'_{r'}}(\phi\chi_{r'}). \end{aligned}$$

The local formula in eq. (12), valid in any open set U_p , implies that the disintegration of $\mu^{cu \otimes s}$ with respect to any partition subordinate to \mathcal{F}^s is given by the Margulis s -conditionals m_x^s for a.e. $x \in M$.

To show f -invariance, observe that for any measurable subset A of $U_p \cap f^{-1}(U_q)$ with $p, q \in M$,

$$\begin{aligned} m(f(A)) &= m_q(f(A)) = \int_{\pi_q^{cu}(f(A))} m_y^s(f(A)) dm_q^{cu}(y) \\ &= D_s \int_{\pi_q^{cu}(f(A))} m_{f^{-1}y}^s(A) dm_q^{cu}(y) = D_s \int_{\pi_p^{cu}(A)} m_x^s(A) df_*^{-1}(m_q^{cu})(x) \\ &= D_s D_u^{-1} m_p(A) = D_s D_u^{-1} m(A). \end{aligned}$$

In the 4th equality above we used that $\pi_p^{cu}(f^{-1}(\pi_q^{cu}(f(A)))) = \pi_p^{cu}(A)$. The same holds for any measurable subset of M . The case $A = M$ implies that $D_s = D_u$. Hence m and therefore $\mu^{cu \otimes s}$ are f -invariant. \square

5.2. Identification of the conditionals. We fix some measure of maximal entropy μ and we show that its disintegration along s - or u -conditionals is given by the Margulis systems and its dilation factor is given by the topological entropy.

We consider the unstable foliation \mathcal{F}^u . We will build an adapted measurable partition ξ satisfying the following properties. First ξ is subordinated to \mathcal{F}^u , i.e., for μ -a.e. x ,

- $\xi(x)$ is a neighborhood of x in $\mathcal{F}(x)$ for the intrinsic topology of $\mathcal{F}(x)$ and with uniformly small diameter,
- $f(\xi(x)) \supset \xi(f(x))$ (i.e., $f\xi$ is less fine than ξ , denoted by $f\xi \prec \xi$).

Second, ξ is generating, that is,

- $\bigvee_{i \geq 0} f^{-i}\xi$ is the partition into points (modulo μ).

To find such a partition ξ we use the following construction from [51, Proposition 3.1 and Lemma 3.2]. For any finite partition \mathcal{P} we denote by \mathcal{P}^u the finer partition obtained by intersecting local plaques of \mathcal{F}^u with elements of \mathcal{P} .

Proposition 5.3. *There exists a finite measurable partition $\mathcal{P} = \{P_i\}$ such that:*

- $\mu(\bigcup_i \text{int}(P_i)) = 1$;
- each P_i is contained in a foliation chart of \mathcal{F}^u ;
- $\bigvee_{i=0}^{\infty} f^i(\mathcal{P}^u)$ is a measurable generating partition subordinated to \mathcal{F}^u .

Apply this proposition to get \mathcal{P} and \mathcal{P}^u and set $\xi := \bigvee_{i=0}^{\infty} f^i(\mathcal{P}^u)$.

Let $\{\mu_{\xi(x)}\}_{x \in M}$ be the disintegration of μ w.r.t. ξ . The entropy with respect to \mathcal{F} is defined [29] as:

$$h(f, \mu, \mathcal{F}) = - \int \log \mu_{\xi(x)}(f^{-1}\xi(fx)) d\mu.$$

Note that one could replace ξ in the above formula by any generating, measurable partition subordinate to \mathcal{F}^u [29].

The next result follows from Theorem C' in [28] and items (i)-(iii) after the statement of the theorem.

Proposition 5.4 (Ledrappier-Young). *Let $f \in \text{Diff}^2(M)$ be partially hyperbolic with strong unstable foliation \mathcal{F}^u . For any $\mu \in \mathbb{P}_{\text{erg}}(f)$,*

$$h(f, \mu, \mathcal{F}^u) \leq h(f, \mu).$$

If all center Lyapunov exponents of μ are nonpositive, then the above inequality is an equality.

