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# THE STATIONARY DISTRIBUTION OF REFLECTED BROWNIAN MOTION IN A WEDGE: DIFFERENTIAL PROPERTIES

M. BOUSQUET-MÉLOU, A. ELVEY PRICE, S. FRANCESCHI, C. HARDOUIN, AND K. RASCHEL

**ABSTRACT.** We consider a semimartingale reflected Brownian motion (SRBM) in a two-dimensional wedge. Under standard assumptions on the parameters of the model (opening of the wedge, angles of the reflections on the axes, drift), we study the algebraic and differential nature of the Laplace transform of its stationary distribution. We derive necessary and sufficient conditions for this Laplace transform to be rational, algebraic, differentially finite or more generally differentially algebraic. These conditions are explicit linear dependencies among the angles involved in the definition of the model.

A complicated integral expression is known for this Laplace transform [20]. In the differentially algebraic case, we go further and provide an explicit, integral-free expression. In particular, we obtain new derivations of the Laplace transform in several well known cases, namely the skew-symmetric case, the orthogonal reflections case and the sum-of-exponential densities case (corresponding to the so-called Dieker-Moriarty conditions on the parameters).

To prove these results, we start from a functional equation that the Laplace transform satisfies, to which we apply tools from diverse horizons. To establish differential algebraicity, a key ingredient is Tutte's *invariant approach*, which originates in enumerative combinatorics. It allows us to express the Laplace transform (or its square) as a rational function of a certain *canonical invariant*, a hypergeometric function in our context. To establish differential transcendence, we turn the functional equation into a difference equation and apply Galoisian results on the nature of the solutions to such equations.

## 1. INTRODUCTION

We consider an obliquely reflected Brownian motion in a two-dimensional convex wedge with opening angle  $\beta \in (0, \pi)$ , defined by its drift  $\tilde{\mu}$  (with associated angle  $\theta = \arg(-\tilde{\mu}) \in (-\pi, \pi]$ ) and two reflections angles  $\delta$  and  $\varepsilon$  in  $(0, \pi)$  (Figure 1). The covariance matrix is taken to be the identity.

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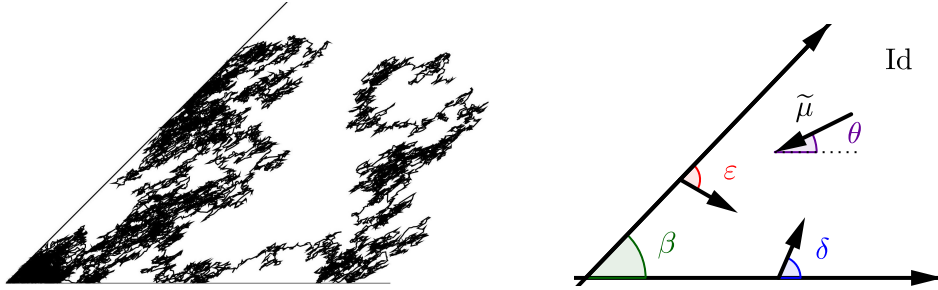


FIGURE 1. A trajectory of the reflected Brownian motion in a wedge, and the parameters  $\beta$ ,  $\theta$ ,  $\delta$  and  $\varepsilon$ .

Since the introduction of the reflected Brownian motion in the eighties [23, 24, 44], the mathematical community has shown a constant interest in this topic. Typical questions deal with the recurrence of the process, the absorption at the corner of the wedge, the existence of stationary distributions... We refer for more details to the introduction of [20] and to Figure 2. The parameter  $\alpha$  occurring there, also central in this paper, is:

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta}. \quad (1)$$

We further introduce the following refinement of the parameter  $\alpha$ :

$$\alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta} \quad \text{and} \quad \alpha_2 = \frac{2\delta - \theta - \pi}{\beta}, \quad (2)$$

so that  $\alpha_1 + \alpha_2 = 2\alpha - 1$ . These numbers also play a key role in this paper, and it seems to be the first time that their importance is acknowledged.

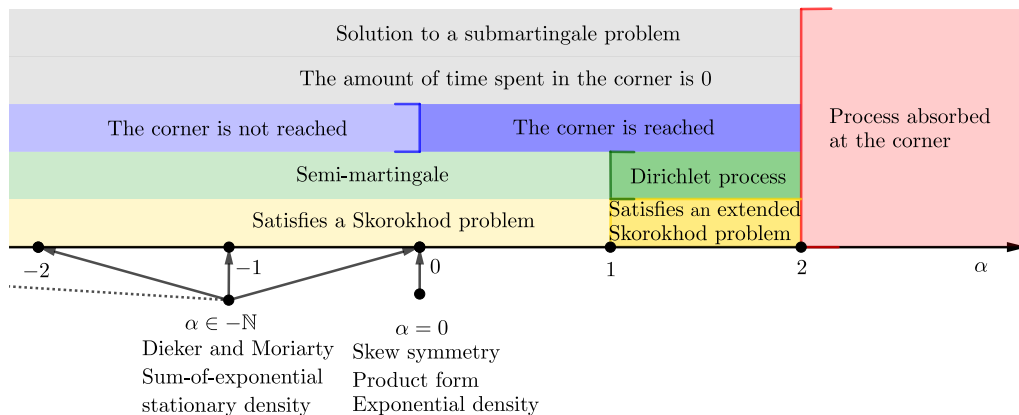


FIGURE 2. Properties of obliquely reflected Brownian motion in terms of  $\alpha = \frac{\delta + \varepsilon - \pi}{\beta}$ . Here are some references: semimartingale property [45, 36, 41]; Skorokhod problem [24, 46]; submartingale problem [44]; amount of time spent at the corner, accessibility of the corner and absorption [44]; Dirichlet process and extended Skorokhod problem [30, 29]; skew symmetry [23, 26]; sum-of-exponential stationary density [11].

It is known [45] that the process is a semimartingale (called SRBM for short) if and only if

$$\delta + \varepsilon - \pi < \beta, \quad \text{or equivalently} \quad \alpha < 1. \quad (3)$$

We assume this to hold in this paper. We denote as before  $\theta = \arg(-\tilde{\mu}) \in (-\pi, \pi]$ , and we also assume that

$$0 < \theta < \beta. \quad (4)$$

The meaning of this condition will be clarified in Section 2.1; see (17). Then, a stationary distribution exists if and only if

$$\beta - \varepsilon < \theta < \delta, \quad (5)$$

and it is in this case unique, see [11, §3] and [27]. We also assume this to hold.

The main object of study in this paper is the Laplace transform  $\Phi(x, y)$  of this two-dimensional stationary distribution. In a recent paper [20], two of the authors gave a (complicated) closed form expression for it, which involves integrals and various trigonometric and algebraic functions. However, it is known that when the parameter  $\alpha$  is a non-positive integer (with an additional non-degeneracy condition), the stationary density is a finite sum of exponentials, of the form  $\sum_i c_i e^{-a_i x - b_i y}$ , which implies that  $\Phi(x, y)$  is a rational function in  $x$  and  $y$  [11]. This drastic simplification raises the following natural question: for which values of the parameters  $\beta, \tilde{\mu}, \delta$  and  $\varepsilon$  does the Laplace transform simplify? The case when it is rational being (mostly) elucidated by [11], when is it an *algebraic* function of  $x$  and  $y$  (meaning that it satisfies a polynomial equation with coefficients in the field  $\mathbb{R}(x, y)$  of rational functions in  $x$  and  $y$ )? When is it *D-finite* (DF)? By this, we mean that it satisfies two linear differential equations with coefficients in  $\mathbb{R}(x, y)$ , one in  $x$  and one in  $y$ . More generally again, when is it *D-algebraic* (DA), that is, when does it satisfy a polynomial differential equation in  $x$ , and another in  $y$ ? In other words, we want to *classify* the parameters of the semimartingale reflected Brownian motion depending on whether, and where, the associated Laplace transform fits in the following natural hierarchy of functions:

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}. \quad (6)$$

A function that does not fit in this hierarchy, that is, is not D-algebraic, is said to be *differentially transcendental* (or D-transcendental for short).

In this paper, we answer completely these questions. Table 1 summarizes the necessary and sufficient conditions that we establish. We denote  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$  and  $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ .

	D-algebraic	D-finite	Algebraic	Rational
$\beta/\pi \notin \mathbb{Q}$	$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ , or $\{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$	$\alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z}$ , or $\{\alpha_1, \alpha_2\} \subset \mathbb{Z} \cup \left(-\mathbb{N} + \frac{\pi}{\beta}\mathbb{Z}\right)$	$\alpha \in -\mathbb{N}_0$ , or $\{\alpha_1, \alpha_2\} \subset \mathbb{Z}$	$\alpha \in -\mathbb{N}_0$
$\beta/\pi \in \mathbb{Q}$	always	$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ , or $\{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$	$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$ , or $\{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$	$\alpha \in -\mathbb{N}_0$

TABLE 1. Nature of the Laplace transform in terms of  $\alpha$ ,  $\alpha_1$  and  $\alpha_2$ .

Observe the key role played by the parameters  $\alpha$ ,  $\alpha_1$  and  $\alpha_2$ , and in particular by the conditions

$$\alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \quad \text{or equivalently} \quad \delta + \varepsilon \in \beta\mathbb{Z} + \pi\mathbb{Z} \quad (7)$$

and

$$\{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \quad \text{or equivalently} \quad \{2\varepsilon + \theta, 2\delta - \theta\} \subset \beta\mathbb{Z} + \pi\mathbb{Z}. \quad (8)$$

We call them the *simple angle condition* and the *double angle condition*, respectively. When one of them holds, we give for  $\Phi(x, y)$  a new, integral-free expression in terms of the classical hypergeometric function  ${}_2F_1$ , from which differential algebraicity follows via classical closure properties of DA functions. Several explicit examples are given in Sections 6.3 and 7.3.

The algebraic and differential properties of the Laplace transform are reflected in various ways on the stationary distribution itself. Let us give two examples, focussing, for simplicity, on the one-dimensional transform  $\Phi(x, 0)$  and the corresponding distribution, denoted by  $\nu$  here.

- **Moments.** If  $\Phi(x, 0)$  is DA, then the differential equation that it satisfies translates into a polynomial recurrence relation for the moments  $M_n$  of  $\nu$ , with polynomial coefficients in  $n$ . This relation becomes linear, and of finite order, as soon as  $\Phi(x, 0)$  is DF. An explicit example is worked out in Section 6.3.
- **Density.** If  $\Phi(x, 0)$  is DF, then the density of  $\nu$  is DF as well. If  $\Phi(x, 0)$  is even rational, the density is a linear combination of terms  $x^k e^{-ax}$ .

For the corresponding *discrete* problem, namely stationary distributions of discrete random walks in a wedge, a number of cases where similar simplifications occur are known: let us cite for instance the famous Jacskon networks [28] and their product form distributions, and works of Latouche and Miyazawa [32] and Chen, Boucherie and Goseling [8], who obtain necessary and sufficient conditions for the stationary distribution of random walks in the quadrant to be sums of geometric terms. See also [15, Chap. 4]. Still, no *uniform* criteria as those that we obtain in this paper in the continuous setting are available.

Let us now describe the methods that we use to prove our results. We believe them to be surprisingly diverse. The starting point is a linear functional equation, established in [9], that characterizes the function  $\Phi(x, y)$ . The proof of D-algebraicity (when (7) or (8) holds) relies on Tutte's *invariant* approach. Between 1973 and 1984, Tutte studied a functional equation that arises in the enumeration of properly colored triangulations [42], and has similarities with the equation defining  $\Phi$ . In order to solve it (and prove that its solution is D-algebraic), Tutte developed an algebraic approach based on the construction of certain *invariants*. This approach has recently been fruitfully applied, first to other map enumeration problems [3, 4], and then in other contexts, as the enumeration of walks confined to the first quadrant [5], or avoiding a quadrant [7]. A first application to the reflected Brownian motion is presented in [19] in the case where  $\beta = \delta = \varepsilon$ . This is clearly a special case of (7), corresponding to orthogonal reflections on the boundaries once the wedge is deformed into a quadrant (see Section 2.1). The present paper goes much further than [19] by finding *the exact applicability of the invariant method* in the determination of the stationary distribution of the reflected Brownian motion. This approach might be applicable to other related problems, such as computation of the Green function and the Martin boundary in the transient case.

The differential transcendence result, proving that  $\Phi(x, y)$  is *not* D-algebraic if  $\beta/\pi \notin \mathbb{Q}$  and neither (7) nor (8) hold, also starts from the functional equation defining  $\Phi(x, y)$ , but relies on a completely different tool, namely difference Galois theory. Analogously to classical Galois theory, difference Galois theory builds a correspondence between the algebraic relations satisfied by the solutions of a linear functional equation and the algebraic dependencies between the coefficients of this equation. Using this theory, one can reduce the question of the D-transcendence to the study of the poles of an explicit rational function. Difference Galois theory has recently been applied to the enumeration of *discrete* walks in the quadrant [13, 14, 12]. To our knowledge, this is the first time that it is applied to a *continuous* random process such as SRBM.

We conclude this introduction by an outline of the paper. In Section 2, we define precisely the process under study and its normalization to a quadrant. We also give the functional equation that characterizes the Laplace transform  $\Phi(x, y)$  (or more precisely, the corresponding transform  $\varphi(x, y)$  on the quadrant). We finally state our results in detail. In Section 3 we introduce the notion of *invariant*, and relate it to a boundary value problem satisfied by  $\varphi$ . In Section 4 we exhibit a simple invariant  $w$ , which is D-finite, explicit, and exists for all values of the parameters. Moreover, we prove that  $w$  is *canonical* in the sense that any invariant is a rational function in  $w$ . In Section 5 we show how to construct an invariant involving  $\varphi$ , provided a certain *decoupling function* exists. We then show that such a function exists if and only if one of the angle conditions (7) or (8) holds. These two cases are then detailed, respectively, in Sections 6 and 7. In particular, we obtain an expression of  $\varphi$  (and  $\Phi$ ) in terms of  $w$ , from which D-algebraicity follows. The D-transcendence condition is established in Section 8. Section 9 is devoted to the case  $\beta/\pi \in \mathbb{Q}$ .

To preserve the reading flow, we have put in appendices the proofs of a number of technical lemmas. Another appendix is devoted to the construction of a meromorphic lifting of the Laplace transform.

## 2. PRELIMINARIES AND MAIN RESULTS

Let us begin with some basic notation. We denote by  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$  the set of natural integers; and by  $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$  the set of positive integers. We denote by  $\mathbb{R}_+$  (resp.  $\mathbb{R}_-$ ) the set of positive (resp. negative) real numbers.

### 2.1. THE SEMIMARTINGALE REFLECTED BROWNIAN MOTION (SRBM) IN A WEDGE

A simple linear transformation maps the reflected Brownian motion discussed in the introduction (with covariance matrix the identity) onto a reflected Brownian motion in the first (non-negative) quadrant with non-trivial covariance matrix. Most of the time we will work in the quadrant, but it will sometimes be important to switch between these two representations, as some quantities are more simply computed or understood in one or the other of the two frameworks. To describe explicitly the quadrant normalization, we first need to define precisely what is a SRBM in the quadrant.

We consider  $(Z_t)_{t \geq 0}$ , an obliquely reflected Brownian motion in the first quadrant, of covariance  $\Sigma$ , drift  $\mu$  and reflection matrix  $R$ , where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

with  $r_{11} > 0$  and  $r_{22} > 0$ . The columns  $R^1$  and  $R^2$  of the matrix  $R$  represent the directions in which the Brownian motion is reflected on the boundaries; see Figure 3, left. The so-called *orthogonal reflections case* corresponds to  $r_{12} = r_{21} = 0$ .

The process  $Z_t$  exists as a semimartingale if and only if

$$\det R > 0 \quad \text{or} \quad (r_{12} > 0 \text{ and } r_{21} > 0). \quad (9)$$

See [41, 36] for a proof of a multidimensional version of this statement, and [46] for a general survey of the SRBM in an orthant. In this case, the reflected Brownian motion may be written as

$$Z_t = Z_0 + B_t + \mu \cdot t + R \cdot \begin{pmatrix} L_t^1 \\ L_t^2 \end{pmatrix}, \quad \forall t \geq 0,$$

where  $Z_0$  is an inner starting point,  $(B_t)_{t \geq 0}$  is a Brownian motion with covariance  $\Sigma$  starting from the origin, and  $(L_t^1)_{t \geq 0}$  (resp.  $(L_t^2)_{t \geq 0}$ ) is (up to a multiplicative constant) the local time on the  $y$ -axis (resp.  $x$ -axis). The process  $(L_t^1)_{t \geq 0}$  is continuous and non-decreasing, starts from 0, and increases only when the process  $Z_t$  touches the vertical boundary, which implies that for all  $t \geq 0$ ,  $\int_0^t \mathbb{1}_{\{Z_s^1 \neq 0\}} dL_s^1 = 0$ . Of course, a similar statement holds for  $L_t^2$ .



FIGURE 3. Transformation of the quadrant into a wedge of opening angle  $\beta$ . The new parameters  $\beta$ ,  $\tilde{\mu}$ ,  $\delta$  and  $\varepsilon$  are given by (10), (12) and (16), respectively.

We now describe the linear transform that maps the Brownian motion in the quadrant with covariance matrix  $\Sigma$  to a Brownian motion with covariance matrix the identity, confined to a wedge of opening  $\beta$  (see Figure 3 and [20, Appendix A]). We take

$$\beta = \arccos\left(-\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right) \in (0, \pi), \quad (10)$$

so that

$$\sin \beta = \sqrt{\frac{\det \Sigma}{\sigma_{11}\sigma_{22}}}.$$

Then we define a linear transformation  $T$ , which depends only on  $\Sigma$ ,

$$T = \begin{pmatrix} 1 & \cot \beta \\ \sin \beta & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{\sigma_{11}} & 1 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\sigma_{22}}{\det \Sigma}} & -\frac{\sigma_{12}}{\sqrt{\sigma_{22}\det \Sigma}} \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} \end{pmatrix}. \quad (11)$$

This is easily inverted into

$$T^{-1} = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix} \begin{pmatrix} \sin \beta & -\cos \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\det \Sigma}{\sigma_{22}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{22}}} \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix}.$$

Under the transformation  $T$ , the reflected Brownian motion  $Z_t$  associated to  $(\Sigma, \mu, R)$  becomes a Brownian motion with covariance matrix the identity in a wedge of angle  $\beta$  with parameters  $(\text{Id}, \tilde{\mu}, \delta, \varepsilon)$ . More explicitly, we first have  $\tilde{\mu} = T\mu$ , that is:

$$\tilde{\mu}_1 = \frac{\mu_1\sigma_{22} - \mu_2\sigma_{12}}{\sqrt{\sigma_{22}\det \Sigma}} \quad \text{and} \quad \tilde{\mu}_2 = \frac{\mu_2}{\sqrt{\sigma_{22}}}, \quad (12)$$

so that

$$\tan \theta = \frac{\tilde{\mu}_2}{\tilde{\mu}_1} = \frac{\mu_2\sqrt{\det \Sigma}}{\mu_1\sigma_{22} - \mu_2\sigma_{12}} = \frac{\sin \beta}{\frac{\mu_1}{\mu_2}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta}, \quad (13)$$

or more precisely

$$\theta := \arg(-\tilde{\mu}) = -\text{sgn}(\tilde{\mu}_2) \arccos\left(\frac{-\tilde{\mu}_1}{\sqrt{\tilde{\mu}_1^2 + \tilde{\mu}_2^2}}\right) \quad (14)$$

$$= -\text{sgn}(\mu_2) \arccos\left(\frac{\mu_2\sigma_{12} - \mu_1\sigma_{22}}{\sqrt{\sigma_{22}\sqrt{\mu_1^2\sigma_{22} - 2\mu_1\mu_2\sigma_{12} + \mu_2^2\sigma_{11}}}}\right). \quad (15)$$

Later we will assume  $\mu_2 < 0$ , thus simplifying the above formula a bit. The new reflection angles  $\delta, \varepsilon \in (0, \pi)$  are given by:

$$\tan \delta = \frac{\sin \beta}{\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta} \quad \text{and} \quad \tan \varepsilon = \frac{\sin \beta}{\frac{r_{21}}{r_{11}}\sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \cos \beta}, \quad (16)$$

see [20, Appendix A]. Then one can prove that the semimartingale conditions (9) for the quadrant translate into the conditions (3) for the  $\beta$ -wedge (see Lemma A.1 (i)). The second result of Lemma A.1 states that Condition (4) is equivalent to the drift  $\mu$  being negative:

$$\mu_1 < 0, \quad \mu_2 < 0. \quad (17)$$

This assumption is standard and appears for instance in [17, 16, 11, 18, 20].



## 2.2. INVARIANT MEASURES AND LAPLACE TRANSFORMS

Assuming that Conditions (9) hold, the reflected Brownian motion in the quadrant has a stationary distribution if and only if [27]:

$$\det R > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0, \quad (18)$$

which strengthens the first condition in (9). Assuming (4), or equivalently (17), Condition (18) can be seen to be equivalent to Conditions (3) and (5) combined (see Lemma A.1 (iii)). From now on, we assume that (18) is satisfied and we denote by  $\Pi$  the stationary distribution, which is an invariant probability measure [25]. Then there exist two finite boundary measures  $\nu_1$  and  $\nu_2$  on the coordinate axes, defined, for  $i = 1, 2$ , by

$$\nu_i(\cdot) = \mathbb{E}_\Pi \left[ \int_0^1 \mathbf{1}_{\{Z_\infty \in \cdot\}} dL_t^i \right],$$

where  $Z_\infty$  follows the law  $\Pi$ . These measures may be considered as invariant measures (or stationary distributions) on the axes, see [25]. The measure  $\nu_1$  (resp.  $\nu_2$ ) has its support on the vertical (resp. horizontal) axis, where  $z_1 = 0$  (resp.  $z_2 = 0$ ). Let  $\varphi$  denote the Laplace transform of  $\Pi$ :

$$\varphi(x, y) = \mathbb{E}_\Pi[\exp((x, y) \cdot Z_\infty)] = \iint_{\mathbb{R}_+^2} \exp((x, y) \cdot z) \Pi(dz),$$

and let  $\varphi_1$  and  $\varphi_2$  the Laplace transforms of  $\nu_1$  and  $\nu_2$ :

$$\varphi_1(y) = \int_{\mathbb{R}_+} \exp(yz_2) \nu_1(dz_2) \quad \text{and} \quad \varphi_2(x) = \int_{\mathbb{R}_+} \exp(xz_1) \nu_2(dz_1). \quad (19)$$

The measures  $\nu_1, \nu_2$  are also bounded [25] and thus these three Laplace transforms exist and are finite at least when  $x$  and  $y$  have non-positive real parts.

It is known that for all values of  $x, y$  for which  $\varphi(x, y)$  is finite, the transforms  $\varphi_1(y)$  and  $\varphi_2(x)$  are finite as well, and that  $\varphi(x, y)$  is then a linear combination of  $\varphi_1(y)$  and  $\varphi_2(x)$  with rational coefficients [9, Lem. 4.1]:

$$-\gamma(x, y)\varphi(x, y) = \gamma_1(x, y)\varphi_1(y) + \gamma_2(x, y)\varphi_2(x), \quad (20)$$

where

$$\begin{cases} \gamma(x, y) = \frac{1}{2}((x, y)\Sigma) \cdot (x, y) + (x, y) \cdot \mu = \frac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1x + \mu_2y, \\ \gamma_1(x, y) = (x, y)R^1 = r_{11}x + r_{21}y, \\ \gamma_2(x, y) = (x, y)R^2 = r_{12}x + r_{22}y. \end{cases} \quad (21)$$

By letting  $x$  and/or  $y$  tend to zero and noticing that  $\varphi(0, 0) = 1$ , we can conversely express  $\varphi_1$  and  $\varphi_2$  in terms of  $\varphi$ :

$$\varphi_1(0) = \frac{\mu_1 r_{22} - \mu_2 r_{12}}{r_{12} r_{21} - r_{11} r_{22}}, \quad \varphi_2(0) = \frac{\mu_2 r_{11} - \mu_1 r_{21}}{r_{12} r_{21} - r_{11} r_{22}}, \quad (22)$$

$$r_{21}\varphi_1(y) = -(\mu_2 + \sigma_{22}y/2)\varphi(0, y) - r_{22}\frac{\mu_2 r_{11} - \mu_1 r_{21}}{r_{12} r_{21} - r_{11} r_{22}},$$

and symmetrically for  $\varphi_2(x)$ . Hence (20) can also be seen as a functional equation in  $\varphi$  only.

Finally, the Laplace transform of  $\Pi$  and the Laplace transform of the corresponding stationary distribution for the Brownian motion in the  $\beta$ -wedge  $T(\mathbb{R}_+^2)$  (with  $T$  given by (11)) are related by a linear change of variables:

$$\varphi(x, y) = \Phi((x, y)T^{-1}).$$

This is proved in [20, Cor. 2] when  $\sigma_{11} = \sigma_{22} = 1$  and still holds in our more general setting. From this, and the above relations between  $\varphi, \varphi_1$  and  $\varphi_2$ , we see that determining the differential and algebraic nature of  $\Phi$  boils down to studying the nature of  $\varphi_1$  and  $\varphi_2$ .



**Proposition 2.1.** *The Laplace transform  $\varphi(x, y)$  is rational (resp. algebraic, D-finite, D-algebraic) if and only if  $\varphi_1$  and  $\varphi_2$  are rational (resp. algebraic, D-finite, D-algebraic). The same holds for the Laplace transform  $\Phi(x, y)$ .*

We have used implicitly various properties of rational/algebraic/D-finite/D-algebraic functions: they include rational functions, form a ring, are closed by specialization of variables, by composition with an affine function... We refer to [33, 34, 40] for classical references on D-finite functions, and to [6, Sec. 6.1] for D-algebraic ones.

### 2.3. MAIN RESULTS

Recall that  $r_{11} > 0$  and  $r_{22} > 0$ , and that we work under the following additional assumptions:

- existence of a stationary distribution:

$$\det R > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0,$$

- negative drift in the quadrant model:

$$\mu_1 < 0, \quad \mu_2 < 0.$$

In terms of the  $\beta$ -wedge, we recall that these combined conditions translate into:

$$\delta - \pi < \beta - \varepsilon < \theta < \delta, \quad 0 < \theta < \beta \tag{23}$$

(see Lemma A.1 (iii); it seems that these equivalences were never formerly established in the SRBM literature.)

We focus on  $\varphi_1(y)$ , since the study of  $\varphi_2(x)$  is obviously symmetric. If we apply an  $x/y$ -symmetry to the quadrant model, thus exchanging  $\varphi_1$  and  $\varphi_2$ , then the angle  $\beta$  is unchanged, while the angles  $\delta$  and  $\varepsilon$  are exchanged (see (10) and (16)). Moreover, the angle  $\theta$  becomes  $\beta - \theta$ . This implies that the parameter  $\alpha$  is unchanged, while  $\alpha_1$  and  $\alpha_2$  are exchanged. Since the angle conditions stated in Theorems 2.2 and 2.3 below are expressed in terms of  $\alpha, \alpha_1$  and  $\alpha_2$ , and are symmetric in  $\alpha_1$  and  $\alpha_2$ , the transforms  $\varphi_1$  and  $\varphi_2$  are always of the same nature, in the sense of the hierarchy (6) — and of the same nature as  $\varphi(x, y)$  and  $\Phi(x, y)$ , by Proposition 2.1.

We distinguish two cases, depending on whether the angle  $\beta$  is a rational multiple of  $\pi$ , or not. As shown already by Table 1, we state our conditions in terms of the Brownian motion in the  $\beta$ -wedge. Note that the conditions are very compact. For comparison, this is not the case when one tries to understand the algebraic/differential nature of the generating function that counts discrete walks in the quadrant [5, 13].

**Theorem 2.2.** *Assume that  $\beta/\pi \notin \mathbb{Q}$ . Then  $\varphi_1(y)$  is differentially algebraic over  $\mathbb{R}(y)$  if and only if one of the following angle conditions holds:*

$$\delta + \varepsilon \in \beta\mathbb{Z} + \pi\mathbb{Z}, \quad \text{that is,} \quad \alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \tag{24}$$

or

$$\{2\varepsilon + \theta, 2\delta - \theta\} \subset \beta\mathbb{Z} + \pi\mathbb{Z}, \quad \text{that is,} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \tag{25}$$

where  $\beta, \delta, \varepsilon$  and  $\theta$  are defined in (10), (16) and (15), while  $\alpha$  and the  $\alpha_i$  are the quantities of (1) and (2). Moreover,  $\varphi_1$  is differentially finite if and only if

$$\delta + \varepsilon \in -\beta\mathbb{N}_0 + \pi\mathbb{Z}, \quad \text{that is,} \quad \alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z},$$

or

$$\{2\varepsilon + \theta - \beta, 2\delta - \theta\} \subset (\beta\mathbb{Z} + \pi) \cup (-\beta\mathbb{N} + \pi\mathbb{Z}), \quad \text{that is,} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z} \cup \left(-\mathbb{N} + \frac{\pi}{\beta}\mathbb{Z}\right).$$

The function  $\varphi_1$  is algebraic if and only if

$$\delta + \varepsilon \in -\beta\mathbb{N}_0 + \pi, \quad \text{that is,} \quad \alpha \in -\mathbb{N}_0,$$

or

$$\{2\varepsilon + \theta, 2\delta - \theta\} \subset \beta\mathbb{Z} + \pi, \quad \text{that is,} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z}.$$

Finally,  $\varphi_1$  is rational if and only if  $\alpha \in -\mathbb{N}_0$ .

We now move to the case where  $\beta$  is a rational multiple of  $\pi$ .

