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Algebraic Structures from Concurrent Constraint Programming Calculi for Distributed Information in Multi-Agent Systems^{*}

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Abstract

Spatial constraint systems (scs) are semantic structures for reasoning about spatial and epistemic information in concurrent systems. We develop the theory of scs to reason about the *distributed information* of potentially *infinite groups*. We characterize the notion of distributed information of a group of agents as the infimum of the set of join-preserving functions that represent the spaces of the agents in the group. We provide an alternative characterization of this notion as the greatest family of join-preserving functions that satisfy certain basic properties. For completely distributive lattices, we establish that the distributed information of c amongst a group is the greatest lower bound of all possible combinations of information in the spaces of the agents in the group that derive c . We show compositionality results for these characterizations and conditions under which information that can be obtained by an infinite group can also be obtained by a finite group. Finally, we provide an application on mathematical morphology where dilations, one of its fundamental operations, define an scs on a powerset lattice. We show that distributed information represents a particular dilation in such scs.

Keywords: Reasoning about Groups, Distributed Knowledge, Infinitely Many Agents, Reasoning about Space, Mathematical Morphology, Algebraic Modeling.

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1. Introduction

In current distributed systems such as social networks, actors behave more as members of a certain *group* than as isolated individuals. Information, opinions, and beliefs of a particular actor are frequently the result of an evolving process of interchanges with other actors in a group. This suggests a reified notion of group as a single actor operating within the context of the collective information of its members. It also conveys two notions of information, one spatial and the other epistemic. In the former, information is localized in

compartments associated with a user or group. In the latter, it refers to something known or believed by a single agent or collectively by a group.

In this paper we pursue the development of a principled account of a reified notion of group by taking inspiration from the epistemic notion of *distributed knowledge* [1]. A group has its information distributed among its member agents. We thus develop a theory about what exactly is the information available to agents as a group when considering all that is distributed among its members.

In our account a group acts itself as an agent carrying the collective information of its members. We can interrogate, for instance, whether there is a potential contradiction or unwanted distributed information that a group might be involved in among its members or by integrating a certain agent. This is a fundamental question since it may predict or prevent potentially dangerous evolutions of the system.

Furthermore, in many real life multi-agent systems, the agents are unknown in advance. New agents can subscribe to the system in unpredictable ways. Thus, there is usually no a-priori bound on the number of agents in the system. It is then often convenient to model the group of agents as an infinite set. In fact, in models from economics and epistemic logic [2, 3], groups of agents have been represented as infinite, even uncountable, sets. In accordance with this fact, in this paper we consider that groups of agents can also be infinite. This raises interesting issues about the distributed information of such groups. In particular, that of group compactness: information that when obtained by an infinite group can also be obtained by one of its finite subgroups.

Context. Constraint systems (cs)² are algebraic structures for the semantics of process calculi from concurrent constraint programming (ccp) [4]. In this paper we shall study cs as semantic structures for distributed information of a group of agents.

A cs is typically formalized as a complete lattice $(\mathbf{C}, \sqsubseteq)$ [5]. The elements of \mathbf{C} represent partial information and we shall think of them as being *assertions*. They are traditionally referred to as *constraints* since they naturally express partial information (e.g., $x > 42$). The relation \sqsubseteq corresponds to information order; $c \sqsubseteq d$, often written $d \sqsupseteq c$, means that c can be derived from d , that d represents at least as much information as c , or that if we assume that d holds true then c also holds true (e.g., $x > 42 \sqsupseteq x > 41$). The join \sqcup , the meet \sqcap , the bottom *true*, and the top *false* of the lattice correspond to conjunction, disjunction, the empty information and, the join of all (possibly inconsistent) information, respectively. For example, $(x > 42 \sqcup x < 42) = \text{false}$ and $(x > 42 \sqcap x < 42) = \text{true}$ ³.

The notion of computational space and the epistemic notion of belief in the spatial ccp (sccp) process calculi [6] is represented as a family of bottom and join-preserving maps $\mathfrak{s}_i : \mathbf{C} \rightarrow \mathbf{C}$ called *space functions*. A cs equipped with space functions is called a *spatial constraint system* (scs). From a *computational point of view*, $\mathfrak{s}_i(c)$ can be interpreted as an assertion specifying that c resides within the space of agent i . Thus, given a constraint $s = \mathfrak{s}_i(c) \sqcup \mathfrak{s}_j(d) \sqcup e$ we may think of c and d as holding within the spaces of agents i and j , respectively. Similarly, $\mathfrak{s}_i(\mathfrak{s}_j(c))$ can be viewed as a hierarchical spatial specification stating that c holds within the space the agent i attributes to agent j . From an *epistemic point of view*, $\mathfrak{s}_i(c)$ specifies that i considers c to be true. An alternative epistemic view is that i interprets c as $\mathfrak{s}_i(c)$. All these interpretations convey the idea of c being local or subjective to agent i ⁴.

This work. In the spatial ccp process calculus *sccp* [6], scs are used to specify the spatial distribution of information in configurations $\langle P, c \rangle$ where P is a process and c is a constraint, called *the store*, representing the current partial information. E.g., a reduction $\langle P, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b) \rangle \longrightarrow \langle Q, \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c) \rangle$ means that P , with a in the space of agent 1 and b in the space of agent 2, can evolve to Q while adding c to the space

We added some clarifications about order in the next paragraphs to address N14.

²For simplicity we use *cs* for both *constraint system* and its plural form.

³The inverse of any boolean algebra is a cs. The opposite is not true since distributivity of meets over joins is not required in cs.

⁴A polymodal algebra [7] is a boolean algebra with top and meet preserving functions over it. By duality, its inverse is a spatial cs.

of agent 2.

Given the above reduction, assume that e is some piece of information resulting from the combination (join) of the three constraints above, i.e., $e = a \sqcup b \sqcup c$, but strictly above the join of any two of them. We are then in the situation where neither agent has e in their spaces, but as a group they could potentially have e by combining their information. Intuitively, e is *distributed* in the spaces of the group $I = \{1, 2\}$. Being able to predict the information that agents 1 and 2 may derive as group is a relevant issue in multi-agent concurrent systems, particularly if e represents unwanted or conflicting information (e.g., $e = \text{false}$).

In this work we develop the theory of group space functions $\mathbb{D}_I : \mathbf{C} \rightarrow \mathbf{C}$ to reason about information distributed among the members of a potentially infinite group I . We shall refer to \mathbb{D}_I as the *distributed space* of group I . In our theory $d \sqsupseteq \mathbb{D}_I(e)$ holds exactly when we can derive from d that e is distributed among the agents in I . For example, for $e = a \sqcup b \sqcup c$ given above, we will have $d = \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c) \sqsupseteq \mathbb{D}_{\{1,2\}}(e)$ meaning that from the information $\mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \sqcup c)$ we can derive that e is distributed among the group $I = \{1, 2\}$. Furthermore, $\mathbb{D}_I(e) \sqsupseteq \mathbb{D}_J(e)$ holds whenever $I \subseteq J$ since if e is distributed among a group I , it should also be distributed in a group that includes the agents of I .

Distributed information of infinite groups can be used to reason about multi-agent computations with unboundedly many agents. For example, a *computation* in sccp is a possibly infinite reduction sequence γ of the form $\langle P_0, c_0 \rangle \longrightarrow \langle P_1, c_1 \rangle \longrightarrow \dots$ with $c_0 \sqsubseteq c_1 \sqsubseteq \dots$. The *result* of γ is $\bigsqcup_{n \geq 0} c_n$, the join of all the stores in the computation. In sccp all fair computations from a configuration have the same result [6]. Thus, the *observable behaviour* of P with initial store c , written $\mathcal{O}(P, c)$, is defined as the result of any fair computation starting from $\langle P, c \rangle$. Now consider a setting where in addition to their sccp capabilities in [6], processes can also create new agents. Hence, unboundedly many agents, say agents $1, 2, \dots$, may be created during an infinite computation. In this case, $\mathcal{O}(P, c) \sqsupseteq \mathbb{D}_{\mathbb{N}}(\text{false})$, where \mathbb{N} is the set of natural numbers, would imply that some (finite or infinite) set of agents in any fair computation from $\langle P, c \rangle$ may reach contradictory local information among them. Notice that from the above-mentioned properties of distributed spaces, the existence of a finite set of agents $H \subseteq \mathbb{N}$ such that $\mathcal{O}(P, c) \sqsupseteq \mathbb{D}_H(\text{false})$ implies $\mathcal{O}(P, c) \sqsupseteq \mathbb{D}_{\mathbb{N}}(\text{false})$. The converse of this implication will be called *group compactness* and we will provide meaningful sufficient conditions for it to hold.

Contributions and Organization.

The paper starts with some background on lattice theory in Section 2 and on spatial constraint systems in Section 3. The main contributions are given in Sections 4 and 5 and are listed below:

1. We characterize the distributed space \mathbb{D}_I as the *greatest* space function below the space functions that represent the spaces (or beliefs) of the agents of a *possibly infinite* group I (Section 4.2).
2. We provide an alternative characterization of a distributed space as the greatest function that satisfies certain basic properties (Section 4.5).
3. We show that distributed spaces have an inherent *compositional* nature: The information of a group is determined by that of its subgroups (Section 4.9).
4. We provide a *group compactness* result: Given an infinite group I , we identify a meaningful condition under which $c \sqsupseteq \mathbb{D}_I(e)$ implies $c \sqsupseteq \mathbb{D}_J(e)$ for some finite group $J \subseteq I$ (Section 4.7).
5. We then show that without this meaningful condition we cannot guarantee that $c \sqsupseteq \mathbb{D}_I(e)$ implies $c \sqsupseteq \mathbb{D}_J(e)$ for some finite group $J \subseteq I$ (Section 4.8).
6. We provide a characterization of distributed spaces for distributive lattices: Given an infinite group I , $\mathbb{D}_I(c)$ can be viewed as the greatest information below all possible combinations of information in the spaces of the agents in I that derive c (Section 4.9).

7. Finally, we investigate applications of the theory developed in this paper to geometry and mathematical morphology (MM) (Section 5). Below we use A, B, \dots to denote sets in a vector space. In geometry, the *Minkowski addition* is given by $A \oplus B = \{a + b \mid a \in A, b \in B\}$ [8]. It is well-known that the distribution law $A \oplus (B \cap C) = (A \cap B) \oplus (A \cap C)$ holds for convex sets⁵ but not in general. As a simple application of our theory, we identify a novel and pleasant law for $A \oplus (B \cap C)$: Namely, $A \oplus (B \cap C) = \bigcap_{X \subseteq A} (X \oplus B) \cup ((A \setminus X) \oplus C)$.