Indeed, as μ has non-positive center exponents there exists $\lambda > 0$ such that the corresponding measurable foliation

$$\mathcal{W}^u(x) := \left\{ y \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < -\lambda \right\}.$$

coincides a.e. with \mathcal{F}^u .

Using the above result we are able to show the following proposition which is an extension of Ledrappier's argument in [26] (see also [30]). The next two results prove Theorem 3.6 and complete the proof of Theorem 3.3.

Proposition 5.5. *Let $f \in \text{Diff}^2(M)$ be a diffeomorphism with a Margulis u -system $\{m_x^u\}_{x \in M \setminus \mathcal{C}^u}$ with dilation factor $D_u > 1$ and μ_x^u fully supported on $\mathcal{F}^u(x)$ for each $x \in M \setminus \mathcal{C}^u$. If $\mu \in \mathbb{P}_{\text{erg}}(f)$ with $\lambda^c(\mu) \leq 0$, then $h(f, \mu) \leq \log D_u$. Moreover, there is equality if and only if the disintegration of μ along \mathcal{F}^u is given by m_x^u , μ -a.e.*

Proof. First suppose that μ gives full measure to \mathcal{C}^u . Item (1) of Lemma 4.11 implies that $h(f, \mu) = 0$ so $h(f, \mu) < \log D_u$. Now suppose that \mathcal{C}^u has zero μ -measure. Following Ledrappier, we will compare the conditional measures $\mu_{\xi(x)}(f^{-1}\xi(fx))$ that appear in the unstable entropy with similarly defined quantities from the unstable Margulis system.

Define a normalized family of measures adapted to the partition ξ , for μ -a.e. $x \in M$,

$$m_x(A) := \frac{m_x^u(A \cap \xi(x))}{m_x^u(\xi(x))}.$$

Observe that the above ratio is well defined, as m_x^u is fully supported and $\xi(x)$ is a neighborhood of x . The dilation property of Margulis measures yields:

$$m_x((f^{-1}\xi)(x)) = D_u^{-1} \frac{m_{f(x)}^u(\xi(fx))}{m_x^u(\xi(x))}.$$

This can be expressed in terms of the function

$$g(x) := -\log m_x((f^{-1}\xi)(x)) \geq 0.$$

Lemma 5.6. *The map*

$$x \mapsto m_x^u(\xi(x))$$

is measurable. Consequently the above function g is measurable.

Proof. A partition by foliated boxes is a finite partition \mathcal{P} modulo μ such that each $A \in \mathcal{P}$ has the form $(-1, +1)^d$ for some u -foliation chart χ_A .

We work with a finer partition \mathcal{P}^u which is measurable and obtained from foliated box partition \mathcal{P} , where

$$\mathcal{P}^u := \{\chi_A((-1, 1)^{d_u} \times \{t\}) : t \in (-1, +1)^{d-d_u}, A \in \mathcal{P}\}.$$

For $n \in \mathbb{N}$ define

$$\mathcal{P}_n^u := \bigvee_{1 \leq k \leq n} f^k(\mathcal{P}^u) \text{ and } \mathcal{P}_n := \bigvee_{1 \leq k \leq n} f^k(\mathcal{P}).$$

Note that $\mathcal{P}_n^u \nearrow \xi$.

Let $m_x^{u, \mathcal{P}}$ be the restriction (not normalized) of m_x^u to the atom $\mathcal{P}^u(x)$ of the partition \mathcal{P}^u . Observe that $m_x^{u, \mathcal{P}}$ may be seen as a measure defined on M . Since $\mathcal{P}^u(x)$ is bounded hence relatively compact in the intrinsic topology of $\mathcal{F}^u(x)$, its m_x^u -measure is finite. As $\xi(x) \subset \mathcal{P}^u(x)$ and $\mathcal{P}_n^u(x) \searrow \xi(x)$, we get:

$$m_x^u(\xi(x)) = m_x^{u, \mathcal{P}}(\xi(x)) = \lim_{n \rightarrow \infty} m_x^{u, \mathcal{P}}(\mathcal{P}_n^u(x)).$$

