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# Canonical forms=Persistence diagrams.

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“Algebraic topology and its applications” seminar, NRU HSE  
Moscow

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24 April 2020

## Chain complexes, filtrations and their invariants

- ▶ Let  $C_*$  is a chain complex over field  $\mathbb{k}$ , a sequence of vector spaces with linear operators

$$\rightarrow C_{j+1} \xrightarrow{\partial_{j+1}} C_j \xrightarrow{\partial_j} C_{j-1} \rightarrow \dots \rightarrow C_0$$

which satisfy

$$\partial_j \circ \partial_{j+1} = 0$$

- ▶ An  $\mathbb{R}$ -filtration  $F$  is an increasing sequence of subcomplexes  $F_{\leq s} C_* \subset F_{\leq r} C_*$ ,  $s < r$ , indexed by finite set of real numbers
- ▶  $\rightarrow$  Canonical Form  $(C_*, F)$  introduced in ([SB1994], "The framed **Morse complex** and its invariants") - combinatorial invariants of filtered complexes
- ▶ Canonical Form  $(C_*, F)$  = Persistence Diagrams  $(C_*, F)$
- ▶ algorithm "bringing to Canonical Form" from loc.cit. = standard algorithm for Persistence Diagrams
- ▶ this gives a simple explanation for the latter algorithm  $\rightarrow$  generalizations of PD and of the algorithm for them

## “Canonical form” of $\mathbb{R}$ -filtered complexes [SB1994]

- ▶ Let  $C_*/k$  is an  $\mathbb{R}$ -filtered chain complex,  $F_{\leq s}C_* \subset F_{\leq r}C_*$ ,  $s < r$ , indexed by finite set of real numbers.
- ▶ It usually comes with a basis compatible with filtration so that each subspace  $F_{\leq r}C_j$  is the span  $\langle e_1^{(j)}, \dots, e_{i_r}^{(j)} \rangle$ ,  $e_i^{(j)} \in F_{\leq r_{ij}}$
- ▶ Chain complex is in “canonical form” in bases  $\{\tilde{e}_i^{(j)}\}_{j \in \{0,1,\dots\}}^{i \in \{1,\dots,\dim_F C_j\}}$  if for any basis element  $\tilde{e}_i^{(j)}$  either  $\partial \tilde{e}_i^{(j)} = 0$  or  $\partial \tilde{e}_i^{(j)} = \tilde{e}_{i'}^{(j-1)}$ , and in the latter case,  $\tilde{e}_i^{(j)} \neq \tilde{e}_{i_1}^{(j)} \Rightarrow \partial \tilde{e}_i^{(j)} \neq \partial \tilde{e}_{i_1}^{(j)}$ .
- ▶ This is the same as decomposition of the set of basis elements  $S = S_{\text{birth}} \sqcup S_{\text{death}} \sqcup S_H$  such that  $\partial \tilde{e}_i^{(j)} = \tilde{e}_{i'}^{(j-1)}$  for  $\tilde{e}_i^{(j)} \in S_{\text{death}}$ ,  $\tilde{e}_{i'}^{(j-1)} \in S_{\text{birth}}$ ,  $\partial \tilde{e}_i^{(j)} = 0$  for  $\tilde{e}_i^{(j)}$  from  $S_H$ .
- ▶ **Theorem (SB, 1994)** One can bring an  $\mathbb{R}$ -filtered complex to canonical form by a linear change of basis *preserving the filtration*. The resulting **canonical form** filtered chain complex is **unique**.

## Category of filtered complexes is semi-simple

- ▶ reformulation of the theorem: Any object in the category of  $\mathbb{R}$ -filtered complexes over field is isomorphic to a *canonically* defined sum of simple objects: 1-dimensional  $\mathbb{R}$ -filtered complex with trivial differential,  $\partial \tilde{e}_i^{(j)} = 0$ ,  $\langle \tilde{e}_i^{(j)} \rangle = F_{\leq r}$ ,  $r \in \mathbb{R}$ , and 2-dimensional  $\mathbb{R}$ -filtered complex with trivial homology  $\partial \tilde{e}_{i_2}^{(j+1)} = \tilde{e}_{i_1}^{(j)}$ ,  
 $\langle \tilde{e}_{i_1}^{(j)} \rangle = F_{\leq s_1}$ ,  $\langle \tilde{e}_{i_1}^{(j)}, \tilde{e}_{i_2}^{(j+1)} \rangle = F_{\leq s_2}$ ,  $s_1, s_2 \in \mathbb{R}$ .
- ▶ Proof of the theorem: bring the complex to the required canonical form by induction, starting from the lowest generators of degrees 1, then 2 etc, the claim is that manipulating degree  $k$  generators does not destroy the canonical form in degree  $k - 1$  and in lower critical values of degree  $k$ .
- ▶ This theorem is somewhat similar in spirit to the Poincaré's definition of the torsion in homology groups.