**Theorem 2.3.** *If  $\beta/\pi \in \mathbb{Q}$ , then  $\frac{1}{\varphi_1} \frac{d\varphi_1}{dy}$ , the logarithmic derivative of  $\varphi_1$ , is D-finite over  $\mathbb{R}(y)$ . In particular,  $\varphi_1$  is differentially algebraic. Moreover,  $\varphi_1$  is algebraic if and only if (7) or (8) holds, and this is the only case where it is D-finite. Finally,  $\varphi_1$  is rational if and only if  $\alpha \in -\mathbb{N}_0$ .*

Moreover, when (7) or (8) holds, we obtain closed form expressions of  $\varphi_1$  in terms of Gauss' hypergeometric function  ${}_2F_1$ . These expressions, which are valid whether  $\beta/\pi$  is rational or not, are given in Theorems 6.3 and 7.4. These theorems could qualify as the third main result of this paper, but their statements requires more definitions than Theorems 2.2 and 2.3 and we will only state them later. Our results apply in particular to three cases in which Condition (7) holds and the Laplace transform is known to take a particularly simple form (see Figure 4 for an illustration):

- The skew symmetric case  $\delta + \varepsilon = \pi$ , or equivalently the case  $\alpha = 0$ , studied for instance in [23, §10], [26] or [10].
- The (more general) Dieker and Moriarty case  $\alpha \in -\mathbb{N}_0$  (see [11]).
- Orthogonal reflections in the quadrant model [19], corresponding to  $r_{12} = r_{21} = 0$ , or equivalently to  $\delta = \varepsilon = \beta$  (see (16)).

The transform  $\varphi_1$  is rational in the first two cases, and  $1/\varphi_1$  is D-finite in the third one. We review these examples in Section 6.3, together with a fourth (D-finite) example,  $\delta + \varepsilon + \beta = 2\pi$ , for which we work out explicitly the recurrence relation satisfied by the moments of  $\varphi_1$ . In Section 7.3 we present other interesting cases, this time where the double angle condition holds, with an emphasis on models where, unexpectedly,  $\varphi_1$  is algebraic while the angle  $\beta$  is not a rational multiple of  $\pi$ .

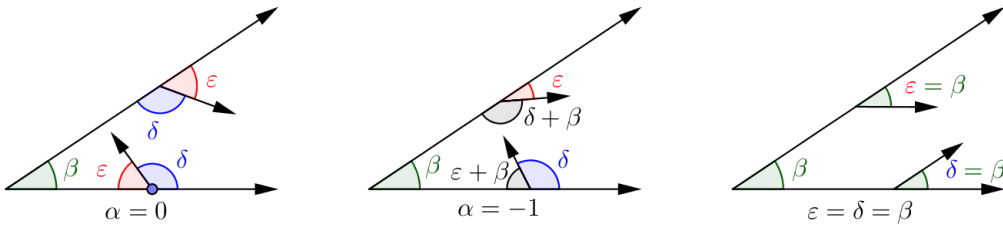


FIGURE 4. Three interesting cases. From left to right: skew symmetry, Dieker and Moriarty condition (for  $\alpha = -1$ ) and orthogonal reflections.

### 3. INVARIANTS

In this section we first study the curve  $\gamma(x, y) = 0$ , where  $\gamma(x, y)$  is the quadratic polynomial involved in the functional equation (20) satisfied by  $\varphi$  (Sections 3.1 and 3.2). Then we introduce the notion of *invariant* and relate it to a characterization of the function  $\varphi_1(y)$  obtained in [20].

#### 3.1. THE KERNEL AND ITS ROOTS

We start from the functional equation (20) relating the Laplace transforms  $\varphi$ ,  $\varphi_1$  and  $\varphi_2$ . The quadratic polynomial  $\gamma(x, y)$  occurring as the coefficient of  $\varphi(x, y)$  is called the *kernel*. The roots of the kernel (when solved for  $x$ , or for  $y$ ) are algebraic functions  $X^\pm(y)$  and  $Y^\pm(x)$  defined by

$$\gamma(X^\pm(y), y) = \gamma(x, Y^\pm(x)) = 0.$$

They can be expressed in closed form:

$$\begin{cases} X^\pm(y) = \frac{-(\sigma_{12}y + \mu_1) \pm \sqrt{y^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2y(\mu_1\sigma_{12} - \mu_2\sigma_{11}) + \mu_1^2}}{\sigma_{11}}, \\ Y^\pm(x) = \frac{-(\sigma_{12}x + \mu_2) \pm \sqrt{x^2(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) + 2x(\mu_2\sigma_{12} - \mu_1\sigma_{22}) + \mu_2^2}}{\sigma_{22}}. \end{cases} \quad (26)$$

Each of the discriminants (that is, the polynomials under the square roots in (26)) admits two roots, which are the branch points of the functions  $X^\pm$  and  $Y^\pm$ . They are respectively given by

$$\begin{cases} y^\pm = \frac{(\mu_1\sigma_{12} - \mu_2\sigma_{11}) \pm \sqrt{(\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2 \det \Sigma}}{\det \Sigma}, \\ x^\pm = \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) \pm \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma}}{\det \Sigma}. \end{cases} \quad (27)$$

Since the covariance matrix  $\Sigma$  is positive-definite, the quantities in the square roots of (27) are positive, so that the roots  $y^\pm$  (as well as  $x^\pm$ ) are real. Moreover, the product  $y^+y^-$  has the sign of  $\frac{\mu_1^2}{\sigma_{12}^2 - \sigma_{11}\sigma_{22}} = -\mu_1^2 / \det \Sigma$ , which is negative. Hence  $y^+$  and  $y^-$  are of opposite signs. We take  $y^- < 0 < y^+$ . Analogous results hold for  $x^\pm$ . The discriminant in the expression of  $X^\pm$  is negative when  $y$  is real and large in modulus, so that the branches  $X^\pm$  are analytic on  $\mathbb{C} \setminus ((-\infty, y^-] \cup [y^+, \infty))$  (we take the principal determination of the square root on  $\mathbb{C} \setminus (-\infty, 0]$ ). Similarly, the branches  $Y^\pm$  are analytic on  $\mathbb{C} \setminus ((-\infty, x^-] \cup [x^+, \infty))$ .

**Remark 3.1.** For  $x \in (-\infty, x^-] \cup [x^+, \infty)$ , the roots of  $\gamma(x, y) = 0$ , solved for  $y$ , are complex conjugate. We still denote them  $Y^\pm(x)$ , but they are only defined up to conjugacy.

The following lemma is proved in Appendix B.1.

**Lemma 3.2.** We have  $0 < Y^\pm(x^-) \leq y^+$ .

### 3.2. PARAMETRIZATION OF THE CURVE $\gamma(x, y) = 0$

It will be very convenient to work with a rational uniformization (or parametrization) of the kernel, rather than with the variables  $x$  and  $y$ . Let us introduce the curve

$$\mathcal{S} := \{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : \gamma(x, y) = 0\},$$

which is a Riemann surface  $\mathcal{S}$  of genus 0, see [18]. The following uniformization of  $\mathcal{S}$  is established in [18, Prop. 5]:

$$\mathcal{S} = \{(x(s), y(s)) : s \in \mathbb{C} \cup \{\infty\}\}, \quad (28)$$

where

$$\begin{cases} x(s) = \frac{x^+ + x^-}{2} + \frac{x^+ - x^-}{4} \left( s + \frac{1}{s} \right), \\ y(s) = \frac{y^+ + y^-}{2} + \frac{y^+ - y^-}{4} \left( \frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right). \end{cases} \quad (29)$$

Recall that the angle  $\beta$  is defined in (10), while  $x^\pm$  and  $y^\pm$  are the branch points given by (27). The points  $s = 0$  and  $s = \infty$  are sent to the (two) points at infinity of the surface  $\mathcal{S}$ . We now introduce the transformations

$$\xi(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}, \quad \zeta(s) = \eta\xi(s) = e^{2i\beta} s. \quad (30)$$

By construction,  $\xi$  (resp.  $\eta$ ) leaves  $x(s)$  (resp.  $y(s)$ ) invariant. By analogy with discrete models, the group  $\langle \xi, \eta \rangle$  generated by  $\xi$  and  $\eta$  will be called the *group of the model*. It is finite if and only if  $\zeta$  has finite order, i.e., if and only if  $\beta/\pi \in \mathbb{Q}$ .

Observe that for any  $s$ , we have the following equality of sets:

$$\{Y^+(x(s)), Y^-(x(s))\} = \{y(s), y(1/s)\} \quad (31)$$

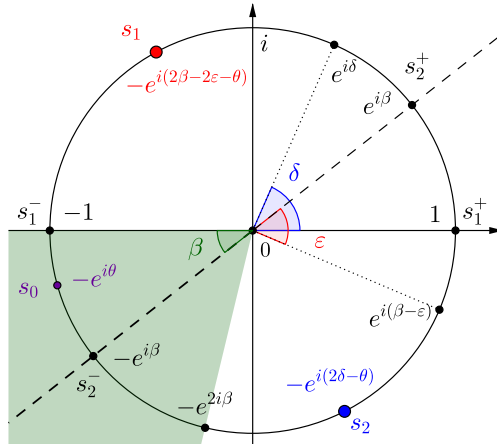


FIGURE 5. The complex  $s$ -plane, from which the uniformization (29) of  $\mathcal{S}$  is expressed. The unit circle corresponds to the real points of  $\mathcal{S}$ .

and analogously,

$$\{X^+(y(s)), X^-(y(s))\} = \left\{x(s), x(e^{2i\beta}/s)\right\}.$$

Also, it follows easily from (29) that

$$x(1) = x^+, \quad x(-1) = x^-, \quad y(e^{i\beta}) = y^+, \quad y(-e^{i\beta}) = y^-.$$

Having in mind that the index 1 (resp. 2) refers to  $x$  (resp.  $y$ ), we will denote

$$s_1^+ = 1, \quad s_1^- = -1, \quad s_2^+ = e^{i\beta}, \quad s_2^- = -e^{i\beta}. \quad (32)$$

These special points are shown in Figure 5. The map  $s \mapsto x(s)$  is 2-to-1 from  $(-\infty, 0)$  onto  $(-\infty, x^-]$ , except at  $s = -1$ . Analogously, it is 2-to-1 from  $(0, +\infty)$  onto  $[x^+, +\infty)$ , except at  $s = 1$ . Similarly, the map  $s \mapsto y(s)$  is 2-to-1 from  $e^{i\beta}\mathbb{R}_-$  (resp.  $e^{i\beta}\mathbb{R}_+$ ) onto  $(-\infty, y^-]$  (resp.  $[y^+, +\infty)$ ), except at the point  $-e^{i\beta}$  (resp.  $e^{i\beta}$ ).

The proofs of the next two lemmas are not especially illuminating, and are given in Appendices B.2 and B.3 respectively.

**Lemma 3.3.** *The pair of equations  $x(s_0) = y(s_0) = 0$  has a unique solution, which is:*

$$s_0 = -e^{i\theta},$$

where we recall that  $\theta = \arg(-\tilde{\mu})$  is given by (15).

Consider now, for  $i \in \{1, 2\}$ , the following equation in  $s$ :

$$\gamma_i(x(s), y(s)) = 0, \quad (33)$$

where  $\gamma_i(x, y)$  is the bilinear function given by (21). Since  $x(s)$  and  $y(s)$  have degree 2, the above equation is quadratic in  $s$  and thus has two solutions. One of them has to be  $s_0$  (because  $(x(s_0), y(s_0)) = (0, 0)$  obviously cancels  $\gamma_i(x, y)$ ). We denote the other solution by  $s_i$ .

**Lemma 3.4.** *The solutions  $s_1$  and  $s_2$  of (33) (distinct from  $s_0$ ) satisfy:*

$$s_0 s_1 = e^{2i(\beta-\varepsilon)} \quad \text{and} \quad s_0 s_2 = e^{2i\delta},$$

where the angles  $\delta$  and  $\varepsilon$  are defined by (16). That is,

$$s_1 = -e^{i(2\beta-2\varepsilon-\theta)} \quad \text{and} \quad s_2 = -e^{i(2\delta-\theta)}.$$

The following lemma will be useful to reformulate in simpler terms a condition occurring in [20]. It is proved in Appendix B.4.

**Lemma 3.5.** *We have  $\gamma_1(x^-, Y^\pm(x^-)) \geq 0$  (resp.  $= 0$ ) if and only if  $2\beta - 2\varepsilon - \theta \geq 0$  (resp.  $= 0$ ).*

### 3.3. BOUNDARY VALUE PROBLEMS AND INVARIANTS

Let us consider the following curve in  $\mathbb{C}$ :

$$\mathcal{R} = \{y \in \mathbb{C} : \gamma(x, y) = 0 \text{ for some } x \in (-\infty, x^-]\}. \quad (34)$$

By Remark 3.1, the curve  $\mathcal{R}$  is symmetric with respect to the real axis (Figure 6). Moreover, as shown in [2, Lem. 9], it is a branch of a hyperbola, which intersects the real axis at the point  $Y^-(x^-) = Y^+(x^-) \in (0, y^+]$  (see Lemma 3.2). We further introduce the domain  $\mathcal{G}_{\mathcal{R}}$ , which is the (open) domain of  $\mathbb{C}$  containing 0 and bounded by  $\mathcal{R}$ . Finally, we denote by  $\bar{\mathcal{G}}_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}} \cup \mathcal{R}$  the closure of  $\mathcal{G}_{\mathcal{R}}$ .

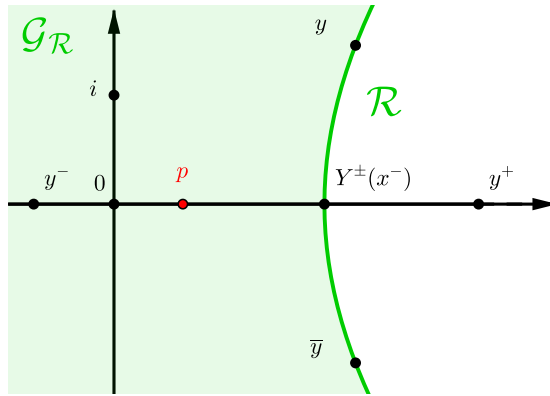


FIGURE 6. The curve  $\mathcal{R}$ , the domain  $\mathcal{G}_{\mathcal{R}}$  and the possible pole  $p$  of  $\varphi_1$ . The branch point  $y^+$  (resp.  $y^-$ ) lies outside (resp. inside)  $\mathcal{G}_{\mathcal{R}}$ .

The following lemma is proved in Appendix B.5.

**Lemma 3.6.** *Let  $y \in \mathcal{R}$ . There exists a unique  $x \in (-\infty, x^-]$  such that  $\{y, \bar{y}\} = \{Y^+(x), Y^-(x)\}$ . Moreover, this value  $x$  coincides with  $X^-(y)$ , but also with  $X^-(\bar{y})$ .*

We now recall, and slightly reformulate, a crucial result from [20], which states that the Laplace transform  $\varphi_1$  satisfies a boundary value problem.

**Proposition 3.7.** *The Laplace transform  $\varphi_1$  is meromorphic in an open domain containing  $\bar{\mathcal{G}}_{\mathcal{R}}$ . It satisfies the boundary condition*

$$\varphi_1(\bar{y}) = G(y)\varphi_1(y), \quad \forall y \in \mathcal{R}, \quad (35)$$

with

$$G(y) = \frac{\gamma_1(X^-(y), y)}{\gamma_2(X^-(y), y)} \frac{\gamma_2(X^-(\bar{y}), \bar{y})}{\gamma_1(X^-(\bar{y}), \bar{y})}, \quad (36)$$

where  $\gamma_1$  and  $\gamma_2$  are the bivariate polynomials of (21).

The function  $\varphi_1$  has at most one pole in  $\bar{\mathcal{G}}_{\mathcal{R}}$ . This pole exists if and only if  $2\beta - 2\varepsilon - \theta \geq 0$ . It is then simple, and its value is

$$p = y(s_1) = \frac{2r_{11}(\mu_1 r_{21} - \mu_2 r_{11})}{r_{11}^2 \sigma_{22} - 2r_{11} r_{21} \sigma_{12} + r_{21}^2 \sigma_{11}}.$$

The pole coincides with  $Y^\pm(x^-) = y(-1)$  if and only if  $2\beta - 2\varepsilon - \theta = 0$ .

Note that the condition (35) is consistent with  $G(\bar{y}) = 1/G(y)$ .

*Proof.* By Lemmas 3 and 5 in [20], the transform  $\varphi_1$  can be extended meromorphically to the set

$$\{\in \mathbb{C} \setminus (y^+, \infty) : \Re(y) \leq 0 \text{ or } \Re(X^-(y)) < 0\},$$

which is open, simply connected, and contains  $\overline{\mathcal{G}}_{\mathcal{R}}$ . This proves the first statement of the proposition.

The boundary condition is established in [20, Prop. 6]. In this proposition it is also stated that  $\varphi_1$  has a pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$  if and only if  $\gamma_1(x^-, Y^\pm(x^-)) \geq 0$ , and that this pole coincides with  $Y^\pm(x^-)$  if and only if  $\gamma_1(x^-, Y^\pm(x^-)) = 0$ . By Lemma 3.5, this gives the conditions stated in the proposition. The simplicity of the pole comes again from [20, Prop. 6], and its value is given by [20, Eqs. (18-19)].  $\square$

A boundary value problem like the one of Proposition 3.7 is said to be *homogeneous* if the function  $G$  in (35) is simply 1. In this case we say that the function that satisfies it is an *invariant*.

**Definition 3.8.** *A function  $I$  from  $\mathcal{G}_{\mathcal{R}}$  to  $\mathbb{C}$  is said to be an invariant if it is meromorphic on a domain containing  $\overline{\mathcal{G}}_{\mathcal{R}}$  and satisfies the boundary (or invariant) condition*

$$I(y) = I(\bar{y}), \quad \forall y \in \mathcal{R}.$$

There exists in the literature a stronger notion of invariant [5, Sec. 5], where one requires that  $I(Y^+(x)) = I(Y^-(x))$  for all  $x$ . This implies the above boundary condition by taking  $x \in (\infty, x^-]$ .

In the following section we exhibit a simple, explicit and D-finite invariant  $w$ . In the next one, we show how to construct another invariant, this time involving the unknown function  $\varphi_1$ , provided one of the angle conditions holds.

#### 4. A CANONICAL INVARIANT

In this section we introduce a key invariant, denoted  $w(y)$ , and study its algebraic and differential properties. It is expressed in terms of an explicit hypergeometric functions  $T_a$ , which generalizes the Chebyshev polynomial of the first kind, obtained for  $a \in \mathbb{N}_0$ .

Below, we use the words “rational”, “algebraic”, “D-finite” and “D-algebraic” without specifying whether we request the coefficients of the corresponding algebraic/differential equations to be real or complex. The reason is that a function which, like  $\varphi_1$ ,  $T_a$  or  $w$ , is analytic in the neighborhood of a real segment, and takes real values on this segment, is, say, D-finite on  $\mathbb{R}(y)$  if and only if it is D-finite on  $\mathbb{C}(y)$  (analogous statements hold for the other three classes of functions).

##### 4.1. A GENERALIZATION OF CHEBYSHEV POLYNOMIALS

For  $x \in \mathbb{C} \setminus (-\infty, -1]$  and  $a \in \mathbb{R}$ , let us define

$$T_a(x) = {}_2F_1\left(-a, a; \frac{1}{2}; \frac{1-x}{2}\right),$$

where  ${}_2F_1$  is the classical Gauss hypergeometric function. In other words,  $T_a$  is the analytic continuation to  $\mathbb{C} \setminus (-\infty, -1]$  of the following series, which converges for  $|1-x| < 2$ :

$$T_a(x) = \sum_{n \geq 0} \frac{a}{a+n} \binom{a+n}{2n} 2^n (x-1)^n.$$

When  $a = m \in \mathbb{N}_0$ , then the above sum ranges from 0 to  $a$ , and  $T_m$  is the classical Chebyshev polynomial. The function  $T_a$  is D-finite for all values of  $a$ , as the hypergeometric function itself. It satisfies the following differential equation:

$$(1-x^2)T_a''(x) - xT_a'(x) + a^2T_a(x) = 0. \quad (37)$$

Other useful expressions are

$$T_a(x) = \cos(a \arccos x) \quad (38)$$

(see [1, 15.1.17]) which is valid for  $x$  in  $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ , and

$$T_a(x) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right) \quad (39)$$

(see [1, 15.1.11]) which is valid for  $x$  in  $\mathbb{C} \setminus (-\infty, -1]$  (here we take  $\sqrt{re^{it}} := \sqrt{r}e^{it/2}$  for  $t \in (-\pi, \pi]$  and  $u^a := \exp(a \log u)$  with the principal determination of the logarithm on  $\mathbb{C} \setminus \mathbb{R}_-$ ). The latter expression shows that  $T_a$  is algebraic when  $a \in \mathbb{Q}$ . The Schwarz list [38] implies that  $T_a$  is transcendental otherwise. This can be also proved by looking at the growth of  $T_a(x)$  at infinity.

We will also use the fact that both  $1 - T_a$  and  $1 + T_a$  are squares of D-finite functions. In fact, it follows from (38) and elementary trigonometry that we can take

$$\sqrt{1 + T_a} = \sqrt{2} T_{a/2}, \quad (40)$$

with the same domain of definition as  $T_a$ . Analogously, we can take  $\sqrt{1 - T_a(x)} = \sqrt{2} \sin(a/2 \arccos x)$ , which satisfies the same differential equation as  $T_{a/2}$ . Moreover,

$$\frac{1}{a} \sqrt{\frac{1 - T_a(x)}{1 - x}} = {}_2F_1 \left( \frac{-a + 1}{2}, \frac{a + 1}{2}; \frac{3}{2}; \frac{1 - x}{2} \right) \quad (41)$$

and this function is analytic on the same domain as  $T_a$ .

Our final property deals with rational functions in  $T_a$ .

**Proposition 4.1.** *Let  $S/R$  be an irreducible fraction with coefficients in  $\mathbb{C}$ . Then  $(S/R)(T_a)$  is D-finite if and only if either  $a \in \mathbb{Q}$  or  $R$  is constant.*

*Proof.* If  $a \in \mathbb{Q}$ , we have seen that  $T_a$  is algebraic, and then so is any fraction in  $T_a$ . If  $R$  is a constant, then  $(S/R)(T_a)$  is D-finite because D-finite functions form a ring. We now assume that  $(S/R)(T_a)$  is D-finite and want to prove that either  $a \in \mathbb{Q}$  or  $R$  is a constant.

Let us first suppose that  $S$  is a constant. In this case, both  $R(T_a)$  and  $1/R(T_a)$  are D-finite. By a result of Harris and Sibuya [22], the function  $(R(T_a))'/R(T_a)$  is algebraic.

If  $R$  is non-constant, there exist  $\kappa \neq 0$ ,  $z_1, \dots, z_n \in \mathbb{C}$  and positive integers  $n, m_1, \dots, m_n$  such that

$$R(z) = \kappa \prod_{i=1}^n (z - z_i)^{m_i}.$$

With this notation, one has

$$\frac{(R(T_a(x)))'}{R(T_a(x))} = T_a'(x) \sum_{i=1}^n \frac{m_i}{T_a(x) - z_i}. \quad (42)$$

In addition to its second order linear differential equation (37), the function  $T_a(x) = \cos(a \arccos x)$  satisfies a first order non-linear differential equation:

$$\frac{1 - x^2}{a^2} T_a'(x)^2 + T_a(x)^2 = 1,$$

which gives

$$T_a'(x)^2 = a^2 \frac{1 - T_a(x)^2}{1 - x^2}.$$

Since (42) is algebraic, we conclude that the function

$$(1 - T_a^2) \left( \sum_{i=1}^n \frac{m_i}{T_a - z_i} \right)^2$$

is also algebraic. This function is a non-trivial fraction in  $T_a$  (because of the multiple poles in the denominator). This implies that  $T_a$  is algebraic, so that  $a \in \mathbb{Q}$ .

To conclude, consider the case of a non-constant polynomial  $S$ . The fraction  $(S/R)(z)$  being irreducible in  $\mathbb{C}(z)$ , it comes from the classical Bézout theorem that there exist two polynomials



$U$  and  $V$  such that  $US + VR = 1$ . Dividing by  $R$ , it follows that  $U \frac{S}{R} + V = \frac{1}{R}$ . This shows that if  $\frac{S}{R}$  is D-finite, then  $\frac{1}{R}$  should also be D-finite.  $\square$

#### 4.2. THE INVARIANT $w$

We now define a function  $w$ , which is analytic on  $\mathbb{C} \setminus [y^+, \infty)$ , by:

$$w(y) := T_{\frac{\pi}{\beta}} \left( -\frac{2y - (y^+ + y^-)}{y^+ - y^-} \right). \quad (43)$$

Note that when  $y = y(s)$  is given by the parametrization (29), the argument simplifies into

$$-\frac{1}{2} \left( \frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right).$$

In particular, we derive from (39) that for  $s \in \mathbb{C} \setminus e^{i\beta}\mathbb{R}_+$ ,

$$w(y(s)) = \frac{1}{2} \left( \left( -\frac{s}{e^{i\beta}} \right)^{\pi/\beta} + \left( -\frac{s}{e^{i\beta}} \right)^{-\pi/\beta} \right), \quad (44)$$

if we define the  $a$ -th power on  $\mathbb{C} \setminus \mathbb{R}_-$  as before (with  $a = \pi/\beta$ ). This can be rewritten as:

$$w(y(s)) = -\frac{1}{2} \left( (-s)^{\pi/\beta} + (-s)^{-\pi/\beta} \right), \quad (45)$$

provided we now take the principal determination of the logarithm on  $\mathbb{C} \setminus e^{i\beta}\mathbb{R}_-$ .

The function  $w$  inherits the algebraic and differential properties of  $T_{\pi/\beta}$ .

**Proposition 4.2.** *If  $\frac{\pi}{\beta} \in \mathbb{Z}$ , the function  $w$  is a polynomial. If  $\frac{\pi}{\beta} \in \mathbb{Q} \setminus \mathbb{Z}$ , the function  $w$  is algebraic but irrational. If  $\frac{\pi}{\beta} \notin \mathbb{Q}$ , the function  $w$  is D-finite but not algebraic.*

*The functions  $\sqrt{1-w}$  and  $\sqrt{1+w}$  are D-finite.*

*Finally, a rational fraction in  $w$ , say  $(S/R)(w)$ , is D-finite if and only if either  $\beta/\pi \in \mathbb{Q}$  or  $S/R$  is a polynomial.*

We will express  $\varphi_1$  in terms of  $w$ , using the fact that  $w$  is, in a certain sense, a *canonical invariant*. The following lemma proves that it is an invariant (in the sense of Definition 3.8), and its canonical properties are described in Proposition 4.4.

**Lemma 4.3.** *The function  $w$  is an invariant in the sense of Definition 3.8. More precisely, it satisfies the following properties:*

- (1) *it is analytic in an open domain containing  $\overline{\mathcal{G}}_{\mathcal{R}}$ , namely  $\mathbb{C} \setminus [y^+, \infty)$ ,*
- (2) *it goes to infinity at infinity, with*

$$w(y) \underset{y \rightarrow \infty}{\sim} \kappa y^{\frac{\pi}{\beta}}$$

*for some constant  $\kappa \neq 0$ ,*

- (3) *it is bijective from  $\mathcal{G}_{\mathcal{R}}$  to  $\mathbb{C} \setminus (-\infty, -1]$ ,*
- (4) *it satisfies the boundary condition*

$$w(y) = w(\overline{y}), \quad \forall y \in \mathcal{R},$$

- (5) *it is 2-to-1 from  $\mathcal{R} \setminus \{Y^\pm(x^-)\}$  to  $(-\infty, -1)$ ,*
- (6) *around the point  $Y^\pm(x^-) = y(-1)$ ,*

$$w(Y^\pm(x^-) - y) = -1 + \frac{2\pi^2}{\beta^2(y^+ - y^-)^2 \sin^2 \beta} y^2 + O(y^3)$$

*as  $y \rightarrow 0$ .*

*We will denote by  $w^{-1}$  the analytic function from  $\mathbb{C} \setminus (-\infty, -1]$  to  $\mathcal{G}_{\mathcal{R}}$  that maps a complex number to its unique preimage lying in  $\mathcal{G}_{\mathcal{R}}$ .*

*Proof.* The first two points follow from known properties of Gauss' hypergeometric function. The next two can be found in [19, Lem. 3.4].

The fifth point follows from (45). Indeed, assume that  $w(y_1) = w(y_2)$  with  $y_1$  and  $y_2$  in  $\mathcal{R}$ . Then there exist  $t_1$  and  $t_2$  in  $\mathbb{R}_-$  such that  $y_i = y(t_i)$ . By (45), and the fact that  $(-t_i)$  is a positive real number, we conclude that either  $t_1 = t_2$ , or  $t_1 = 1/t_2$ . In the former case,  $y_1 = y_2$ . In the latter one, we have  $x := x(t_1) = x(t_2) \in (-\infty, x^-]$ , hence  $y_1$  and  $y_2$  are the two conjugate solutions of  $\gamma(x, y) = 0$ .

For the last point, we first recall that  $Y^\pm(x^-) = y(-1)$ , so that

$$-\frac{2Y^\pm(x^-) - (y^+ + y^-)}{y^+ - y^-} = \cos \beta.$$

In particular,  $w(Y^\pm(x^-)) = T_{\pi/\beta}(\cos \beta) = \cos(\pi) = -1$  by (38). More generally, by differentiating (38) twice, we obtain:

$$T'_{\pi/\beta}(\cos \beta) = 0, \quad T''_{\pi/\beta}(\cos \beta) = \frac{\pi^2}{\beta^2 \sin^2 \beta},$$

and the final property follows.  $\square$

Let us now explain in what sense the invariant  $w$  is canonical.