To conclude, it is enough to see that for each $n \geq 1$, the map $x \mapsto m_x^{u, \mathcal{P}}(\mathcal{P}_n^u(x))$ is measurable. But Lemma 4.13 shows that $x \mapsto m_x^{u, \mathcal{P}}(A)$ is measurable for each element A of the finite partition \mathcal{P}_n . \square

By the pointwise ergodic theorem, we know $\lim_n \frac{1}{n} S_n g(x)$ (possibly $+\infty$) exists almost everywhere. To identify this limit, observe that it is also a limit in probability. Taking the logarithm of the previous identity, we see that $g(x) = h(fx) - h(x) + \log D_u$ for a measurable function h . Therefore the limit in probability and therefore almost everywhere is the constant $\log D_u$. Thus g is integrable with:

$$-\int \log m_x((f^{-1}\xi)(x)) d\mu = \log D_u.$$

Now recall that $h(f, \mu, \mathcal{F}^u) = -\int \log \mu_{\xi(x)}((f^{-1}\xi)(x)) d\mu$ and so,

$$-\int \log \mu_{\xi(x)}((f^{-1}\xi)(x)) d\mu \leq -\int \log m_x((f^{-1}\xi)(x)) d\mu = \log(D_u).$$

The inequality comes from Jensen's inequality and the (strict) concavity of the logarithm. The case of equality for Jensen's inequality yields that this is an equality if and only if $\mu_{\xi(x)}((f^{-1}\xi)(x)) = m_x((f^{-1}\xi)(x))$ for μ -a.e. $x \in M$. Replacing ξ by $f^{-n}\xi$, we obtain that

$$\mu_{f^{-n}\xi(x)}(f^{-n-1}\xi)(x) = \frac{m_x^u((f^{-n-1}\xi)(x))}{m_x^u((f^{-n}\xi)(x))}$$

so

$$\begin{aligned} \mu_{\xi(x)}((f^{-n-1}\xi)(x)) &= \prod_{k=0}^n \frac{\mu_{\xi(x)}((f^{-k-1}\xi)(x))}{\mu_{\xi(x)}((f^{-k}\xi)(x))} = \prod_{k=0}^n \mu_{(f^{-k}\xi)(x)}((f^{-k-1}\xi)(x)) \\ &= \prod_{k=0}^n \frac{m_x^u((f^{-k-1}\xi)(x))}{m_x^u((f^{-k}\xi)(x))} = \frac{m_x^u((f^{-n-1}\xi)(x))}{m_x^u(\xi(x))}. \end{aligned}$$

Since ξ is generating and increasing, the disintegration of μ along ξ is given by the Margulis u -conditionals as claimed. \square

The next result identifies the dilation factor and proves the Addendum 3.4.

Proposition 5.7. *Let $f \in \text{Diff}^2(M)$ be a flow-type diffeomorphism with minimal strong foliations and $D_u = D_s^{-1}$ as in Proposition 5.1, then $D_u = \exp(h_{\text{top}}(\phi))$.*

Proof. Let μ be an ergodic measure for f . If $\lambda^c(\mu) \leq 0$. Then by Proposition 5.5 we know that $h(f, \mu) \leq \log D_u$. On the other hand, if $\lambda^c(\mu) > 0$, then for f^{-1} we see that $h(f, \mu) = h(f^{-1}, \mu) \leq \log D_s^{-1} = \log D_u$. Hence, for any ergodic measure we have $h(f, \mu) \leq \log D_u$ and by the Variational Principle we have $h_{\text{top}}(f) \leq \log D_u$.