## Proof of the theorem

- ▶ Let for  $p = j$  and  $m \leq i$ , or  $p < j$  and all  $m$ ,  $\partial e_m^{(p)}$  has the required form. Let's simplify  $\partial e_{i+1}^{(j)}$

$$\partial e_{i+1}^{(j)} = \sum_k e_k^{(j-1)} \alpha_k. \quad (1)$$

- ▶ Move all the terms with  $e_k^{(j-1)} = \partial e_q^j$ ,  $q \leq i$ , from the right to the left in (1):  $\partial(e_{i+1}^{(j)} - \sum_{q \leq i} e_q^{(j)} \alpha_{k(q)}) = \sum_k e_k^{(j-1)} \beta_k$
- ▶ If  $\beta_k = 0$  for all  $k$ , let  $\tilde{e}_{i+1}^{(j)} = e_{i+1}^{(j)} - \sum_{q \leq i} e_q^{(j)} \alpha_{k(q)}$ ,  $\partial \tilde{e}_{i+1}^{(j)} = 0$
- ▶ Otherwise let  $k_0$  be the maximal  $k$  with  $\beta_k \neq 0$ :

$$\partial(e_{i+1}^{(j)} - \sum_{q \leq i} e_q^{(j)} \alpha_{k(q)}) = e_{k_0}^{(j-1)} \beta_{k_0} + \sum_{k < k_0} e_k^{(j-1)} \beta_k, \beta_{k_0} \neq 0.$$

where  $k_0 \neq k(q)$  for  $q \leq i$ . Define

$$\tilde{e}_{i+1}^{(j)} = \frac{1}{\beta_{k_0}} \left( e_{i+1}^{(j)} - \sum_{q \leq i} e_q^{(j)} \alpha_{k(q)} \right), \tilde{e}_{k_0}^{(j-1)} = e_{k_0}^{(j-1)} + \sum_{k < k_0} e_k^{(j-1)} \frac{\beta_k}{\beta_{k_0}}$$

- ▶ Then  $\partial \tilde{e}_{i+1}^{(j)} = \tilde{e}_{k_0}^{(j-1)}$ , with  $k_0 \neq k(q)$  for  $q \leq i$ .  $\square$

## Uniqueness of the “canonical form”

- ▶ Let  $\{a_i^{(j)}\}$ ,  $\{b_i^{(j)} = \sum_{k \leq i} a_k^{(j)} \alpha_k\}$ , be two bases of  $C_*$  for the two canonical forms. Assume that for all indexes  $p < j$  and all  $n$ , and  $p = j$  and  $n \leq i$  the canonical forms agree. Let  $\partial a_{i+1}^{(j)} = a_m^{(j-1)}$  and  $\partial b_{i+1}^{(j)} = b_l^{(j-1)}$  with  $m > l$ .
- ▶ It follows that