In MM, a *dilation* by a *structuring element* A can be seen as a function δ_A that transforms every input image X into the image $\delta_A(X) = A \oplus X$ [9]. We show that dilations are space functions and that the distributed space corresponding to these dilations is the dilation that arises from the intersection of their structuring elements: i.e., if $\mathfrak{s}_1 = \delta_A$ and $\mathfrak{s}_2 = \delta_B$ then $\mathbb{D}_{\{1,2\}} = \delta_{A \cap B}$.

All in all, in this paper we put forward an algebraic theory for group reasoning in the context of ccp that can also be applied to other domains. The theory here developed can be used in the semantics of the spatial ccp process calculus to reason about or prevent potential unwanted evolutions of ccp processes. One could imagine the incorporation of group reasoning in a variety of process algebraic settings and indeed we expect that such formalisms will appear in due course. We will also show that our algebraic theory can be applied to prove new results in other realms such as geometry and mathematical morphology.

Remark 1. This paper is the extended version of the CONCUR'19 paper in [10] with full proofs and the contributions described above in the points 5, 6 and 7. For the sake of the reviewers we also include contents, index and subject tables and for Sections 4 and 5, which contain the main contributions, we include a summary at the end of each section. \square

2. Background

We presuppose knowledge of basic notions from domain theory and order theory [11, 12, 13]. In this section we present notation and definitions that we use throughout the paper.

Notation 1. Let (P, \sqsubseteq_P) be a poset and let $S \subseteq P$. We use $\bigsqcup_P S$ to denote the *least upper bound (lub)* (or *supremum* or *join*) of the elements in S , and $\bigsqcap_P S$ is the *greatest lower bound (glb)* (*infimum* or *meet*) of the elements in S . We shall often omit the index P from \sqsubseteq_P , \bigsqcup_P and \bigsqcap_P when no confusion arises. As usual, if $S = \{c, d\}$, $c \sqcup d$ and $c \sqcap d$ represent $\bigsqcup S$ and $\bigsqcap S$, respectively. If $S = \emptyset$, we denote $\bigsqcup S = \text{true}$ and $\bigsqcap S = \text{false}$.

An element $e \in S$ is the *greatest element* of S if and only if for every element $e' \in S$, $e' \sqsubseteq e$. If such an e exists, we denote it by $\max S$.

The dual (or inverse) of a poset (P, \sqsubseteq) is the poset (P, \sqsubseteq^{op}) where $c \sqsubseteq^{op} d$ holds if and only if $d \sqsubseteq c$ holds.

We added the following notation.

The next definition introduces complete lattices and some of their basic properties.

Definition 1 ([14]). Let (P, \sqsubseteq) be a poset

- (i) P is said to be a *complete lattice* if and only if $\bigsqcup S$ and $\bigsqcap S$ exist for every $S \subseteq P$.
- (ii) P is *distributive* if and only if for every $a, b, c \in P$, $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

⁵A convex set is a set of points such that, given any two points in that set, the line segment joining them lies entirely within that set.

- (iii) A non-empty set $D \subseteq P$ is *directed* if and only if for every pair of elements $x, y \in D$, there exists $z \in D$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, or, equivalently, if and only if every *finite* subset of D has an upper bound in D .
- (iv) An element $c \in P$ is *compact* if and only if for any directed set $D \subseteq P$, $c \sqsubseteq \bigsqcup D$ implies that $c \sqsubseteq d$ for some $d \in D$.
- (v) P is said to be a *completely distributive lattice* if it is a complete lattice and for any doubly indexed subset $\{x_{ij}\}_{i \in I, j \in J_i}$ of P

$$\bigsqcup_{i \in I} \left(\prod_{j \in J_i} x_{ij} \right) = \prod_{f \in F} \left(\bigsqcup_{i \in I} x_{if(i)} \right)$$

where F is the class of choice functions f choosing for each index $i \in I$ some index $f(i) \in J_i$.

Our space functions will be defined as self-maps with some structural properties intended to capture our notion of space.

Definition 2 ([13]). Let (L, \sqsubseteq) be a complete lattice. A *self-map* on L is a function f from L to L . Let f be a self-map on L .

- (i) f is *monotonic* if for every $a, b \in L$ such that $a \sqsubseteq b$, then $f(a) \sqsubseteq f(b)$.
- (ii) We say that f *preserves the join* of a set $S \subseteq L$ if and only if $f(\bigsqcup S) = \bigsqcup \{f(c) \mid c \in S\}$.
- (iii) We say that f *preserves arbitrary joins* if and only if it preserves the join of any arbitrary set.
- (iv) f is *continuous* if and only if it preserves the join of any directed set on L .

We conclude this background section with a well-known fact about continuous functions.

Proposition 1 ([15]). Let (P, \sqsubseteq) be a poset where P is a countable set. Let f be a self-map that preserves the join of increasing chains, i.e., for every $S = \{c_1, c_2, \dots\} \subseteq P$ such that $c_1 \sqsubseteq c_2 \sqsubseteq \dots$, we have $f(\bigsqcup S) = \bigsqcup \{f(c) \mid c \in S\}$. Then f is continuous.

3. Spatial and Standard Constraint Systems

In this section we recall the notion of constraint system and its spatial extension [6]. Furthermore, we generalize this extension to allow for infinitely many agents and state some results that will be used in later sections.

3.1. Constraint Systems

Constraint systems (cs) [4] are semantic structures to specify partial information. Following the traditional approach in [5], a cs can be formalized as a complete lattice $(\mathbf{C}, \sqsubseteq)$. The elements of \mathbf{C} are called *constraints* and they represent (partial) information.

We changed this paragraph.

Definition 3 (Constraint Systems [5]). A *constraint system* (cs) is a complete lattice $(\mathbf{C}, \sqsubseteq)$. The elements of \mathbf{C} are called *constraints*. The symbols \sqcup , *true* and *false* will be used to denote the join operation, the bottom, and the top element of \mathbf{C} .

Remark 2. Some readers familiar with boolean algebras may feel uneasy about referring to the bottom and top as *true* and *false*, respectively. Nevertheless, this is the traditional information order in cs [16]. One can think of cs as the the dual of boolean algebras without the distributivity requirement. In fact the Herbrand cs [5] given below is not distributive. \square

We added the next remark.

The term *constraints* has been traditionally used in the realm of concurrent constraint programming to refer to partial information for historical reasons; they were developed as a generalization of assertions or restrictions over variables. For example, constraints can be instantiated as arithmetic assertions such as $x > 42$ or as a set of equalities between Herbrand terms such as $\{x = a, y = b\}$ where x, y are variables and a, b are constants [5]. In fact, much of the terminology for the abstract definition of cs is derived from *concrete* constraint systems such as the Herbrand cs illustrated below in Ex. 1. We will use this concrete cs to give some intuitions about the intended meaning of cs.

We added the next four paragraphs to address N12, N13.

The relation \sqsubseteq represents information order. Thus $c \sqsubseteq d$, alternatively written $d \sqsupseteq c$, means that the information represented by c can be derived from the information represented by d , that the assertion d represents at least as much information as c , or that if d holds then c must also hold. For example, we shall see that $\{x = y\} \sqsubseteq \{x = a, y = a\}$ in the Herbrand cs below. This realizes the intuition that if $x = a \wedge y = a$ holds then $x = y$ holds.

The operator \sqcup represents join of information; $c \sqcup d$ results from joining the information from both c and d . Notice that $c \sqcup d$ is the least constraint that allows us to derive *both* c and d , i.e., $c \sqcup d \sqsupseteq c$ and $c \sqcup d \sqsupseteq d$. The join is typically interpreted as the conjunction of information and in the concrete case of Herbrand cs below is obtained by set union. For instance, in the Herbrand cs below for $e = \{x = a\} \sqcup \{y = b\} = \{x = a, y = b\}$, we have $e \sqsupseteq \{x = a\}$ and $e \sqsupseteq \{x = b\}$. This realizes the intuition that if $x = a \wedge y = b$ holds the both $x = a$ and $y = b$ also hold.

The top element of the cs represents the join of all, possibly inconsistent, information, hence it is traditionally referred to as *false*. For example $\{x = a\} \sqcup \{x = b\} = \text{false}$ where a and b are different constants. This realizes the intuition that assertion $x = a \wedge x = b$ is inconsistent. The bottom element *true* represents *empty* or *null* information, i.e., from *true* you cannot derive anything else, $\text{true} \not\sqsupseteq e$ for any $e \neq \text{true}$. In the particular case of the Herbrand cs below, *true* is the empty set.

The following examples recall two standard concrete cs. The former captures syntactic equality between terms based on a first-order alphabet and, the later shows that boolean assignments can be made into a cs and logic propositions can be interpreted on it.

The next example is new, to address N2.

Example 1 (Herbrand Constraint System [5]). The Herbrand cs captures *syntactic* equality between terms t, t', \dots built from a first-order alphabet \mathcal{L} with variables x, y, \dots , function symbols, and equality $=$. The constraints are (equivalence classes of) sets of equalities over the terms of \mathcal{L} : E.g., $\{x = t, y = t\}$ is a constraint. The relation $c \sqsubseteq d$ holds if the equalities in c follow from those in d : E.g., $\{x = y\} \sqsubseteq \{x = t, y = t\}$. The constraint *false* is the set of all term equalities in \mathcal{L} and *true* is (the equivalence class of) the empty set. The compact elements are the (equivalence class of) finite sets of equalities. The join is the (equivalence class of) set union. Fig. 1 depicts the Herbrand cs for variables x and y and, terms (constants) a and b . \square

The next paragraph is associated to the above example. We added this also to clarify the order in cs.

In the above example constraints are represented as set of equations and thus the join of constraints corresponds to the equivalence class of *union* of their equations. We can also view a constraint c as a representation of a set of variable assignments [17]. For instance, a constraint $x > 42$ can be thought of as the set of assignments mapping x to a value greater than 42; i.e., the solutions to (or models of) $x > 42$. In this case the join of constraints naturally corresponds to the *intersection* of their assignments, *false* as the empty set of assignments, and *true* as the set of all assignments. For example, the interpretation of the constraint $x > 42 \wedge x < 50$ corresponds to the intersection, hence the join, of the interpretation of the constraints $x > 42$ and $x < 50$. Below we illustrate this in more detail for the case of boolean assignments.



Figure 1: Herbrand cs for variables x and y and, constants a and b .

Example 2 (Boolean Constraint System [18]). Let Φ be a set of *primitive propositions*. A boolean (or truth) assignment π over Φ is a total map from Φ to the set $\{0, 1\}$. We use $\mathcal{A}(\Phi)$ to denote the set of all such boolean assignments. We can now define the boolean cs $\mathbf{B}(\Phi)$ as $(\mathcal{P}(\mathcal{A}(\Phi)), \supseteq)$: The powerset of assignments ordered by \supseteq . Thus constraints in $\mathbf{B}(\Phi)$ are sets of assignments, \sqsubseteq is \supseteq , *false* is \emptyset , *true* is $\mathcal{A}(\Phi)$, the join operator \sqcup is \cap , and the meet operator \sqcap is \cup . A constraint c in $\mathbf{B}(\Phi)$ is compact iff $\mathcal{A}(\Phi) \setminus c$ is a finite set.