**Proposition 4.4.** *Assume that  $I$  is an invariant which has a finite number of poles in  $\overline{\mathcal{G}}_{\mathcal{R}}$  and grows at most polynomially at infinity. Then there exist polynomials  $R$  and  $S$  in  $\mathbb{C}[z]$  such that*

$$I = \frac{S \circ w}{R \circ w}.$$

*Proof.* Let us consider the function  $I \circ w^{-1}$ , where  $w^{-1}$  is the analytic function in Lemma 4.3. By composition,  $I \circ w^{-1}$  is meromorphic on  $\mathbb{C} \setminus (-\infty, -1]$ . Moreover, for  $z \in (-\infty, -1]$  we can define  $I \circ w^{-1}(z) := I(y) = I(\bar{y})$ , where  $y$  and  $\bar{y}$  are the two values of  $\mathcal{R}$  such that  $w(y) = z$ . By Morera's theorem,  $I \circ w^{-1}$  is analytic at  $z$  unless  $y$  (and  $\bar{y}$ ) is a pole of  $I$ . If  $y_0 \in \mathcal{R}$  is one of the poles of  $I$  then  $z_0 = w(y_0)$  is an isolated singularity of  $I \circ w^{-1}$ . Let  $\ell \in \mathbb{N}$  be such that  $(w(y) - w(y_0))^\ell I(y)$  tends to 0 as  $y$  tends to  $y_0$  or  $\bar{y}_0$  in  $\mathcal{G}_{\mathcal{R}}$ . Such an  $\ell$  exists since  $I$  is meromorphic and  $w$  analytic in neighborhoods of  $y_0$  and  $\bar{y}_0$ . If a sequence  $(z_n)_n$  tends to  $z_0$  in  $\mathbb{C} \setminus (-\infty, -1]$ , then for  $n$  large enough,  $w^{-1}(z_n)$  is arbitrarily close to  $y_0$  or arbitrarily close to  $\bar{y}_0$  (since these are the only two pre-images of  $z_0$  by  $w$ ). Hence  $(z_n - z_0)^\ell I \circ w^{-1}(z_n)$  tends to 0. This proves that  $z_0$  is a pole of  $I \circ w^{-1}$ .

The function  $I \circ w^{-1}$  is thus meromorphic on  $\mathbb{C}$ . Each of its poles is the image by  $w$  of a pole of  $I$  lying in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . Hence  $I \circ w^{-1}$  has finitely many poles, and can be written as  $S/R$  where  $R$  is a polynomial and  $S$  is entire. Finally, since  $w^{-1}$  has polynomial growth at infinity by Lemma 4.3, the same holds for  $S = R \cdot (I \circ w^{-1})$ . This implies, by a standard extension of Liouville's theorem, that  $S$  is a polynomial.  $\square$

## 5. AN INVARIANT INVOLVING THE LAPLACE TRANSFORM

Let us now return to the inhomogeneous problem of Proposition 3.7. A natural idea to transform it into an homogeneous one is to observe that the function  $G$  in (36) may be written as a ratio

$$G(y) = \frac{F_0(y)}{F_0(\bar{y})}, \tag{46}$$

with  $F_0(y) = \frac{\gamma_1}{\gamma_2}(X^-(y), y)$ . Then the boundary condition (35) rewrites as:

$$(F_0 \cdot \varphi_1)(y) = (F_0 \cdot \varphi_1)(\bar{y}), \quad \forall y \in \mathcal{R}.$$

The function  $F_0 \varphi_1$  would thus be an invariant if  $F_0$  was meromorphic in a neighbourhood of  $\overline{\mathcal{G}}_{\mathcal{R}}$ . But this is *a priori* not the case, as the algebraic function  $X^-$  which enters the definition of  $F_0$  is not analytic in the neighbourhood of the interval  $(-\infty, y^-)$ , which is contained in  $\mathcal{G}_{\mathcal{R}}$ .

Solving the boundary value problem of Proposition 3.7 in full generality is the main contribution of [20], where an explicit expression for  $\varphi_1$  is obtained in terms of contour integrals. Our point in the present paper is different: we want to characterize the cases for which the boundary condition (35) may be transformed into an homogeneous one, which is then easy to solve in terms of the canonical invariant of Section 4. This transformation relies on the notion of *decoupling functions*.

### 5.1. DECOUPLING FUNCTIONS

**Definition 5.1.** *Let  $m$  be a positive integer. A quadrant model with parameters  $\mu$ ,  $\Sigma$  and  $R$  is  $m$ -decoupled if there exist rational functions  $F$  and  $L$  such that*

$$\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, y) = \frac{F(y)}{L(x)}$$

whenever  $\gamma(x, y) = 0$ . By this, we mean that any of the two following equivalent identities between algebraic functions holds:

$$\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^+(x)) = \frac{F(Y^+(x))}{L(x)}, \quad \left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^-(x)) = \frac{F(Y^-(x))}{L(x)}, \quad (47)$$

The functions  $F(y)$  and  $L(x)$  are then said to form a decoupling pair for the model, and more precisely an  $m$ -decoupling pair.

A few remarks are in order:

- First, the two identities of (47) are equivalent because any rational relation between  $x$  and  $Y^-(x)$  must hold as well with  $x$  and  $Y^+(x)$ , by irreducibility of the quadratic polynomial  $\gamma(x, y)$  (recall that  $Y^\pm(x)$  are the two roots of this polynomial).
- As will be seen in Theorem 5.6, there may exist several decoupling pairs.
- In the enumeration of discrete walks confined to the quadrant [5], decoupling pairs are defined as solving the equation  $xy = F(y) + L(x)$  on a certain curve. In contrast, we have here a multiplicative version of this notion.

We now relate the decoupling property to factorizations of  $G(y)$  (or even  $G^m(y)$ ) of the form (46).

**Lemma 5.2.** *A model is  $m$ -decoupled if and only if one of the following equivalent assertions holds:*

- *There exists a rational function  $F$  such that the following identity between algebraic functions holds:*

$$\frac{\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^-(x))}{\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^+(x))} = \frac{F(Y^-(x))}{F(Y^+(x))}. \quad (48)$$

- *There exists a rational function  $F$  such that for all  $y \in \mathcal{R}$ ,*

$$G^m(y) = \frac{F(y)}{F(\bar{y})}. \quad (49)$$

Moreover, any rational function  $F$  satisfying (48) satisfies (49), and vice-versa. We call  $F$  an  $m$ -decoupling function, and define

$$L(x) = \frac{F(Y^-(x))}{\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^-(x))}.$$

Then  $L$  is a rational function in  $x$  and  $(F, L)$  is an  $m$ -decoupling pair in the sense of Definition 5.1.

Note that Condition (48) is left unchanged upon exchanging  $Y^+(x)$  and  $Y^-(x)$ .

*Proof.* Assuming that the model is decoupled, we obtain (48) by taking the ratio of the two identities in (47).

Now assume that  $F$  satisfies (48). Let  $y \in \mathcal{R}$ , and let  $x = X^-(y) = X^-(\bar{y})$  be the unique real number in  $(-\infty, x^-]$  such that  $\{y, \bar{y}\} = \{Y^+(x), Y^-(x)\}$  (Lemma 3.6). Writing (48) for this pair  $(x, y)$  gives (49), by definition (36) of  $G(y)$ .

Now assume that (49) holds. As we have just observed, this means that (48) holds for  $x \in (-\infty, x^-]$ . Since  $Y^\pm(x)$  are the roots of a quadratic polynomial over  $\mathbb{R}(x)$ , there exist rational functions  $L(x)$  and  $M(x)$  such that

$$\frac{F(Y^\pm(x))}{\left(\frac{\gamma_1}{\gamma_2}\right)^m(x, Y^\pm(x))} = L(x) + M(x)Y^\pm(x).$$

Specializing this to  $x \in (-\infty, x^-]$  (using (48)) shows that  $M(x) = 0$  on this half-line, and thus everywhere since  $M$  is rational. Hence (47) holds, and the model is decoupled with  $(F, L)$  as decoupling pair.  $\square$

The following simple observation underlines that decoupling functions yields invariants.

**Lemma 5.3.** *If  $F$  an  $m$ -decoupling function, then the product function  $F\varphi_1^m$  is an invariant in the sense of Definition 3.8.*

*Proof.* This follows directly from (49) and Proposition 3.7.  $\square$

## 5.2. THE RATIONAL FUNCTION $E(s)$

In Section 5.3, we will prove that decoupling functions exist if and only if one of the angle conditions (24) or (25) holds (Theorem 5.6). One important tool is the rational parametrization (29) of the kernel.

Let us return to the function  $G$  given by (36). By the definition of  $s_1$  and  $s_2$  given above Lemma 3.4, there exist constants  $c_1$  and  $c_2$  such that

$$s\gamma_1(x(s), y(s)) = c_1(s - s_1)(s - s_0) \quad \text{and} \quad s\gamma_2(x(s), y(s)) = c_2(s - s_2)(s - s_0).$$

Let us introduce the following rational function:

$$E(s) = \frac{\gamma_1(x(s), y(s))}{\gamma_2} \frac{\gamma_2(x(1/s), y(1/s))}{\gamma_1} \tag{50}$$

$$= \frac{s_2(s - s_1)(s - \frac{1}{s_2})}{s_1(s - s_2)(s - \frac{1}{s_1})}. \tag{51}$$

**Lemma 5.4.** *For  $s \in (-\infty, 0)$ , we have  $y(s) \in \mathcal{R}$ ,*

$$X^-(y(s)) = x(s), \quad \overline{y(s)} = y(1/s) \quad \text{and} \quad X^-(y(1/s)) = x(s).$$

*Moreover, for these values of  $s$ ,*

$$G(y(s)) = E(s), \tag{52}$$

*where  $E(s)$  is the above rational function.*

*Proof.* When  $s \in (-\infty, 0)$ , then  $x(s) \in (-\infty, x^-)$ . Of course,  $\gamma(x(s), y(s)) = 0$ , hence  $y(s)$  lies on the curve  $\mathcal{R}$  defined by (34). Thanks to Lemma 3.6 we have  $X^-(y(s)) = X^-(\overline{y(s)}) = x(s)$ . Moreover  $\{y(s), \overline{y(s)}\} = \{Y^+(x(s)), Y^-(x(s))\}$ , so that by (31), we must have  $\overline{y(s)} = y(1/s)$ . The first part of the lemma is thus proved.

Now in order to prove (52), we start from the definition (36) of  $G(y)$  and apply the three identities that we have just established, as well as  $x(s) = x(1/s)$ . We thus end with the expression (50) of  $E(s)$ .  $\square$

For ease of notation, we denote by  $q$  the complex number

$$q = e^{2i\beta}.$$

Note that the condition  $\beta/\pi \in \mathbb{Q}$  translates into the fact that  $q$  is a root of unity.

Now assume that the model is  $m$ -decoupled, and take  $s \in (-\infty, 0)$ . By Lemma 5.4,  $y(s)$  and  $y(1/s) = \overline{y(s)}$  lie in  $\mathcal{R}$ . Hence by (49), there exists a rational function  $F$  such that

$$G^m(y(s)) = \frac{F(y(s))}{F(y(1/s))}.$$

But by Lemma 5.4, this is also  $E^m(s)$ . Hence the rational fractions  $E^m(s)$  and  $\frac{F(y(s))}{F(y(1/s))}$ , which coincide on  $(-\infty, 0)$  must be equal, which gives

$$\begin{aligned} E^m(s) &= \frac{F(y(s))}{F(y(1/s))} = \frac{F(y(s))}{F(y(qs))}, & \text{since } y(1/s) = y(qs), \\ &= \frac{H(s)}{H(qs)} & \text{with } H(s) = F(y(s)). \end{aligned} \quad (53)$$

It is thus natural to ask when the rational function  $E^m$  can be written in the form  $\frac{H(s)}{H(qs)}$ . This is answered by the following elementary lemma, which shows how Conditions (24) and (25) naturally occur. This lemma, proved in Appendix C, follows from a more general result, Lemma C.4, that applies to an arbitrary rational function.

**Lemma 5.5.** *Let  $E(s)$  be the rational function given by (51):*

$$E(s) = \frac{s_2(s - s_1)(s - \frac{1}{s_2})}{s_1(s - s_2)(s - \frac{1}{s_1})}.$$

The following statements are equivalent:

- there exists  $m \in \mathbb{N}$  and  $H \in \mathbb{C}(s)^*$  such that  $E^m(s) = \frac{H(s)}{H(qs)}$ ,
- the elliptic divisor of  $E$  relative to  $q$  (defined in Definition C.3) is zero,
- one of the conditions (24) and (25) holds; that is,

$$\frac{s_1}{s_2} \in q^{\mathbb{Z}} \quad \text{or} \quad \left( s_1^2 \in q^{\mathbb{Z}} \text{ and } s_2^2 \in q^{\mathbb{Z}} \right). \quad (54)$$

Moreover, if these conditions hold, the first one holds for  $m = 2$ .

The reader should not worry about the terminology *elliptic divisor*, as the third condition is a straightforward translation of the second. In the following subsection, we construct  $H(s)$  (and even  $F(y)$ ) explicitly assuming that (24) or (25) holds. Hence the part of the proof that is really delayed till Appendix C is the fact that these conditions are necessary. The equivalence between the two conditions in (54) and Conditions (24) and (25), respectively, directly follow from Lemma 3.4, and (54) turns out to be a convenient formulation.

### 5.3. EXPLICIT DECOUPLING FUNCTIONS

We can now establish the equivalence between the simple and double angle conditions and the existence of decoupling functions, and provide explicit decoupling functions.

For  $r \in \mathbb{Z}$  and  $\sigma \in \mathbb{C}$ , let us define a rational function  $F_{r,\sigma}$  by:

$$F_{r,\sigma}(y) = \begin{cases} P_{r,\sigma}(y) = \prod_{j=0}^{r-1} (y - y(\sigma q^{-j})) & \text{if } r \geq 0, \\ \frac{1}{Q_{r,\sigma}(y)} = \prod_{j=1}^{|r|} \frac{1}{(y - y(\sigma q^j))} & \text{if } r < 0. \end{cases} \quad (55)$$

We further define  $P_{r,\sigma}(y) = 1$  when  $r < 0$  and  $Q_{r,\sigma}(y) = 1$  when  $r \geq 0$ , so that we can write in full generality

$$F_{r,\sigma} = \frac{P_{r,\sigma}}{Q_{r,\sigma}}.$$

Moreover, we note that for any  $r$ , we have  $P_{r,\sigma q^r} = Q_{-r,\sigma}$  and  $Q_{r,\sigma q^r} = P_{-r,\sigma}$ , so that

$$F_{r,\sigma q^r} = \frac{1}{F_{-r,\sigma}}. \quad (56)$$

For  $e$  and  $\varepsilon$  in  $\{0, 1\}$ , let us define the polynomial  $f_{e,\varepsilon}$  by

$$f_{e,\varepsilon}(y) = \begin{cases} 1 & \text{if } e = 0, \\ y - y((-1)^\varepsilon \sqrt{q}) & \text{if } e = 1, \end{cases} \quad (57)$$

with  $\sqrt{q} = e^{i\beta}$ . Equivalently, returning to the definition (29) of  $y(s)$ :

$$f_{e,\varepsilon}(y) = \begin{cases} 1 & \text{if } e = 0, \\ y - y^+ & \text{if } e = 1 \text{ and } \varepsilon = 0, \\ y - y^- & \text{if } e = 1 \text{ and } \varepsilon = 1. \end{cases}$$

When Condition (24) holds, we choose  $r \in \mathbb{Z}$  such that

$$s_1/s_2 = q^r. \quad (58)$$

Analogously, when Condition (25) holds, we choose integers  $r_1$  and  $r_2$ , and numbers  $e_1, e_2, \varepsilon_1, \varepsilon_2$  in  $\{0, 1\}$ , such that

$$s_1 = (-1)^{\varepsilon_1} \sqrt{q}^{\varepsilon_1} q^{r_1} \quad \text{and} \quad s_2 = (-1)^{\varepsilon_2} \sqrt{q}^{\varepsilon_2} q^{r_2}. \quad (59)$$

**Theorem 5.6.** *There exists an  $m$ -decoupling pair in the sense of Definition 5.1 if and only if any of the following equivalent conditions holds:*

- (1) *there exists a rational function  $F$  such that  $E^m(s) = \frac{F(y(s))}{F(y(1/s))}$ ,*
- (2) *there exists a rational function  $H$  such that  $E^m(s) = \frac{H(s)}{H(qs)}$ ,*
- (3) *one of the angle conditions (24) or (25) holds.*

*If these conditions hold, the function  $F$  of Assertion (1) is an  $m$ -decoupling function in the sense of Lemma 5.2.*

*Moreover, when (24) holds and  $r$  satisfies (58), we can take*

$$m = 1 \quad \text{and} \quad F = F_{r,s_1}. \quad (60)$$

*When (25) holds and the  $r_i, \varepsilon_i$  and  $e_i$  satisfy (59), we can take*

$$m = 2 \quad \text{and} \quad F = \left( \frac{F_{r_1,s_1}}{F_{r_2,s_2}} \right)^2 \cdot \frac{f_{e_1,\varepsilon_1}}{f_{e_2,\varepsilon_2}}. \quad (61)$$

Notice that the roots and poles of these decoupling functions are real, since  $q, s_1$  and  $s_2$  have modulus 1 and  $s \mapsto y(s)$  sends the unit circle on the real line.

*Proof of Theorem 5.6.* We have already explained in the previous subsection that if the model is  $m$ -decoupled in the sense of Definition 5.1, then Assertion (1) holds (see (53)). Conversely, if this assertion holds, we can work out the same argument backwards to conclude that (49) holds (because  $\mathcal{R} = \{y(s), s \in (-\infty, 0)\}$ ), so that the model is decoupled by the second point of Lemma 5.2.

We now focus on the assertions (1), (2), (3), and prove that (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1).

Assume that (1) holds, and define  $H(s) = F(y(s))$ . Then (2) follows from the fact that  $y(1/s) = y(qs)$ .

Now assume that (2) holds. By Lemma 5.5, one of the angle conditions (24) or (25) holds. This gives (3).

It remains to check that if one of these conditions holds, the functions  $F$  given by (60) and (61) are indeed  $m$ -decoupling functions, for  $m = 1$  and  $m = 2$  respectively. We first observe that  $y(s) - y(\sigma q^{-j})$  is a Laurent polynomial in  $s$ , of degree 1 and valuation  $-1$ , that vanishes for  $s = \sigma q^{-j}$  and  $s = q^{j+1}/\sigma$  (because  $y(s) = y(q/s)$ ). Hence there exists a constant  $\kappa$  (depending on  $\sigma$  and  $j$ ) such that

$$y(s) - y(\sigma q^{-j}) = \kappa (sq^j - \sigma) \left( \frac{q^{j+1}}{s} - \sigma \right).$$

It follows that, for any  $r \geq 0$ ,

$$\frac{F_{r,\sigma}(y(s))}{F_{r,\sigma}(y(qs))} = \prod_{j=0}^{r-1} \frac{(sq^j - \sigma) \left( \frac{q^{j+1}}{s} - \sigma \right)}{(sq^{j+1} - \sigma) \left( \frac{q^j}{s} - \sigma \right)} = \frac{1}{q^r} \cdot \frac{s - \sigma}{s - 1/\sigma} \cdot \frac{s - q^r/\sigma}{s - \sigma/q^r}. \quad (62)$$

A similar calculation, or more directly the identity (56), proves that this still holds for  $r < 0$ . Given that

$$E(s) = \frac{s_2 (s - s_1)(s - \frac{1}{s_2})}{s_1 (s - s_2)(s - \frac{1}{s_1})},$$

this already establishes (60) when  $s_1 = q^r s_2$ .

Furthermore, if  $\sigma = (-1)^\varepsilon \sqrt{q}^\varepsilon q^r$ , with  $r \in \mathbb{Z}$  and  $e, \varepsilon \in \{0, 1\}$ , we derive from (62) that

$$\frac{F_{r,\sigma}(y(s))}{F_{r,\sigma}(y(qs))} = \frac{1}{\sigma} \cdot \frac{s - \sigma}{s - 1/\sigma} \cdot \frac{(-1)^\varepsilon \sqrt{q}^\varepsilon s - 1}{s - (-1)^\varepsilon \sqrt{q}^\varepsilon}. \quad (63)$$

The rightmost ratio reduces to  $(-1)^\varepsilon$  if  $e = 0$ , and its square is thus 1. If  $e = 1$ , its square is

$$\left( \frac{(-1)^\varepsilon \sqrt{q}^\varepsilon s - 1}{s - (-1)^\varepsilon \sqrt{q}^\varepsilon} \right)^2 = \frac{qs + \frac{1}{s} - 2(-1)^\varepsilon \sqrt{q}^\varepsilon}{s + \frac{q}{s} - 2(-1)^\varepsilon \sqrt{q}^\varepsilon} = \frac{y(qs) - y((-1)^\varepsilon \sqrt{q}^\varepsilon)}{y(s) - y((-1)^\varepsilon \sqrt{q}^\varepsilon)} = \frac{f_{e,\varepsilon}(y(qs))}{f_{e,\varepsilon}(y(s))},$$

where  $f_{e,\varepsilon}$  is defined by (57). Hence, whether  $e = 0$  or  $e = 1$ , we obtain, by squaring (63):

$$\left( \frac{1}{\sigma} \cdot \frac{s - \sigma}{s - 1/\sigma} \right)^2 = \left( \frac{F_{r,\sigma}(y(s))}{F_{r,\sigma}(y(qs))} \right)^2 \frac{f_{e,\varepsilon}(y(s))}{f_{e,\varepsilon}(y(qs))}.$$

Returning to the above expression of  $E(s)$ , this implies that the function  $F$  given by (61) is indeed a 2-decoupling function when the double angle condition (59) holds.  $\square$

We have thus proved some parts of Theorems 2.2 and 2.3, which we state as a corollary.

**Corollary 5.7.** *Assume that one of the angle conditions (24) or (25) holds. Then there exist  $m \in \{1, 2\}$ , a rational function  $F$ , and polynomials  $S$  and  $R$ , such that*

$$F(y)\varphi_1^m(y) = \frac{S}{R} \circ w(y),$$

where  $w$  is the canonical invariant of Section 4. In particular,  $\varphi_1$  is  $D$ -algebraic, and algebraic if  $\beta/\pi \in \mathbb{Q}$ .

*Proof.* The function  $F\varphi_1^m$  is an invariant (Lemma 5.3), to which Proposition 4.4 applies. This gives the expression of  $\varphi_1^m$ . The algebraic/differential properties of  $\varphi_1$  come from those of  $w$  (Proposition 4.2) and classical closure properties.  $\square$

In the next two sections, we will make the expression of  $\varphi_1^m$  more explicit, by giving the degrees of  $S$  and  $R$  in terms of the parameters of the model. Since every pole or root of  $S/R$  comes from a pole or root of  $F$  or  $\varphi_1$  lying in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , we need to clarify how many of the roots or poles of the function  $F_{r,\sigma}$  defined by (55) lie in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . In the following lemma, we focus on those that lie in  $\mathcal{G}_{\mathcal{R}}$ .



Indeed, since all roots and poles of  $F_{r,\sigma}$  are real, lying on the curve  $\mathcal{R}$  simply means being equal to  $y(-1)$ . It will be convenient to use the following notation: for  $\sigma \in \mathbb{C}$ , and  $a, b \in \mathbb{Z}$ :

$$m_{\pm}(\sigma; a, b) = \#\{j : a \leq j \leq b : \sigma q^{\pm j} = -1\}. \quad (64)$$

Note that  $m_{\pm}(\sigma; a, b) = 0$  if  $b < a$ .

**Lemma 5.8.** *Let  $\sigma = e^{i\omega}$  be a complex number of modulus 1. For  $r \geq 0$ , the number of roots of the polynomial  $F_{r,\sigma}(y) = P_{r,\sigma}(y)$  that lie in the open domain  $\mathcal{G}_{\mathcal{R}}$  is*

$$\left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} \right\rfloor - \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - r \frac{\beta}{\pi} \right\rfloor - m_{-}(\sigma; 0, r-1).$$

For  $r \leq 0$ , the function  $F_{r,\sigma}(y) = 1/Q_{r,\sigma}(y)$  is the reciprocal of a polynomial, and the number of poles of  $F_{r,\sigma}(y)$  that lie in the open domain  $\mathcal{G}_{\mathcal{R}}$  is

$$\left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - r \frac{\beta}{\pi} \right\rfloor - \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} \right\rfloor - m_{+}(\sigma; 1, |r|).$$

*Proof.* First, we note that

$$y(s) \in \mathcal{G}_{\mathcal{R}} \iff \arg(s) \in (\pi, \pi + 2\beta) \pmod{2\pi} \quad (65)$$

$$y(s) \in \mathcal{R} \iff \arg(s) \in \{\pi, \pi + 2\beta\} \pmod{2\pi}. \quad (66)$$

In other words, the preimage by  $y$  of  $\mathcal{R}$  is  $\mathbb{R}_{-} \cup e^{2i\beta}\mathbb{R}_{-}$  and the preimage by  $y$  of  $\mathcal{G}_{\mathcal{R}}$  is the green/shaded area in Figure 5 (in particular,  $y(-e^{i\beta}) = y^{-} \in \mathcal{G}_{\mathcal{R}}$ ).

Hence, when  $r \geq 0$ , the question is to determine how many of the points  $\sigma q^{-j}$ , for  $0 \leq j \leq r-1$ , have their argument in  $(\pi, \pi + 2\beta)$  modulo  $2\pi$ . This argument is  $\omega - 2j\beta$ .

This kind of counting problem is standard in the study of *Sturmian* or *mecanical* sequences [35, Chap. 2]. Denoting by  $\{x\} := x - \lfloor x \rfloor$  the fractional part of  $x$ , we want to determine

$$\#\left\{j : 0 \leq j < r, \left\{ \frac{\omega}{2\pi} - \frac{1}{2} - j \frac{\beta}{\pi} \right\} \in (0, \beta/\pi) \right\}.$$

Let us begin by counting those values of  $j$  for which

$$\left\{ \frac{\omega}{2\pi} - \frac{1}{2} - j \frac{\beta}{\pi} \right\} \in [0, \beta/\pi). \quad (67)$$

Observing that the difference

$$\left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - j \frac{\beta}{\pi} \right\rfloor - \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - (j+1) \frac{\beta}{\pi} \right\rfloor$$

takes values in  $\{0, 1\}$ , and equals 1 if and only if (67) holds, we conclude that

$$\begin{aligned} \#\left\{j : 0 \leq j < r, \left\{ \frac{\omega}{2\pi} - \frac{1}{2} - j \frac{\beta}{\pi} \right\} \in [0, \beta/\pi) \right\} &= \sum_{j=0}^{r-1} \left( \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - j \frac{\beta}{\pi} \right\rfloor - \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - (j+1) \frac{\beta}{\pi} \right\rfloor \right) \\ &= \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} \right\rfloor - \left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - r \frac{\beta}{\pi} \right\rfloor. \end{aligned}$$

We need to subtract the number of  $j$  for which the fractional part shown in (67) takes the value 0, which is equivalent to saying that  $\sigma q^{-j} = -1$ . This proves the lemma when  $r \geq 0$ .

When  $r < 0$ , the result follows by observing that the poles of  $F_{r,\sigma}$  are the zeroes of  $F_{-r,\sigma q^{-r}}$ , and applying the above result.  $\square$

## 6. EXPRESSION OF THE LAPLACE TRANSFORM WHEN $\alpha \in \mathbb{Z} + \pi\mathbb{Z}/\beta$

Let us assume that the simple angle Condition (24) holds. Then by Theorem 5.6, there exists a 1-decoupling function  $F$ , and by Corollary 5.7, the function  $F\varphi_1$  can be written  $(S/R)\circ w$  for some polynomials  $S$  and  $R$ . In this section we determine the degrees and roots of these polynomials. Recall that we have chosen an integer  $r$  such that  $s_1 = q^r s_2$ . Returning to Lemma 3.4, this defines an integer  $k$  such that

$$\delta + \varepsilon = (1 - r)\beta + (1 + k)\pi.$$

Equivalently,

$$\pi + \beta - \delta - \varepsilon = \beta(1 - \alpha) = r\beta - k\pi. \quad (68)$$

If  $q$  is not a root of unity, that is, if  $\beta/\pi \notin \mathbb{Q}$ , then the choice of  $r$  and  $k$  is unique. Otherwise, write  $\beta = n\pi/d$ , with  $n$  and  $d$  relatively prime and  $0 < n < d$ . Then if  $(r, k)$  is a solution, all other solutions are of the form  $(r + jd, k + jn)$ , for  $j \in \mathbb{Z}$ . In particular, there always exist solutions such that  $|r| < d$ , or even  $-\lfloor d/2 \rfloor \leq r < \lfloor (d+1)/2 \rfloor$ . Such a choice of  $r$  will sometimes simplify certain expressions, but what follows holds for any choice of  $r$ .

### 6.1. PRELIMINARIES

A first simple task is to determine the *difference* of the degrees of  $S$  and  $R$ , that is, the behaviour of  $S/R$  at infinity.

**Lemma 6.1.** *There exists a constant  $\kappa \neq 0$  such that*

$$\frac{S}{R}(z) = (\varphi_1 F)(w^{-1}(z)) \underset{y \rightarrow \infty}{\sim} \kappa z^k.$$

*Proof.* By Proposition 19 in [20], there exists a constant  $\kappa \neq 0$  such that

$$\varphi_1(y) \underset{y \rightarrow \infty}{\sim} \kappa y^{\alpha-1}, \quad (69)$$

where  $\alpha = (\delta + \varepsilon - \pi)/\beta$  is the key parameter introduced in (1). Since  $\alpha - 1 = k\frac{\pi}{\beta} - r$  (see (68)) and  $F(y) = F_{r,s_1}(y) \underset{y \rightarrow \infty}{\sim} y^r$  (see (55)), it follows that for some  $\kappa \neq 0$ ,

$$(\varphi_1 F)(y) \underset{y \rightarrow \infty}{\sim} \kappa y^{k\frac{\pi}{\beta}}.$$

Then as  $w(y) \underset{y \rightarrow \infty}{\sim} \kappa' y^{\frac{\pi}{\beta}}$  (see Lemma 4.3), the inverse function  $w^{-1}(z)$  grows as  $z^{\beta/\pi}$ , and we deduce that for some constant  $\kappa$ ,

$$(\varphi_1 F)(w^{-1}(z)) \underset{y \rightarrow \infty}{\sim} \kappa z^k.$$

□

We now apply Lemma 5.8 to the decoupling function  $F = F_{r,s_1}$ , to determine how many poles and roots of  $F$  lie in  $\mathcal{G}_{\mathcal{R}}$ . When  $r \geq 0$  (resp.  $r < 0$ ), we write  $F = P$  (resp.  $F = 1/Q$ ) to lighten the notation  $P_{r,s_1}$  (resp.  $1/Q_{r,s_1}$ ). Recall the notation  $m_{\pm}(\sigma; a, b)$  defined in (64).