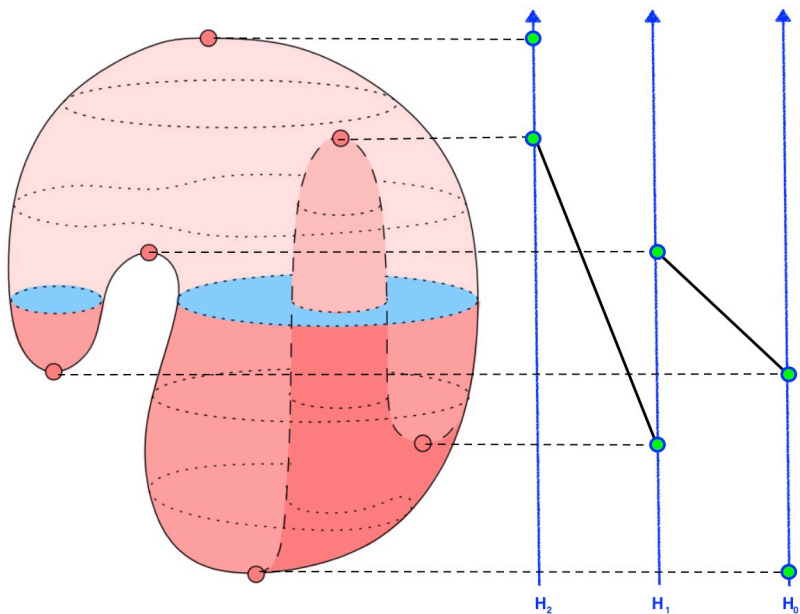
$$\partial \left( \sum_{k \leq i+1} a_k^{(j)} \alpha_k \right) = \sum_{n \leq l} a_n^{(j-1)} \beta_n,$$

where  $\alpha_{i+1} \neq 0$ ,  $\beta_l \neq 0$ . Therefore

$$\partial a_{i+1}^{(j)} = \sum_{n \leq l} a_n^{(j-1)} \beta_n / \alpha_{i+1} - \sum_{k \leq i} \partial a_k^{(j)} \alpha_k / \alpha_{i+1}.$$

- ▶ On the other hand  $\partial a_{i+1}^{(j)} = a_m^{(j-1)}$ , with  $m > l$ , and  $\partial a_k^{(j)}$  for  $k \leq i$  are either zero or some basis elements different from  $a_m^{(j-1)}$ . This gives a contradiction and the canonical forms agree for  $p = j$  and  $n = i + 1$ .
- ▶ Similarly if  $\partial b_{i+1}^{(j)} = 0$ , then  $\partial a_{i+1}^{(j)} = -\sum_{k \leq i} \partial a_k^{(j)} \alpha_k / \alpha_{i+1} \square$

## Example of the canonical form





## Homology $H_*(F_{\leq s} C_*)$

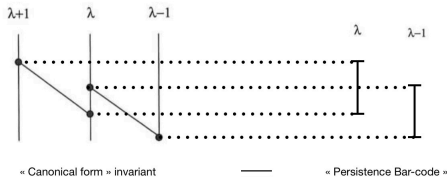
- ▶ From the “Canonical Form” one can immediately read the homology of any subcomplex  $H_*(F_{\leq s} C_*)$  as well as the images of  $H_i(F_{\leq s_j} C_*) \rightarrow H_i(F_{\leq s_k} C_*)$
- ▶ For each 2-dim piece, or pair  $\partial \tilde{e}_{i_2}^{(j)} = \tilde{e}_{i_1}^{(j-1)}$  in the “canonical form”, with filtration indexes  $s_1, s_2 \in \mathbb{R}$  a new homology class of  $H_*(F_{\leq s} C_*)$  is born at  $s_1$  and dies at  $s_2$ .
- ▶ Similarly each 1-dim piece  $\partial \tilde{e}_i^{(j)} = 0, e_i \in S_H$ , with filtration index  $r \in \mathbb{R}$  in the “canonical form”, corresponds to a new homology class of  $H_*(F_{\leq s} C_*)$  born at  $r$  which never dies.

## “Canonical form” invariants = “Persistence Diagrams”

- ▶ This is precisely the definition of the persistence diagram:
- ▶ 2-dim pieces in canonical forms or pair  $\partial \tilde{e}_{i_2}^{(j)} = \tilde{e}_{i_1}^{(j-1)}$  in the “canonical form”, with filtration indexes  $s_1, s_2 \in \mathbb{R}$  are in 1-to-1 correspondence precisely with the points  $(s_1, s_2)$  in the persistent diagram
- ▶ 1-dim pieces  $\partial \tilde{e}_i^{(j)} = 0, e_i \in S_H$ , with filtration index  $r \in \mathbb{R}$  in the “canonical form”, are in 1-to-1 correspondence precisely with points  $(r, +\infty)$  in the persistence diagram