The next example is new, this addresses N2.

Notice that logic propositions can be straightforwardly interpreted as constraints in $\mathbf{B}(\Phi)$. Let $\mathcal{L}_0(\Phi)$ be the propositional language built from Φ by the grammar

$$\phi, \psi, \dots := p \mid \phi \wedge \psi \mid \neg\phi \quad (1)$$

where $p \in \Phi$. As usual $\mathbf{F} \equiv p \wedge \neg p$ for some $p \in \Phi$, $\mathbf{T} \equiv \neg\mathbf{F}$, $\phi \vee \psi \equiv \neg(\neg\phi \wedge \neg\psi)$ and, $\psi \Rightarrow \phi \equiv \neg\phi \vee \psi$. A boolean assignment π *satisfies* ϕ iff $\pi \models \phi$ where \models is defined inductively as follows: $\pi \models p$ iff $\pi(p) = 1$, $\pi \models \phi \wedge \psi$ iff $\pi \models \phi$ and $\pi \models \psi$, and $\pi \models \neg\phi$ iff $\pi \not\models \phi$. We interpret each formula ϕ as the constraint $\mathbf{B}[\![\phi]\!] \stackrel{\text{def}}{=} \{\pi \in \mathcal{A}(\Phi) \mid \pi \models \phi\}$ in $\mathbf{B}(\Phi)$. Under this interpretation, we clearly have the following equalities:

$$\begin{aligned} \mathbf{B}[\![\mathbf{F}]\!] &= \text{false} = \emptyset \\ \mathbf{B}[\![\mathbf{T}]\!] &= \text{true} = \mathcal{A}(\Phi) \\ \mathbf{B}[\![p]\!] &= \{\pi \in \mathcal{A}(\Phi) \mid \pi(p) = 1\} \\ \mathbf{B}[\![\phi \wedge \psi]\!] &= \mathbf{B}[\![\phi]\!] \sqcup \mathbf{B}[\![\psi]\!] = \mathbf{B}[\![\phi]\!] \cap \mathbf{B}[\![\psi]\!] \\ \mathbf{B}[\![\neg\phi]\!] &= \mathcal{A}(\Phi) \setminus \mathbf{B}[\![\phi]\!]. \end{aligned}$$

Also notice that $\mathbf{B}[\![\phi]\!] \sqsubseteq \mathbf{B}[\![\psi]\!]$ holds iff $\psi \Rightarrow \phi$ is *valid*, i.e., satisfied by every truth assignment. \square

Other typical examples include constraint system for streams (the Kahn cs), rational intervals, and first-order theories [4].

3.2. Distributive Constraint Systems

Distributivity is ubiquitous in order theory and it plays a fundamental role in the results of this paper. We consider three forms of distribution.

Definition 4 (Distributive cs). A cs $(\mathbf{C}, \sqsubseteq)$ is said to be *distributive (completely distributive)* iff it is a distributive (completely distributive) lattice. It is said to be a *constraint frame* iff its finite joins distribute over arbitrary meets. More precisely, $c \sqcup \prod S = \prod \{c \sqcup e \mid e \in S\}$ for every $c \in \mathbf{C}$ and $S \subseteq \mathbf{C}$.

Clearly every completely distributive cs is a constraint frame and every constraint frame is also distributive cs. For finite constraint systems all the three notions of distributivity are equivalent.

Constraint frames allow us to include the subtraction operator from co-Heyting Algebras, the dual of Heyting algebras [19]. The *subtraction operator* $d \ominus c$ in our setting corresponds to the *weakest constraint one needs to join c with to derive d* .

The following definition and proposition have changed to address N26, N27.

Definition 5 (Subtraction [20]). Let $(\mathbf{C}, \sqsubseteq)$ be a cs. Define $d \ominus c$ as $\prod \{e \in \mathbf{C} \mid c \sqcup e \sqsupseteq d\}$.

The following properties of subtraction correspond to standard logical properties if we interpret \sqcup as conjunction, \sqsupseteq as entailment, and $d \ominus c$ as the logical implication of d by c .

Proposition 2 ([18]). Let $(\mathbf{C}, \sqsubseteq)$ be a constraint frame. Then for every $c, d \in \mathbf{C}$ the following properties hold: (1) $c \sqcup (d \ominus c) = c \sqcup d$, (2) $d \sqsupseteq (d \ominus c)$, (3) $d \ominus c = \text{true}$ iff $c \sqsupseteq d$.

Notice that under the above logical interpretation, Prop. 2 (1) corresponds to *Modus Ponens*. Let us now illustrate a simple application of this proposition.

Example 3. Constraint systems of the form $(\mathcal{P}(U), \supseteq)$ as $\mathbf{B}(\Phi)$ in Ex. 2 are completely distributive lattices, hence constraint frames. It is easy to see that $\mathbf{B}[\phi \Rightarrow \psi] = \mathbf{B}[\psi] \ominus \mathbf{B}[\phi]$. From Ex. 2, we have the following equality $\mathbf{B}[\phi \wedge (\phi \Rightarrow \psi)] = \mathbf{B}[\phi] \sqcup \mathbf{B}[\phi \Rightarrow \psi]$. Thus using Prop. 2 (1) we obtain $\mathbf{B}[\phi \wedge (\phi \Rightarrow \psi)] = \mathbf{B}[\phi \wedge \psi]$ as expected. \square

We added the next example.

The powerset ordered by inclusion is a stereotypical example of completely distributive lattice.

Example 4 (Powerset Constraint System). The power set of any set S ordered by inclusion $(\mathcal{P}(S), \subseteq)$ is a cs. In fact it is the stereotypical example of *completely distributive cs*. In this case $\sqsubseteq = \subseteq$, $\text{true} = \emptyset$, $\text{false} = S$, for every $A, B \subseteq S$, $A \sqcup B = A \cup B$, $A \sqcap B = A \cap B$, $B \ominus A = B \setminus A$. \square

The next example has changed according to Prop. 2 and comment N27.

We conclude this section by discussing the distributivity of the cs of our previous examples.

Remark 3. Notice that the cs in Ex. 2 and Ex. 4 are distributive as they are powersets ordered by (reversed) inclusion, whereas the Herbrand cs in Ex. 1 is not. To see this consider the constraints $c = \{x = a\}$, $d = \{x = a, y = a\}$ and $e = \{x = b\}$ in Fig. 1. We have

We added the next remark.

$$c \sqcup (d \sqcap e) = c \sqcup \text{true} = c \neq (c \sqcup d) \sqcap (c \sqcup e) = d \sqcap \text{false} = d.$$

\square

3.3. Spatial Constraint Systems

The authors of [6] extended the notion of cs to account for distributed and multi-agent scenarios with a finite number of agents, each having their own space for local information and their computations. The extended structures are called spatial constraint systems (scs). In this section we adapt scs to reason about possibly infinite groups of agents. First we recall the intuition about scs given in [6].

Remark 4. In Remark 2 we pointed out that (distributive) cs are the dual of boolean algebras, the algebraic generalization of propositional logic. The extension of cs to scs from [6] is much like the extension of boolean algebras to (poly)modal algebras [21], the algebraic generalization of the K_n propositional modal logic. A modal algebra is a boolean algebra equipped with unary operators, called modalities, that preserve meets and top. Dually, a scs is a cs equipped with functions, called *space functions*, that preserve joins and bottom.

This remark is new associated to N9, N14, N15, N23.

Roughly speaking, the preservation of joins by a space function f , i.e., $f(c \sqcup d) = f(c) \sqcup f(d)$, corresponds to the Distribution Axiom **K**, of modal logic which states that box modality \Box distributes over conjunction. The preservation of bottom, i.e., $f(\text{true}) = \text{true}$, corresponds to Generalization Axiom **N**, that states that if a formula ϕ is valid, hence equivalent to true, then the formula $\Box\phi$ is also valid.

Furthermore, if these space functions are also *closure operators* (i.e., idempotent and extensive space functions), the corresponding scs is called *epistemic scs* [6] and they correspond to the $S4$ modal logic of knowledge and belief. In fact epistemic scs are dual to *closure algebras* [22] which are modal algebras whose modalities are closure operators. Closure algebras are the generalization of the $S4$ modal logic for knowledge. The extensiveness of a space function f , i.e., $f(c) \sqsupseteq c$, corresponds to the Truth Axiom **T** of $S4$, $\Box\phi \rightarrow \phi$, stating that if ϕ is known then it must be true. Idempotence of a space function f , $f(c) = f(f(c))$, corresponds to the Axiom **4** of $S4$, $\Box\phi \rightarrow \Box\Box\phi$, stating that it is known what it is known (the implication $\Box\Box\phi \rightarrow \Box\phi$ follows from Axiom **T**).

Modal logics have been widely used to reason about space, knowledge, and belief [23, 24, 25, 26]. They typically arise as restrictions over the Kripke models of the above-mentioned modal logic K_n . For this reason we will focus on scs in general rather than in the more restrictive epistemic scs. This will allow us to interpret scs as structures for space, belief or knowledge.

Below we shall often use the term *epistemic* to refer to both belief or knowledge. When necessary we will use the more specific term *doxastic* to refer to belief. \square

A scs is defined over a set (or group) of agents G . Each agent $i \in G$ has a *space function* $\mathfrak{s}_i : \mathbf{C} \rightarrow \mathbf{C}$ that preserve joins and bottom. Recall that constraints can be viewed as assertions. Thus given $c \in \mathbf{C}$, we can then think of the constraint $\mathfrak{s}_i(c)$ as an assertion stating that c is a piece of information residing *within the space of agent i* . Some alternative *epistemic* interpretations of $\mathfrak{s}_i(c)$ is that it is an assertion stating that agent i *believes* c , that c holds within the space of agent i , or that agent i *interprets* c as the constraint $\mathfrak{s}_i(c)$. All these interpretations convey the idea that c is local or subjective to agent i .

We added the next paragraph to address N1, N14, N15.

The reader familiar with modal logic may notice that the above intuition about representation of information in scs is fundamentally different from the representation of information in epistemic/doxastic logic, or in modal logic in general. In epistemic logic, the statement $w \models K_i(\phi)$ means “at world w , agent i knows fact ϕ ,” or “at world w , agent i possesses information ϕ .” Here w is a world (or state), part of the model, describing a potential situation or state of affairs, and ϕ and $K_i(\phi)$ are formulas, representing pieces of information. In scs, on the other hand, the statement $\mathfrak{s}_i(d) \sqsubseteq c$ means that when c is actually true, agent i knows/believes d , or possesses information d . In this case, unlike in epistemic logic, one constraint (c) represents a (possibly partial) description of the actual state of the world, and another constraint (d) represents a description of the information that an agent possesses, and a third constraint, $\mathfrak{s}_i(d)$, represents the fact that agent i possesses information d .