**Lemma 6.2.** *Let us write as before  $\delta + \varepsilon = (1 - r)\beta + (1 + k)\pi$ . Then either  $r < 0$  and  $k < 0$ , or  $r > 0$  and  $k \geq 0$ . The 1-decoupling function  $F$  given in (60) is rational with real roots and poles.*

(1) *If  $r > 0$ , so that  $F(y) = P(y)$ , the number of roots of  $F$  (counted with multiplicity) lying in the open region  $\mathcal{G}_{\mathcal{R}}$  (or equivalently in  $(-\infty, y(-1)) = (-\infty, Y^{\pm}(x^-))$ ) is:*

$$r_F := k + \mathbb{1}_{2\beta - 2\varepsilon - \theta \geq 0} - m_{-}(s_1; 0, r - 1).$$

(2) *If  $r < 0$ , so that  $F(y) = 1/Q(y)$ , the number of poles of  $F$  lying in  $\mathcal{G}_{\mathcal{R}}$  is*

$$p_F := -k - \mathbb{1}_{2\beta - 2\varepsilon - \theta \geq 0} - m_{+}(s_1; 1, |r|).$$

If  $q$  is not a root of unity, the cardinalities  $m_{\pm}$  occurring in this lemma are 0 or 1. Otherwise, as discussed at the beginning of the section, we can always choose  $r$  so that this property still holds.

*Proof.* Recall from (23) that  $0 < \delta + \varepsilon - \beta < \pi$ . This implies that  $r$  cannot be 0, and that  $r < 0$  implies  $k < 0$ , while  $r > 0$  implies  $k \geq 0$ .

Now recall that  $F = F_{r,s_1}$ , and let us apply Lemma 5.8 with  $\sigma = s_1$ , or equivalently

$$\omega = \pi + 2\beta - 2\varepsilon - \theta = -\pi + 2\delta - \theta + 2r\beta - 2k\pi.$$

From the first expression of  $\omega$ , we derive

$$\left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{2\beta - 2\varepsilon - \theta}{2\pi} \right\rfloor = -\mathbb{1}_{2\beta - 2\varepsilon - \theta < 0},$$

by the angle assumptions (23). Moreover, the second expression of  $\omega$  given above leads to

$$\left\lfloor \frac{\omega}{2\pi} - \frac{1}{2} - r\frac{\beta}{\pi} \right\rfloor = -k - 1 + \left\lfloor \frac{2\delta - \theta}{2\pi} \right\rfloor = -k - 1,$$

again by (23). Then Lemma 5.8 gives the announced expressions for the number of roots of  $Q$  and  $P$  lying in  $\mathcal{G}_{\mathcal{R}}$ .  $\square$

## 6.2. EXPRESSION OF $\varphi_1$

We can now describe precisely the polynomials  $S$  and  $R$  such that  $F\varphi_1 = \frac{S}{R} \circ w$ .

**Theorem 6.3.** *Let us assume that Condition (24) holds, that is,  $\alpha \in \mathbb{Z} + \pi\mathbb{Z}/\beta$ , and let  $k, r \in \mathbb{Z}$  be chosen so that  $\delta + \varepsilon = (1 - r)\beta + (1 + k)\pi$ . Let  $F$  be the decoupling function of Theorem 5.6. Depending on the sign of  $r$ , we have  $F = P$  (when  $r > 0$ ) or  $F = 1/Q$  (when  $r < 0$ ), where  $P$  and  $Q$  are (real-rooted) polynomials of degree  $|r|$ .*

*The Laplace transform  $\varphi_1$  defined by (19) can be meromorphically continued to  $\mathbb{C} \setminus [y^+, \infty)$ . Moreover,*

(1) *if  $r < 0$ , then  $k < 0$  and*

$$\varphi_1(y) = \frac{Q(y)}{R(w(y))},$$

*where  $R$  is a polynomial of degree  $|k|$  whose roots (taken with multiplicity) are*

- *the  $w(y(s_1q^j))$  for  $j = 1, \dots, r$  such that  $y(s_1q^j) \in \mathcal{G}_{\mathcal{R}}$ ,*
- *plus  $w(y(s_1))$  if  $2\beta - 2\varepsilon - \theta > 0$ ,*
- *and finally  $w(y(-1)) = -1$ , with multiplicity  $m_+(s_1; 0, |r| - 1)$ .*

(2) *if  $r > 0$ , then  $k \geq 0$  and*

$$\varphi_1(y) = \frac{S(w(y))}{P(y)},$$

*where  $S$  is a polynomial of degree  $k$  whose roots (taken with multiplicity) are*

- *the  $w(y(s_1q^{-j}))$ , for  $j = 1, \dots, r - 1$  such that  $y(s_1q^{-j}) \in \mathcal{G}_{\mathcal{R}}$ ,*
- *plus  $w(y(s_1))$  if  $y(s_1) < y(-1)$  and  $2\beta - 2\varepsilon - \theta \leq 0$ ,*
- *and finally  $w(y(-1)) = -1$ , with multiplicity  $m_-(s_1; 1, r - 1)$ .*

*In particular,  $\varphi_1$  is always  $D$ -algebraic. It is  $D$ -finite if and only if either  $\beta/\pi \in \mathbb{Q}$  or  $\alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z}$ . It is algebraic if and only if  $\beta/\pi \in \mathbb{Q}$  or  $\alpha \in -\mathbb{N}_0$ . It is rational if and only if  $\alpha \in -\mathbb{N}_0$ .*

### Remarks

1. As discussed earlier, there is always a choice of  $r$  that gives the value 0 or 1 to the multiplicities  $m_{\pm}(s_1; a, b)$  that occur in the theorem.

2. The above theorem characterizes the polynomials  $R$  and  $S$  up to multiplicative constants that can be adjusted thanks to the value  $\varphi_1(0)$  given in (22). Thus we can compute  $\varphi_1(y)$  explicitly, using the simple characterization of points  $s$  such that  $y(s) \in \mathcal{G}_{\mathcal{R}}$  given by (65). Several examples are worked out in Section 6.3.

*Proof of Theorem 6.3.* By Corollary 5.7, there exist two polynomials  $R$  and  $S$  such that

$$F\varphi_1 = \frac{S}{R} \circ (w).$$

Given the properties of the map  $w$  (Lemma 4.3), any zero (resp. pole)  $z_0$  of  $S/R$  is the image by  $w$  of a zero (resp. pole)  $y_0$  of  $F\varphi_1$  lying in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . Moreover, if  $y_0 \in \mathcal{G}_{\mathcal{R}}$ , its multiplicity in  $F\varphi_1$  (as a zero or pole) is the same as the multiplicity of  $z_0$  in  $S/R$ . If  $y_0 = y(-1) = Y^{\pm}(x^-)$ , its multiplicity in  $F\varphi_1$  is twice the multiplicity of  $z_0 = -1$  in  $S/R$  (by Lemma 4.3 (6)).

**First case:  $r < 0$ .** Then  $k < 0$  and  $F = \frac{1}{Q}$ . The decoupling function  $F$  has  $p_F$  poles in  $\mathcal{G}_{\mathcal{R}}$ , where  $p_F$  is given by Lemma 6.2. Moreover, the multiplicity of  $y(-1)$  as a pole of  $F$  is

$$\begin{aligned} & \# \{j : 1 \leq j \leq |r|, s_1 q^j \in \{-1, -q\}\} \\ &= \# \{j : 1 \leq j \leq |r|, s_1 q^j = -1\} + \# \{j : 0 \leq j < |r|, s_1 q^j = -1\} \\ &= 2\# \{j : 1 \leq j < |r|, s_1 q^j = -1\} + \mathbb{1}_{s_1 q^{|r|} = -1} + \mathbb{1}_{s_1 = -1} \\ &= 2m_+(s_1; 0, |r| - 1) + \mathbb{1}_{s_1 = -1}, \end{aligned}$$

as  $s_1 q^{|r|} = s_2$  is never equal to  $-1$  due to the assumptions (23). This multiplicity is not always even, but we should remember that  $\varphi_1$  has a (simple) pole at  $y(-1)$  if  $2\beta - 2\varepsilon - \theta = 0$ . Due to the assumptions (23), this condition is equivalent to  $s_1 = -1$ . Consequently, the multiplicity of  $y(-1)$  as a pole of  $F\varphi_1$  is  $2m_+(s_1; 0, |r| - 1)$ . Finally, recall that  $\varphi_1$  has a simple pole at  $y(s_1)$  when  $2\beta - 2\varepsilon - \theta > 0$ . These considerations lead us to introduce the polynomial  $R$  described in the theorem. Its degree is

$$p_F + \mathbb{1}_{2\beta - 2\varepsilon - \theta > 0} + m_+(s_1; 0, |r| - 1) = -k.$$

We have used again the fact that  $s_1 q^{|r|} \neq -1$  and that  $s_1 = -1 \Leftrightarrow 2\beta - 2\varepsilon - \theta = 0$ . Now consider the invariant  $F(y)\varphi_1(y)R(w(y))$ . By construction, it has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . Hence by Proposition 4.4, it is of the form  $S(w(y))$  as claimed. Finally, since  $R$  has degree  $-k$ ,  $S$  must be a constant by Lemma 6.1.

**Second case:  $r > 0$ .** Then  $k \geq 0$  and  $F = P$ . The decoupling function  $F$  has  $r_F$  roots in  $\mathcal{G}_{\mathcal{R}}$ , where  $r_F$  is given by Lemma 6.2. If  $y(s_1) < y(-1)$ , one of them is  $y(s_1)$ , which coincides with the pole of  $\varphi_1$  when  $2\beta - 2\varepsilon - \theta > 0$ . Moreover, the multiplicity of  $y(-1)$  as a root of  $F$  is

$$\begin{aligned} & \# \{j : 0 \leq j < r, s_1 q^{-j} \in \{-1, -q\}\} \\ &= \# \{j : 0 \leq j < r, s_1 q^{-j} = -1\} + \# \{j : 1 \leq j \leq r, s_1 q^{-j} = -1\} \\ &= 2\# \{j : 1 \leq j < r, s_1 q^{-j} = -1\} + \mathbb{1}_{s_1 = -1} + \mathbb{1}_{s_1 q^{-r} = -1} \\ &= 2m_-(s_1; 1, r - 1) + \mathbb{1}_{s_1 = -1}, \end{aligned}$$

by the same arguments as in the case  $r < 0$ . This multiplicity is not always even, but we should remember that  $\varphi_1$  has a (simple) pole at  $(-1)$  if  $2\beta - 2\varepsilon - \theta = 0$ , that is, if  $s_1 = -1$ . Consequently, the multiplicity of  $y(-1)$  as a root of  $F\varphi_1$  is  $m_-(s_1; 1, r - 1)$ . These considerations lead us to introduce the polynomial  $S$  described in the theorem. Its degree is

$$r_F - \mathbb{1}_{2\beta - 2\varepsilon - \theta > 0} + m_-(s_1; 1, r - 1) = k.$$

Now consider the invariant  $F(y)\varphi_1(y)/S(w(y))$ . By construction, it has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . Hence by Proposition 4.4, it is of the form  $\tilde{S}(w(y))$  for some polynomial  $\tilde{S}$ . Finally, since  $S$  has degree  $k$ , the polynomial  $\tilde{S}$  must be a constant by Lemma 6.1.

Now that we have given expressions for  $\varphi_1$ , its meromorphicity on  $\mathbb{C} \setminus [y^+, \infty)$  follows from the fact that the canonical invariant  $w$  is analytic on this domain. Moreover,  $\varphi_1$  is D-algebraic because  $w$  is D-finite.

Let us now discuss the other differential/algebraic properties of  $\varphi_1$ , starting from rational cases. If  $k = 0$ , that is,  $\alpha \in -\mathbb{N}_0$ , then  $\varphi_1$  is the reciprocal of a polynomial, hence a rational

function. Conversely, if  $\varphi_1$  is rational, then (69) implies that  $\alpha$  is an integer, and thus belongs to  $-\mathbb{N}_0$  since we have assumed  $\alpha < 1$ . If  $k \neq 0$ , it follows from the expressions of  $\varphi_1$  that if  $\varphi_1$  is algebraic then so is  $w$ , which forces  $\beta/\pi \in \mathbb{Q}$  by Proposition 4.2. Conversely, if  $\beta/\pi \in \mathbb{Q}$  then  $w$  is algebraic and so is  $\varphi_1$ . Finally, the characterization of D-finite cases stems from Proposition 4.2. Indeed,  $\varphi_1$  is D-finite if and only if  $(S/R)(w)$  is D-finite, and  $R$  is non-trivial as soon as  $r < 0$ . Hence  $\varphi_1$  is D-finite if and only if either  $\beta/\pi \in \mathbb{Q}$  or  $r > 0$ . The latter condition translates into  $\alpha \in -\mathbb{N}_0 + \frac{\pi}{\beta}\mathbb{Z}$ .  $\square$

### 6.3. EXAMPLES

We now give four applications of Theorem 6.3. We start with the three already known cases mentioned in Section 2.3 and illustrated in Figure 4, and we conclude with a case that is entirely new. Recall that we choose integers  $r$  and  $k$  such that

$$\delta + \varepsilon = (1 - r)\beta + (1 + k)\pi.$$

**The skew symmetric case.** The model is said to skew symmetric if  $\delta + \varepsilon = \pi$ , that is,  $\alpha = 0$ . It can be shown thanks to (10) and (16) that this is equivalent to

$$2\sigma_{12} = \frac{r_{21}}{r_{11}}\sigma_{11} + \frac{r_{12}}{r_{22}}\sigma_{22}.$$

One can take in this case  $r = 1$  and  $k = 0$ . Then Theorem 5.6 gives the decoupling function  $P(y) = y - y(s_1)$ . Theorem 6.3 (2) implies that for some constant  $\kappa$ ,

$$\varphi_1(y) = \frac{\kappa}{P(y)} = \frac{y(s_1)\varphi_1(0)}{y(s_1) - y},$$

where  $\varphi_1(0)$  is given by (22) and  $y(s_1)$  by Proposition 3.7.

If we invert the Laplace transform, we find that the density of the invariant measure  $\nu_1$  is exponential. This result is very well known and can be found for instance in [23, 26, 10].

**The Dieker and Moriarty condition** [11]. It reads  $\alpha = \frac{\delta + \varepsilon - \pi}{\beta} \in -\mathbb{N}_0$  and generalizes the previous case. We can take  $r = 1 - \alpha > 0$  and  $k = 0$ . The decoupling function  $F(y) = F_{r,s_1}$  is given by (55), and it is a polynomial  $P(y)$ . Theorem 6.3 (2) implies that for some constant  $\kappa$ ,

$$\varphi_1(y) = \frac{\kappa}{P(y)}.$$

When all roots of  $P$  are simple, we obtain by inverting the Laplace transform that the density of  $\nu_1$  is a sum of exponentials. In fact, we can show using (55) and the expression of  $s_1$  given in Lemma 3.4 that  $P$  has a multiple root if and only if  $2\varepsilon + \theta + j\beta = 0 \pmod{\pi}$  for some  $j \in \llbracket 0, 2r-4 \rrbracket$ . Equivalently, since  $\alpha = -r + 1$ , this is equivalent to saying that  $\theta - 2\delta - j\beta = 0 \pmod{\pi}$  for some  $j \in \llbracket 2, -2\alpha \rrbracket$ . This condition for all roots to be simple is equivalent to the condition “ $\theta \in \Theta_i$ ” occurring in [11, Thm. 1].

**The orthogonal case.** In this case  $R$  is a diagonal matrix, or equivalently  $\delta = \varepsilon = \beta$ . Thus we can take  $r = -1$  and  $k = -1$ . The decoupling function  $F_{r,s_1} = 1/Q$  of Theorem 5.6 reads  $F = 1/(y - y(s_1q)) = 1/(y - y(s_2))$ . Since  $\gamma_2(x, y) = r_{22}y$  in the orthogonal case, we have  $y(s_2) = 0$  by definition of  $s_2$ . Hence  $Q(y)$  is simply  $y$ . Theorem 6.3(1) gives  $R(w(y)) = \kappa(w(y) - w(y(s_2))) = \kappa(w(y) - w(0))$  for some constant  $\kappa$ . Using the identity  $\varphi_1(0) = -\mu_1/r_{11}$  derived from (22), we obtain

$$\varphi_1(y) = \frac{Q(y)}{R(w(y))} = -\frac{\mu_1}{r_{11}} \frac{w'(0)y}{w(y) - w(0)},$$

which is the main result of [19].

**A  $D$ -finite example: recurrence for the moments.** Suppose now that

$$\delta + \varepsilon + \beta = 2\pi, \quad (70)$$

that is,  $\alpha = \pi/\beta - 1$ . Then we can take  $r = 2$  et  $k = 1$ . Applying Theorems 5.6 and 6.3, we obtain

$$\varphi_1(y) = \kappa \frac{w(y) - w_0}{(y - y(s_1))(y - y(s_1/q))},$$

where the constant  $\kappa$  can be derived from the normalisation (22), and

$$w_0 = \begin{cases} w(y(s_1/q)) & \text{if } y(s_1/q) \in \mathcal{G}_{\mathcal{R}}, \\ w(y(s_1)) & \text{if } y(s_1) < y(-1) \text{ and } 2\beta - 2\varepsilon - \theta \leq 0, \\ w(y(-1)) = -1 & \text{if } s_1 = -q. \end{cases}$$

These three cases are those of Theorem 6.3. Using (65) and the expression of  $s_1$ , as well as (70) and the basic conditions (23), they can be rewritten respectively as

$$2\varepsilon + \theta < 2\pi, \quad 2\pi < 2\varepsilon + \theta, \quad \text{and} \quad 2\varepsilon + \theta = 2\pi.$$

These three cases actually occur, for instance with the three following values of  $(\beta, \delta, \varepsilon, \theta)$ :

$$(2\pi/3, 5\pi/6, \pi/2, \pi/4), \quad (2\pi/3, 4\pi/9, 8\pi/9, \pi/4), \quad (2\pi/3, 7\pi/12, 3\pi/4, \pi/2).$$

Starting from the expression (43) of  $w$  in terms of  $T_a$  (with  $a = \pi/\beta$ ), the differential equation (37) satisfied by  $T_a$  leads to a (non-homogeneous) second order linear differential equation satisfied by  $\varphi_1(y)$ . Upon expanding it in  $y$ , it gives a linear recurrence relation between the moments

$$M_n = \int_{\mathbb{R}_+} t^n \nu_1(dt) = n! [y^n] \varphi_1(y).$$

This recurrence is found to be of fourth order. Its coefficients are polynomials in  $n$ , of degree 4. We refer to the MAPLE worksheet available on MBM's webpage for details. We have used the GFUN package to derive the recurrence relation from the differential equation [37].

To show that this is perfectly explicit, let us focus on the simplest case, that is,  $2\varepsilon + \theta = 2\pi$ . Then

$$\varphi_1(y) = \kappa \frac{w(y) + 1}{(y - y(-1))^2},$$

or equivalently, denoting  $z := 2y/(y^+ - y^-)$ ,

$$\varphi_1(y) = \kappa' \frac{T_a(c_2 - z) + 1}{(c_2 - c_1 - z)^2}, \quad (71)$$

with

$$a = \pi/\beta, \quad c_2 = \frac{y^+ + y^-}{y^+ - y^-} = \cos(\beta - \theta), \quad c_1 = \cos \beta.$$

We have derived this expression of  $c_2$  from the fact that  $y(s_0) = 0$ . Setting  $z = 0$  gives

$$\kappa' = \varphi_1(0) \frac{(c_2 - c_1)^2}{T_a(c_2) + 1} = \varphi_1(0) \frac{(c_2 - c_1)^2}{1 - \cos(\pi\theta/\beta)},$$

where  $\varphi_1(0)$  is given by (22). Then, writing

$$\varphi_1(y) = \sum_{n \geq 0} \frac{M_n}{n!} y^n = \sum_{n \geq 0} \frac{\widetilde{M}_n}{n!} z^n$$

that is,

$$M_n = \frac{2^n}{(y^+ - y^-)^n} \widetilde{M}_n,$$

we obtain the first two coefficients  $\widetilde{M}_n$  from (71):

$$\widetilde{M}_0 = \varphi_1(0), \quad \widetilde{M}_1 = \varphi_1(0) \left( \frac{2}{c_2 - c_1} - \frac{a \sin(\pi\theta/\beta)}{\sin(\beta - \theta)(1 - \cos(\pi\theta/\beta))} \right),$$

and then the sequence  $\widetilde{M}_n$  satisfies the following recurrence relation, valid for  $n \geq 0$ :

$$\begin{aligned} (1 - c_2^2)(c_1 - c_2)^2 \widetilde{M}_{n+2} &= a^2 \kappa' \mathbb{1}_{n=0} + (c_2 - c_1) (2(c_2 c_1 - 2c_2^2 + 1)n + c_2 c_1 - 5c_2^2 + 4) \widetilde{M}_{n+1} \\ &+ \left( (c_1^2 - 6c_2 c_1 + 6c_2^2 - 1)n^2 - 3(2c_2 c_1 - 3c_2^2 + 1)n - (c_1 - c_2)^2 a^2 - 2c_2 c_1 + 4c_2^2 - 2 \right) \widetilde{M}_n \\ &- n(2(2c_2 - c_1)n^2 + 3c_2 n + 2(c_1 - c_2)a^2 + c_2) \widetilde{M}_{n-1} + n(n-1)(n^2 - a^2) \widetilde{M}_{n-2}. \end{aligned}$$

## 7. EXPRESSION OF THE LAPLACE TRANSFORM WHEN $\alpha_1, \alpha_2 \in \mathbb{Z} + \pi\mathbb{Z}/\beta$

We now assume that the double angle Condition (25) holds. Then by Theorem 5.6, there exists a 2-decoupling function  $F$ , and by Corollary 5.7, the function  $F\varphi_1^2$  can be written  $S/R \circ w$  for some polynomials  $S$  and  $R$ . In this section we determine the rational function  $S/R$ . Recall that we have chosen integers  $r_1$  and  $r_2$ , and  $e_1, e_1, \varepsilon_1, \varepsilon_2$  in  $\{0, 1\}$ , such that  $s_i = (-1)^{\varepsilon_i} \sqrt{q}^{e_i} q^{r_i}$  for  $i = 1, 2$ . Returning to Lemma 3.4, this defines two integers  $k_1$  and  $k_2$  such that the arguments of  $s_1$  and  $s_2$  are respectively:

$$\omega_1 := \pi + 2\beta - 2\varepsilon - \theta = (2r_1 + e_1)\beta - (2k_1 + \varepsilon_1)\pi, \quad (72)$$

$$\omega_2 := -\pi + 2\delta - \theta = (2r_2 + e_2)\beta - (2k_2 + \varepsilon_2)\pi. \quad (73)$$

Equivalently, given the definition (2) of  $\alpha_1$  and  $\alpha_2$ ,

$$\begin{aligned} 1 - \alpha_1 &= 2r_1 + e_1 - (2k_1 + \varepsilon_1) \frac{\pi}{\beta}, \\ \alpha_2 &= 2r_2 + e_2 - (2k_2 + \varepsilon_2) \frac{\pi}{\beta}. \end{aligned}$$

Note that we now have

$$\delta + \varepsilon = \left( 1 - r_1 + r_2 - \frac{e_1 - e_2}{2} \right) \beta + \left( 1 + k_1 - k_2 + \frac{\varepsilon_1 - \varepsilon_2}{2} \right) \pi,$$

which should be compared to the condition  $\delta + \varepsilon = (1 - r)\beta + (1 + k)\pi$  of the previous section.

As in the previous section, the numbers  $r_i$  (and  $e_i$ , and  $k_i$ , and  $\varepsilon_i$ ) are uniquely defined when  $q$  is not a root of unity. Otherwise, if  $\beta = \pi n/d$  with  $0 < n < d$ , we may always choose each  $r_i$  such that  $|r_i| < d$ , or even  $-\lfloor d/2 \rfloor \leq r_i < \lfloor (d+1)/2 \rfloor$ . Such a choice of  $r_i$  will sometimes simplify certain expressions, but what follows holds for any choice.

### 7.1. PRELIMINARIES

A first simple task is again to determine the difference of the degrees of  $S$  and  $R$ , that is, the behaviour of  $S/R$  at infinity.

**Lemma 7.1.** *Then there exists a constant  $\kappa \neq 0$  such that*

$$\frac{S}{R}(z) = (F\varphi_1^2)(w^{-1}(z)) \underset{y \rightarrow \infty}{\sim} \kappa z^{2k_1 - 2k_2 + \varepsilon_1 - \varepsilon_2}.$$

*Proof.* The arguments are the same as in the proof of Lemma 6.1. Ignoring multiplicative factors, the behaviour at infinity of  $\varphi_1(y)$  is  $y^{\alpha-1}$ , where now  $\alpha = (\delta + \varepsilon - \pi)/\beta$  satisfies

$$\alpha - 1 = \frac{1}{2} \left( 2r_2 - 2r_1 + e_2 - e_1 + (2k_1 - 2k_2 + \varepsilon_1 - \varepsilon_2) \frac{\pi}{\beta} \right).$$

The behaviour at infinity of the decoupling function  $F(y)$  of (61) is  $y^{2r_1 - 2r_2 + e_1 - e_2}$ , hence it follows that

$$(F\varphi_1^2)(y) \underset{y \rightarrow \infty}{\sim} y^{(2k_1 - 2k_2 + \varepsilon_1 - \varepsilon_2) \frac{\pi}{\beta}}.$$



The result then follows from the fact that  $w^{-1}(z)$  grows as  $z^{\beta/\pi}$ .  $\square$

Our next lemma prevents several polynomials from having common factors, under an additional assumption.

**Lemma 7.2.** *Assume that the simple angle condition (24) does not hold, that is,  $s_1/s_2 \notin q^{\mathbb{Z}}$ . Then there exist no integers  $i$  and  $j$  such that  $y(s_1q^i) = y(s_2q^j)$ .*

*Proof.* Recall that  $y(s) = y(s')$  if and only if  $s' = s$  or  $s' = q/s$ . Hence if  $y(s_1q^i) = y(s_2q^j)$ , then either  $s_1/s_2 \in q^{\mathbb{Z}}$ , or  $s_1s_2 \in q^{\mathbb{Z}}$ . Since  $s_2^2 \in q^{\mathbb{Z}}$ , in both cases we would have  $s_1/s_2 \in q^{\mathbb{Z}}$ , which we have excluded.  $\square$

Let us denote, for short:

$$F_i = F_{r_i, s_i}, \quad P_i = P_{r_i, s_i}, \quad Q_i = Q_{r_i, s_i}. \quad (74)$$

We now apply Lemma 5.8 to determine how many poles and roots of  $F_i$  lie in  $\mathcal{G}_{\mathcal{R}}$ , for  $i = 1, 2$ .

**Lemma 7.3.** *Let us write the arguments  $\omega_1$  and  $\omega_2$  of  $s_1$  and  $s_2$  as in (72) and (73).*

- When  $r_1 \geq 0$ , the number of roots of  $F_1 = P_1$  (counted with multiplicity) lying in the open region  $\mathcal{G}_{\mathcal{R}}$  is

$$r_{F_1} := k_1 + \mathbb{1}_{2\beta - 2\varepsilon - \theta \geq 0} - m_-(s_1; 0, r_1 - 1).$$

- When  $r_1 \leq 0$ , the number of poles of  $F_1 = 1/Q_1$  lying in  $\mathcal{G}_{\mathcal{R}}$  is

$$p_{F_1} := -k_1 - \mathbb{1}_{2\beta - 2\varepsilon - \theta \geq 0} - m_+(s_1; 1, |r_1|).$$

- When  $r_2 \geq 0$ , the number of poles of  $1/F_2 = 1/P_2$  lying in  $\mathcal{G}_{\mathcal{R}}$  is

$$p_{F_2} := k_2 - m_-(s_2; 0, r_2 - 1).$$

- When  $r_2 \leq 0$ , the number of roots of  $1/F_2 = Q_2$  lying in  $\mathcal{G}_{\mathcal{R}}$  is

$$r_{F_2} := -k_2 - m_+(s_2; 1, |r_2|).$$

Observe that if  $q$  is not a root of unity, all the sets occurring in this lemma have cardinality 0 or 1. If  $q$  is a root of unity, we may choose the  $r_i$ 's so as to minimize  $|r_i|$ , and then each of these sets has cardinality 0 or 1.