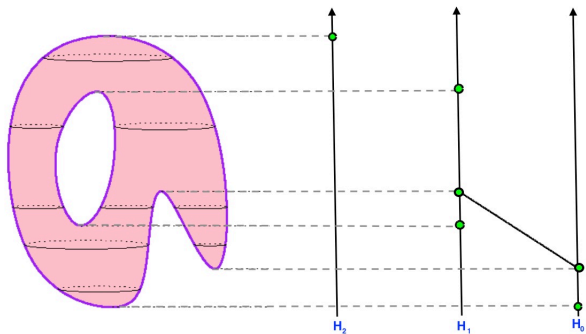
# “Canonical form” invariants = “Persistence Bar-codes”

There are three equivalent visualizations



of the same invariants. “Persistence Bar-codes”/“Persistence diagrams” were introduced in applied mathematics in the beginning of 2000s (H.Edelsbrunner, J.Harer, A.Zamorodian “Hierarchical **Morse complexes** for piecewise linear 2-manifolds” Proc. of Symp on Comput Geometry, June 2001, A.Zamorodian “Persistence and hierarchical **Morse complexes**, PhD Thesis, University of Illinois, 2001). There are several software packages for computing these invariants of a finite filtration. The principal algorithm is based on the bringing of the filtered complex to its canonical form by upper-triangular matrices from [SB1994].

Partition of critical values into pairs “birth-death” plus homological critical values (“births” paired with  $+\infty$ )



## Arnold's problem on extension of smooth function inside a ball

- ▶ Given  $f \in C^\infty(\partial B^n \times [-\varepsilon, \varepsilon]) \rightarrow$  how many critical points of given index must a generic smooth extension of  $f$  inside the ball  $B$  have?
- ▶ Example: function on closed manifold and a ball containing all critical points of the function. Then a restriction of the function to the neighborhood of the boundary of this ball must contain information on the *Betti numbers of the manifold*.

## Morse complex

$f : M^n \rightarrow \mathbb{R}$ ,  $f \in C^\infty$ , generic,  $\{x \mid f(x) \leq c\}$  –compact. Then  $p_\alpha$ –critical points,  $df|_{T_{p_\alpha}} = 0$ , are isolated, near

$p_\alpha: f = \sum_{l=1}^j -(x^l)^2 + \sum_{l=j}^n (x^l)^2$ . Let  $g$  is a generic metric. Then define

$$C_j = \bigoplus_{\text{index}(p_\alpha)=j} [p_\alpha, \text{or}(T_{p_\alpha}^-)]$$

where  $T_{p_\alpha} = T_{p_\alpha}^- \oplus T_{p_\alpha}^+$  is wrt  $\partial^2 f$  and  $g$ .

The differential is

$$\partial_j [p_\alpha, \text{or}] = \sum_{\text{index}(p_\beta)=j-1} [p_\beta, \text{or}] \# \mathcal{M}(p_\alpha, p_\beta)$$

$$\mathcal{M}(p_\alpha, p_\beta) = \{ \gamma : \mathbb{R} \rightarrow M^n \mid$$

$$\dot{\gamma} = -(\text{grad}_g f)(\gamma(t)), \lim_{t \rightarrow -\infty} \gamma = p_\alpha, \lim_{t \rightarrow +\infty} \gamma = p_\beta \} / \mathbb{R}$$