To finish the proof, it is enough to show that, $\mu^{cu \otimes s}$ being the measure constructed in Proposition 5.1,

$$h(\mu^{cu \otimes s}) \geq \log(D^u).$$

Since the conditional measures of $\mu^{cu \otimes s}$ along \mathcal{F}^s are given by the Margulis system $\{m^s\}_{x \in M \setminus \mathcal{C}^s}$, the previous proposition shows that:

$$h(\mu^{cu \otimes s}, f^{-1}, \mathcal{F}^s) = -\log(D_s).$$

Note that \mathcal{F}^s is the strong unstable foliation of f^{-1} . By Proposition 5.5,

$$h(\mu^{cu \otimes s}, f) = h(\mu^{cu \otimes s}, f^{-1}) \geq -\log(D_s) = \log(D_u).$$

\square

Proof of Theorem 3.6. Let f be a C^2 diffeomorphism of a compact manifold M . Assume that f is a flow-type diffeomorphism and minimal strong foliations. The dilation factor D_u is greater than 1 by Lemma 4.7. By Proposition 5.7, $D_u = D_s^{-1} = \exp h_{\text{top}}(f)$.

Therefore Proposition 5.5 shows that any ergodic MME μ with a nonpositive central exponent has unstable conditionals given by the unstable Margulis system $\{m_x^u\}_{x \in M^u}$.

Finally we prove that μ has full support. Observe that by Proposition 5.5 the conditional measures of μ with respect to a subordinated partition ξ are equal to (normalized restrictions of the) Margulis measures to the atom of ξ . So, there exists an atom of partition such that almost every (with respect to normalized Margulis measure) point is contained in the support of μ . By construction, each atom of ξ contains an open set inside unstable leaf. As Margulis measures are fully supported

we get an open set (inside unstable leaf) totally contained in the support of μ . By invariance of support and minimality of unstable foliation we get $\text{supp}(\mu) = M$. \square

5.3. Hyperbolic MMEs. In this section we assume the existence of some hyperbolic MME. We build its twin measure, which is a MME with opposite central exponent. We get a uniqueness result from a version of Hopf's argument.

Proof of Proposition 3.9. By assumption (1), the exponent along $\mathcal{F}^c(x)$ is negative for almost every x . Let $\mathcal{W}^c(x) := \{y \in \mathcal{F}^c(x) : \lim_{n \rightarrow \infty} d(f^n x, f^n y) = 0\}$. Since the central foliation is one-dimensional, it is not difficult to see that $\mathcal{W}^c(x)$ is an open arc that contains, for μ -a.e. $x \in M$, a neighborhood of x in its central leaf. Moreover, the C^1 submanifold $\mathcal{W}^c(x)$ depends measurably on x . These two facts are immediate consequences of [1, Theorem 3.11] which deals more generally with stable manifolds in the C^1 setting with dominated splitting). By item (2), this curve is bounded. Thus, the following is well-defined μ -almost everywhere (recall that $\mathcal{F}^c(x)$ is oriented):

$$\beta : M \rightarrow M, \quad x \mapsto \sup \mathcal{W}^c(x).$$

Note that $\beta(x)$ must be one of the endpoints of $\mathcal{W}^c(x)$. Moreover, β is measurable and satisfies $\beta \circ f = f \circ \beta$ and $\beta(x) \in \mathcal{F}^c(x)$ for all $x \in M$.

Claim 5.8. *There is a measurable subset $Z \subset M$ with $\mu(Z) = 1$ such that the restriction $\beta|_Z$ is injective.*

We defer the proof of the claim and first deduce the proposition from it. Let $\nu := \beta_*(\mu)$. Note that the claim implies that β is a measure-preserving conjugacy between (f, μ) and (f, ν) . If $\lambda_c(\nu)$ was negative, for μ -a.e. $x \in M$, $\beta(x)$ would be contained in the open $\mathcal{W}^c(\beta(x))$ contradicting that $\beta(x) \in \partial \mathcal{W}^c(x)$. Thus $\lambda_c(\nu) \geq 0$ and the proposition is established. \square