We now introduce the notion of space function.

Definition 6 (Space Functions). A *space function* over a constraint system $(\mathbf{C}, \sqsubseteq)$ is a *continuous* self-map $f : \mathbf{C} \rightarrow \mathbf{C}$ such that for every $c, d \in \mathbf{C}$:

- (S.1) $f(\text{true}) = \text{true}$, and
- (S.2) $f(c \sqcup d) = f(c) \sqcup f(d)$.

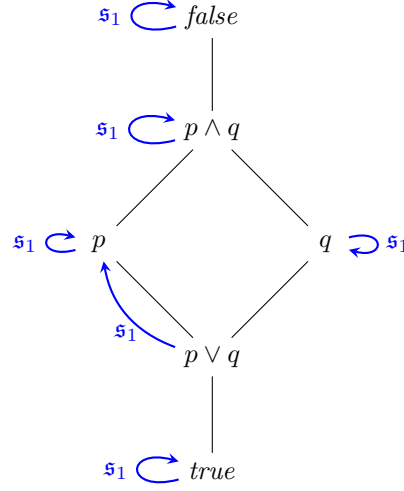


Figure 2: Cs ordered by logical implication and space function \mathfrak{s}_1 .

We shall use $\mathcal{S}(\mathbf{C})$ to denote the *set of all space functions* over \mathbf{C} .

In Remark 4 we pointed out that S.1 and S.2 correspond to Axioms **N** and **K** of modal logic. From a *spatial point of view*, the assertion $f(c)$ can be interpreted as saying that c is in the space represented by f . Property S.1 states that having an empty local space amounts to nothing. Property S.2 allows us to join and distribute the information in the space represented by f .

We added the following clarification

A *spatial constraint system* is a constraint system with a possibly infinite group of agents each one having a space function. We specify such a group as a tuple of space functions.

Definition 7 (Spatial Constraint Systems). A *spatial constraint system (scs)* is a constraint system $(\mathbf{C}, \sqsubseteq)$ equipped with a possibly infinite tuple $\mathfrak{s} = (\mathfrak{s}_i)_{i \in G}$ of space functions in $\mathcal{S}(\mathbf{C})$.

We shall use $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ to denote an scs with a tuple $(\mathfrak{s}_i)_{i \in G}$. We refer to G and \mathfrak{s} as the *group of agents* and *space tuple* of \mathbf{C} and to each \mathfrak{s}_i as the *space function* in \mathbf{C} of agent i . Subsets of G are also referred to as groups of agents (or sub-groups of G).

We added the next paragraph to clarify the interpretation of space functions.

As mentioned before we can think of scs as the dual of polymodal boolean algebras [21] without the distributivity requirement, where each space function corresponds to a modal operator.

In Remark 4 we pointed out that *epistemic scs* are those scs whose space functions are idempotent and extensive. They were used in [6] to reason about knowledge. Here is an example of such a scs.

Example 5 (Knowledge). Consider the cs in Fig. 2 and the space function \mathfrak{s}_1 defined on it. It is easy to see that \mathfrak{s}_1 is an idempotent and extensive space function. If we want to determine what agent 1 knows at q , we have no problem because $q = \mathfrak{s}_1(q)$, meaning that agent 1 knows q . On the other hand, for $p \vee q$, it is more difficult to determine what agent 1 knows because there is no constraint c such that $\mathfrak{s}_1(c) = p \vee q$. We solve this problem by considering the constraints *below* $p \vee q$. Since $c \sqsubseteq p \vee q$ means that $p \vee q$ logically implies c , if we can find a constraint c where $c \sqsubseteq p \vee q$ and $c = \mathfrak{s}_1(c')$ for some constraint c' , we know that $p \vee q$ logically implies $\mathfrak{s}_1(c')$, so at $p \vee q$, agent 1 has information c' . Finally, by taking the join (logical conjunction) of all such c' , we take into account all of the information that agent 1 possesses at $p \vee q$. \square

We added the next example and Fig. 2.

The idempotence or extensiveness of space functions from epistemic scs allows for knowledge interpretations but it may be too restrictive. For example, notice that extensiveness implies that $\mathfrak{s}_i(\text{false}) = \text{false}$ and $\mathfrak{s}_i(c) \sqcup \mathfrak{s}_i(d) = \text{false}$ if $c \sqcup d = \text{false}$. This would rule out interpretations we wish to allow for general scs as those given below.

Doxastic and Spatial Interpretations. We could have a scs where $\mathfrak{s}_i(\text{false}) \neq \text{false}$ for some agent i . This is referred to as *inconsistency confinement* and, intuitively, can be seen as if inconsistencies generated within an agent's space are confined to its own space. We also have the possibility that given two different agents i and j , $\mathfrak{s}_i(x > 42) \sqcup \mathfrak{s}_j(x < 42) \neq \text{false}$, despite $x > 42 \sqcup x < 42 = \text{false}$. The spatial interpretation is that $x > 42$ is in the space of i while $x < 42$ is in the space of j , so there is no conflict or contradiction. The doxastic interpretation is that agent i believes $x > 42$ while agent j believes that $x < 42$. We refer to this as *freedom of opinion*. Agents could have different information or perception about the same subject. Another aspect is *information blindness*. It is possible that an agent cannot distinguish between two different constraints, e.g., a color blind agent i that cannot distinguish red from green, i.e., $\mathfrak{s}_i(\text{red}) = \mathfrak{s}_i(\text{green})$ as shown in the next example.

Example 6 (Color Perception). Table 1 describes color perception for four agents. The underlying cs is based on four colors ordered by color brightness. We define space functions $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ and \mathfrak{s}_4 as shown in the table. In these situations, agent 1 perceives colors accurately, but agent 2 is colorblind and cannot tell any difference between yellow, red and green, i.e., $\mathfrak{s}_2(\text{yellow}) = \mathfrak{s}_2(\text{red}) = \mathfrak{s}_2(\text{green})$. Similarly, agent 3 cannot distinguish between red and yellow, i.e., $\mathfrak{s}_3(\text{red}) = \mathfrak{s}_3(\text{yellow})$. Finally, agent 4 is totally blind, he perceives all the colors as black. \square

The next example is new associated to N1, N3. We also added Table 1.

Our next example illustrate a simple scs that will be used throughout the paper.

Example 7 (Proposition Perception). The scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in \{1,2\}})$ in Fig. 3 is given by the four-element boolean algebra (isomorphic to the complete lattice \mathbf{M}_2) and two agents. We have $\mathbf{C} = \{p \vee \neg p, p, \neg p, p \wedge \neg p\}$ and $c \sqsubseteq d$ holds if c is a logical consequence of d . The top element (*false*) is $p \wedge \neg p$, the bottom element (*true*) is $p \vee \neg p$, and the constraints p and $\neg p$ are incomparable with each other. The set of agents is $\{1, 2\}$ with space functions \mathfrak{s}_1 and \mathfrak{s}_2 depicted in Fig. 3. The intuition is that the agent 2 sees no difference between p and *false* while agent 1 interprets $\neg p$ as p and vice versa. \square

We conclude this section with the Kripke scs from [18] which can be used to interpret modal, doxastic and epistemic logics. Other examples of scs for epistemic reasoning are *Aumann structures* and they will be illustrated in Sec. 4.4.

The next subsection is new. We present the Kripke scs.

3.4. Kripke Spatial Constraint Systems

We now extend Ex. 2 by moving from the set boolean assignments to the set of (pointed) *Kripke structures*.

Definition 8 (Kripke Structures [24]). An n -agent Kripke structure (model) (KS) M over a set of atomic propositions Φ is a tuple $M = (S, \pi, \mathcal{R}_1, \dots, \mathcal{R}_n)$ where

- S is a nonempty set of states,
- $\pi : S \rightarrow (\Phi \rightarrow \{0, 1\})$ is an interpretation that associates with each state a truth assignment to the primitive propositions in Φ , and
- \mathcal{R}_i is a binary relation on S .

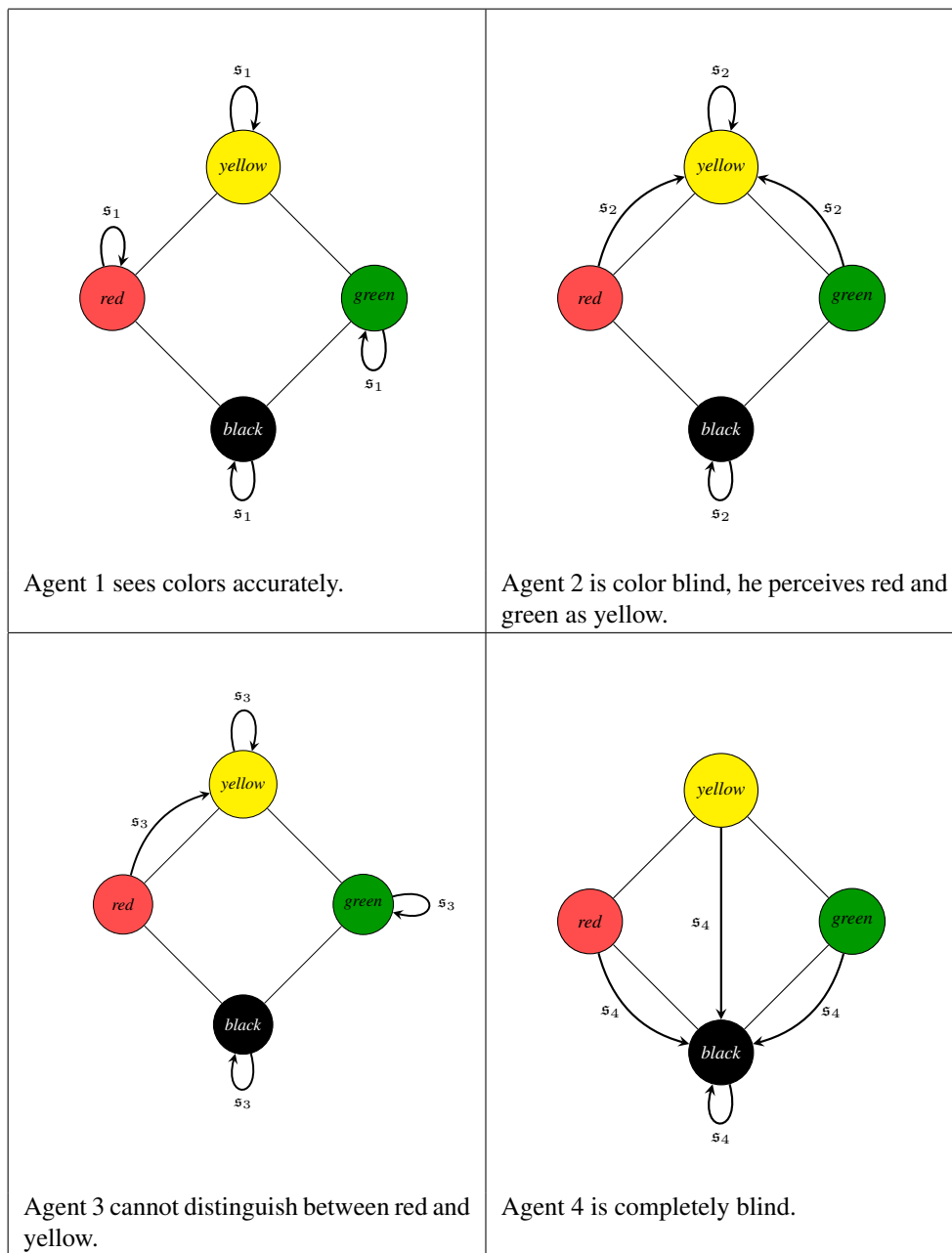


Table 1: Cs (isomorphic to the four-element boolean algebra) of colors ordered by color brightness. Space functions s_1 , s_2 , s_3 and s_4 . Perception of color by agents 1, 2, 3 and 4.