*Proof.* We recall the assumptions (23) on the angles involved in the definition of the model, and apply Lemma 5.8. We have

$$\begin{aligned} \left\lfloor \frac{\omega_1}{2\pi} - \frac{1}{2} \right\rfloor &= \left\lfloor \frac{2\beta - 2\varepsilon - \theta}{2\pi} \right\rfloor = -\mathbb{1}_{2\beta - 2\varepsilon - \theta < 0}, \\ \left\lfloor \frac{\omega_1}{2\pi} - \frac{1}{2} - r_1 \frac{\beta}{\pi} \right\rfloor &= -k_1 - 1 + \left\lfloor \frac{e_1\beta}{2\pi} + \frac{1 - \varepsilon_1}{2} \right\rfloor = -k_1 - 1, \\ \left\lfloor \frac{\omega_2}{2\pi} - \frac{1}{2} \right\rfloor &= \left\lfloor \frac{2\delta - \theta}{2\pi} - 1 \right\rfloor = -1, \\ \left\lfloor \frac{\omega_2}{2\pi} - \frac{1}{2} - r_2 \frac{\beta}{\pi} \right\rfloor &= -k_2 - 1 + \left\lfloor \frac{e_2\beta}{2\pi} + \frac{1 - \varepsilon_2}{2} \right\rfloor = -k_2 - 1, \end{aligned}$$

and this gives the announced formulas.  $\square$

## 7.2. EXPRESSION OF $\varphi_1$

We can now describe precisely polynomials  $S$  and  $R$  such that  $F\varphi_1^2 = \frac{S}{R} \circ w$ . In order to avoid having four different cases depending on the signs of the  $r_i$ 's, we will use a compact form. In the simple case where  $\alpha \in \mathbb{Z} + \pi/\beta\mathbb{Z}$ , studied in the previous section, we could have written, for compactness:

$$\varphi_1(y) = \frac{Q(y) S(w(y))}{P(y) R(w(y))},$$

where  $P = P_{r,s_1}$ ,  $Q = Q_{r,s_1}$ , and

$$\text{either } (P = 1 \text{ and } S = 1) \text{ or } (Q = 1 \text{ and } R = 1).$$

When  $\alpha_1, \alpha_2 \in \mathbb{Z} + \pi/\beta\mathbb{Z}$ , we will express  $\varphi_1$  in compact form as well. First, for  $i = 1, 2$  we use again the notation (74), as well as  $f_i = f_{e_i, \varepsilon_i}$ . We further denote

$$a^- = e_1\varepsilon_1 - e_2\varepsilon_2 \quad \text{and} \quad a^+ = e_1(1 - \varepsilon_1) - e_2(1 - \varepsilon_2), \quad (75)$$

so that

$$\frac{f_1(y)}{f_2(y)} = (y - y^-)^{a^-} (y - y^+)^{a^+}. \quad (76)$$

Also, let

$$b = \varepsilon_1(1 - e_1) - \varepsilon_2(1 - e_2). \quad (77)$$

Observe that  $a^+, a^-$  and  $b$  take their values in  $\{-1, 0, 1\}$ .

We finally introduce four polynomials denoted  $R_i$  and  $S_i$ , for  $i = 1, 2$ . In fact we only define them up to a constant factor, by giving the list of their roots. The values of their degrees easily follow Lemma 7.3, as will be established in the proof of Theorem 7.4 below.

- If  $r_1 \leq 0$ , we take  $R_1$  to be a polynomial of degree  $-k_1$  whose roots (taken with multiplicity) are
  - the  $w(y(s_1q^j))$ , for  $j = 1, \dots, |r_1|$  such that  $y(s_1q^j) \in \mathcal{G}_{\mathcal{R}}$ ,
  - plus  $w(y(s_1))$  if  $2\beta - 2\varepsilon - \theta > 0$ ,
  - plus  $w(y(-1)) = -1$  with multiplicity  $m_+(s_1; 0, |r_1|)$ .
 If  $r_1 > 0$ , we take  $R_1$  to be constant.
- If  $r_1 > 0$ , we take  $S_1$  to be a polynomial of degree  $k_1$  whose roots are
  - the  $w(y(s_1q^{-j}))$ , for  $j = 1, \dots, r_1 - 1$ , such that  $y(s_1q^{-j}) \in \mathcal{G}_{\mathcal{R}}$ ,
  - plus  $w(y(s_1))$  if  $y(s_1) < y(-1)$  and  $2\beta - 2\varepsilon - \theta \leq 0$ ,
  - plus  $w(y(-1)) = -1$  with multiplicity  $m_-(s_1; 1, r_1 - 1)$ .
 If  $r_1 \leq 0$ , we take  $S_1$  to be constant.
- If  $r_2 \leq 0$ , we take  $R_2$  to be a polynomial of degree  $-k_2$  whose roots are
  - the  $w(y(s_2q^j))$ , for  $j = 1, \dots, |r_2|$ , such that  $y(s_2q^j) \in \mathcal{G}_{\mathcal{R}}$ ,
  - plus  $w(y(-1)) = -1$  with multiplicity  $m_+(s_2; 1, |r_2|)$ .
 If  $r_2 > 0$ , we take  $R_2$  to be constant.
- If  $r_2 > 0$ , we take  $S_2$  to be a polynomial of degree  $k_2$  whose roots are
  - the  $w(y(s_2q^{-j}))$  for  $j = 0, \dots, r_2 - 1$ , such that  $y(s_2q^{-j}) \in \mathcal{G}_{\mathcal{R}}$ ,
  - plus  $w(y(-1)) = -1$  with multiplicity  $m_-(s_2; 1, r_2 - 1)$ .
 If  $r_2 \leq 0$ , we take  $S_2$  to be constant.

**Theorem 7.4.** *Let us assume that Condition (25) holds, that is,  $\alpha_1, \alpha_2 \in \mathbb{Z} + \pi\mathbb{Z}/\beta$ . Let the integers  $r_i, k_i, e_i, \varepsilon_i$ , for  $i = 1, 2$ , satisfy (72) and (73). The 2-decoupling function of Theorem 5.6 reads:*

$$F = \left( \frac{P_1}{Q_1} \cdot \frac{Q_2}{P_2} \right)^2 \cdot \frac{f_1}{f_2},$$

and, if  $\varphi_1$  denotes the Laplace transform defined by (19), the function  $I := F\varphi_1^2$  is an invariant.

The function  $\varphi_1$  can be meromorphically continued to  $\mathbb{C} \setminus [y^+, \infty)$ . Moreover, the polynomials  $R_i$  and  $S_i$  defined above (up to a multiplicative constant) can be chosen to satisfy

$$\varphi_1(y) = \left( \frac{Q_1}{P_1} \cdot \frac{P_2}{Q_2} \right)(y) \cdot \left( \frac{S_1}{R_1} \cdot \frac{R_2}{S_2} \right)(w(y)) \cdot \sqrt{\frac{1-w(y)}{y-y^-}}^{a^-} \cdot \frac{\sqrt{1+w(y)}^b}{\sqrt{y^+-y}^{a^+}}, \quad (78)$$

where  $a^+$ ,  $a^-$  and  $b$  are defined by (75) and (77) and take their values in  $\{-1, 0, 1\}$ .

The function  $\varphi_1$  is always  $D$ -algebraic. It is  $D$ -finite if

$$\beta/\pi \in \mathbb{Q} \quad \text{or} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z} \cup \left( -\mathbb{N} + \frac{\pi}{\beta} \mathbb{Z} \right), \quad (79)$$

and algebraic if

$$\beta/\pi \in \mathbb{Q} \quad \text{or} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z}. \quad (80)$$

If the simple angle condition (24) does not hold, that is,  $s_1/s_2 \notin q^{\mathbb{Z}}$ , Condition (79) (resp. (80)) is also necessary for  $\varphi_1$  to be  $D$ -finite (resp. algebraic), and moreover  $\varphi_1$  is never rational.

As in the simple case studied in the previous section, the polynomials  $R_i$  and  $S_i$  are only determined up to a multiplicative constant. These constants can be adjusted using the value  $\varphi_1(0)$  given in (22). Several examples are worked out in Section 7.3.

*Proof of Theorem 7.4.* We consider each element of the decoupling function  $F$  separately, starting with  $P_2$  and  $Q_2$ , which are a bit simpler than their counterparts  $P_1$  and  $Q_1$ .

• When  $r_2 > 0$ , the fraction  $1/P_2$  has  $p_{F_2}$  poles in  $\mathcal{G}_{\mathcal{R}}$ , where  $p_{F_2}$  is given by Lemma 7.3. Moreover, the multiplicity of  $y(-1)$  as a pole of  $1/P_2$  is

$$\begin{aligned} & \# \{j : 0 \leq j < r_2, s_2 q^{-j} \in \{-1, -q\}\} \\ &= \# \{j : 0 \leq j < r_2, s_2 q^{-j} = -1\} + \# \{j : 1 \leq j \leq r_2, s_2 q^{-j} = -1\} \\ &= 2\# \{j : 1 \leq j < r_2, s_2 q^{-j} = -1\} + \mathbb{1}_{s_2 q^{-r_2} = -1}, \end{aligned}$$

since  $s_2$  never equals  $-1$  by the assumptions (23). This leads us to introduce the polynomial  $S_2$  defined above. Its degree is

$$p_{F_2} + m_-(s_2; 1, r_2 - 1) = -k_2.$$

By construction,  $S_2(w(y))/P_2(y)$  has at most one pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , namely a simple pole at  $y(-1)$  if  $s_2 q^{-r_2} = (-1)^{\varepsilon_2} \sqrt{q}^{\varepsilon_2} = -1$ , or equivalently if  $\varepsilon_2(1 - e_2) = 1$  (recall that  $q \neq 1$ ). Let us finally recall that  $w(y(-1)) = -1$ . Consequently,

$$\left( \frac{S_2(w(y))}{P_2(y)} \right)^2 (w(y) + 1)^{\varepsilon_2(1-e_2)} \quad (81)$$

has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ .

• When  $r_2 \leq 0$ , the polynomial  $Q_2$  has  $r_{F_2}$  roots in  $\mathcal{G}_{\mathcal{R}}$ , where  $r_{F_2}$  is given by Lemma 7.3. Moreover, the multiplicity of  $y(-1)$  as a root of  $Q_2$  is

$$\begin{aligned} & \# \{j : 1 \leq j \leq |r_2|, s_2 q^j \in \{-1, -q\}\} \\ &= \# \{j : 1 \leq j \leq |r_2|, s_2 q^j = -1\} + \# \{j : 0 \leq j < |r_2|, s_2 q^j = -1\} \\ &= 2\# \{j : 1 \leq j \leq |r_2|, s_2 q^j = -1\} - \mathbb{1}_{s_2 q^{-r_2} = -1}, \end{aligned}$$

since  $s_2$  never equals  $-1$  by the assumptions (23). This leads us to introduce the polynomial  $R_2$  defined above. Its degree is

$$r_{F_2} + m_+(s_2; 1, |r_2|) = k_2.$$

By construction,  $Q_2(y)/R_2(w(y))$  has at most one pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , namely a simple pole at  $y(-1)$  if  $s_2 q^{-r_2} = -1$ , or equivalently if  $\varepsilon_2(1 - e_2) = 1$ . Consequently,

$$\left( \frac{Q_2(y)}{R_2(w(y))} \right)^2 (w(y) + 1)^{\varepsilon_2(1-e_2)} \quad (82)$$

has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ .

• When  $r_1 > 0$ , the polynomial  $P_1$  has  $r_{F_1}$  roots in  $\mathcal{G}_{\mathcal{R}}$ , where  $r_{F_1}$  is given by Lemma 7.3. If  $y(s_1) < y(-1)$ , one of them is  $y(s_1)$ , which cancels with the pole of  $\varphi_1$  at this point when  $2\beta - 2\varepsilon - \theta > 0$ . Moreover, the multiplicity of  $y(-1)$  as a root of  $P_1$  is

$$\begin{aligned} \sharp \{j : 0 \leq j < r_1, s_1 q^{-j} \in \{-1, -q\}\} \\ = 2\sharp \{j : 1 \leq j < r_1, s_1 q^{-j} = -1\} + \mathbb{1}_{s_1 = -1} + \mathbb{1}_{s_1 q^{-r_1} = -1}. \end{aligned}$$

Recall that  $\varphi_1$  has a pole at  $y(-1)$  if  $s_1 = -1$ . This leads us to introduce the polynomial  $S_1$  defined above. Its degree is

$$r_{F_1} - \mathbb{1}_{2\beta - 2\varepsilon - \theta > 0} + m_-(s_1; 1, r_1 - 1) = k_1.$$

By construction,  $P_1(y)\varphi_1(y)/S_1(w(y))$  has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , but has a simple pole at  $y(-1)$  if  $s_1 q^{-r_1} = -1$ , or equivalently if  $\varepsilon_1(1 - e_1) = 1$ . Consequently,

$$\left( \frac{P_1(y)\varphi_1(y)}{S_1(w(y))} \right)^2 \frac{1}{(w(y) + 1)^{\varepsilon_1(1 - e_1)}} \quad (83)$$

has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ .

• When  $r_1 \leq 0$ , the fraction  $1/Q_1$  has  $p_{F_1}$  poles in  $\mathcal{G}_{\mathcal{R}}$ , where  $p_{F_1}$  is given by Lemma 7.3. Recall that  $\varphi_1$  also has a pole in  $\mathcal{G}_{\mathcal{R}}$ , located at  $y(s_1)$ , if  $2\beta - 2\varepsilon - \theta > 0$ . Moreover, the multiplicity of  $y(-1)$  as a pole of  $1/Q_1$  is

$$\begin{aligned} \sharp \{j : 1 \leq j \leq |r_1|, s_1 q^j \in \{-1, -q\}\} \\ = 2\sharp \{j : 0 \leq j \leq |r_1|, s_1 q^j = -1\} - \mathbb{1}_{s_1 = -1} - \mathbb{1}_{s_1 q^{-r_1} = -1}. \end{aligned}$$

Recall that  $\varphi_1$  has a pole at  $y(-1)$  if  $s_1 = -1$ . This leads us to introduce the polynomial  $R_1$  defined above. Its degree is

$$p_{F_1} + \mathbb{1}_{2\beta - 2\varepsilon - \theta > 0} + m_+(s_1; 0, |r_1|) = -k_1.$$

By construction,  $\varphi_1(y)R_1(w(y))/Q_1(y)$  has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , but has a zero at  $y(-1)$  if  $s_1 q^{-r_1} = -1$ , or equivalently if  $\varepsilon_1(1 - e_1) = 1$ . Consequently,

$$\left( \frac{\varphi_1(y)R_1(w(y))}{Q_1(y)} \right)^2 \frac{1}{(w(y) + 1)^{\varepsilon_1(1 - e_1)}} \quad (84)$$

has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ .

We have now constructed polynomials  $R_i$  and  $S_i$  from  $\varphi_1$ , the  $P_i$ 's and the  $Q_i$ 's. We still need to investigate the term  $f_1/f_2$ , given by (76). Recall that  $y^-$  lies in  $\mathcal{G}_{\mathcal{R}}$ , but not  $y^+$ . Indeed,  $y(-1) \leq y^+$  by Lemma 3.2, and we cannot have  $y(-1) = y^+ = y(\sqrt{q})$  because the only values  $s$  such that  $y(s) = y(-1)$  are  $-1$  and  $-q$ . Moreover, it follows from (44) that  $w(y^-) = w(y(-e^{i\beta})) = 1$ . This leads us to include a factor  $(1 - z)^{a^-}$  in  $S/R$ . By construction,

$$\frac{f_1}{f_2}(y) \cdot \frac{1}{(1 - w(y))^{a^-}} \quad (85)$$

has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ .

So let us now define the rational function

$$\frac{S}{R}(z) := \left( \frac{S_1}{R_1}(z) \frac{R_2}{S_2}(z) \right)^2 (1 - z)^{a^-} (1 + z)^b$$

where  $b = \varepsilon_1(1 - e_1) - \varepsilon_2(1 - e_2)$ . It follows from the definition of the invariant  $I = F\varphi_1^2$ , and from the fact that the functions (81), (82), (83), (84) and (85) have no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , that  $I(y)R(w(y))/S(w(y))$  is an invariant that has no pole in  $\overline{\mathcal{G}}_{\mathcal{R}}$ . By Proposition 4.4, it is a polynomial  $\tilde{S}$  in  $w(y)$ . But the behaviour at infinity of  $R/S(z)$  is in  $z^d$ , where

$$d = 2k_1 - 2k_2 + a^- + b = 2k_1 - 2k_2 + \varepsilon_1 - \varepsilon_2.$$

By Lemma 7.1,  $\tilde{S}$  is thus a constant, and we have obtained the expression of  $\varphi_1^2$ . Recall now that both  $\sqrt{1+w(y)}$  and  $\sqrt{(1-w(y))/(y-y^-)}$  are defined analytically on  $\mathbb{C} \setminus [y^+, \infty)$  (see (40), (41) and the definition (43) of  $w$  in terms of  $T_{\pi/\beta}$ ). The announced expression of  $\varphi_1$  follows, and  $\varphi_1$  is meromorphic on  $\mathbb{C} \setminus [y^+, \infty)$ .

Let us now discuss the differential/algebraic nature of  $\varphi_1$ . Since  $w$  is D-finite, and any algebraic function of a D-finite (or even D-algebraic) function is D-algebraic. Hence  $\varphi_1$  is D-algebraic.

**D-finiteness.** For  $\varphi_1$  to be D-finite, it is necessary that its square is D-finite, that is, that

$$\left(\frac{S_1 R_2}{R_1 S_2}\right)^2 (w) \cdot (1-w)^{a^-} (1+w)^b$$

is D-finite. By Proposition 4.2, this holds if and only if either  $\beta/\pi \in \mathbb{Q}$  (in which case  $\varphi_1$  is algebraic), or the rational function

$$\left(\frac{S_1 R_2}{R_1 S_2}\right)^2 (z) \cdot (1-z)^{a^-} (1+z)^b \quad (86)$$

has no pole. Recall that one of  $R_1$  and  $S_1$  (resp.  $R_2$  and  $S_2$ ) is always a constant, and observe that all roots of  $R_1$  and  $S_1$  (resp.  $R_2$  and  $S_2$ ) are of the form  $w(y(s_1 q^j))$  (resp.  $w(y(s_2 q^j))$ ) for some integer  $j$ .

Let us assume that  $s_1/s_2 \notin q^{\mathbb{Z}}$ . Then  $y(s_1 q^i) \neq y(s_2 q^j)$  for all integers  $i, j$  (Lemma 7.2), and, since  $w$  is injective on  $(-\infty, 1]$  (Lemma 4.3), we conclude that  $S_1 R_2$  and  $R_1 S_2$  have no common root. Since  $a^-$  and  $b$  are at most 1, and  $R_1 S_2$  is squared in (86), we can rephrase the conditions for  $\varphi_1^2$  to be D-finite as follows: either  $\beta/\pi \in \mathbb{Q}$ , or the three following conditions hold:

- (i)  $R_1$  and  $S_2$  are constants,
- (ii) if  $a^- = -1$ , that is,  $\varepsilon_2 = e_2 = 1$ , then 1 must be a root of  $S_1 R_2$ ,
- (iii) if  $b = -1$ , that is,  $\varepsilon_2 = 1$  and  $e_2 = 0$ , then  $-1$  must be a root of  $S_1 R_2$ .

By definition of  $R_1$  and  $S_2$ , the conditions (i) read:

$$r_1 > 0 \text{ or } (r_1 \leq 0 \text{ and } k_1 = 0), \quad (87)$$

and

$$r_2 \leq 0 \text{ or } (r_2 > 0 \text{ and } k_2 = 0). \quad (88)$$

It follows from (72) and the assumptions (23) that if  $r_1 \leq 0$  and  $k_1 = 0$ , then we cannot have  $\varepsilon_1 = 1$ . This adds to the second condition in (87) the condition  $\varepsilon_1 = 0$ . We end up with  $r_1 > 0$  or  $k_1 = \varepsilon_1 = 0$ , which translates into  $\alpha_1 \in -\mathbb{N} + \pi/\beta\mathbb{Z}$  or  $\alpha_1 \in \mathbb{Z}$ .

If  $\varepsilon_2 = e_2 = 1$ , that is,  $s_2 = -q^{r_2} \sqrt{q}$ , we want one of the roots of  $S_1 R_2$  to be  $1 = w(y^-) = w(y(-\sqrt{q})) = w(y(s_2 q^{-r_2}))$ . By Lemma 7.2, this cannot be true for any root of  $S_1$ . The description of  $R_2$  shows that it contains a factor  $(z-1)$  if and only if  $r_2 < 0$ .

Similarly, if  $\varepsilon_2 = 1$  and  $e_2 = 0$ , that is,  $s_2 = -q^{r_2}$ , we want  $-1 = w(y(-1)) = w(y(s_2 q^{-r_2}))$  to be a root of  $S_1 R_2$ . Again, this is the case if and only if  $r_2 < 0$ . Hence, in summary, we want  $r_2 < 0$  if  $\varepsilon_2 = 1$ . This adds to the second condition occurring in (88) the condition  $\varepsilon_2 = 0$ . Thus the conditions on  $r_2, e_2, k_2$  and  $\varepsilon_2$  now read

$$r_2 < 0 \text{ or } (r_2 = 0 \text{ and } \varepsilon_2 = 0) \text{ or } (r_2 > 0 \text{ and } k_2 = \varepsilon_2 = 0).$$

Let us examine the case  $r_2 = 0$  and  $\varepsilon_2 = 0$ . The assumptions (23) then imply  $k_2 = 0$ . Hence we can summarize the above conditions into  $r_2 < 0$  or  $k_2 = \varepsilon_2 = 0$ , that is,  $\alpha_2 \in -\mathbb{N} + \pi/\beta\mathbb{Z}$  or  $\alpha_2 \in \mathbb{Z}$ .

We have thus proved that if  $s_1/s_2 \notin q^{\mathbb{Z}}$  and  $\varphi_1$  is D-finite, then either  $\beta/\pi \in \mathbb{Q}$  (in which case  $\varphi_1$  is algebraic) or

$$\{\alpha_1, \alpha_2\} \subset \mathbb{Z} \cup \left(-\mathbb{N} + \frac{\pi}{\beta}\mathbb{Z}\right),$$

as claimed in the theorem.

Now assume that the latter condition holds, that is,

$$(r_1 > 0 \text{ or } k_1 = \varepsilon_1 = 0) \text{ and } (r_2 < 0 \text{ or } k_2 = \varepsilon_2 = 0).$$

Then we can check that the above conditions (i), (ii) and (iii) hold, so that (86) is a polynomial. Moreover, all its roots have an even multiplicity, with the possible exceptions of  $-1$  and  $1$ . But  $\sqrt{1 \pm w}$  is D-finite. Hence

$$\left( \frac{S_1 R_2}{R_1 S_2} \right) (w) \sqrt{1-w}^{a^-} \sqrt{1+w}^b$$

is D-finite, and  $\varphi_1$  is D-finite as well.

**Algebraicity.** It follows from the expression (78) of  $\varphi_1$  that  $\varphi_1$  is algebraic if and only if either  $w$  is algebraic (that is,  $\beta/\pi \in \mathbb{Q}$ ) or

$$\left( \frac{S_1 R_2}{R_1 S_2} \right) (z) \sqrt{1-z}^{a^-} \sqrt{1+z}^b \quad (89)$$

is in fact independent of  $z$ . This forces  $a^- = b = 0$ , that is,  $\varepsilon_1 = \varepsilon_2 = 0$ . If  $s_1/s_2 \notin q^{\mathbb{Z}}$ , then  $S_1 R_2$  and  $R_1 S_2$  have no common root, and this forces the  $R_i$ 's and  $S_i$ 's to be constants. That is,  $k_1 = k_2 = 0$ . Hence  $\alpha_1$  and  $\alpha_2$  must be both in  $\mathbb{Z}$ . Conversely, one readily checks that these conditions imply that (89) is independent of  $z$ , so that  $\varphi_1$  is algebraic.

**Rationality.** If  $\varphi_1$  is rational, then (69) implies that  $\alpha$  is an integer. But then the simple angle condition (24) holds.  $\square$

### 7.3. EXAMPLES

We now give several applications of Theorem 7.4, focussing on cases where  $\varphi_1$  is algebraic, even if  $\beta$  is not a rational multiple of  $\pi$ .

**Algebraic cases.** We have seen that  $\varphi_1$  is algebraic if  $\alpha_1$  and  $\alpha_2$  are both integers. That is,  $k_i = \varepsilon_i = 0$  for  $i = 1, 2$ . We focus here on this case:

$$\beta(1 - \alpha_1) = \pi + 2\beta - 2\varepsilon - \theta = (2r_1 + e_1)\beta, \quad \beta\alpha_2 = -\pi + 2\delta - \theta = (2r_2 + e_2)\beta.$$

We assume that  $e_1 \neq e_2$ , otherwise  $\alpha = (\alpha_1 + \alpha_2 + 1)/2$  is an integer and we are in the rational case, studied in Section 6. The condition  $\alpha < 1$  translates into  $2r_2 + e_2 < 2r_1 + e_1$ . As seen in the proof of Theorem 7.4, the expression of  $\varphi_1$  does not involve  $w(y)$  and reduces to

$$\varphi_1(y) = \left( \frac{Q_1}{P_1} \cdot \frac{P_2}{Q_2} \right) (y) \cdot \sqrt{y^+ - y}^{e_2 - e_1}.$$

because  $a^- = b = 0$  and  $a^+ = e_1 - e_2$ . Observe that if  $e_1 = 1$  and  $r_1 < 0$ , then  $Q_1$  contains a factor  $(y - y(s_1 q^{-r_1})) = (y - y(\sqrt{q})) = (y - y^+)$ , and symmetrically for  $e_2$  and  $Q_2$ .

Let us be more explicit in five cases where the polynomials  $P_i$  and  $Q_i$  have small degrees.

• Take  $r_1 = 0, r_2 = 0, e_1 = 1$  and  $e_2 = 0$ . That is,  $s_1 = \sqrt{q}$  and  $s_2 = 1$ . Then  $P_1 = Q_1 = P_2 = Q_2 = 1$ . There exists a constant  $\kappa$  such that

$$\varphi_1(y) = \frac{\kappa}{\sqrt{y^+ - y}}.$$

It follows that the measure  $\nu_1/\varphi_1(0)$  is a Gamma distribution of parameters  $1/2$  and  $1/y^+$ , with density

$$\sqrt{\frac{y^+}{\pi}} \cdot \frac{e^{-ty^+}}{\sqrt{t}}$$

for  $t \geq 0$ .

• Take  $r_1 = 1, r_2 = 0, e_1 = 1$  and  $e_2 = 0$ . That is,  $s_1 = q\sqrt{q}$  and  $s_2 = 1$ . Then  $P_1 = (y - y(s_1))$  and  $Q_1 = P_2 = Q_2 = 1$ . There exists a constant  $\kappa$  such that

$$\varphi_1(y) = \frac{\kappa}{y - y(q\sqrt{q})} \cdot \frac{1}{\sqrt{y^+ - y}}.$$

- Take  $r_1 = 1, r_2 = 0, e_1 = 0$  and  $e_2 = 1$ . That is,  $s_1 = q$  and  $s_2 = \sqrt{q}$ . Again  $P_1 = (y - y(s_1))$  and  $Q_1 = P_2 = Q_2 = 1$ . There exists a constant  $\kappa$  such that

$$\varphi_1(y) = \kappa \frac{\sqrt{y^+ - y}}{y - y(q)}.$$

- Take  $r_1 = 0, r_2 = -1, e_1 = 1$  and  $e_2 = 0$ . That is,  $s_1 = \sqrt{q}$  and  $s_2 = 1/q$ . Then  $Q_2 = (y - y(s_2q)) = (y - y(1))$  and  $P_1 = Q_1 = P_2 = 1$ . There exists a constant  $\kappa$  such that

$$\varphi_1(y) = \frac{\kappa}{y - y(1)} \cdot \frac{1}{\sqrt{y^+ - y}}.$$

- Take  $r_1 = 0, r_2 = -1, e_1 = 0$  and  $e_2 = 1$ . That is,  $s_1 = 1$  and  $s_2 = 1/\sqrt{q}$ . Then  $Q_2 = (y - y(s_2q)) = (y - y^+)$  and  $P_1 = Q_1 = P_2 = 1$ . There exists a constant  $\kappa$  such that

$$\varphi_1(y) = \frac{\kappa}{\sqrt{y^+ - y}}.$$

As in the first example, the measure  $\nu_1/\varphi_1(0)$  is a Gamma distribution of parameters  $1/2$  and  $1/y^+$ .

## 8. DIFFERENTIAL TRANSCENDENCE WHEN $\beta \notin \pi\mathbb{Q}$

The aim of this section is to complete the proof of Theorem 2.2, which deals with the case  $\beta/\pi \notin \mathbb{Q}$ . Comparing its statement with the conclusions of Theorems 6.3 and 7.4, we see that the only point that remains to be proven is the following:

if neither of the angle conditions (24) or (25) holds, the Laplace transform  $\varphi_1$  is not D-algebraic.

To prove this, a key tool is difference Galois theory. This theory builds a dictionary between the algebraic relations satisfied by the solutions of a linear difference equation and the algebraic dependencies among the coefficients of the difference equation. We refer to [43] for a complete introduction. This theory also has applications to the study of the *differential* properties of the solutions, which is what we use here.

Our strategy will be first to transform the boundary value condition (35) into a finite difference equation (Section 8.2; see (92)) and then to apply a Galoisian criterion for differential transcendence (Section 8.3). This criterion is presented in Section 8.1.

### 8.1. GALOISIAN CRITERIA FOR DIFFERENTIAL TRANSCENDENCE

The classical difference Galois theory studies algebraic relations between solutions  $g_0, \dots, g_n$  of difference equations of the form

$$\sigma(g_i) = g_i + b_i, \tag{90}$$

for  $0 \leq i \leq n$ , where the coefficients  $b_i$  lie in a field  $K$  endowed with an automorphism  $\sigma$ . In particular, a theorem due to Ostrowski gives a necessary and sufficient condition for the algebraic independence of  $g_0, \dots, g_n$  in terms of algebraic relations satisfied by the coefficients  $b_i$ .

This setting allows one to study as well the *differential* algebraicity of a function  $g$  satisfying  $\sigma(g) = g + b$ , provided that the derivation (denoted  $\partial$ ) commutes with  $\sigma$ . Indeed, the functions  $g_i = \partial^i g$  then satisfy a system of the form (90), with  $b_i = \partial^i b$ , and are algebraically related if and only if  $g$  satisfies a differential equation of order at most  $n$ .

We will use as a black box the following theorem, proved in [12]. It relaxes some assumptions of [21] (in particular, it does not require that  $f$  belongs to a difference field).