To prepare the proof of the claim, recall from Section 2.3 that one can find a measurable disintegration of μ along the foliation \mathcal{F}^c into Radon measures $\{\bar{\mu}_x^c\}_{x \in M}$ (well-defined up to a factor, possibly depending on the leaf). Note that μ is atomless since otherwise it would be carried by a periodic orbit hence by compact center leaves, in contradiction to assumption (3). Therefore $f(x) \in \mathcal{F}^c(x)$ with $x \neq f(x)$ for μ -a.e. x and one can define Radon measures $\{\mu_x^c\}_{x \in M}$ by using the invariant normalization $\mu_x^c([y, f(y))_c) = 1$ for μ -a.e. $x \in M$ and for all $y \in \mathcal{F}^c(x)$. We first show:

$$(14) \quad \text{for } \mu\text{-a.e. } x \in M \quad \mu_x^c|_{\mathcal{W}^c(x)} = c_x \delta_x \quad \text{for some } c_x > 0.$$

For each $\epsilon > 0$, let $\widehat{\mathcal{W}}_\epsilon^c(x) := \{y \in \mathcal{W}^c(x) : d_c(y, \partial \mathcal{W}^c(x)) > \epsilon\}$ where $d_c(\cdot, \cdot)$ is the induced distance on the center foliation. Note that for any $\delta > 0$, there are a set S of positive μ -measure of points x and some arbitrarily small $\epsilon > 0$ such that

$$\forall x \in S \quad \mu_x^c(\widehat{\mathcal{W}}_\epsilon^c(x)) \geq (1 - \delta) \mu_x^c(\mathcal{W}^c(x)).$$

The last measure is positive for μ -a.e. x . Now, for μ -a.e. x , for all large $n \geq 0$,

$$f^n(\widehat{\mathcal{W}}_\epsilon^c(f^{-n}x)) \subset \{y \in \mathcal{W}^c(x) : d_c(y, x) < \epsilon\}.$$

The ergodicity of μ implies that, for μ -a.e. x , $f^{-n}x \in S$ for some arbitrarily large integers n . Hence, by invariance of μ :

$$\begin{aligned} \mu_x^c(\{y \in \mathcal{W}^c(x) : d_c(y, x) < \epsilon\}) &\geq \mu_{f^{-n}x}^c(\widehat{\mathcal{W}}_\epsilon^c(f^{-n}x)) \\ &\geq (1 - \delta)\mu_{f^{-n}x}^c(\mathcal{W}^c(f^{-n}x)) = (1 - \delta)\mu_x^c(\mathcal{W}^c(x)). \end{aligned}$$

Since $\epsilon, \delta > 0$ were arbitrarily small, eq. (14) follows.

Proof of Claim 5.8. To prove the claim, let Z be the set of $x \in M$ for which μ_x^c satisfies eq. (14). This is a measurable set with full measure μ . Now, let $x, y \in Z$. If $\beta(x) = \beta(y)$ then $\mathcal{W}^c(x) = \mathcal{W}^c(y)$. In particular $\mu_x^c|_{\mathcal{W}^c(x)} = \mu_y^c|_{\mathcal{W}^c(y)}$. By eq. (14), this implies $x = y$, concluding the proof of the claim. \square

We now turn to the Hopf argument.

Proof of Proposition 3.8. Let μ, ν be ergodic MME with $\lambda_c(\mu) < 0$ and $\lambda_c(\nu) \leq 0$. Proposition 5.5 implies that their conditional measures along unstable foliation are both given by the u -Margulis system $\{m_x^u\}_{x \in M \setminus \mathcal{C}^u}$. Let $B_\mu := \{x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \rightarrow \mu\}$ be the ergodic basin of μ (the convergence is in the weak star topology as $n \rightarrow +\infty$). By ergodicity $\mu(M \setminus B_\mu) = 0$ and so $m_x^u(M \setminus B_\mu) = 0$ for μ -a.e. x . Similarly, letting B_ν be the ergodic basin of ν , for ν -a.e. y we have $m_y^u(M \setminus B_\nu) = 0$.