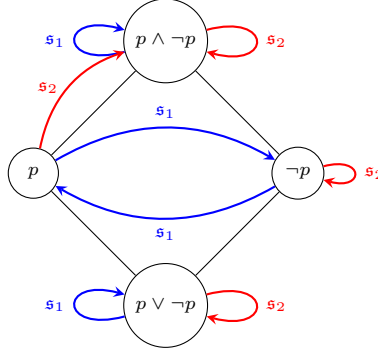


Figure 3: Cs given by the four-element boolean algebra ordered by logical implication and space functions s_1 and s_2 .

Notation 2. The states of a KS are often referred to as *worlds*. Each \mathcal{R}_i is referred to as the *accessibility* or *possibility* relation for agent i : $(s, t) \in \mathcal{R}_i$ is meant to capture that agent i considers world t possible given its information in world s . We use $s \xrightarrow{i}_M t$ to denote $(s, t) \in \mathcal{R}_i$ in the KS M . We use $\mathcal{W}_i(M, s) = \{t \mid s \xrightarrow{i}_M t\}$ to denote the worlds agent i considers possible from a state s of KS M . The interpretation function π tells us what primitive propositions are true at a given world: p holds at state s iff $\pi(s)(p) = 1$. We use π_M to denote the interpretation π of the KS M .

Recall that in Ex. 2 constraints are sets of boolean assignments. This allowed us to interpret each propositional formula as a constraint; the set of assignments that are models of (or satisfy) the formula. Similarly, in the following example (spatial) constraints are sets of (pointed) KS models. A *pointed KS* is a pair (M, s) where M is a KS and s , called the *actual world*, is a state of M . This will allow us to interpret each modal formula as its set of pointed KS models; i.e., a spatial constraint.

Definition 9 (Kripke scs [18]). Let $\mathcal{S}_n(\Phi)$ be a non-empty set of n -agent Kripke structures over Φ . Let Δ be the set of all pointed Kripke structures (M, s) such that $M \in \mathcal{S}_n(\Phi)$. We define the Kripke n -scs for $\mathcal{S}_n(\Phi)$ as the tuple

$$\mathbf{K}(\mathcal{S}_n(\Phi)) = (\mathbf{C}, \sqsubseteq, K_1, \dots, K_n)$$

where $\mathbf{C} = \mathcal{P}(\Delta)$, and for every $X, Y \in \mathbf{C}$: $X \sqsubseteq Y$ iff $Y \subseteq X$, and

$$K_i(X) \stackrel{\text{def}}{=} \{(M, s) \in \Delta \mid \forall t : s \xrightarrow{i}_M t \text{ implies } (M, t) \in X\} \quad (2)$$

for every agent $i \in \{1, \dots, n\}$.

The Kripke n -scs $\mathbf{K}(\mathcal{S}_n(\Phi))$ is a complete lattice given by a powerset ordered by \sqsupseteq . The \sqcap is set intersection, the top element *false* is \emptyset , and the bottom *true* is the set Δ of all pointed Kripke structures (M, s) with $M \in \mathcal{S}_n(\Phi)$. It is easy to verify that $K_i(\text{true}) = \text{true}$ and $K_i(c_1 \sqcap c_2) = K_i(c_1) \sqcap K_i(c_2)$. Similar to Ex. 2, a constraint c in $\mathbf{K}(\mathcal{S}_n(\Phi))$ is compact iff $\Delta \setminus c$ is a finite set [18].

Proposition 3 ([18]). Let $\mathbf{K}(\mathcal{S}_n(\Phi)) = (\mathbf{C}, \sqsubseteq, K_1, \dots, K_n)$ be as in Definition 9. Then $\mathbf{K}(\mathcal{S}_n(\Phi))$ is a scs.

A modal language. Modal formulae can be interpreted as constraints in the scs $\mathbf{K}(\mathcal{S}_n(\Phi))$.

The modal language $\mathcal{L}_n(\Phi)$ is obtained by extending the grammar for the propositional language $\mathcal{L}_0(\Phi)$ in Eq. 1 with modalities $\Box_i\phi$ in the standard way.

Definition 10 (Modal Language). Let Φ be a set of primitive propositions. The language $\mathcal{L}_n(\Phi)$ is given by the following grammar:

$$\phi, \psi, \dots := p \mid \phi \wedge \psi \mid \neg\phi \mid \Box_i\phi \quad (3)$$

where $p \in \Phi$ and $i \in \{1, \dots, n\}$.

The semantics of modal logics is typically given using KS's. We say that a pointed KS (M, s) satisfies ϕ iff $(M, s) \models \phi$ where \models is defined inductively as follows: $(M, s) \models p$ iff $\pi_M(s)(p) = 1$, $(M, s) \models \phi \wedge \psi$ iff $(M, s) \models \phi$ and $(M, s) \models \psi$, $(M, s) \models \neg\phi$ iff $(M, s) \not\models \phi$, and $(M, s) \models \Box_i\phi$ iff $(M, t) \models \phi$ for every t such that $s \xrightarrow{i}_M t$.

As in Ex. 2 we can interpret each formula ϕ as constraints in Kripke constraint systems.

Definition 11 (Kripke Constraint Interpretation). Let \mathcal{C} be a Kripke scs $\mathbf{K}(\mathcal{S}_n(\Phi))$. Given a modal formula ϕ in the language $\mathcal{L}_n(\Phi)$, its interpretation in the Kripke scs \mathcal{C} is the constraint $\mathcal{C}[\![\phi]\!]$ inductively defined as follows:

$$\begin{aligned} \mathcal{C}[\![p]\!] &= \{(M, s) \in \Delta \mid \pi_M(s)(p) = 1\} \\ \mathcal{C}[\![\phi \wedge \psi]\!] &= \mathcal{C}[\![\phi]\!] \sqcup \mathcal{C}[\![\psi]\!] \\ \mathcal{C}[\![\neg\phi]\!] &= \Delta \setminus \mathcal{C}[\![\phi]\!] \\ \mathcal{C}[\![\Box_i\phi]\!] &= K_i(\mathcal{C}[\![\phi]\!]) \end{aligned}$$

where Δ is the set of all pointed Kripke structures (M, s) such that $M \in \mathcal{S}_n(\Phi)$.

Notation 3. Notice that the interpretation of $\Box_i(\phi)$, $\mathcal{C}[\![\Box_i(\phi)]\!]$, is equal to the constraint $K_i(\mathcal{C}[\![\phi]\!])$ in $\mathbf{K}(\mathcal{S}_n(\Phi))$. Often, by abuse of notation, we shall suppress the semantic symbols $\mathcal{C}[\![\]\!]$ from formulae—e.g., we write $K_i(\phi)$ for the constraint $K_i(\mathcal{C}[\![\phi]\!])$.

Following our intended meaning of constraints, we think of $K_i(\phi)$ as stating that ϕ holds in the space of agent i , or as an epistemic assertion stating that agent i considers/believes ϕ to be true.

3.5. Continuity

In [6] space functions were not required to be continuous. Nevertheless, we will argue later, in Remark 7, that continuity comes naturally in the intended phenomena we wish to capture: modeling information of possibly *infinite* groups. In fact, in [6] scs could only have finitely many agents. In this work we also extend scs to allow arbitrary, *possibly infinite*, sets of agents. We illustrate scs with infinite groups in the next section.

Notice that the continuity and preservation of finite joins by space functions will provide us with their preservation of arbitrary joins. The following proposition gives us sufficient conditions for the existence of the join of an arbitrary set on a given poset. A sketch of its proof is presented in [27], for the sake of completeness we present our own proof of it.

Proposition 4 ([27]). Let (P, \sqsubseteq) be a poset. Suppose that $\bigsqcup F$ and $\bigsqcup D$ exist for every finite set $F \subseteq P$ and for every directed set $D \subseteq P$. Then $\bigsqcup A$ also exists for every $A \subseteq P$.

We reorganized the next two paragraphs.

Proof. Let $A \subseteq P$ be an arbitrary set and let $D = \{\sqcup F \mid F \subseteq A \text{ and } F \text{ is finite}\}$. First we prove that D is directed. For $F_1, F_2 \subseteq A$, both finite sets, the elements $\sqcup F_1, \sqcup F_2 \in D$ and the set $F_3 = F_1 \cup F_2 \subseteq A$ is finite. Then $\sqcup F_3 \in D$ and, both $\sqcup F_1 \sqsubseteq \sqcup F_3$ and $\sqcup F_2 \sqsubseteq \sqcup F_3$ hold.

To complete the proof we show that for every $c \in P$:

c is an upper bound of D if and only if c is an upper bound of A .

For the “only if” direction, we prove its contrapositive. Let $c \in P$. If c is not an upper bound of A , there is an $a \in A$ such that $a \not\sqsubseteq c$. Notice that $\{a\} \subseteq A$ and thus $\sqcup\{a\} = a \in D$ but $a \not\sqsubseteq c$. Hence c is not an upper bound of D .

For the other direction, assume that $c \in P$ is an upper bound of A . Let F be any finite subset of A . Since $e \sqsubseteq c$ for every $e \in F$, then $\sqcup F \sqsubseteq c$. Therefore, c is an upper bound of D .

We then conclude $\sqcup A = \sqcup D$ as wanted. \square

The following proposition states two useful properties of space functions: monotonicity and preservation of arbitrary joins.