**Theorem 8.1** (Thm. C.8 in [12], case  $\Delta = 0$ ). *Let  $K$  be a field endowed with a field automorphism  $\sigma$  and a derivation  $\partial$  commuting with  $\sigma$ . We assume that the field  $K^\sigma = \{f \in K : \sigma(f) = f\}$ , called the field of constants, is relatively algebraically closed in  $K$ , that is, there is no proper algebraic extension of  $K^\sigma$  in  $K$ .*



Let  $L$  be a ring extension of  $K$  endowed with an automorphism  $\sigma_L$  extending  $\sigma$  and a derivation  $\partial_L$  extending  $\partial$ . Let  $g \in L$  satisfy  $\sigma_L(g) = g + b$  for some  $b \in K$ . If  $g$  is  $\partial_L$ -algebraic over  $K$  then there exist  $N \in \mathbb{N}_0$ , constants  $c_0, \dots, c_N \in K^\sigma$ , not all zero, and finally  $h \in K$  such that

$$c_0 b + c_1 \partial b + \dots + c_N \partial^N(b) = \sigma(h) - h.$$

The difference equation that we will derive from the boundary condition satisfied by  $G$  is not additive as above, but multiplicative, of the form  $\sigma(f) = af$  (see (92)). But a simple logarithmic transformation yields an additive equation for  $g := \frac{\partial f}{f}$ :

$$\sigma(g) = \sigma\left(\frac{\partial f}{f}\right) = \frac{\partial(af)}{af} = g + b,$$

with  $b = \partial a/a$ . Moreover,  $g$  is D-algebraic if and only if  $f$  is D-algebraic. We thus obtain the following result.

**Theorem 8.2.** *Under the same assumptions on  $K$  and  $L$  as in Theorem 8.1, let  $f$  be invertible in  $L$  and satisfy  $\sigma_L(f) = af$  for some  $a \in K$ . If  $f$  is  $\partial_L$ -algebraic over  $K$  then there exist  $N \in \mathbb{N}_0$ , constants  $c_0, \dots, c_N \in K^\sigma$ , not all zero, and finally  $h \in K$  such that*

$$c_0 \frac{\partial a}{a} + c_1 \partial \left(\frac{\partial a}{a}\right) + \dots + c_N \partial^N \left(\frac{\partial a}{a}\right) = \sigma(h) - h.$$

## 8.2. A FINITE DIFFERENCE EQUATION

Recall that  $q$  is defined to be  $e^{2i\beta}$ , and is thus a root of unity if and only if  $\beta/\pi \in \mathbb{Q}$ . All results of this subsection also hold whether this is the case, or not.

By Proposition 3.7, the function  $\varphi_1$  is meromorphic in a domain containing  $\overline{\mathcal{G}}_{\mathcal{R}}$ . As observed in (65–66), the map  $s \mapsto y(s)$  defined in (29) sends the closed wedge  $\arg(s) \in [\pi, \pi + 2\beta]$  to  $\overline{\mathcal{G}}_{\mathcal{R}}$ . Hence we can define  $\tilde{\varphi}_1(s) := \varphi_1(y(s))$ , at least in a neighbourhood of this wedge.

Let us specialize the boundary value condition (35) to  $y = y(s)$  with  $s \in (-\infty, 0)$ . By Lemma 5.4, this gives

$$\varphi_1(y(1/s)) = E(s)\varphi_1(y(s)),$$

where  $E(s)$  is the simple rational function (51). Equivalently, since  $y(1/s) = y(qs)$  and  $\arg(qs) = \pi + 2\beta$ ,

$$\tilde{\varphi}_1(qs) = E(s)\tilde{\varphi}_1(s). \tag{91}$$

However, we cannot apply directly Galois theory techniques to this  $q$ -difference equation, because we would need  $\tilde{\varphi}_1$  to be defined in a domain closed under the rotation  $s \mapsto qs$ . We prove in Appendix D (Corollary D.6) that  $\tilde{\varphi}_1$  may be continued as a meromorphic function on the slit plane  $\mathbb{C} \setminus e^{i\beta} \mathbb{R}_+$ , but again, this is not enough. We will remedy this by a second parametrization, this time of the  $s$ -plane, in terms of a new variable  $\omega$ .

Let us write  $s = e^{i\omega}$  with  $\omega \in \mathbb{C}$ . This transformation of  $\omega$  into  $s$  sends the strip  $[\pi, \pi + 2\beta] + i\mathbb{R}$  in the wedge  $\arg(s) \in [\pi, \pi + 2\beta]$ . Hence we can define, at least in a neighbourhood of this strip,

$$\psi_1(\omega) = \tilde{\varphi}_1(e^{i\omega}) = \varphi_1(y(e^{i\omega})).$$

Note that for  $\omega \in \pi + i\mathbb{R}$ , we have  $\psi_1(\omega + 2\beta) = \tilde{\varphi}_1(qe^{i\omega})$ , so that the  $q$ -difference equation (91) becomes a finite difference equation  $\psi_1(\omega + 2\beta) = E(e^{i\omega})\psi_1(\omega)$ . Moreover, we show in Appendix D how to extend  $\psi_1$  (and its counterpart  $\psi_2$ ) meromorphically to the whole complex plane, starting from the basic functional equation (20) (see Theorem D.5 and (119)).

**Proposition 8.3.** *The function  $\psi_1$  can be continued as a meromorphic function on  $\mathbb{C}$  that satisfies*

$$\psi_1(\omega + 2\beta) = M(\omega)\psi_1(\omega), \tag{92}$$

where  $M(\omega) = E(e^{i\omega})$ , and  $E(s)$  is the rational function given by (50–51).

We now relate the differential properties of  $\varphi_1$ ,  $\tilde{\varphi}_1$  and  $\psi_1$ . The proof of the following lemma is analogous to the proof of [14, Prop. 2.3] and is therefore omitted.

**Lemma 8.4.** *The following statements are equivalent:*

- $\varphi_1$  is  $\frac{d}{dy}$ -algebraic over  $\mathbb{C}(y)$ ,
- $\tilde{\varphi}_1$  is  $\partial$ -algebraic over  $\mathbb{C}(s)$  with  $\partial = is \frac{d}{ds}$ ,
- $\psi_1$  is  $\frac{d}{d\omega}$ -algebraic over  $\mathbb{C}(e^{i\omega})$ .

**Remarks**

1. Of course, since  $y$  is  $\frac{d}{dy}$ -algebraic over  $\mathbb{C}$ , we could replace in the first statement the field  $\mathbb{C}(y)$  by  $\mathbb{C}$ , or even  $\mathbb{R}$ . A similar remark applies to the other two statements. However, we choose to keep this formulation to emphasize the fact that (91) and (92) have coefficients in these base fields.

2. The choice of the derivation  $\partial$  might seem peculiar since  $\frac{d}{ds}$  would do as well. But it will be crucial in Section 9.2 because  $\partial$  commutes with the multiplication of  $s$  by  $q$ . That is, if we define the operator  $\zeta^*$ , acting on  $\mathbb{C}(s)$ , by  $\zeta^* f(s) = f(sq)$ , we have

$$\partial \zeta^* = \zeta^* \partial. \quad (93)$$

(The notation  $\zeta^*$  has been chosen so as to match the map  $\zeta$  defined by (30).) Moreover, for any analytic function  $f(s)$ , we have

$$\frac{d}{d\omega}(f(e^{i\omega})) = (\partial f)(e^{i\omega}). \quad (94)$$

### 8.3. DIFFERENTIAL TRANSCENDENCE OF THE LAPLACE TRANSFORM

We now assume that  $\beta/\pi$  is irrational, that is, that  $q$  is not a root of unity. We will prove the “only if” part in the first statement of Theorem 2.2.

**Proposition 8.5.** *Assume that  $q$  is not a root of unity, and that  $\varphi_1(y)$  is  $\frac{d}{dy}$ -algebraic. Then one of Conditions (24) or (25) holds.*

*Proof.* Assume that  $\varphi_1(y)$  is  $\frac{d}{dy}$ -algebraic. Then Lemma 8.4 implies that  $\psi_1$  is  $\frac{d}{d\omega}$ -algebraic. Recall from Proposition 8.3 that  $\psi_1$  satisfies the difference equation  $\psi_1(\omega + 2\beta) = M(\omega)\psi_1(\omega)$  with  $M(\omega) = E(e^{i\omega})$ . We now apply Theorem 8.2 to this equation, with  $a = M(\omega)$  and the following algebraic setting:

- the field  $K$  is  $\mathbb{C}(e^{i\omega})$ , endowed with the automorphism  $\sigma(h)(\omega) = h(\omega + 2\beta)$ , and the derivation  $\frac{d}{d\omega}$ ,
- the ring  $L$  is  $\mathbb{C}(e^{i\omega})[1/\psi_1, \psi_1, \frac{d}{d\omega}(\psi_1), \dots, \frac{d^j}{d\omega^j}(\psi_1), \dots]$  (derivatives of  $\psi_1$  of any order exist since  $\psi_1$  is meromorphic), with the same automorphism and derivation (we prove stability of  $L$  by  $\sigma$  below).

Let us check that the assumptions of Theorem 8.2 (or Theorem 8.1) hold. We begin with the assumptions on  $K$ . First,  $\sigma$  clearly commutes with the derivation. Then, since  $q$  is not a root of unity, it is well-known, and easy to see, that  $K^\sigma = K$ . This field is algebraically closed, hence relatively algebraically closed in  $K$ . Regarding  $L$ , we first note that this ring is obviously closed by the derivation  $\frac{d}{d\omega}$ . Moreover, since  $\frac{d}{d\omega}$  and  $\sigma$  commute, the following formula holds for all  $j \in \mathbb{N}_0$ :

$$\sigma \left( \frac{d^j}{d\omega^j} \psi_1 \right) = \frac{d^j}{d\omega^j} (\sigma(\psi_1)) = \frac{d^j}{d\omega^j} (M(\omega)\psi_1)$$

by Proposition 8.3. It follows that  $L$  is fixed by  $\sigma$ . Finally,  $\varphi_1$  is invertible in  $L$  by definition of  $L$ .

The functional equation of Proposition 8.3 now reads  $\sigma(f) = af$  with  $f = \psi_1$  and  $a = M(\omega) = E(e^{i\omega}) \in K$ . Since we have assumed that  $\varphi_1$  is D-algebraic, Theorem 8.2 implies that there exist  $N \in \mathbb{N}_0$ , constants  $c_0, \dots, c_N \in \mathbb{C}$ , not all zero, and finally  $h \in \mathbb{C}(e^{i\omega})$ , such that

$$c_0 \frac{M'}{M}(\omega) + c_1 \frac{d}{d\omega} \left( \frac{M'}{M} \right) (\omega) + \dots + c_N \frac{d^N}{d\omega^N} \left( \frac{M'}{M} \right) (\omega) = h(e^{i\omega+2i\beta}) - h(e^{i\omega}),$$

where  $M' = \frac{dM}{d\omega}$ . Upon replacing  $e^{i\omega}$  by  $s$ , and recalling that  $\frac{d}{d\omega}(f(e^{i\omega})) = (\partial f)(e^{i\omega})$  (see (94)), we conclude that

$$c_0 \frac{\partial E}{E} + c_1 \partial \left( \frac{\partial E}{E} \right) + \cdots + c_N \partial^N \left( \frac{\partial E}{E} \right) = \zeta^*(h) - h,$$

where we recall that  $\zeta^*(h)(s) = h(qs)$ .

We now apply to this equation Lemma 3.8 of [21] (with  $n = 1$ ), and conclude that the *elliptic divisor* of  $E$  must be zero (this lemma is only proved in [21] when  $|q| \neq 1$ , but the proof works *verbatim* as long as  $q$  is not a root of unity, which we have assumed here). By Lemma 5.5, this means that one of Conditions (24) or (25) holds. This completes the proof of Proposition 8.5, and of Theorem 2.2.  $\square$

## 9. WHEN $\beta \in \pi\mathbb{Q}$

We now consider the case where  $q = e^{2i\beta}$  is a root of unity, and prove Theorem 2.3 in two steps.

### 9.1. D-FINITENESS OF THE LOG-DERIVATIVE

The first part of Theorem 2.3 will follow from an explicit integral expression of  $\varphi_1$  given in [20, Thm. 1]. This theorem states that there exists constants  $c$  in  $\mathbb{C}^*$  and  $\chi \in \{0, 1\}$  such that

$$\varphi_1(y) = c \left( \frac{w(0) - w(p)}{w(y) - w(p)} \right)^{-\chi} \exp \left( \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(t) \left[ \frac{w'(t)}{w(t) - w(y)} - \frac{w'(t)}{w(t) - w(0)} \right] dt \right),$$

where  $p$  is the pole of  $\varphi_1$  lying in  $\overline{\mathcal{G}}_{\mathcal{R}}$ , if any (see Proposition 3.7),  $\mathcal{R}^- = \mathcal{R} \cap \{y \in \mathbb{C} : \Im y \leq 0\}$  is the bottom part of the the branch of hyperbola  $\mathcal{R}$  defined by (34),  $G$  is the algebraic function (36), and  $w$  is the canonical invariant (43). When  $\beta/\pi \in \mathbb{Q}$ , then  $w$  is algebraic (Proposition 4.2). As observed in [20, Prop. 22], we can rewrite the above expression as

$$\varphi_1(y) = g(y) \exp(h(y)),$$

where  $g$ , as  $w$  itself, is algebraic and  $h$  is D-finite (because the integrand is D-finite, and the integral of a D-finite function along an algebraic curve remains D-finite. Thus

$$\frac{\varphi_1'}{\varphi_1} = \frac{g'}{g} + h'$$

is clearly D-finite.

### 9.2. D-FINITE/ALGEBRAIC/RATIONAL CASES

Assume now that  $\varphi_1(y)$  is D-finite, and let us prove that it is in fact algebraic, and that one of the angle conditions (24) or (25) holds. Since  $\varphi_1'/\varphi_1$  is D-finite as we have just established, we can apply the final statement of [39], and conclude that  $\varphi_1'/\varphi_1$  is in fact algebraic. This implies that  $f = \frac{\partial \tilde{\varphi}_1}{\tilde{\varphi}_1}$  is algebraic over  $\mathbb{C}(s)$ , where as before  $\tilde{\varphi}_1(s) = \varphi_1(y(s))$  and  $\partial = is \frac{d}{ds}$ . Recall that for  $s \in (-\infty, 0)$ , the following equation holds (see (91)):

$$\zeta^*(\tilde{\varphi}_1) := \tilde{\varphi}_1(qs) = E(s)\tilde{\varphi}_1(s).$$

Since  $\partial$  commutes with  $\zeta^*$  (see (93)), the above defined function  $f$  satisfies

$$\zeta^*(f) = \zeta^* \left( \frac{\partial \tilde{\varphi}_1}{\tilde{\varphi}_1} \right) = \frac{\partial \zeta^* \tilde{\varphi}_1}{\zeta^* \tilde{\varphi}_1} = \frac{\partial(E\tilde{\varphi}_1)}{E\tilde{\varphi}_1} = f + \frac{\partial E}{E}.$$

Write the minimal polynomial equation satisfied by  $f$  over  $\mathbb{C}(s)$  as

$$c_0 + c_1 f + \cdots + f^d = 0,$$

and apply the operator  $\zeta^*$ . This gives

$$\zeta^*(c_0) + \zeta^*(c_1) \left( f + \frac{\partial E}{E} \right) + \cdots + \zeta^*(c_{d-1}) \left( f + \frac{\partial E}{E} \right)^{d-1} + \left( f + \frac{\partial E}{E} \right)^d = 0.$$

Comparing with the previous equation, and using minimality, gives, for  $j = 0, \dots, d-1$ :

$$c_j = \sum_{i=j}^d \zeta^*(c_i) \binom{i}{j} \left( \frac{\partial E}{E} \right)^{i-j}$$

with  $c_d = 1$ . In particular,

$$c_{d-1} = \zeta^*(c_{d-1}) + d \frac{\partial E}{E}.$$

Hence we can write  $\frac{\partial E}{E} = \zeta^*(h) - h$  where  $h = -c_{d-1}/d \in \mathbb{C}(s)$ . Let  $n > 0$  be such that  $q^n = 1$ . Applying  $\zeta^{*j}$  with  $j = 0, \dots, n-1$  to the latter equation, we obtain

$$\zeta^{*j} \left( \frac{\partial E}{E} \right) = \zeta^{*j+1}(h) - \zeta^{*j}(h).$$

Summing these identities and noting that  $\zeta^{*n}$  is the identity on  $\mathbb{C}(s)$ , we find

$$0 = \sum_{j=0}^{n-1} \zeta^{*j} \left( \frac{\partial E}{E} \right) = \frac{\partial(\prod_{j=0}^{n-1} \zeta^{*j}(E))}{\prod_{j=0}^{n-1} \zeta^{*j}(E)}.$$

The latter identity uses the fact that  $\partial$  and  $\zeta^*$  commute (see (93)). Thus there exists  $c \in \mathbb{C}^*$  such that  $\prod_{j=0}^{n-1} \zeta^{*j}(E) = c$ . By specializing  $s$  to 0, we find that  $c = E(0)^n = (s_1/s_2)^n$ . If instead we let  $s$  tend to infinity, we find that  $c = (s_2/s_1)^n$ . Hence  $(s_2/s_1)^n = \pm 1$ . Moreover, the norm of  $(s_2/s_1)E$  with respect to the field extension  $\mathbb{C}(s^n) \subset \mathbb{C}(s)$  is equal to 1. By Hilbert's Theorem 90 [31, p. 288], there exists  $H \in \mathbb{C}(s)$  such that  $(s_2/s_1)E(s) = \frac{H(s)}{H(sq)}$ . Since  $(s_2/s_1)^{2n} = 1$ , it follows that  $E(s)^{2n}$  can be written as  $\frac{\tilde{H}(s)}{\tilde{H}(sq)}$ . We conclude, using Lemma 5.5, that one of the angle conditions (24) or (25) holds. But then Theorem 6.3 or Theorem 7.4 applies, and tells us that  $\varphi_1$  is in fact algebraic (since  $\beta/\pi \in \mathbb{Q}$ ).

We have thus proved that  $\varphi_1$  is D-finite if and only if it is algebraic, and that in this case one of Conditions (24) or (25) holds. To conclude the proof of Theorem 2.3, we now investigate the rational cases.

If  $\alpha \in -\mathbb{N}_0$ , then Condition (24) holds, and  $\varphi_1$  is rational by Theorem 6.3. Conversely, if  $\varphi_1$  is rational, then  $\alpha \in -\mathbb{N}_0$  by (69).

The proof of Theorem 2.3 is now complete.  $\square$

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## APPENDIX A. BASIC CONDITIONS: FROM THE QUADRANT TO THE $\beta$ -WEDGE

In this appendix, we prove the equivalence between various conditions on the parameters of the model, considered in a quadrant or in the  $\beta$ -wedge. Surprisingly, we could not find these proofs in the literature.

Let us recall the angle conditions met in Sections 1 and 2 of the paper. The first four are relevant to the  $\beta$ -wedge:

$$(3) : \delta + \varepsilon - \pi < \beta, \quad \text{or equivalently} \quad \alpha < 1,$$

$$(4) : 0 < \theta < \beta,$$

$$(5) : \beta - \varepsilon < \theta < \delta,$$

$$(3) + (4) + (5) = (23) : \delta - \pi < \beta - \varepsilon < \theta < \delta, \quad 0 < \theta < \beta,$$

while the next three are relevant to the quadrant:

$$(9) : \det R > 0 \quad \text{or} \quad (r_{12} > 0 \text{ and } r_{21} > 0),$$

$$(17) : \mu_1 < 0, \quad \mu_2 < 0,$$

$$(18) : \det R > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0.$$

Recall finally the natural reflection conditions, which hold throughout the paper:

- in the  $\beta$ -wedge, the reflection angles  $\delta$  and  $\varepsilon$  lie in  $(0, \pi)$ ,
- in the quadrant, the reflection matrix  $R$  satisfies  $r_{11} > 0$  and  $r_{22} > 0$ .

**Lemma A.1.** *We have the following equivalences:*

- The semimartingale condition for the  $\beta$ -cone (3) is equivalent to the semimartingale condition for the quadrant (9).*
- Condition (4) is equivalent to the drift condition (17).*
- Condition (23) = (3) + (4) + (5) is equivalent to (17) + (18).*

*Proof.* First of all recall that  $\beta, \delta$  and  $\varepsilon$  lie in  $(0, \pi)$ . Also, recall from (16) and (13) that:

$$\tan \delta = \frac{\sin \beta}{\frac{r_{12}}{r_{22}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta}, \quad \tan \varepsilon = \frac{\sin \beta}{\frac{r_{21}}{r_{11}} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \cos \beta}, \quad \tan \theta = \frac{\sin \beta}{\frac{\mu_1}{\mu_2} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta}. \quad (95)$$

This implies that

$$\tan(\beta - \varepsilon) = \frac{\sin \beta}{\frac{r_{11}}{r_{21}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta}, \quad \tan(\beta - \theta) = \frac{\sin \beta}{\frac{\mu_2}{\mu_1} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \cos \beta}. \quad (96)$$

We will use repeatedly the fact that the cotangent function is decreasing in  $(0, \pi)$ . For instance, since by (95),

$$\cot \varepsilon - \cot \beta = \frac{1}{\tan \varepsilon} - \frac{1}{\tan \beta} = \frac{r_{21}}{r_{11}} \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} \frac{1}{\sin \beta},$$

we have

$$\varepsilon \leq \beta \Leftrightarrow r_{21} \geq 0. \quad (97)$$

Recall also that  $\cot x = \cot(x + \pi)$ .

We now begin with the proof of i). We have the following sequence of equivalences, starting from Condition (3):

$$\begin{aligned} \delta + \varepsilon - \pi < \beta &\Leftrightarrow (0 < \delta < \beta - \varepsilon + \pi < \pi) \text{ or } (\beta - \varepsilon \geq 0) && \text{as } \delta \in (0, \pi) \\ &\Leftrightarrow \left( \frac{1}{\tan(\beta - \varepsilon)} < \frac{1}{\tan \delta} \text{ and } r_{21} < 0 \right) \text{ or } r_{21} \geq 0 && \text{by (97)} \\ &\Leftrightarrow \left( \frac{r_{11}}{r_{21}} < \frac{r_{12}}{r_{22}} \text{ and } r_{21} < 0 \right) \text{ or } r_{21} \geq 0 && \text{by (95-96)} \\ &\Leftrightarrow (r_{11}r_{22} - r_{12}r_{21} > 0 \text{ and } r_{21} < 0) \text{ or } r_{21} \geq 0. \end{aligned}$$

A case analysis reveals that this is equivalent to Condition (9), namely  $\det R > 0$  or  $(r_{21} > 0$  and  $r_{12} > 0)$ .

We go on with the proof of ii). First, it follows from (14) and (12) that  $\theta > 0$  is equivalent to  $\mu_2 < 0$ . Hence we will now assume that  $\theta > 0$  and  $\mu_2 < 0$ , and prove, under these assumptions, that  $\theta < \beta$  is equivalent to  $\mu_1 < 0$ . We have the following sequence of equivalences:

$$\begin{aligned} \theta < \beta &\Leftrightarrow \frac{1}{\tan \beta} < \frac{1}{\tan \theta} \\ &\Leftrightarrow \frac{\mu_1}{\mu_2} > 0 && \text{by (95)} \\ &\Leftrightarrow \mu_1 < 0, && \text{since } \mu_2 < 0. \end{aligned}$$

Let us now prove iii). We have already seen that (4) and (17) are equivalent. We thus assume that they hold, that is, that  $0 < \theta < \beta$  and  $\mu_1, \mu_2 < 0$ , and prove the equivalence of (3) + (5) and

(18) under this assumption. Starting from the first part of (5), we have the following sequence of equivalences:

$$\begin{aligned} \beta - \varepsilon < \theta &\Leftrightarrow \beta - \theta < \varepsilon \Leftrightarrow \frac{1}{\tan \varepsilon} < \frac{1}{\tan(\beta - \theta)} \\ &\Leftrightarrow \frac{r_{21}}{r_{11}} < \frac{\mu_2}{\mu_1} && \text{by (95) and (96)} \\ &\Leftrightarrow r_{11}\mu_2 < r_{21}\mu_1 && \text{as } \mu_1 < 0. \end{aligned}$$

We recognize the third part of (18). In a similar fashion we can show that the second part of (5) is equivalent to the second part of (18):

$$\begin{aligned} \theta < \delta &\Leftrightarrow \frac{1}{\tan \delta} < \frac{1}{\tan \theta} \\ &\Leftrightarrow \frac{r_{12}}{r_{22}} < \frac{\mu_1}{\mu_2} && \text{by (95)} \\ &\Leftrightarrow r_{22}\mu_1 < r_{12}\mu_2 && \text{as } \mu_2 < 0. \end{aligned}$$

Assume now that (18) holds. The above two calculations show that (5) holds as well. Moreover, the first part of (18) implies (9), which implies (3) by item i).

Conversely, assume that Conditions (3) and (5) hold. The above two calculations show that the second and third parts of (18) hold as well:

$$r_{22}\mu_1 < r_{12}\mu_2 \quad \text{and} \quad r_{11}\mu_2 < r_{21}\mu_1. \quad (98)$$

Moreover, by item i) we have  $\det R > 0$  or ( $r_{12} > 0$  and  $r_{21} > 0$ ). If  $\det R > 0$  we are done, since this is the first part of (18). If ( $r_{12} > 0$  and  $r_{21} > 0$ ), it follows from (98) and the fact that  $\mu_1 < 0$  and  $\mu_2 < 0$  that

$$r_{22}/r_{12} > \mu_2/\mu_1 \quad \text{and} \quad r_{11}/r_{21} > \mu_1/\mu_2.$$

This implies that  $r_{22}r_{11}/(r_{21}r_{12}) > 1$ , so that  $\det R > 0$  again. This concludes the proof.  $\square$

## APPENDIX B. PROOFS OF SOME KERNEL LEMMAS

### B.1. PROOF OF LEMMA 3.2

We have

$$x^- = \frac{(\mu_2\sigma_{12} - \mu_1\sigma_{22}) - \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma}}{\det \Sigma}$$

and

$$Y^\pm(x^-) = -\frac{\sigma_{12}x^- + \mu_2}{\sigma_{22}}.$$

Recall that  $x^- < 0$ ,  $\mu_1 < 0$ ,  $\mu_2 < 0$ ,  $\sigma_{22} > 0$ ,  $\sigma_{11} > 0$  and  $\det \Sigma > 0$ .

We first want to prove that  $Y^\pm(x^-)$  is positive. Let us first assume that  $\sigma_{12} \geq 0$ . It follows that  $\sigma_{12}x^- + \mu_2 < 0$  and then  $Y^\pm(x^-) > 0$ .

Now assume that  $\sigma_{12} < 0$ . We have the following equivalences:

$$\begin{aligned} Y^\pm(x^-) > 0 &\Leftrightarrow \sigma_{12}x^- + \mu_2 < 0 \\ &\Leftrightarrow -\frac{\sigma_{12}}{\det \Sigma} \sqrt{(\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2 \det \Sigma} < -\mu_2 - \frac{\sigma_{12}}{\det \Sigma} (\mu_2\sigma_{12} - \mu_1\sigma_{22}). \end{aligned}$$

Both sides of the inequality are positive. By squaring them, we obtain

$$\begin{aligned}
Y^\pm(x^-) > 0 &\Leftrightarrow \frac{\sigma_{12}^2}{\det \Sigma^2} ((\mu_2 \sigma_{12} - \mu_1 \sigma_{22})^2 + \mu_2^2 \det \Sigma) \\
&< \mu_2^2 + \frac{\sigma_{12}^2}{\det \Sigma^2} (\mu_2 \sigma_{12} - \mu_1 \sigma_{22})^2 + 2\mu_2 \frac{\sigma_{12}}{\det \Sigma} (\mu_2 \sigma_{12} - \mu_1 \sigma_{22}) \\
&\Leftrightarrow \sigma_{12}^2 \mu_2^2 < \mu_2^2 \det \Sigma + 2\mu_2 \sigma_{12} (\mu_2 \sigma_{12} - \mu_1 \sigma_{22}) \\
&\Leftrightarrow 0 < \mu_2^2 \sigma_{11} \sigma_{22} - 2\mu_1 \mu_2 \sigma_{22} \sigma_{12}.
\end{aligned}$$

But the last equality holds because  $\mu_1 \mu_2 \sigma_{12} < 0$ .

Let us now prove that  $Y^\pm(x^-) \leq y^\pm$ . The equation (in  $x$ )  $\gamma(x, Y^\pm(x^-)) = 0$  admits a real root, namely  $x^-$ . This implies that the value  $Y^\pm(x^-)$  is not in the cuts of the functions  $X^+$  and  $X^-$ , that is, that  $y^- \leq Y^\pm(x^-) \leq y^+$ .  $\square$

### B.2. PROOF OF LEMMA 3.3

The existence of  $s_0$  is guaranteed by the fact that  $\gamma(0, 0) = 0$ , and that  $(x(s), y(s))$  is a uniformization of the curve  $\gamma(x, y) = 0$ . Using (29), we are going to derive the following two explicit expressions of  $s_0$ :

$$s_0 = \frac{i}{\sin \beta} \left( \frac{e^{i\beta}(x^+ + x^-)}{x^+ - x^-} - \frac{y^+ + y^-}{y^+ - y^-} \right) \quad (99)$$

$$= \frac{x^+ + x^- + 2i\sqrt{-x^+x^-}}{x^- - x^+}. \quad (100)$$

The first equality follows from eliminating  $\frac{1}{s_0}$  between the two equations  $x(s_0) = 0$  and  $y(s_0) = 0$ . The second arises when solving  $x(s) = 0$  using the first equation of (29). The sign in front of the square root is chosen so that  $\Im s_0 < 0$  (recall that  $x^+$  and  $x^-$  have opposite signs), an inequality that follows from (99). Indeed, since  $\sin \beta > 0$  and  $x^+ > x^-$ , we have

$$\operatorname{sgn}(\Im s_0) = \operatorname{sgn} \left( \cos \beta (x^+ + x^-) - \frac{x^+ - x^-}{y^+ - y^-} (y^+ + y^-) \right).$$

We now use (10) to express  $\cos \beta$ , and (27) for the numbers  $x^\pm$  and  $y^\pm$ . In particular, we find:

$$\frac{x^+ - x^-}{y^+ - y^-} = \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}. \quad (101)$$

This leads us to

$$\operatorname{sgn}(\Im s_0) = \operatorname{sgn} \left( \frac{2\mu_2}{\sqrt{\sigma_{11}\sigma_{22}}} \right),$$

which is negative since the drift  $\mu$  is negative (see (17)). So we have indeed chosen the sign correctly in (100).