As the center Lyapunov exponent of μ is negative, for μ -a.e. $x \in B_\mu$, $m_x^u(M \setminus B_\mu) = 0$ and there is a subset $K \subset \mathcal{F}^u(x) \cap B_\mu$ with $m_x^u(K) > 0$ and such that the size of the Pesin local stable manifolds $\mathcal{W}_{loc}^s(z)$ of points $z \in K$ is uniformly bounded from below.

Pick $y \in B_\nu$ with $m_y^u(M \setminus B_\nu) = 0$. The density of $\mathcal{F}^u(y)$ implies that there is a local cs -holonomy $h : K \rightarrow \mathcal{F}^u(y)$ (perhaps after replacing K with a smaller subset). This holonomy is absolutely continuous from (K, m_x^u) to $(\mathcal{F}^u(y), m_y^u)$, hence:

$$m_y^u(h(K)) > 0.$$

By the choice of K , $h(z)$ belongs to the Pesin stable manifold of z . Since the ergodic basin is saturated by stable manifolds, $h(K) \subset B_\nu$ and therefore $m_y^u(B_\nu) > 0$. As $m_y^u(M \setminus B_\nu) = 0$, we conclude that $B_\nu \cap B_\mu \neq \emptyset$ and consequently $\mu = \nu$. \square

Remark 5.9. *Let f be a flow-type C^2 diffeomorphism with minimal strong foliations. Whenever μ is an ergodic MME with $\lambda_c(\mu) < 0$, it is the only ergodic invariant probability measure whose u -conditionals are given by an unstable Margulis system.*

Proof of Theorem 3.10. Let f be a flow-type C^2 diffeomorphism and minimal strong foliations. Observe that since f is partially hyperbolic with a one-dimensional center, there must exist an ergodic MME μ [15, 19, 31]. For instance, assume that $\lambda^c(\mu) < 0$ (if not, use f^{-1}).

Proposition 3.8 shows that there is no other ergodic MME with nonpositive central exponent. Now Proposition 3.9 gives some ergodic MME ν with $\lambda^c(\nu) \geq 0$. In particular $\nu \neq \mu$. The previous uniqueness shows that the case $\lambda^c(\nu) = 0$ cannot occur so $\lambda^c(\nu) > 0$. We see that there are exactly two ergodic MME. The dichotomy is proved.

It remains to prove that μ is Bernoulli. The symbolic dynamics of [36] implies that (f, μ) is isomorphic to the product of a Bernoulli measure with a circular permutation of some order $p \geq 1$. It follows that $\mu = (\mu_1 + \dots + \mu_p)/p$ where the measures μ_1, \dots, μ_p are distinct ergodic MMEs for f^p such that $\lambda^c(f^p, \mu_k) = p \cdot \lambda^c(f, \mu)$ does not depend on k and is not zero. Observe that f^p like f is a flow-type diffeomorphism with minimal strong foliations. Hence, the previous uniqueness result applies to f^p showing that $p = 1$, i.e., (f, μ) is Bernoulli.

The theorem is established. \square

6. PERTURBING TIME-ONE MAPS TO GET FLOW-TYPE WITH MINIMAL STRONG FOLIATIONS

We prove Theorem 3.11. Let $\varphi^t : M \rightarrow M$ be a topologically transitive Anosov flow on a compact manifold and let $T > 0$. We find an open set of diffeomorphisms that are flow-type and minimal strong foliations accumulating on φ^T . We first use structural stability results, mainly from [24], to see that all C^1 perturbations of φ^T are flow-type. A C^1 perturbation will then be used to get the robust minimality of the strong foliations.

Proposition 6.1. *Let $\varphi^T : M \rightarrow M$ be the time $T > 0$ map of a Anosov flow. Then there is a C^1 -neighborhood \mathcal{V} of φ^T such that $f \in \mathcal{V}$ is flow-type.*

Proof. We must prove (I), (II), (III), and (IV) for all diffeomorphisms C^1 -close to φ^T . Observe that these properties are well-known for φ^T itself. Let us see that they hold for all C^1 -close diffeomorphisms using the structural stability theory in [24].