Proposition 5 ([28]). *Let $f : \mathbf{C} \rightarrow \mathbf{C}$ be a function over a cs $(\mathbf{C}, \sqsubseteq)$. Then*

1. *If f is space function then f is monotonic.*
2. *f is space function if and only if it preserves arbitrary joins.*

4. Distributed Information

This section contains the main technical contributions of this paper. In particular, we will characterize the notion of collective information of a group of agents. Roughly speaking, the *distributed (or collective) information* of a group I is the join of each piece of information that resides in the space of an agent $i \in I$. For each constraint c , the distributed information of I w.r.t c is the distributive information of I that can be derived from c . We wish to formalize whether a given constraint e can be derived from the collective information of the group I w.r.t c .

The following examples, which we will use throughout this paper, illustrate the above intuition.

Example 8. Consider an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $G = \mathbb{N}$ and $(\mathbf{C}, \sqsubseteq)$ is a constraint frame. Let $c = \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \ominus a) \sqcup \mathfrak{s}_3(e \ominus b)$. The constraint c specifies the situation where $a, b \ominus a$ and $e \ominus b$ are in the spaces of agent 1, 2 and 3, respectively. Neither agent necessarily holds e in their space w.r.t c . Nevertheless, the information e can be derived from the collective information of the three agents w.r.t c , since from Prop. 2 we have $a \sqcup (b \ominus a) \sqcup (e \ominus b) \sqsupseteq e$. **Let us now consider an example with infinitely many agents.**

Infinitely Many Agents. Suppose that there exists an element $e' \in \mathbf{C}$ and an increasing chain $a_0 \sqsubseteq a_1 \sqsubseteq \dots$ such that $\sqcup_{i \in \mathbb{N}} a_i \sqsupseteq e'$ and $e' \not\sqsubseteq a_i$ for every $i \in \mathbb{N}$. Therefore e' is not *compact* (see Def. 1). Let $c' \stackrel{\text{def}}{=} \sqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i)$. Notice that for no agent $i \in \mathbb{N}$ holds (or can derive) e' in their space since $e' \not\sqsubseteq a_i$. Yet, from our assumption, e' can be derived from the collective information w.r.t c' of all the agents in \mathbb{N} , i.e., $\sqcup_{i \in \mathbb{N}} a_i \sqsupseteq e'$. \square

We changed this paragraph.

The above example may suggest that distributed information can be obtained by joining individual local information derived from c . Such information can be characterized as the i -projection of agent i w.r.t c .

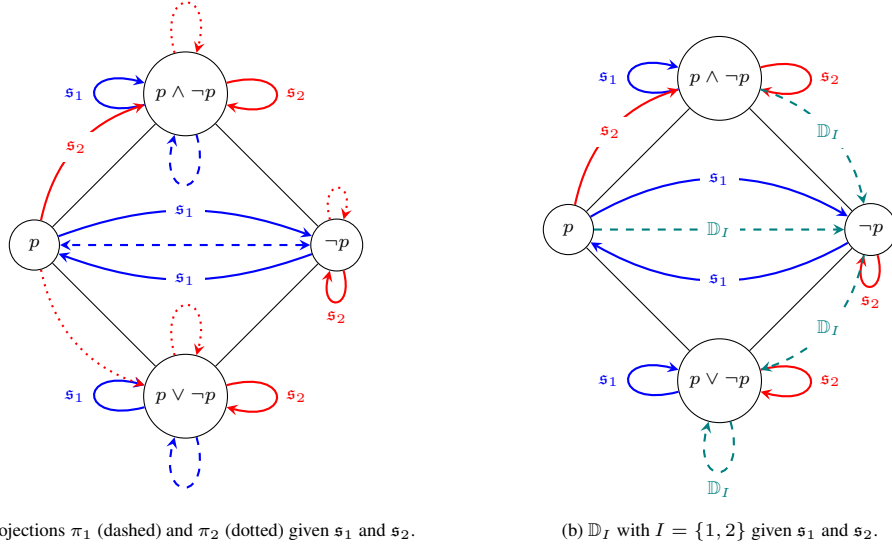


Figure 4: Projections (see Def. 12) (a) and distributed space function (see Def. 15) (b) over four-element boolean algebra.

Definition 12 (Agent and Join Projections). Let $(\mathbf{C}, \sqsubseteq, (s_i)_{i \in G})$ be an scs. Given $i \in G$, the i -agent projection of $c \in \mathbf{C}$ is defined as $\pi_i(c) \stackrel{\text{def}}{=} \bigsqcup \{e \mid c \sqsupseteq s_i(e)\}$. We say that e is i -agent derivable from c if and only if $\pi_i(c) \sqsupseteq e$. Given $I \subseteq G$ the I -join projection of a group I of c is defined as $\pi_I(c) \stackrel{\text{def}}{=} \bigsqcup \{\pi_i(c) \mid i \in I\}$. Similarly, we say that e is I -join derivable from c if and only if $\pi_I(c) \sqsupseteq e$.

The i -agent projection of $i \in G$ of c naturally represents the join of all the information that agent i has in c . The I -join projection of group I joins individual i -agent projections of c for $i \in I$. This projection can be used as a sound mechanism for reasoning about distributed-information: If e is I -join derivable from c then it follows from the distributed-information of I w.r.t c .

Example 9. Let c be as in Ex. 8. We have $\pi_1(c) \sqsupseteq a$, $\pi_2(c) \sqsupseteq (b \ominus a)$ and $\pi_3(c) \sqsupseteq (e \ominus b)$. Indeed e is I -join derivable from c since $\pi_{\{1,2,3\}}(c) = \pi_1(c) \sqcup \pi_2(c) \sqcup \pi_3(c) \sqsupseteq e$. Similarly, we conclude that e' is I -join derivable from c' in Ex. 8 since $\pi_{\mathbb{N}}(c') = \bigsqcup_{i \in \mathbb{N}} \pi_i(c) \sqsupseteq \bigsqcup_{i \in \mathbb{N}} a_i \sqsupseteq e'$. \square

Nevertheless, I -join projections do not provide a complete mechanism for reasoning about distributed information as illustrated below.

Example 10. Let $d \stackrel{\text{def}}{=} s_1(b) \sqcap s_2(b)$. Recall that we think of \sqcup and \sqcap as conjunction and disjunction of assertions: d specifies that b is present in the space of agent 1 or in the space of agent 2 though not exactly in which one. Thus from d we should be able to conclude that b belongs to the space of *some* agent in $\{1, 2\}$. Nevertheless, b is not necessarily I -join derivable from d since from $\pi_{\{1,2\}}(d) = \pi_1(d) \sqcup \pi_2(d)$ we cannot, in general, derive b . To see this consider the scs in Fig. 4a, taking $b = \neg p$. We have $\pi_{\{1,2\}}(d) = \pi_1(d) \sqcup \pi_2(d) = \text{true} \sqcup \text{true} = \text{true} \not\sqsupseteq b$. One can generalize this example to infinitely many agents.

Infinite Many Agents. Consider the scs in Ex. 8 and let $d' \stackrel{\text{def}}{=} \bigsqcap_{i \in \mathbb{N}} s_i(b')$. We should be able to conclude from d' that b' is in the space of *some* agent in \mathbb{N} but, in general, b' is not \mathbb{N} -join derivable from d' . \square

We changed this paragraph.

4.1. Distributed Spaces

We have just illustrated in Ex. 10 that the I -join projection of c , $\pi_I(c)$, the join of individual projections, may not project all distributed information of a group I . To solve this problem we develop the notion of I -group projection of c , written as $\Pi_I(c)$. We will first define a space function \mathbb{D}_I called the distributed space of group I . The function \mathbb{D}_I can be thought of as a virtual space including all the information that can be in the space of a member of I . We will then define an I -projection, Π_I , in terms of \mathbb{D}_I much like π_i is defined in terms of s_i .

Set of Space Functions

We now introduce a new partial order induced by \mathbf{C} : The set of space functions ordered point-wise. Recall that $\mathcal{S}(\mathbf{C})$ denotes the set of all space functions over a cs \mathbf{C} (Def. 6). For notational convenience, we shall use $(f_I)_{I \subseteq G}$ to denote the tuple $(f_I)_{I \in \mathcal{P}(G)}$ of elements of $\mathcal{S}(\mathbf{C})$.

Definition 13 (Function Order). Let $(\mathbf{C}, \sqsubseteq)$ be a cs. Given $f, g : \mathbf{C} \rightarrow \mathbf{C}$ define $f \sqsubseteq_s g$ iff $f(c) \sqsubseteq g(c)$ for every $c \in \mathbf{C}$.

An important design aspect of our structure is that the set of space functions $\mathcal{S}(\mathbf{C})$ can be made into a complete lattice.

Lemma 1 ([29]). *Let $(\mathbf{C}, \sqsubseteq)$ be a cs. Then $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$ is a complete lattice.*

In the next section we use the properties of $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$ to formalize the distributed space of a group I as the greatest space function below every space function s_i with $i \in I$.

4.2. Distributed Spaces as Max Spaces

We can now give the definition of distributed spaces. It is convenient to give the following intuition first.

Remark 5. Suppose that f and g are space functions in $\mathcal{S}(\mathbf{C})$. Intuitively, $f(c)$ means c is within the space represented by f . By definition, $f \sqsubseteq_s g$ means $f(c) \sqsubseteq g(c)$ for every $c \in \mathbf{C}$. Intuitively, from the assertion that c is in the space represented by g we can derive that c is also in the space represented by f . This can be interpreted as saying that the space represented by g is included in the space represented by f ; in other words *the bigger the space, the smaller the function that represents it*. Thus every c in g is also in f , hence f is a bigger space. \square

We changed the next remark according to comment N38.

Following the above intuition, the order relation \sqsubseteq_s of $\mathcal{S}(\mathbf{C})$ represents (reverse) space inclusion and the join and meet operations in $\mathcal{S}(\mathbf{C})$ represent intersection and union of spaces. The biggest and the smallest spaces are represented by the bottom and the top elements of the lattice $\mathcal{S}(\mathbf{C})$, here called λ_{\perp} and λ_{\top} , respectively, and defined as follows.

Definition 14 (Top and Bottom Spaces). Let $\mathcal{S}(\mathbf{C})$ be the lattice of space functions. Define λ_{\perp} and λ_{\top} in $\mathcal{S}(\mathbf{C})$ as follows: $\lambda_{\perp}(c) \stackrel{\text{def}}{=} \text{true}$ for every $c \in \mathbf{C}$; and $\lambda_{\top}(c) \stackrel{\text{def}}{=} \text{true}$ if $c = \text{true}$ and $\lambda_{\top}(c) \stackrel{\text{def}}{=} \text{false}$ if $c \neq \text{true}$.

The distributed space \mathbb{D}_I of a group I can be viewed as the function that represents the smallest space that includes all the local information of the agents in I . From Remark 5, \mathbb{D}_I should be the *greatest space function* below the space functions of the agents in I . The existence of such a function follows from completeness of $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$ stated in Lemma 1.