## What are the invariants of Morse complexes independent of metrics?

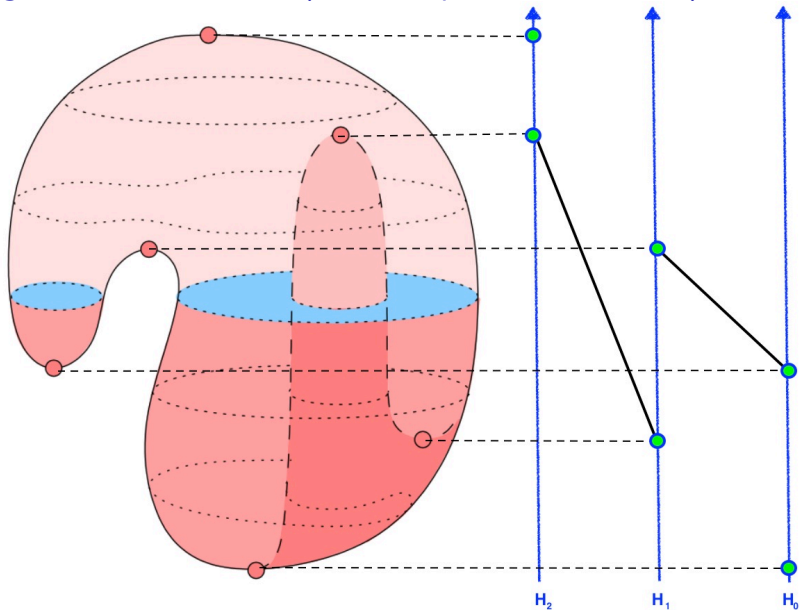
- ▶ The Morse complex is naturally filtered  $F_{\leq s} C_* \subset F_{\leq r} C_*$ ,  $s < r$ , by the set  $\{f(p_\alpha)\} \subset \mathbb{R}$  of critical values of  $f$ :  
 $[p_\alpha] \in F_s C_*$  if  $f(p_\alpha) \leq s$
- ▶ Claim: under generic perturbation of the metrics, the anti-gradient trajectory exceptionnaly goes from the critical point  $p_\alpha$  to the critical point  $p_{\tilde{\alpha}_{\text{lower}}}$  with  $\text{index}(p_\alpha) = \text{index}(p_{\tilde{\alpha}_{\text{lower}}})$
- ▶ The change of the Morse complex is described then by the change of the basis:  $[p_\alpha] \rightarrow [p_\alpha] \pm [p_{\tilde{\alpha}_{\text{lower}}}]$ .
- ▶ What can be done with the complex using such *upper-triangular* change of bases?

## “Canonical form” invariants of Morse complexes

- ▶ The Morse complex is naturally filtered by the set  $\{f(p_\alpha)\}$  of critical values of  $f$ :  $[p_\alpha, \text{or}] \in F_{\leq s} C_*$  if  $f(p_\alpha) \leq s$
- ▶  $\rightarrow$  canonical partition of the set of critical values  $\{f(p_\alpha)\}$  into pairs “birth-death”, plus separate set giving a basis in  $H_*(M, \mathbb{k})$  or “births” paired with  $+\infty$
- ▶ Claim: the “canonical form” of Morse complex does not depend on the metrics: under generic perturbation of the metrics the complex changes via series of change of bases:  $e_i^{(j)} \rightarrow e_i^{(j)} \pm e_{i_{\text{lower}}}^{(j)}$
- ▶ when the function is deformed the “canonical form” invariant changes naturally in continuous way. This can be expressed in  $\epsilon - \delta$  language.

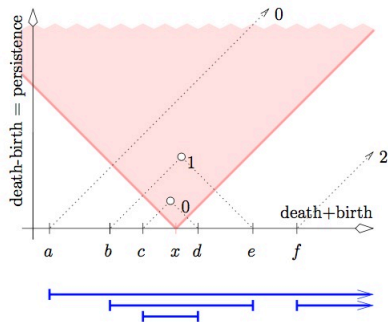
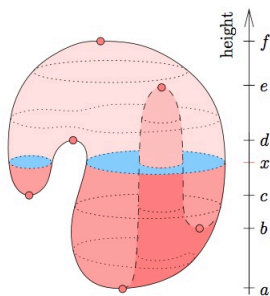


Partition of critical values into pairs “birth-death” plus homological critical values (“births” paired with  $+\infty$ )



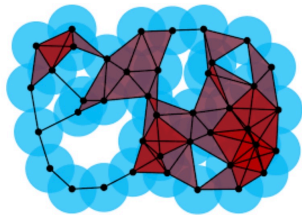
## Another comparison

This illustration is taken from the plenary talk “Persistent Homology: Theory and Practice.” H. Edelsbrunner, D. Morozov, at the *European Congress of Mathematics*, 2012:



## Point clouds and Čech Complex

- ▶  $f =$  distance to a set of points (point cloud),  $M = \mathbb{R}^n$ , sublevel sets  $\{x \mid f(x) \leq d\}$  are unions of balls, their intersections define the Čech complex for each  $d \in \mathbb{R}$ , increasing the distance  $d \rightarrow$  more intersections  $\rightarrow$  the filtered

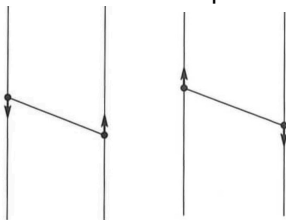


complex.