Now starting from (100), one readily checks that  $s_0$  has modulus 1. Since  $\Im(-s_0)$  is positive and  $\theta \in (0, \beta) \subset (0, \pi)$ , proving that  $-s_0 = e^{i\theta}$  boils down to proving that the tangent of  $\theta$ , given by (13), coincides with  $2\sqrt{-x^-x^+}/(x^+ + x^-)$ , which is readily done using (27).  $\square$

### B.3. PROOF OF LEMMA 3.4

Using the relations between roots and coefficients in the quadratic equation (33), we obtain:

$$s_0 s_1 = \frac{r_{11}(x^+ - x^-) + r_{21}(y^+ - y^-)e^{i\beta}}{r_{11}(x^+ - x^-) + r_{21}(y^+ - y^-)e^{-i\beta}} = \frac{(r_{11}(x^+ - x^-) + r_{21}(y^+ - y^-)e^{i\beta})^2}{|r_{11}(x^+ - x^-) + r_{21}(y^+ - y^-)e^{i\beta}|^2}.$$

Hence we can write  $s_0 s_1 = e^{2i\omega_1}$ , where  $\omega_1$  is defined (modulo  $\pi$ ) by:

$$\tan \omega_1 = \frac{r_{21}(y^+ - y^-) \sin \beta}{r_{11}(x^+ - x^-) + r_{21}(y^+ - y^-) \cos \beta} = \frac{r_{21} \sin \beta}{r_{11} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + r_{21} \cos \beta}$$



by (101). This coincides with  $\tan(\beta - \varepsilon)$ , which is given by (96). This concludes the proof for  $s_1$ .

Analogously,

$$s_0 s_2 = \frac{(r_{12}(x^+ - x^-) + r_{22}(y^+ - y^-)e^{i\beta})^2}{|r_{12}(x^+ - x^-) + r_{22}(y^+ - y^-)e^{i\beta}|^2} = e^{2i\omega_2},$$

where

$$\tan \omega_2 = \frac{r_{22}(y^+ - y^-) \sin \beta}{r_{12}(x^+ - x^-) + r_{22}(y^+ - y^-) \cos \beta} = \frac{r_{22} \sin \beta}{r_{12} \sqrt{\frac{\sigma_{22}}{\sigma_{11}} + r_{22} \cos \beta}} = \tan \delta$$

by (95). The proof is now complete.  $\square$

#### B.4. PROOF OF LEMMA 3.5

We first want to write  $\gamma_1(x(s), y(s))$  explicitly. The definition of  $s_1$ , given just above Lemma 3.4, implies that for some constant  $c_1$  we have

$$s\gamma_1(x(s), y(s)) = c_1(s - s_1)(s - s_0).$$

The constant can be determined looking an equivalent of  $s\gamma_1(x(s), y(s))$  at  $s = \infty$ . We thus obtain:

$$c_1 = r_{11} \frac{x^+ - x^-}{4} + r_{21} \frac{y^+ - y^-}{4} e^{-i\beta}.$$

It follows from (101) and (95) that

$$\frac{r_{21}}{r_{11}} \cdot \frac{y^+ - y^-}{x^+ - x^-} = \frac{\sin \beta}{\tan \varepsilon} - \cos \beta.$$

Hence

$$\begin{aligned} c_1 &= r_{11} \frac{x^+ - x^-}{4} \left( 1 + \left( \frac{\sin \beta}{\tan \varepsilon} - \cos \beta \right) (\cos \beta - i \sin \beta) \right) \\ &= r_{11} \frac{x^+ - x^-}{4} \frac{\sin \beta}{\sin \varepsilon} ((\sin \beta \sin \varepsilon + \cos \beta \cos \varepsilon) + i(\cos \beta \sin \varepsilon - \sin \beta \cos \varepsilon)) \\ &= r_{11} \frac{x^+ - x^-}{4} \frac{\sin \beta}{\sin \varepsilon} e^{i(\varepsilon - \beta)}. \end{aligned} \tag{102}$$

Hence  $c_1$  lies in  $e^{i(\varepsilon - \beta)} \mathbb{R}_+$ . Observe that for  $s = -1$ , we have  $(x(s), y(s)) = (x^-, Y^\pm(x^-))$ . Thus

$$\begin{aligned} \gamma_1(x^-, Y^\pm(x^-)) \geq 0 &\Leftrightarrow \gamma_1(x(-1), y(-1)) \geq 0 \\ &\Leftrightarrow c_1(-1 - s_1)(-1 - s_0) \leq 0 \\ &\Leftrightarrow c_1(e^{i(2\beta - 2\varepsilon - \theta)} - 1)(e^{i\theta} - 1) \leq 0 && \text{by Lemmas 3.3 and 3.4} \\ &\Leftrightarrow c_1 e^{i(\beta - \varepsilon - \theta/2)} i \sin(\beta - \varepsilon - \theta/2) e^{i\theta/2} i \sin(\theta/2) \leq 0 \\ &\Leftrightarrow \sin(\beta - \varepsilon - \theta/2) \sin(\theta/2) \geq 0 && \text{by (102)} \\ &\Leftrightarrow \sin(\beta - \varepsilon - \theta/2) \geq 0 && \text{as } \theta > 0 \\ &\Leftrightarrow \beta - \varepsilon - \theta/2 \geq 0 && \text{as } -\pi < \beta - \varepsilon - \theta/2 < \pi/2 \text{ by (23)}. \end{aligned}$$

The same calculation proves that  $\gamma_1(x^-, Y^\pm(x^-)) = 0$  if and only if  $\beta - \varepsilon - \theta/2 = 0$ .  $\square$

#### B.5. PROOF OF LEMMA 3.6

By definition of  $\mathcal{R}$  in (34) there exists  $x \in (-\infty, x^-]$  such that  $\gamma(x, y) = 0$ , that is,  $y = Y^\pm(x)$ . Let us assume that  $y = u + iv$ , with  $u$  and  $v$  real. Then, returning to the expression (26) of  $Y^\pm$ , we can write:

$$y = \frac{-(\sigma_{12}x + \mu_2) + i \operatorname{sgn}(v) \sqrt{(x - x^-)(x - x^+) \det \Sigma}}{\sigma_{22}} = u + iv. \tag{103}$$

The fact that  $\gamma(x, y) = 0$  implies that  $x$  is one of the values  $X^+(y)$  or  $X^-(y)$ , and exactly one unless  $y = y^+$  (in which case  $X^+(y) = X^-(y)$  and there is nothing to prove). We want to determine which one. Given that

$$X^\pm(y) = \frac{-(\sigma_{12}y + \mu_1) \pm \sqrt{-(y - y^-)(y - y^+) \det \Sigma}}{\sigma_{11}}$$

we have

$$\Im X^\pm(y) = -\frac{\sigma_{12}}{\sigma_{11}}v \pm \Im \frac{\sqrt{-(y - y^-)(y - y^+) \det \Sigma}}{\sigma_{11}}.$$

Let us first assume that  $v \neq 0$ . One of the values  $X^\pm(y)$ , being  $x$ , has imaginary part 0, but the above formula shows that the other has non-zero imaginary part. Given that  $\det \Sigma > 0$ , it now suffices to compare the signs of  $\sigma_{12}v$  and  $\Im \sqrt{-(y - y^-)(y - y^+) \det \Sigma}$ : if they have the same sign then  $\Im X^-(y) \neq 0$  and thus  $x = X^+(y)$ , and otherwise  $x = X^-(y)$ . Since we always take the principal determination of the square root on  $\mathbb{C} \setminus \mathbb{R}_-$ , we have

$$\operatorname{sgn} \Im \sqrt{-(y - y^-)(y - y^+) \det \Sigma} = -\operatorname{sgn} \Im ((y - y^-)(y - y^+)).$$

Now, writing  $y = u + iv$ , we first obtain:

$$\Im (y - y^-)(y - y^+) = v(2u - y^+ - y^-),$$

and, using (103) to express  $u$ , and the expressions of  $y^\pm$  and  $x^\pm$ , we obtain:

$$\Im (y - y^-)(y - y^+) = 2v \frac{\sigma_{12}}{\sigma_{22}} \left( \frac{x^+ + x^-}{2} - x \right).$$

This has the same sign as  $\sigma_{12}v$ , hence  $\sigma_{12}v$  and  $\Im \sqrt{-(y - y^-)(y - y^+) \det \Sigma}$  have opposite signs and  $x = X^-(y)$  as announced.

Finally, if  $v = 0$ , that is,  $y = Y^\pm(x^-)$ , then the two values  $X^\pm(y)$  are real, one of them being  $x = x^-$ . The other one cannot be less than  $x^-$  (because  $y = Y^\pm(x)$  would not be real). Then  $x^- = X^-(y)$  by continuity of  $X^-$  as  $x$  approaches  $x^-$  from below.  $\square$

### APPENDIX C. PROOF OF LEMMA 5.5

Several of the results that we prove here were already proved in [21], but assuming that  $|q| \neq 1$ . Unfortunately, our value of  $q$  in this paper is  $e^{2i\beta}$ , which has modulus 1. However, the proofs of [21] apply *verbatim* as soon as  $q$  is not a root of unity, so we often focus here on the case where  $q$  is a root of unity, that is, when  $\beta/\pi \in \mathbb{Q}$ .

We choose an arbitrary system  $\mathcal{S} \subset \mathbb{C}^*$  of representatives of  $\mathbb{C}^*/q^{\mathbb{Z}}$ . In other words,  $\mathcal{S}$  is a subset of  $\mathbb{C}^*$  such that for any  $z \in \mathbb{C}^*$  there exists a unique  $z' \in \mathcal{S}$  such that  $z = q^\ell z'$  for some  $\ell \in \mathbb{Z}$ . We denote by  $[z]$  the equivalence class of  $z$  for the relation defined by  $z \sim z'$  if  $z/z' \in q^{\mathbb{Z}}$ .

**Definition C.1.** Let  $a \in \mathbb{C}(s)$ . We say that  $a$  is standard if for any  $z \in \mathbb{C}^*$ , at most one element of  $[z]$  is a zero or a pole of  $a$ , possibly of multiple order.

**Lemma C.2.** Let  $a \in \mathbb{C}(s)$ . There exist  $f \in \mathbb{C}(s)^*$  and  $\bar{a} \in \mathbb{C}(s)$  standard such that  $a = \bar{a} \frac{f(qs)}{f(s)}$ .

*Proof.* We refer to [21, Lem. 3.3] for a proof that holds when  $q$  is not a root of unity. If  $q$  is a root of unity of order  $n$ , let us write

$$a = \kappa s^\ell \prod_{z \in \mathcal{S}} \prod_{k=0}^{n-1} (q^k s - z)^{m_{k,z}}, \quad (104)$$

where  $\ell \in \mathbb{Z}$ ,  $m_{k,z} \in \mathbb{Z}$  and  $\kappa \in \mathbb{C}$ . Define

$$\bar{a} = \kappa s^\ell \prod_{z \in \mathcal{S}} (s - z)^{\sum_{k=0}^{n-1} m_{k,z}}.$$

Then  $\bar{a}$  is clearly standard and one easily checks that  $a = \bar{a} \frac{f(qs)}{f(s)}$ , with

$$f(s) = \prod_{z \in \mathcal{S}} \prod_{k=0}^{n-1} \prod_{j=0}^{k-1} (q^j s - z)^{m_{k,z}}.$$

□

**Definition C.3.** Let  $a \in \mathbb{C}(s)$ . If  $q$  is a root of unity of order  $n$ , let us write  $a$  as in (104). The elliptic divisor of  $a$  is defined as the formal sum

$$\operatorname{div}_q(a) = \sum_{z \in \mathcal{S}} \left( \sum_{k=0}^{n-1} m_{k,z} \right) [z].$$

If  $q$  is not a root of unity, let us write

$$a = \kappa s^\ell \prod_{z \in \mathcal{S}} \prod_{k \in \mathbb{Z}} (q^k s - z)^{m_{k,z}},$$

where  $\ell \in \mathbb{Z}$ ,  $\kappa \in \mathbb{C}$  and finitely many of the  $m_{k,z} \in \mathbb{Z}$  are non-zero. We define the elliptic divisor of  $a$  as the formal sum

$$\operatorname{div}_q(a) = \sum_{z \in \mathcal{S}} \left( \sum_{k \in \mathbb{Z}} m_{k,z} \right) [z].$$

One can easily prove that

- for  $a$  and  $b$  in  $\mathbb{C}(s)$ ,  $\operatorname{div}_q(ab) = \operatorname{div}_q(a) + \operatorname{div}_q(b)$ ,
- for  $f \in \mathbb{C}(s)^*$ ,  $\operatorname{div}_q\left(\frac{f(qs)}{f(s)}\right) = 0$ .

**Lemma C.4.** Let  $a \in \mathbb{C}(s)$ . The following statements are equivalent:

- there exist  $\kappa \in \mathbb{C}$ ,  $\ell \in \mathbb{Z}$  and  $f \in \mathbb{C}(s)$  such that  $a = \kappa s^\ell \frac{f(qs)}{f(s)}$ ,
- the elliptic divisor  $\operatorname{div}_q(a) = \sum_{z \in \mathcal{S}} n_z [z]$  is zero, that is,  $n_z = 0$  for all  $z \in \mathcal{S}$ .

*Proof.* It is clear from the above observations that the first condition implies the second, as  $\operatorname{div}_q(s^\ell) = 0$ . We now assume that  $\operatorname{div}_q(a) = 0$ , and prove that the first condition holds. As before, we refer to [21, Lem. 3.5] when  $q$  is not a root of unity, and assume that  $q$  is a root of unity of order  $n$ . Let us write  $a = \bar{a} f(qs)/f(s)$  as in Lemma C.2, with  $\bar{a}$  standard. Using again the remark above, and the assumption  $\operatorname{div}_q(a) = 0$ , we find that  $\operatorname{div}_q(\bar{a}) = 0$ . Let us write  $\bar{a} = \kappa s^\ell \prod_{z \in \mathcal{S}} (q^{k_z} s - z)^{m_z}$  where  $k_z \in \mathbb{Z}$  and only finitely many of the  $m_z \in \mathbb{Z}$  are non-zero. Then

$$0 = \operatorname{div}_q(\bar{a}) = \sum_{z \in \mathcal{S}} m_z [z],$$

which implies that all exponents  $m_z$  are zero, so that  $\bar{a} = \kappa s^\ell$ . This concludes the proof. □

We are not ready to prove Lemma 5.5.

*Proof of Lemma 5.5.* Assume that there exists  $m \in \mathbb{N}$  and  $H \in \mathbb{C}(s)$  such that  $E(s)^m = \frac{H(s)}{H(qs)}$ . By Lemma C.4, one has  $0 = \operatorname{div}_q(E^m) = m \operatorname{div}_q(E)$ , so that  $\operatorname{div}_q(E) = 0$ .

Now let us prove the equivalence between the second and third conditions of Lemma 5.5. Since  $\operatorname{div}_q(E) = [s_1] + [\frac{1}{s_2}] - [s_2] - [\frac{1}{s_1}]$ , saying that the elliptic divisor of  $E$  is zero means that either  $[s_i] = [\frac{1}{s_i}]$  for  $i = 1, 2$  or that  $[s_1] = [s_2]$  (in which case  $[\frac{1}{s_1}] = [\frac{1}{s_2}]$ ). The first case can be restated by saying that  $s_1^2$  and  $s_2^2$  belong to  $q^\mathbb{Z}$ , and the second by saying that  $s_1/s_2 \in q^\mathbb{Z}$ .

Finally, if the elliptic divisor of  $E$  is zero, then Lemma C.4 implies that  $E(s) = \kappa s^\ell f(qs)/f(s)$  for some  $\kappa \in \mathbb{C}$ ,  $\ell \in \mathbb{Z}$  and  $f(s) \in \mathbb{Q}(s)$ . We can assume without loss of generality that  $f(s)$  tends to a non-zero finite limit as  $s$  tends to infinity. Indeed, if  $f(s)$  grows like  $s^e$  for some  $e \in \mathbb{Z}$ , the function  $\tilde{f}(s) := s^{-e} f(s)$  also satisfies  $E(s) = \tilde{\kappa} s^\ell \tilde{f}(qs)/\tilde{f}(s)$  for another constant  $\tilde{\kappa}$ . So let us assume that  $f(s)$  tends to a non-zero finite limit as  $s$  tends to infinity. By letting  $s$  tend to

infinity, we see that  $\ell = 0$  and  $s_2/s_1 = \kappa$ . By setting  $s = 0$ , we see that  $s_1/s_2 = \kappa$ . Hence  $\kappa = 1/\kappa = \pm 1$ , and  $E^2(s) = f^2(qs)/f^2(s)$ .  $\square$

#### APPENDIX D. LIFTING OF $\varphi_1$ AND $\varphi_2$ ON THE UNIVERSAL COVERING OF $\mathcal{S} \setminus \{0, \infty\}$

In this section, we explain how the function

$$\psi_1(\omega) := \varphi_1(y(e^{i\omega}))$$

that has been used in Section 8, originally defined as a meromorphic function in a neighbourhood of the strip  $\Re\omega \in [\pi, \pi + \beta]$ , can be extended to a meromorphic function on  $\mathbb{C}$ , together with its counterpart  $\psi_2$  defined by  $\psi_2(\omega) := \varphi_2(x(e^{i\omega}))$ . The key idea is to use the relation

$$\gamma_1(x(s), y(s))\varphi_1(y(s)) + \gamma_2(x(s), y(s))\varphi_2(x(s)),$$

derived from the basic functional equation (20), to construct  $\psi_1$  and  $\psi_2$  on larger and larger domains.

##### D.1. CONSTRUCTION OF THE UNIVERSAL COVERING

The uniformization (29) allows us to constructively and explicitly identify  $\mathcal{S}$ , the Riemann surface of genus 0 defined in (28), with  $\mathbb{C} \cup \{\infty\}$ . Let us consider  $\mathbb{C}^* \equiv \mathcal{S} \setminus \{0, \infty\}$ , which is mapped by (29) to the finite points of the surface. This surface is then homeomorphic to a cylinder and may be considered as an infinite vertical strip whose opposite edges are identified (Figure 7, top). Informally, the universal covering  $\widehat{\mathcal{S}}$  of  $\mathcal{S} \setminus \{0, \infty\}$  may be viewed as infinitely many such strips glued together and covering the complex plane (Figure 7, bottom).

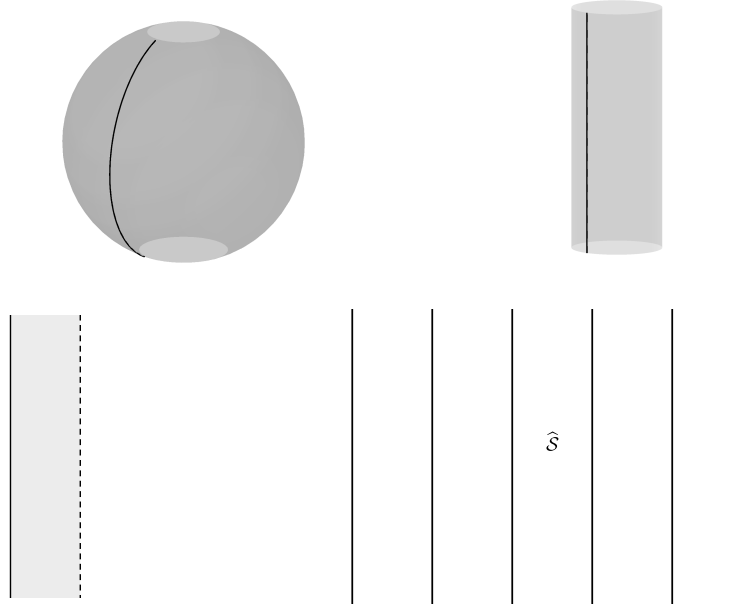


FIGURE 7. An informal construction of the universal covering  $\widehat{\mathcal{S}}$  of  $\mathcal{S} \setminus \{0, \infty\}$ .

More precisely, let us define the map  $\lambda$  by

$$\begin{aligned} \lambda: \widehat{\mathcal{S}} \equiv \mathbb{C} &\longrightarrow \mathbb{C}^* \equiv \mathcal{S} \setminus \{0, \infty\} \\ \omega &\longmapsto \lambda(\omega) := e^{i\omega}. \end{aligned}$$

This is a  $2\pi$ -periodic, non-branching covering map from  $\mathbb{C}$  to  $\mathcal{S} \setminus \{0, \infty\}$ . Every segment of the form  $[a + ib, a + 2\pi + ib]$ , with  $a$  and  $b$  real, is projected onto a closed curve of  $\mathcal{S} \setminus \{0, \infty\}$  homologous to a curve going around the cylinder.

Given  $s \in \mathcal{S} \setminus \{0, \infty\}$  and  $S \subset \mathcal{S} \setminus \{0, \infty\}$ , we will use the notation  $\widehat{s}$  and  $\widehat{S}$  for their preimages by  $\lambda$  in some prescribed vertical strip of width  $2\pi$  (which is often taken to be  $\{0 \leq \Re \omega < 2\pi\}$ , but not always). In particular, given that  $\lambda(\pi + \theta) = s_0 = -e^{i\theta}$ ,  $\lambda(\pi) = s_1^- = -1$ ,  $\lambda(\pi + \beta) = s_2^- = -e^{i\beta}$  (see Lemma 3.3 and (32)), we will write:

$$\widehat{s}_0 = \pi + \theta, \quad \widehat{s}_1^- = \pi, \quad \widehat{s}_2^- = \pi + \beta.$$

Every conformal automorphism  $\chi$  of  $\mathcal{S} \setminus \{0, \infty\}$  may be lifted to a conformal automorphism  $\widehat{\chi} = \lambda^{-1}\chi\lambda$  of the universal covering  $\mathbb{C}$ . The function  $\lambda^{-1}$  being multivalued, this continuation is uniquely defined if we fix the image by  $\widehat{\chi}$  of given point  $\omega_0 \in \mathbb{C}$ .

Recall the definitions (30) of the maps  $\xi$ ,  $\eta$  and  $\zeta$ . Recall in particular that  $\xi$  fixes 1 and  $-1$ , while  $\eta$  fixes  $\pm e^{i\beta}$ . Let us define  $\widehat{\xi}$  (resp.  $\widehat{\eta}$ ) by choosing its fixed point to be  $\widehat{s}_1^- = \pi$  (resp.  $\widehat{s}_2^- = \pi + \beta$ ). Using (30) we have

$$\widehat{\xi}(\omega) = -\omega + 2\pi \quad \text{and} \quad \widehat{\eta}(\omega) = -\omega + 2(\pi + \beta).$$

These are central symmetries of respective centers  $\widehat{s}_1^-$  and  $\widehat{s}_2^-$ . It follows that  $\widehat{\eta}\widehat{\xi}$  and  $\widehat{\xi}\widehat{\eta}$  are just translations by  $2\beta$  and  $-2\beta$ :

$$\widehat{\eta}\widehat{\xi}(\omega) = \omega + 2\beta \quad \text{and} \quad \widehat{\xi}\widehat{\eta}(\omega) = \omega - 2\beta. \quad (105)$$

## D.2. SOME IMPORTANT DOMAINS AND CURVES

The initial domain of definition of the Laplace transform  $\varphi_2(x)$  is  $\{x \in \mathbb{C} : \Re x \leq 0\}$ . Returning to the uniformization (29) of the curve  $\gamma(x, y) = 0$  by the variable  $s$ , we define  $\Delta_1 := \{s \in \mathcal{S} \setminus \{0, \infty\} : \Re x(s) \leq 0\}$  and we introduce its lifting

$$\widehat{\Delta}_1 := \{\omega \in \mathbb{C} : 0 \leq \Re \omega < 2\pi \text{ and } \Re x(\lambda\omega) \leq 0\}. \quad (106)$$

The goal of this subsection is to study the curves that bound this lifted convergence domain. We denote by  $\widehat{\mathcal{L}}_x$  the curve where the value  $x(e^{i\omega})$  is purely imaginary:

$$\widehat{\mathcal{L}}_x := \{\omega \in \mathbb{C} : 0 \leq \Re \omega < 2\pi \text{ and } \Re x(\lambda\omega) = 0\}.$$

The following lemma is illustrated in Figure 8, which we have completed by more examples in Figure 9.

**Lemma D.1.** *The curve  $\widehat{\mathcal{L}}_x$  consists of two connected branches, with vertical asymptotes at  $\Re \omega \in \{\pi/2, 3\pi/2\}$ . Denoting  $\omega = u + iv$ , with  $u$  and  $v$  real, these branches lie in two disjoint vertical strips and are defined by the equation:*

$$\cosh v = -\frac{\cos \theta}{\cos u}, \quad \text{for} \quad \begin{cases} u \in (\pi/2, \pi - \theta) \cup [\pi + \theta, 3\pi/2) & \text{if } \theta < \pi/2, \\ u \in [\pi - \theta, \pi/2) \cup (3\pi/2, \pi + \theta] & \text{if } \theta > \pi/2. \end{cases} \quad (107)$$

The case  $\theta = \pi/2$  is degenerate, with  $\widehat{\mathcal{L}}_x$  consisting of two vertical lines at abscissas  $\pi/2$  and  $3\pi/2$ .

We denote by  $\widehat{\mathcal{L}}_x^-$  the rightmost branch, which goes through  $\widehat{s}_0 = \pi + \theta$ , and by  $\widehat{\mathcal{L}}_x^+$  the leftmost one, which goes through  $\widehat{s}'_0 := \widehat{\xi}(\widehat{s}_0) = \pi - \theta$ . The notation  $\widehat{\mathcal{L}}_x^\pm$  comes from the fact that  $\Re y(\lambda\omega)$  is positive (resp. non-positive) on  $\widehat{\mathcal{L}}_x^+$  (resp.  $\widehat{\mathcal{L}}_x^-$ ). The automorphism  $\widehat{\xi}$  exchanges the branches  $\widehat{\mathcal{L}}_x^+$  and  $\widehat{\mathcal{L}}_x^-$ . Finally, the domain  $\widehat{\Delta}_1$  is the area lying between the two branches of  $\widehat{\mathcal{L}}_x$ .

*Proof.* Let us observe that, for  $\omega = u + iv$ ,

$$\begin{aligned} x(\lambda\omega) &= \frac{x^+ + x^-}{2} + \frac{x^+ - x^-}{2} (\cos u \cosh v - i \sin u \sinh v), \\ &= \frac{x^+ - x^-}{2} (\cos \theta + \cos u \cosh v - i \sin u \sinh v). \end{aligned} \quad (108)$$

The latter expression is obtained by specializing  $\omega$  to  $\pi + \theta$ , using the fact that  $x(s_0) = x(-e^{i\theta}) = 0$  (Lemma 3.3). An elementary study, based on (108), establishes equation (107).

Since  $x(\widehat{\xi}s) = x(s)$ , the curve  $\widehat{\mathcal{L}}_x$  is fixed by the automorphism  $\widehat{\xi}$ . Since  $\widehat{\xi}$  swaps the two vertical strips that contain the branches of  $\widehat{\mathcal{L}}_x$ , we conclude that it exchanges these two branches.

Let us now justify the notation  $\widehat{\mathcal{L}}_x^\pm$ . First, we derive for  $y(\lambda\omega)$  the following counterpart of (108):

$$\begin{aligned} y(\lambda\omega) &= \frac{y^+ + y^-}{2} + \frac{y^+ - y^-}{2} (\cos(u - \beta) \cosh v - i \sin(u - \beta) \sinh v) \\ &= \frac{y^+ - y^-}{2} (\cos(\theta - \beta) + \cos(u - \beta) \cosh v - i \sin(u - \beta) \sinh v). \end{aligned}$$

By combining this equation with (107), we see that on the curve  $\widehat{\mathcal{L}}_x$ , the value  $\Re y(\lambda\omega)$  has the sign of

$$\cos(\theta - \beta) - \cos(u - \beta) \frac{\cos \theta}{\cos u} = \sin \theta \frac{\sin(\theta - u)}{\cos u},$$

that is, the sign of  $\sin(\theta - u)/\cos u$ . The result follows by considering separately the two vertical strips of (107) and the two cases  $\theta \leq \pi/2$  or  $\theta \geq \pi/2$ .

The domain  $\widehat{\Delta}_1$  is the area lying between the two branches of  $\widehat{\mathcal{L}}_x$ : indeed, one can check that at  $\omega = \pi$ , we have  $\Re x(\lambda\omega) \leq 0$ .  $\square$

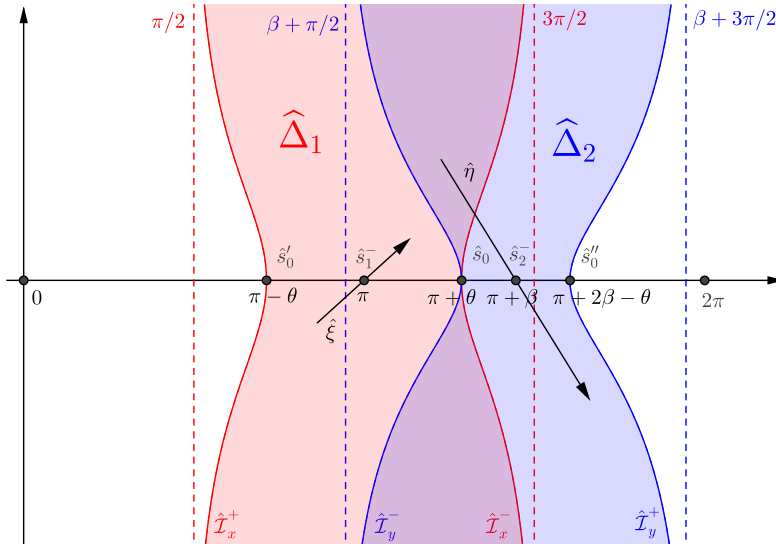


FIGURE 8. In red, the domain  $\widehat{\Delta}_1$ , and its boundary  $\widehat{\mathcal{L}}_x$  show with its asymptotes. In blue,  $\widehat{\Delta}_2$ , and its boundary  $\widehat{\mathcal{L}}_y$  shown with its asymptotes. In this example,  $\theta < \pi/2$  and  $\beta - \theta < \pi/2$ . More examples are shown in Figure 9.