Definition 15 (Distributed Space). Let $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs. The *distributed spaces* of \mathbf{C} are given by $\mathbb{D} = (\mathbb{D}_I)_{I \subseteq G}$ where

$$\mathbb{D}_I \stackrel{\text{def}}{=} \max \{ f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_i \text{ for every } i \in I \}.$$

We shall say that e is *distributed among* $I \subseteq G$ w.r.t c if and only if $c \sqsupseteq \mathbb{D}_I(e)$. We shall refer to each \mathbb{D}_I as the (*distributed*) *space* of the group I .

Remark 6. From Lemma 1, $\mathbb{D}_I = \bigsqcup_{\mathcal{S}(\mathbf{C})} \{ f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_i \text{ for each } i \in I \} = \prod_{\mathcal{S}(\mathbf{C})} \{ \mathfrak{s}_i \mid i \in I \}$ where $\bigsqcup_{\mathcal{S}(\mathbf{C})}$ and $\prod_{\mathcal{S}(\mathbf{C})}$ are the join and meet in the complete lattice $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$. \square

Let us consider a concrete example.

Example 11. Fig. 4b illustrates an scs with space functions \mathfrak{s}_1 and \mathfrak{s}_2 , and their distributed space $\mathbb{D}_{\{1,2\}}$. The reader can verify that $\mathbb{D}_{\{1,2\}}$ is indeed the greatest function such that $\mathbb{D}_{\{1,2\}} \sqsubseteq_s \mathfrak{s}_1$ and $\mathbb{D}_{\{1,2\}} \sqsubseteq_s \mathfrak{s}_2$. Notice that $\mathfrak{s}_1(p) \sqcup \mathfrak{s}_2(\neg p) \sqsupseteq \mathbb{D}_{\{1,2\}}(p \sqcup \neg p) = \mathbb{D}_{\{1,2\}}(\text{false})$ meaning that if agents 1 and 2 had p and $\neg p$ in their corresponding spaces, as a group they could derive an inconsistency. \square

In [6], shared information of a group I specifies the fact that a given constraint e resides within the space of every agent in a group I . In contrast, distributed information specifies that e is distributed among the members of I . The next example illustrates this difference.

We added the next paragraph and example to address N41.

Example 12. Let $I = \{1, \dots, n\}$ and let $\mathfrak{s}_I(e) \stackrel{\text{def}}{=} \bigsqcup_{i \in I} \mathfrak{s}_i(e)$ as in [6]. Here $\mathfrak{s}_I(e)$ specifies that e is shared by the agents in I : e is present in the space of i for every $i \in I$. Instead, $\mathbb{D}_I(e)$ means that the information e is distributed among the spaces of the members in I . To see the difference, let $I = \{1, 2\}$ and consider the scs in Fig. 4b. Let $c = \mathfrak{s}_1(p) \sqcup \mathfrak{s}_2(\neg p)$. Notice that $p \wedge \neg p$ is distributed in I w.r.t c : $c \sqsupseteq \mathbb{D}_I(p \wedge \neg p)$. However, $p \wedge \neg p$ is not shared information of I w.r.t c , namely, $c \not\sqsupseteq \mathfrak{s}_I(p \wedge \neg p)$. \square

4.3. Compositionality of Distributed Spaces

Distributed spaces have pleasant compositional properties. They capture the intuition that the *distributed information* of a group I can be obtained from the distributive information of its subgroups.

Proposition 6. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $K, J \subseteq I \subseteq G$.

1. $\mathbb{D}_I = \lambda_{\top}$ if $I = \emptyset$.
2. $\mathbb{D}_I = \mathfrak{s}_i$ if $I = \{i\}$.
3. $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \sqsupseteq \mathbb{D}_I(a \sqcup b)$.
4. $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \sqsupseteq \mathbb{D}_I(c)$ if $(\mathbf{C}, \sqsubseteq)$ is a constraint frame.

Proof. 1. It follows directly from Def. 14 and Def. 15.

2. Let $I = \{i\}$, from Def. 15, $\mathbb{D}_I = \max \{ f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_i \} = \mathfrak{s}_i$.
3. Assume $K, J \subseteq I$. From Def. 15 we conclude $\mathbb{D}_I \sqsubseteq_s \mathbb{D}_J$ and $\mathbb{D}_I \sqsubseteq_s \mathbb{D}_K$. Thus $\mathbb{D}_J(a) \sqsupseteq \mathbb{D}_I(a)$, $\mathbb{D}_K(b) \sqsupseteq \mathbb{D}_I(b)$ and therefore $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \sqsupseteq \mathbb{D}_I(a) \sqcup \mathbb{D}_I(b)$. Since \mathbb{D}_I is a space function, $\mathbb{D}_I(a) \sqcup \mathbb{D}_I(b) = \mathbb{D}_I(a \sqcup b)$, then we obtain $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \sqsupseteq \mathbb{D}_I(a \sqcup b)$ as wanted.
4. It follows from part (3) with $a = a$ and $b = c \ominus a$, and Prop. 2.

\square

Recall that λ_{\top} corresponds to the empty space (see Def. 14). The first property realizes the intuition that the empty subgroup \emptyset *does not* have any information whatsoever distributed w.r.t a consistent c : for if $c \sqsupseteq \mathbb{D}_{\emptyset}(e)$ and $c \neq \text{false}$ then $e = \text{true}$. Intuitively, the second property says that the function \mathbb{D}_I for the group of one agent must be the agent's space function. The third property states that a group can join the information of its subgroups. The last property uses subtraction, hence the constraint frame condition, to express that by joining the information a and $c \ominus a$ of their subgroups, the group I can obtain c .

Let us illustrate how to derive information of a group from smaller ones using Prop. 6.

Example 13. Let $c = \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \ominus a) \sqcup \mathfrak{s}_3(e \ominus b)$ as in Ex. 8. We want to prove that e is distributed among $I = \{1, 2, 3\}$ w.r.t c , i.e., $c \sqsupseteq \mathbb{D}_{\{1,2,3\}}(e)$. Using Properties (2) and (4) in Prop. 6 we obtain $c \sqsupseteq \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \ominus a) = \mathbb{D}_{\{1\}}(a) \sqcup \mathbb{D}_{\{2\}}(b \ominus a) \sqsupseteq \mathbb{D}_{\{1,2\}}(b)$, and then $c \sqsupseteq \mathbb{D}_{\{1,2\}}(b) \sqcup \mathfrak{s}_3(e \ominus b) = \mathbb{D}_{\{1,2\}}(b) \sqcup \mathbb{D}_{\{3\}}(e \ominus b) \sqsupseteq \mathbb{D}_{\{1,2,3\}}(e)$ as wanted.

Remark 7 (Continuity and Infinitely Many Agents). The example with infinitely many agents in Ex. 8 illustrates well why we require our spaces to be continuous in the presence of possibly infinite groups. Clearly $c' = \bigsqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i) \sqsupseteq \bigsqcup_{i \in \mathbb{N}} \mathbb{D}_{\mathbb{N}}(a_i)$. By continuity, $\bigsqcup_{i \in \mathbb{N}} \mathbb{D}_{\mathbb{N}}(a_i) = \mathbb{D}_{\mathbb{N}}(\bigsqcup_{i \in \mathbb{N}} a_i)$ which indeed captures the idea that each a_i is in the distributed space $\mathbb{D}_{\mathbb{N}}$. \square

4.4. Distributed Spaces in Aumann and Kripke Structures

We now consider an important structure from mathematical economics used for group epistemic reasoning: Aumann structures [3]. We illustrate that the notion of distributed knowledge in these structures is an instance of a distributed space.

Example 14. *Aumann Constraint Systems* [6]. Aumann structures are an *event-based* approach to modelling knowledge. An Aumann structure is a tuple $\mathcal{A} = (S, \mathcal{P}_1, \dots, \mathcal{P}_n)$ where S is a set of states and each \mathcal{P}_i is a partition on S for agent i [3]. The sets of each partition \mathcal{P}_i are called *information sets*. If two states t and u are in the same information set for agent i , it means that in state t agent i considers state u possible, and vice versa. An *event* in an Aumann structure is any subset of S . Event e holds at state t if $t \in e$. The set $\mathcal{P}_i(s)$ denotes the information set of \mathcal{P}_i containing s . The event of *agent i knowing e* is defined as

$$K_i(e) = \{s \in S \mid \mathcal{P}_i(s) \subseteq e\},$$

and the *distributed knowledge of an event e among the agents in a group $I \subseteq \{1, \dots, n\}$* is defined as

$$D_I(e) = \left\{ s \in S \mid \bigcap_{i \in I} \mathcal{P}_i(s) \subseteq e \right\}.$$

An Aumann structure induces the spatial constraint system $\mathbf{C}(\mathcal{A})$ with events as constraints, i.e., $\mathbf{C} = \{e \mid e \text{ is an event in } \mathcal{A}\}$, and for every $e_1, e_2 \in \mathbf{C}$, $e_1 \sqsubseteq e_2$ iff $e_2 \subseteq e_1$. The operators join (\sqcup) and meet (\sqcap) are intersection (\cap) and union (\cup) of events, respectively; *true* = S and *false* = \emptyset^6 . The space functions are the knowledge operators, i.e., $\mathfrak{s}_i(c) = K_i(c)$.

The next proposition states that in fact distributed knowledge and distributed information coincide.

Proposition 7 (Distributed Spaces in Aumann Structures). *Let $\mathcal{A} = (S, \mathcal{P}_1, \dots, \mathcal{P}_n)$ be an Aumann structure and let $\mathbf{C}(\mathcal{A})$ be its induced scs. Then $\mathbb{D}_I = D_I$ for every $I \subseteq \{1, \dots, n\}$.*

⁶Notice that in this cs we use the reverse inclusion which is the dual of the powerset example in Ex. 4. Then, the join and meet operators and, the top and bottom elements, are swapped.

We added a footnote in this example to include a comment in N43

We added the next proposition to address comment N43. IMPORTANT: Its proof uses one of the main results of this paper –Th. 4.

Therefore distributed knowledge in Aumann structures is indeed an instance of distributed information. The proof uses one of the main results of this paper (Th. 4). For the sake of the presentation, we postpone it to Sec. 6.1. \square

The next example shows that distributed knowledge in epistemic logic is also an instance of distributed information.

We added the next example.

Example 15 (Distributed Spaces in Kripke Structures). Distributed knowledge in logic is expressed with the formula $D_I(\phi)$ whose intended meaning is that the knowledge ϕ is distributed among the members of I . Its semantics is expressed in terms of its Kripke models; i.e., the models of $D_I(\phi)$ are given by $\mathcal{D}_I(X)$ defined next where X are the models of ϕ [24].