- ▶ the “canonical form” invariants=“persistence bar-codes/diagrams” of the filtered complex  $\rightarrow$  main tool in the *topological data analysis*.
- ▶ these invariants permit to calculate the homology of the manifold which is approximated by the point cloud, since for some  $d$  the sublevel set of the distance is homotopically equivalent to this manifold.

## Arnold's problem on extension of smooth function

- ▶ Given  $f \in C^\infty(\partial B^n \times [-\varepsilon, \varepsilon]) \rightarrow$  how many are there critical points of generic smooth extension of  $f$  inside the ball  $B^n$ ?
- ▶ Theorem (SB,1994) These pairs in “canonical form”  $f|_{\partial B^n}$  indicate the critical points of certain index in  $B^n$  :



They can cancel each other in certain configurations, details are in [SB1994].

# Small eigenvalues of twisted Laplacian

These “canonical form” invariants were also applied in Le Peutrec D., Nier F., Viterbo C. *“The Witten Laplacian and Morse–Barannikov Complex”* [LNV2011] to find formulas for small eigenvalues of the Witten Laplacian of  $d_{f,h} = hd + df$  and identification of critical points with eigenforms

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^* = \bigoplus_{\rho=0}^{\dim M} \Delta_{f,h}^{(\rho)}.$$

There is a one to one correspondence  $j_\rho$  between  $\mathcal{U}^{(\rho)}$  and the set of eigenvalues (counted with multiplicities) of  $\Delta_{f,h}^{(\rho)}$  lying in  $[0, h^{3/2})$  such that

$$j_\rho(U^{(\rho)}) = 0 \quad \text{if } U^{(\rho)} \in \mathcal{U}_H^{(\rho)}$$

$$j_\rho(U^{(\rho)}) = \kappa^2(U^{(\rho+1)}) \frac{h}{\pi} \frac{|\lambda_1^{(\rho+1)} \dots \lambda_{\rho+1}^{(\rho+1)}|}{|\lambda_1^{(\rho)} \dots \lambda_\rho^{(\rho)}|} \frac{|\text{Hess}f(U^{(\rho)})|^{1/2}}{|\text{Hess}f(U^{(\rho+1)})|^{1/2}} (1 + \mathcal{O}(h)) e^{-2 \frac{f(U^{(\rho+1)}) - f(U^{(\rho)})}{h}}$$

if  $\partial_B U^{(\rho+1)} = U^{(\rho)}$

$$j_\rho(U^{(\rho)}) = \kappa^2(U^{(\rho)}) \frac{h}{\pi} \frac{|\lambda_1^{(\rho)} \dots \lambda_\rho^{(\rho)}|}{|\lambda_1^{(\rho-1)} \dots \lambda_{\rho-1}^{(\rho-1)}|} \frac{|\text{Hess}f(U^{(\rho-1)})|^{1/2}}{|\text{Hess}f(U^{(\rho)})|^{1/2}} (1 + \mathcal{O}(h)) e^{-2 \frac{f(U^{(\rho)}) - f(U^{(\rho-1)})}{h}}$$

if  $\partial_B U^{(\rho)} = U^{(\rho-1)}$

Here the  $\lambda$ 's denote the negative eigenvalues of the  $\text{Hess}f$  at the corresponding points.

- [SB1994] S.Barannikov, "*The Framed Morse complex and its invariants*" volume 21 of Adv. Soviet Math., pages 93–115. Amer. Math. Soc., Providence, RI, 1994.
- [EHZ2001] H.Edelsbrunner, J.Harer, A.Zamorodian "*Hierarchical Morse complexes for piecewise linear 2-manifolds*" Proc. of Symp on Comput Geometry, June 2001
- [EM2012] H.Edelsbrunner, D. Morozov, "*Persistent Homology: Theory and Practice.*" Proceedings of the European Congress of Mathematics, 2012.
- [LNV2011] Le Peutrec, D., Nier, F., Viterbo, C. "*Precise Arrhenius Law for  $p$ -forms: The Witten Laplacian and Morse–Barannikov Complex*" Ann. Henri Poincaré (2013) 14: 567.