In a similar fashion we now define  $\Delta_2 = \{s \in \mathcal{S} \setminus \{0, \infty\} : \Re y(s) \leq 0\}$  and

$$\widehat{\Delta}_2 := \{\omega \in \mathbb{C} : \beta \leq \Re \omega < \beta + 2\pi \text{ and } \Re y(\lambda\omega) \leq 0\}. \quad (109)$$

Note that we have taken this lifting in the translate of the vertical strip  $[0, 2\pi]$  by  $\beta$ . We define  $\widehat{\mathcal{L}}_y$  as  $\widehat{\mathcal{L}}_x$ , but again in the translated strip:

$$\widehat{\mathcal{L}}_y := \{\omega \in \mathbb{C} : \beta \leq \Re \omega < \beta + 2\pi \text{ and } \Re y(\lambda\omega) = 0\}.$$

Recall that  $0 < \theta < \beta < \pi$  by assumption. We have the following counterpart of Lemma D.1 (see again Figures 8 and 9 for various illustrations).

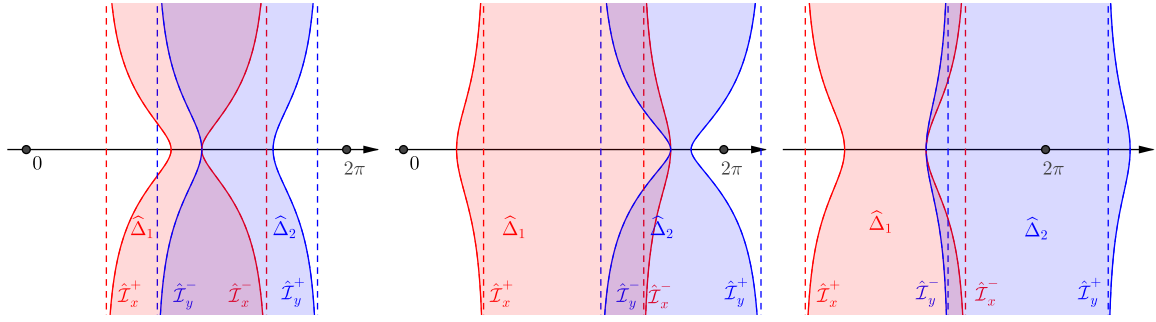


FIGURE 9. Some examples of the domains  $\widehat{\Delta}_1$  (red) and  $\widehat{\Delta}_2$  (blue), for various values of  $\theta$  and  $\beta$ : left,  $\theta < \pi/2$  and  $\beta - \theta < \pi/2$ ; middle,  $\theta > \pi/2$  and  $\beta - \theta < \pi/2$ ; right,  $\theta < \pi/2$  and  $\beta - \theta > \pi/2$ . Given that  $\beta < \pi$ , it is not possible to have  $\theta \geq \pi/2$  and  $\beta - \theta \geq \pi/2$ .

**Lemma D.2.** *The curve  $\widehat{\mathcal{I}}_y$  consists of two branches, with vertical asymptotes at  $\Re\omega \in \{\beta + \pi/2, \beta + 3\pi/2\}$ . Denoting  $\omega = u + iv$ , with  $u$  and  $v$  real, these two branches lie in two disjoint vertical strips and are defined by:*

$$\cosh v = -\frac{\cos(\theta - \beta)}{\cos(u - \beta)}, \quad \text{for } \begin{cases} u \in (\beta + \pi/2, \pi + \theta) \cup [\pi + 2\beta - \theta, \beta + 2\pi) & \text{if } \beta - \theta < \pi/2, \\ u \in [\pi + \theta, \beta + \pi/2) \cup (\beta + 3\pi/2, \pi + 2\beta - \theta] & \text{if } \beta - \theta > \pi/2. \end{cases}$$

The case  $\beta - \theta = \pi/2$  is degenerate, with  $\widehat{\mathcal{I}}_y$  consisting of two vertical lines at abscissas  $\beta + \pi/2$  and  $\beta + 3\pi/2$ . We denote by  $\widehat{\mathcal{I}}_y^-$  the leftmost branch, which goes through  $\hat{s}_0 = \pi + \theta$ , and by  $\widehat{\mathcal{I}}_y^+$  the rightmost one, which goes through  $\hat{s}_0'' := \hat{\eta}(\hat{s}_0) = \pi + 2\beta - \theta$ . The curve  $\widehat{\mathcal{I}}_y$  is fixed by the automorphism  $\hat{\eta}$ , which swaps its two branches. The value of  $\Re x(\lambda\omega)$  is positive on  $\widehat{\mathcal{I}}_y^+$  and non-positive on  $\widehat{\mathcal{I}}_y^-$ . The domain  $\widehat{\Delta}_2$  lies between  $\widehat{\mathcal{I}}_y^-$  and  $\widehat{\mathcal{I}}_y^+$ .

The proof is a straightforward adaptation of the one of Lemma D.1. Finally, the following lemma is illustrated in Figure 8, and can be checked further on the examples of Figure 9.

**Lemma D.3.** *The domain  $\widehat{\Delta}_2$  contains the branch  $\widehat{\mathcal{I}}_x^-$ . The domain  $\widehat{\Delta}_1$  contains the branch  $\widehat{\mathcal{I}}_y^-$ .*

*Proof.* By Lemma D.1, the value of  $\Re y(\lambda\omega)$  is non-positive in  $\widehat{\mathcal{I}}_x^-$ : hence this branch is included in  $\widehat{\Delta}_2$  by definition (109) of this domain. In the same way, the second property follows from Lemma D.2.  $\square$

### D.3. LIFTING OF $\varphi_1$ AND $\varphi_2$ TO THE UNIVERSAL COVERING

As the Laplace transform  $x \mapsto \varphi_2(x)$  is defined and continuous in the set  $\{x \in \mathbb{C} : \Re x \leq 0\}$  and analytic in its interior, we can lift it to  $\Delta_1 = \{s \in \mathcal{S} \setminus \{0, \infty\} : \Re x(s) \leq 0\}$  by setting  $\tilde{\varphi}_2(s) = \varphi_2(x(s))$  for all  $s \in \Delta_1$ . The set  $\Delta_1$  is the image by  $\lambda$  of  $\widehat{\Delta}_1$ . Since

$$x(s) = \frac{x^+ - x^-}{2} \left( s + \frac{1}{s} + 2 \cos \theta \right)$$

(we have used again the fact that  $x(s_0) = 0$ , with  $s_0 = -e^{i\theta}$ ), it is the part of the plane bounded by the curve  $\Re(s + 1/s) + 2 \cos \theta = 0$  that contains  $-1 = \lambda(\pi)$ . This curve consists of two branches, symmetric in the  $x$ -axis, which come infinitely close to 0, and have a vertical asymptote at  $\Re s = -2 \cos \theta$ . See Figure 10 for examples.

Since the map  $x$  is analytic on  $\mathbb{C}^*$ , the lifting  $\tilde{\varphi}_2$  is analytic in the interior of  $\Delta_1$  and continuous at each point of the curve  $\Re(s + 1/s) + 2 \cos \theta = 0$ .

In the same way we can lift  $\varphi_1$  by setting  $\tilde{\varphi}_1(s) = \varphi_1(y(s))$  for all  $s \in \Delta_2$ . This set is the part of the plane containing  $e^{i(\pi+\beta)}$  that is bounded by the curve  $\Re(se^{-i\beta} + e^{i\beta}/s) + 2 \cos(\beta -$



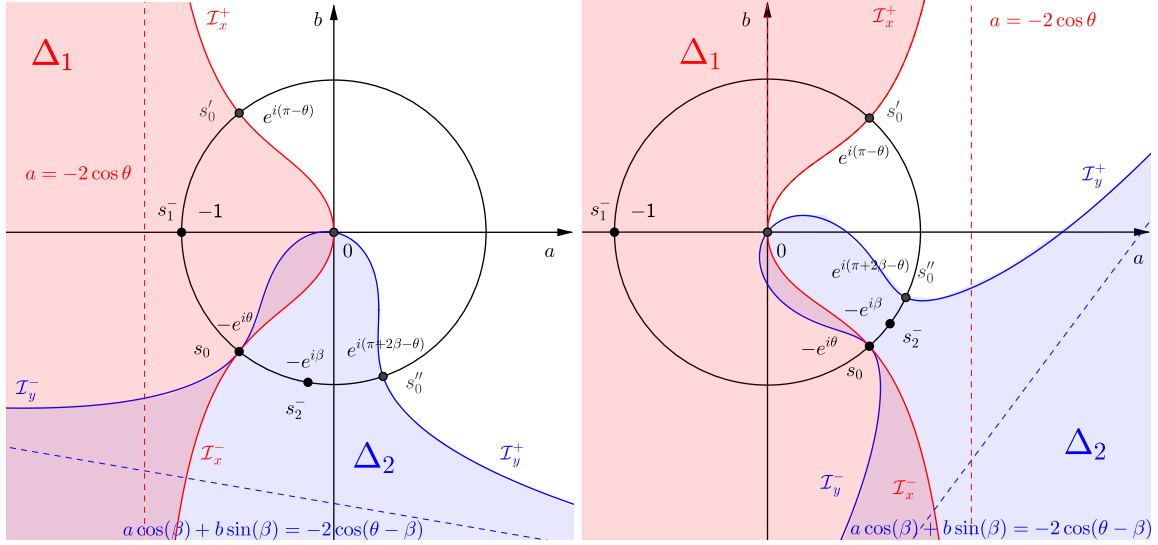


FIGURE 10. In red, the set  $\Delta_1$  for  $\theta < \pi/2$  (left) and  $\theta > \pi/2$  (right). In blue, the set  $\Delta_2$ .

$\theta) = 0$ . It is symmetric in the line that goes through 0 and  $e^{i(\pi+\beta)}$ . Now the asymptote is  $\Re(se^{-i\beta}) = a \cos \beta + b \sin \beta = -2 \cos(\beta - \theta)$  with  $s = a + ib$ . Recall that we have already defined  $\tilde{\varphi}_1$  in Section 8.2, in fact in a larger set than  $\Delta_2$ . Indeed, we had defined it in the wedge  $\arg s \in [\pi, \pi + 2\beta]$ , which is sent by the map  $y$  on  $\overline{\mathcal{G}}_{\mathcal{R}}$  while  $\Delta_2$  is only sent on  $\Re y < 0$ .

We then lift the functions  $\tilde{\varphi}_1(s)$  and  $\tilde{\varphi}_2(s)$  analytically to the sets  $\hat{\Delta}_2$  and  $\hat{\Delta}_1$  defined in the previous subsection. That is,

$$\begin{aligned} \psi_1(\omega) &:= \tilde{\varphi}_1(\lambda\omega) = \varphi_1(y(e^{i\omega})), & \forall \omega \in \hat{\Delta}_2, \\ \psi_2(\omega) &:= \tilde{\varphi}_2(\lambda\omega) = \varphi_2(x(e^{i\omega})), & \forall \omega \in \hat{\Delta}_1. \end{aligned} \quad (110)$$

Again, these maps are continuous at each point of the boundary of their domains. Then for all  $\omega$  inside  $\hat{\Delta}_1 \cap \hat{\Delta}_2$ , the main functional equation (20) yields

$$\gamma_1(x(\lambda\omega), y(\lambda\omega))\psi_1(\omega) + \gamma_2(x(\lambda\omega), y(\lambda\omega))\psi_2(\omega) = 0. \quad (111)$$

Note that the intersection  $\hat{\Delta}_1 \cap \hat{\Delta}_2$  is non-empty: in fact it is the area lying between the curves  $\hat{\mathcal{I}}_y^-$  (to the left) and  $\hat{\mathcal{I}}_x^-$  (to the right).

#### D.4. MEROMORPHIC CONTINUATION ON THE UNIVERSAL COVERING

We can now extend meromorphically  $\psi_1$  and  $\psi_2$  to the interior of

$$\hat{\Delta} := \hat{\Delta}_1 \cup \hat{\Delta}_2$$

by means of the formulas

$$\psi_1(\omega) = -\frac{\gamma_2}{\gamma_1}(x(\lambda\omega), y(\lambda\omega))\psi_2(\omega) \quad \text{if } \omega \in \hat{\Delta}_1, \quad (112)$$

$$\psi_2(\omega) = -\frac{\gamma_1}{\gamma_2}(x(\lambda\omega), y(\lambda\omega))\psi_1(\omega) \quad \text{if } \omega \in \hat{\Delta}_2, \quad (113)$$

see [20, Lem. 3] or [18, Lem. 6]. Note that (111) guarantees that the values (110) and (112) actually coincide on  $\hat{\Delta}_1 \cap \hat{\Delta}_2$ . We also extend these functions on the boundary of  $\hat{\Delta}$  by continuity. A similar statement holds for  $\psi_2$ . The fact that this extension is only meromorphic, rather than analytic, comes from the division by  $\gamma_1$  and  $\gamma_2$ , which may create poles. The functional equation (111) now holds on the whole interior of  $\hat{\Delta}$ , and also on its boundary by continuity.

In order to extend  $\psi_1$  and  $\psi_2$  to  $\mathbb{C} \equiv \widehat{\mathcal{S}}$ , we will need the following lemma, which states in particular that the complex plane is completely covered by translates of the set  $\widehat{\Delta}$  by shifts of  $2\beta$ . We recall that a translation by  $2\beta$  is precisely the effect of  $\widehat{\eta}\widehat{\xi}$  (see (105)).

**Lemma D.4.** *Recall that  $\widehat{\Delta} = \widehat{\Delta}_1 \cup \widehat{\Delta}_2$ , where  $\widehat{\Delta}_1$  and  $\widehat{\Delta}_2$  are defined by (106) and (109) respectively. The set  $\widehat{\Delta}$  is bounded by  $\widehat{\mathcal{I}}_x^+$  (on the left) and  $\widehat{\mathcal{I}}_y^+$  (on the right). Moreover,  $\widehat{\mathcal{I}}_x^+ + 2\beta \subset \widehat{\Delta}_2$  and  $\widehat{\mathcal{I}}_y^+ - 2\beta \subset \widehat{\Delta}_1$ . This implies that*

$$\mathbb{C} = \bigcup_{n \in \mathbb{Z}} (\widehat{\Delta} + 2n\beta) = \bigcup_{n \in \mathbb{Z}} (\widehat{\eta}\widehat{\xi})^n \widehat{\Delta}.$$

Moreover,  $\widehat{\Delta} \cap \widehat{\eta}\widehat{\Delta} = \widehat{\Delta}_2$ , and  $\widehat{\Delta} \cap \widehat{\xi}\widehat{\Delta} = \widehat{\Delta}_1$ .

*Proof.* The first statement follows from the discussion in Section D.2, see Figures 8 and 9. Now

$$\begin{aligned} \widehat{\mathcal{I}}_x^+ + 2\beta &= \widehat{\eta}\widehat{\xi}\widehat{\mathcal{I}}_x^+ \\ &= \widehat{\eta}\widehat{\mathcal{I}}_x^- && \text{since } \widehat{\xi} \text{ exchanges the two branches of } \widehat{\mathcal{I}}_x \text{ (Lemma D.1)} \\ &\subset \widehat{\eta}\widehat{\Delta}_2 && \text{since } \widehat{\mathcal{I}}_x^- \subset \widehat{\Delta}_2 \text{ (Lemma D.3)} \\ &\subset \widehat{\Delta}_2 && \text{since } \widehat{\Delta}_2 \text{ is left invariant by } \widehat{\eta} \text{ (Lemma D.2)}. \end{aligned}$$

A similar argument proves that  $\widehat{\xi}\widehat{\eta}\widehat{\mathcal{I}}_y^+ = \widehat{\mathcal{I}}_y^+ - 2\beta \subset \widehat{\Delta}_1 \subset \widehat{\Delta}$ . The above properties are illustrated in Figures 11 and 12. They imply that  $\mathbb{C}$  can be covered by translates of  $\widehat{\Delta}$  by multiples of  $2\beta$ .

Let us now prove that  $\widehat{\Delta} \cap \widehat{\eta}\widehat{\Delta} = \widehat{\Delta}_2$ . Since  $\widehat{\eta}\widehat{\Delta}_2 = \widehat{\Delta}_2$ , we have  $\widehat{\Delta}_2 \subset \widehat{\Delta} \cap \widehat{\eta}\widehat{\Delta}$ . Now let us take  $\omega \in \widehat{\Delta}_1 \setminus \widehat{\Delta}_2$ . This point thus lies to the left of the curve  $\widehat{\mathcal{I}}_y^-$ . Then  $\widehat{\eta}\omega$  lies to the right of  $\widehat{\eta}\widehat{\mathcal{I}}_y^- = \widehat{\mathcal{I}}_y^+$ , and thus cannot be in  $\widehat{\Delta}$ . We prove the final statement of the lemma in a similar fashion.  $\square$

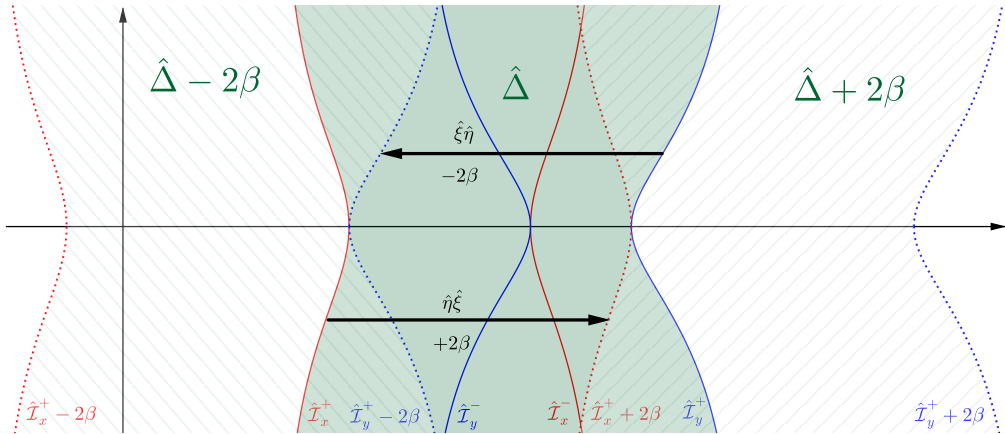


FIGURE 11. The set  $\widehat{\Delta}$ , in grey/green, and its translates by  $\pm 2\beta$  (dashed areas).

To lighten notation in the functional equation (111), we will denote

$$\widehat{\gamma}_1(\omega) := \gamma_1(x(\lambda\omega), y(\lambda\omega)) \quad \text{and} \quad \widehat{\gamma}_2(\omega) := \gamma_2(x(\lambda\omega), y(\lambda\omega)).$$

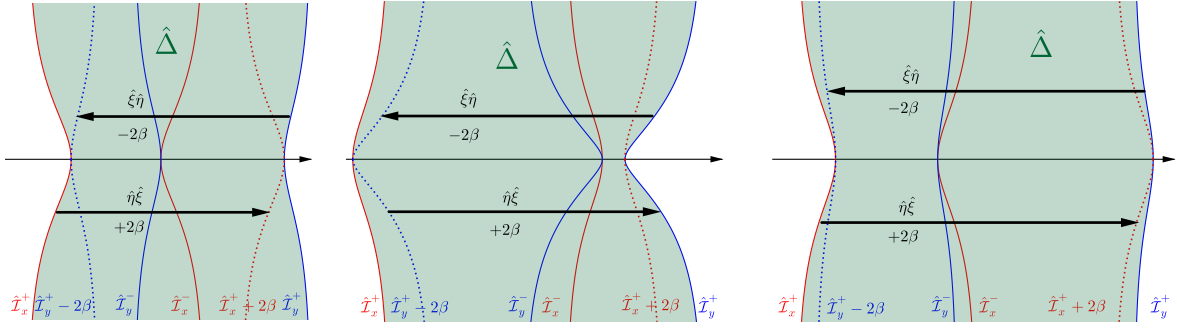


FIGURE 12. The set  $\hat{\Delta}$  and, in dotted lines, the translates of the branches  $\hat{\mathcal{I}}_x^+$  (red) and  $\hat{\mathcal{I}}_y^+$  (blue) by  $2\beta$  and  $-2\beta$ , respectively. In all cases, these translated branches fit in the domain  $\hat{\Delta}$ .

**Theorem D.5.** *The functions  $\psi_1$  and  $\psi_2$ , which are so far defined on the interior of  $\hat{\Delta}$ , can be continued meromorphically to the whole of  $\mathbb{C}$ . For all  $\omega \in \mathbb{C}$ , these functions satisfy*

$$\begin{cases} \hat{\gamma}_1(\omega)\psi_1(\omega) + \hat{\gamma}_2(\omega)\psi_2(\omega) = 0, & (\text{functional equation}) & (114) \\ \psi_1(\hat{\eta}\omega) = \psi_1(2\pi + 2\beta - \omega) = \psi_1(\omega), & (\text{invariance of } \psi_1 \text{ by } \hat{\eta}) & (115) \\ \psi_2(\hat{\xi}\omega) = \psi_2(2\pi - \omega) = \psi_2(\omega), & (\text{invariance of } \psi_2 \text{ by } \hat{\xi}) & (116) \\ \psi_1(\hat{\eta}\hat{\xi}\omega) = \psi_1(\omega + 2\beta) = \frac{\hat{\gamma}_2(\hat{\xi}\omega)}{\hat{\gamma}_1(\hat{\xi}\omega)} \frac{\hat{\gamma}_1(\omega)}{\hat{\gamma}_2(\omega)} \psi_1(\omega), & (\text{shift of } 2\beta) & (117) \\ \psi_2(\hat{\xi}\hat{\eta}\omega) = \psi_2(\omega - 2\beta) = \frac{\hat{\gamma}_1(\hat{\eta}\omega)}{\hat{\gamma}_2(\hat{\eta}\omega)} \frac{\hat{\gamma}_2(\omega)}{\hat{\gamma}_1(\omega)} \psi_2(\omega), & (\text{shift of } -2\beta). & (118) \end{cases}$$

Note that (117) can be rewritten as

$$\begin{aligned} \psi_1(\omega + 2\beta) &= \frac{\gamma_2(x(e^{-i\omega}), y(e^{-i\omega}))}{\gamma_1(x(e^{-i\omega}), y(e^{-i\omega}))} \frac{\gamma_1(x(e^{i\omega}), y(e^{i\omega}))}{\gamma_2(x(e^{i\omega}), y(e^{i\omega}))} \psi_1(\omega), \\ &= E(e^\omega)\psi_1(\omega), \end{aligned} \quad (119)$$

where  $E(s)$  is defined by (50). This is the formula announced in Proposition 8.3.

*Proof of Theorem D.5.* We have constructed  $\psi_1$  and  $\psi_2$  meromorphically inside  $\hat{\Delta}$ . Our first task will be to prove the identities (114-118) where they are well defined. We have already seen that the functional equation (114) holds in  $\hat{\Delta}$  (see the discussion following (113)).

Let us now prove that the invariance formula (115) holds where it is well-defined, that is, for  $\omega \in \hat{\Delta} \cap \hat{\eta}\hat{\Delta} = \hat{\Delta}_2$  (Lemma D.4). But for  $\omega \in \hat{\Delta}_2$ , the value  $\psi_1(\omega)$  only depends on  $y(\lambda\omega)$  (see (110)), and  $y \circ \lambda$  is invariant by  $\hat{\eta}$ . Hence (115) holds. In the same way, we prove that (116) holds for  $\omega \in \hat{\Delta} \cap \hat{\xi}\hat{\Delta} = \hat{\Delta}_1$ .

Let us now establish the translation formula (117) where it is well-defined, that is, for  $\omega \in \hat{\Delta} \cap \hat{\xi}\hat{\eta}\hat{\Delta} = \hat{\Delta} \cap (\hat{\Delta} - 2\beta)$ . This is the area located between the branches  $\hat{\mathcal{I}}_x^+$  and  $\hat{\mathcal{I}}_y^+ - 2\beta$  (see

Figures 11 and 12). In particular,  $\omega \in \widehat{\Delta}_1$  (Lemma D.4), hence  $\widehat{\xi}\omega \in \widehat{\Delta}_1$  as well. We have:

$$\begin{aligned} \psi_1(\widehat{\eta}\widehat{\xi}\omega) &= \psi_1(\widehat{\xi}\omega) && \text{by (115), given that } \widehat{\eta}\widehat{\xi}\omega \in \widehat{\Delta} \text{ and } \xi\omega \in \widehat{\Delta}, \\ &= -\frac{\widehat{\gamma}_2(\widehat{\xi}\omega)}{\widehat{\gamma}_1(\widehat{\xi}\omega)}\psi_2(\widehat{\xi}\omega) && \text{by (114), given that } \widehat{\xi}\omega \in \widehat{\Delta}, \\ &= -\frac{\widehat{\gamma}_2(\widehat{\xi}\omega)}{\widehat{\gamma}_1(\widehat{\xi}\omega)}\psi_2(\omega) && \text{by (116), given that } \widehat{\xi}\omega \in \widehat{\Delta} \text{ and } \omega \in \Delta, \\ &= \frac{\widehat{\gamma}_2(\widehat{\xi}\omega)}{\widehat{\gamma}_1(\widehat{\xi}\omega)}\frac{\widehat{\gamma}_1(\omega)}{\widehat{\gamma}_2(\omega)}\psi_1(\omega) && \text{by (114), given that } \omega \in \widehat{\Delta}. \end{aligned}$$

This completes the proof of (117). We prove (118) in a similar fashion for  $\omega \in \widehat{\Delta} \cap \widehat{\eta}\widehat{\xi}\widehat{\Delta}$ .

We will now continue  $\psi_1$  meromorphically on successive translates of  $\widehat{\Delta}$  by multiples of  $\pm 2\beta$ , using the translation formulas (117) and (118). Let us first define  $\psi_1$  on  $\widehat{\eta}\widehat{\xi}\widehat{\Delta} = \widehat{\Delta} + 2\beta$ . For  $\omega \in \widehat{\Delta}$  we set

$$\psi_1(\omega + 2\beta) = \psi_1(\widehat{\eta}\widehat{\xi}\omega) := \frac{\widehat{\gamma}_2(\widehat{\xi}\omega)}{\widehat{\gamma}_1(\widehat{\xi}\omega)}\frac{\widehat{\gamma}_1(\omega)}{\widehat{\gamma}_2(\omega)}\psi_1(\omega).$$

Since (117) holds on  $\widehat{\Delta} \cap \widehat{\xi}\widehat{\eta}\widehat{\Delta}$ , this is consistent with the already defined values of  $\psi_1$ . We thus obtained a meromorphic extension of  $\psi_1$  on  $\widehat{\Delta} \cup \widehat{\eta}\widehat{\xi}\widehat{\Delta}$  (recall that  $\widehat{\Delta} \cap \widehat{\eta}\widehat{\xi}\widehat{\Delta} = \widehat{\Delta} \cap (\widehat{\Delta} + 2\beta)$  has a non-empty interior). Let us now define  $\psi_1$  on  $\widehat{\xi}\widehat{\eta}\widehat{\Delta} = \widehat{\Delta} - 2\beta$ . For  $\omega \in \widehat{\Delta} - 2\beta$ , we set:

$$\psi_1(\omega) := \frac{\widehat{\gamma}_1(\widehat{\xi}\omega)}{\widehat{\gamma}_2(\widehat{\xi}\omega)}\frac{\widehat{\gamma}_2(\omega)}{\widehat{\gamma}_1(\omega)}\psi_1(\widehat{\eta}\widehat{\xi}\omega).$$

Again, the fact that (117) holds on  $\widehat{\Delta} \cap \widehat{\xi}\widehat{\eta}\widehat{\Delta}$  guarantees that we have indeed an extension of  $\psi_1$ .

With the same translation procedure we now propagate the construction of  $\psi_1$  to

$$(\widehat{\xi}\widehat{\eta})^n \widehat{\Delta} \cup \dots \cup \widehat{\xi}\widehat{\eta}\widehat{\Delta} \cup \widehat{\Delta} \cup \widehat{\eta}\widehat{\xi}\widehat{\Delta} \cup \dots \cup (\widehat{\eta}\widehat{\xi})^n \widehat{\Delta} = \bigcup_{k \in [-n, n]} (\widehat{\Delta} + 2k\beta)$$

for all  $n \in \mathbb{N}$ . Lemma D.4 guarantees that we finally cover the whole of  $\mathbb{C}$ .

We continue  $\psi_2$  meromorphically to  $\mathbb{C}$  using a similar procedure, based now on (118). The principle of analytic/meromorphic continuation implies that the equations (114-118) are satisfied on the whole of  $\mathbb{C}$ .  $\square$

**Corollary D.6.** *The function  $\widetilde{\varphi}_1$  may be continued meromorphically on the slit plane  $\mathbb{C} \setminus e^{i\beta}\mathbb{R}_+$ .*

*Proof.* Let us recall from (110) that  $\widetilde{\varphi}_1(e^{i\omega}) = \psi_1(\omega)$  for  $\omega \in \widehat{\Delta}_2$ . The domain  $\widehat{\Delta}_2$  contains in particular a neighbourhood of  $\pi + \beta + i\mathbb{R}$ . Let  $\log$  be the determination of the complex logarithm in the slit plane  $\mathbb{C} \setminus e^{i\beta}\mathbb{R}_+$  that satisfies  $\log(e^{i\omega}) = i\omega$  when  $\Re \omega = \pi + \beta$ . In a neighbourhood of  $\arg s = \pi + \beta$ , we now have

$$\widetilde{\varphi}_1(s) = \psi_1(-i \log s).$$

But Theorem D.5 states that  $\psi_1$  can be continued meromorphically to  $\mathbb{C}$ . Then the above formula allows us to continue  $\widetilde{\varphi}_1$  meromorphically to  $\mathbb{C} \setminus e^{i\beta}\mathbb{R}_+$ .  $\square$

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