Partial hyperbolicity with the center subbundle of a given dimension is well-known to be robust. Since the center foliation \mathcal{F}^c of φ^T is the partition into the orbits of the flows, it is smooth, hence it is plaque expansive [24, (7.2)]. Therefore $(\varphi^T, \mathcal{F}^c)$ is structurally stable [24, (7.1)], and dynamical coherence (II) holds robustly. Indeed, this theorem yields a center foliation \mathcal{F}_g^c for g , whereas its proof, especially [24, thm. (6.8)], gives stable and unstable manifolds of center leaves, these provide the required invariant foliations \mathcal{F}_g^{cu} and \mathcal{F}_g^{cs} tangent to E^{cu} and E^{cs} .

The center foliation of φ^T is obviously oriented by the vector field. The flow being expansive it has at most countably many closed orbits. The Anosov Closing Lemma yields (infinitely many) closed orbits. The structural stability of $(\varphi^T, \mathcal{F}^c)$ implies that (III) holds robustly.

To establish (IV), we need to refer to the proof of the structural stability of $(\varphi^T, \mathcal{F}^c)$ [24, Thm. (7.1)] and especially of [24, Thm. (6.8)]. We use terminology and notations from [24, chap. 6, 7]. On page 107 of [24], it is shown that the perturbed diffeomorphism f' has a center foliation \mathcal{F}' whose every leaf is close to the corresponding leaf of \mathcal{F} in the sense that $\mathcal{F}'(h(x))$ is represented by section of the formal normal bundle to $\mathcal{F}(x)$ which is close to zero when f' is close to $f = \varphi^T$. Since the lifts i^*f, i^*f' of f, f' to this formal bundle are close to each other, we see that $f'(x) = F'(\tau(x), x)$ for some function $\tau : M \rightarrow \mathbb{R}$ with $\sup |\tau - T| < \epsilon$ where $\epsilon > 0$ can be taken arbitrarily small by assuming f' to be close enough to $f = \varphi^T$.

The previous reasoning shows that the minimum of the lengths of the closed loops for \mathcal{F}' is close to that of \mathcal{F} . In particular it can be assumed to be larger than 3ϵ .

If τ is not continuous, then there are two sequences of points $x_n, y_n \in M$ that converge to the same limit z and such that $s := \lim_{n \rightarrow \infty} \tau(x_n)$ and $t := \lim_{n \rightarrow \infty} \tau(y_n)$ exist and are distinct. This implies that $F^s(z) = F^t(z)$ with $|t - s| < 2\epsilon$, i.e., \mathcal{F}' contains a loop of length less than 2ϵ . The contradiction proves the continuity of τ . \square

The robust minimality of the strong foliations follows from works of Bonatti, Díaz and Urès [6, 7]. Let \mathcal{T} be the set of time T maps of topologically transitive Anosov flows on some compact manifold. We need to find an open set \mathcal{U} of diffeomorphisms whose strong stable and strong unstable foliations are both minimal and such that $\mathcal{T} \subset \overline{\mathcal{U}}$. We proceed in two steps.

Step 1. *Any diffeomorphism in \mathcal{T} can be C^1 -approximated by robustly transitive diffeomorphisms.*

This follows from the work of Bonatti and Diaz in [6].

Step 2. *Any diffeomorphism in \mathcal{T} has a C^1 -neighborhood \mathcal{V} with the following property. Any robustly transitive diffeomorphism in \mathcal{V} can be C^1 -approximated by diffeomorphisms whose strong stable and strong unstable foliations are both minimal.*

This follows from the work of Bonatti, Díaz and Ures [7] (it is in fact a simpler situation since in our setting hyperbolic periodic points are dense and contained in invariant compact leaves).

This completes the proof of Theorem 3.11.

Remark 6.2. *A recent work of Ures, Viana, and Yang [49] extends [7] to a larger class of systems among the C^∞ volume preserving diffeomorphisms. More precisely, they prove that in a neighborhood of the time one map of the geodesic flow of any hyperbolic surface, there is an open and dense set of diffeomorphisms with both strong foliations minimal.*

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