Let $\mathbf{K}(\mathcal{S}_n(\Phi)) = (\mathbf{C}, \sqsubseteq, K_1, \dots, K_n)$ be a Kripke n -scs as in Def. 9. The function $\mathcal{D} : \mathbf{C} \rightarrow \mathbf{C}$ is defined as

$$\mathcal{D}_I(X) \stackrel{\text{def}}{=} \left\{ (M, s) \in \Delta \mid \forall t : (s, t) \in \bigcap_{i \in I} \mathcal{R}_i \text{ implies } (M, t) \in X \right\}.$$

The following proposition states that $\mathbb{D}_I = \mathcal{D}_I$. As for Prop. 7, we use Th. 4 to prove it. The proof is given in Sec. 6.2.

Proposition 8 (Distributed Spaces in Kripke Structures). *Let $\mathbf{K}(\mathcal{S}_n(\Phi)) = (\mathbf{C}, \sqsubseteq, K_1, \dots, K_n)$ be a Kripke n -scs as in Def. 9. Then $\mathbb{D}_I = \mathcal{D}_I$ for every $I \subseteq \{1, \dots, n\}$.*

We added the next proposition. IMPORTANT: Its proof uses one of the main results of this paper –Th. 4.

Hence we can extend the semantics in Def.11 to distributed knowledge formulae $D_I(\phi)$ as $\mathcal{C}[\![D_I(\phi)]\!] = \mathbb{D}_I(\mathcal{C}[\![\phi]\!])$. \square

In Prop. 6 we listed some useful properties about $(\mathbb{D}_I)_{I \subseteq G}$. In the next section we shall see that $(\mathbb{D}_I)_{I \subseteq G}$ is the greatest solution of three basic properties.

4.5. Distributed Spaces as Group Distributions Candidates.

We now wish to single out a few fundamental properties on tuples of self-maps that can be used to characterize distributed spaces.

Definition 16 (Distribution Candidates). Let $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ an scs. A *group distribution candidate* (gdc) of \mathbf{C} is a tuple $\mathfrak{d} = (\mathfrak{d}_I)_{I \subseteq G}$ of self-maps on \mathbf{C} such that for each $I, J \subseteq G$:

- (D.1) \mathfrak{d}_I is a space function in \mathbf{C} ,
- (D.2) $\mathfrak{d}_I = \mathfrak{s}_i$ if $I = \{i\}$,
- (D.3) $\mathfrak{d}_I \sqsupseteq_s \mathfrak{d}_J$ if $I \subseteq J$.

Property D.1 requires each \mathfrak{d}_I to be a space function. This is trivially met for $\mathfrak{d}_I = \mathbb{D}_I$. Property D.2 says that the function \mathfrak{d}_I for a group of one agent must be the agent's space function. Clearly, $\mathfrak{d}_{\{i\}} = \mathbb{D}_{\{i\}}$ satisfies D.2; indeed the distributed space of a single agent is their own space. Finally, Property D.3 states that $\mathfrak{d}_I(c) \sqsupseteq \mathfrak{d}_J(c)$, if $I \subseteq J$. This is also trivially satisfied if we take $\mathfrak{d}_I = \mathbb{D}_I$ and $\mathfrak{d}_J = \mathbb{D}_J$. Indeed if a group I has some distributed information c then any group J , that includes I , should also have c . This realizes the intuition in Remark 5: The bigger the group, the bigger the space and thus the smaller the space function that represents it.

Properties D.1-D.3, however, do not determine \mathbb{D} uniquely. In fact, there could be infinitely-many tuples of space functions that satisfy them. For example, if we were to chose $\mathfrak{d}_\emptyset = \lambda_\top$, $\mathfrak{d}_{\{i\}} = \mathfrak{s}_i$ for every $i \in G$, and $\mathfrak{d}_I = \lambda_\perp$ whenever $|I| > 1$ then D.1-D.3 would be trivially met. But these space functions would not

capture our intended meaning of distributed spaces: E.g., we would have $true \sqsupseteq \mathbb{d}_I(e)$ for every e thus implying that any e would be distributed in the empty information $true$ amongst the agents in $I \neq \emptyset$.

Nevertheless, we prove that $(\mathbb{D}_I)_{I \subseteq G}$ is the greatest solution satisfying D.1-D.3.

Theorem 1 (Max gcd). *Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Then*

1. $(\mathbb{D}_I)_{I \subseteq G}$ is a gcd of \mathbf{C} .
2. If $(\mathbb{d}_I)_{I \subseteq G}$ is a gcd of \mathbf{C} then $\mathbb{d}_I \sqsubseteq_s \mathbb{D}_I$ for each $I \subseteq G$.

Proof. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of \mathbf{C} .

1. We need to prove that $(\mathbb{D}_I)_{I \subseteq G}$ satisfies properties D.1-D.3 in Def. 16.
Property D.1 follows from definition of \mathbb{D}_I (see Def. 15). Property D.2 is proven in Prop. 6 part (2).
For property D.3, let $I, J \subseteq G$ such that $I \subseteq J$. Notice that $\{f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_i \text{ for every } i \in I\} \subseteq \{f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_j \text{ for every } j \in J\}$. Then $\mathbb{D}_J \sqsubseteq_s \mathbb{D}_I$.
2. Since $(\mathbb{d}_I)_{I \subseteq G}$ is a gcd, we have $\mathbb{d}_I \sqsubseteq_s \mathbb{d}_{\{i\}} = \mathfrak{s}_i$, for every $i \in I$. Then $\mathbb{d}_I \in \{f \in \mathcal{S}(\mathbf{C}) \mid f \sqsubseteq_s \mathfrak{s}_i \text{ for every } i \in I\}$ which implies $\mathbb{d}_I \sqsubseteq_s \mathbb{D}_I$.

□

Therefore, Th. 1 tells us that distributed spaces could have been equivalently defined as the *greatest space functions* satisfying Properties D.1-D.3. We shall use the characterization of distributed spaces in Th. 1 in the proofs of Prop.9 and Th.5 in the next sections. Let us first illustrate the use of such properties in the following example.

Example 16. Let $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ and $c = \mathfrak{s}_1(a) \sqcup \mathfrak{s}_2(b \ominus a) \sqcup \mathfrak{s}_3(e \ominus b)$ as in Ex. 8. Here we shall show that e can be derived from the distributed information among $I = \{1, 2, 3\}$.

We want to prove $c \sqsupseteq \mathbb{D}_I(e)$ for $I = \{1, 2, 3\}$. Since \mathbf{C} is a constraint frame, by Def. 16 and Prop. 2 we have

$$\begin{aligned}
c &= \mathbb{D}_{\{1\}}(a) \sqcup \mathbb{D}_{\{2\}}(b \ominus a) \sqcup \mathbb{D}_{\{3\}}(e \ominus b) && \text{(Def. 16 D.2)} \\
&\sqsupseteq \mathbb{D}_I(a) \sqcup \mathbb{D}_I(b \ominus a) \sqcup \mathbb{D}_I(e \ominus b) && \text{(Def. 16 D.3)} \\
&= \mathbb{D}_I(a \sqcup (b \ominus a) \sqcup (e \ominus b)) && \text{(Def. 16 D.1)} \\
&\sqsupseteq \mathbb{D}_I(e). && \text{(Prop. 2)}
\end{aligned}$$

Thus $c \sqsupseteq \mathbb{D}_I(e)$ as wanted.

Infinitely Many Agents. Recall the case with infinitely many agents from Ex. 8. Similarly, we can show that e' is distributed in \mathbb{N} w.r.t c' :

$$c' = \bigsqcup_{i \in \mathbb{N}} \mathbb{D}_{\{i\}}(a_i) \sqsupseteq \bigsqcup_{i \in \mathbb{N}} \mathbb{D}_I(a_i) = \mathbb{D}_I\left(\bigsqcup_{i \in \mathbb{N}} a_i\right) \sqsupseteq \mathbb{D}_I(e').$$

Now consider our counter-example in Ex. 10 where we have $d = \mathfrak{s}_1(b) \sqcap \mathfrak{s}_2(b)$. Recall that d specifies that b resides either within the space of agent 1 or in the space of agent 2, although we do not know exactly in which one. Here we wish to prove that b is distributed in the group $\{1, 2\}$, i.e., b can be derived from d as being in a space of a member of $\{1, 2\}$. We want to prove $d \sqsupseteq \mathbb{D}_I(b)$ for $I = \{1, 2\}$:

$$\begin{aligned}
d = \mathfrak{s}_1(b) \sqcap \mathfrak{s}_2(b) &= \mathbb{D}_{\{1\}}(b) \sqcap \mathbb{D}_{\{2\}}(b) && \text{(Def. 16 D.2)} \\
&\sqsupseteq \mathbb{D}_{\{1,2\}}(b) \sqcap \mathbb{D}_{\{1,2\}}(b) && \text{(Def. 16 D.3)} \\
&= \mathbb{D}_{\{1,2\}}(b) && \text{(Prop. of } \sqcap \text{)}
\end{aligned}$$

We updated the next example according to N44.

Then $d \sqsupseteq \mathbb{D}_{\{1,2\}}(b)$ as wanted.

Infinitely Many Agents. If we have $d' \stackrel{\text{def}}{=} \prod_{i \in \mathbb{N}} \mathfrak{s}_i(b')$ we can show that b' is distributed in the group \mathbb{N} w.r.t d' : $d' = \prod_{i \in \mathbb{N}} \mathbb{D}_{\{i\}}(b') \sqsupseteq \prod_{i \in \mathbb{N}} \mathbb{D}_{\mathbb{N}}(b') = \mathbb{D}_{\mathbb{N}}(b')$. \square

The above example shows the capability of \mathbb{D}_I to express the distributed information of a group I in different scenarios. In the later sections we shall provide properties of \mathbb{D}_I and its characterization for completely distributive lattices.

4.6. Group Projections

As promised at the beginning of Section 4.2 we now give a definition of *Group Projection*. The function $\Pi_I(c)$ extracts exactly all information that the group I may have distributed w.r.t c .

Definition 17 (Group Projection). Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Given the set $I \subseteq G$, the I -group projection of $c \in \mathbf{C}$ is defined as $\Pi_I(c) \stackrel{\text{def}}{=} \bigsqcup \{e \mid c \sqsupseteq \mathbb{D}_I(e)\}$. We say that e is I -group derivable from c if and only if $\Pi_I(c) \sqsupseteq e$.

Much like space functions and agent projections, group projections and distributed spaces also form a pleasant correspondence: a *Galois connection* [11].

Proposition 9. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. For every $c, e \in \mathbf{C}$,

1. $c \sqsupseteq \mathbb{D}_I(e)$ if and only if $\Pi_I(c) \sqsupseteq e$.
2. $\Pi_I(c) \sqsupseteq \Pi_J(c)$ if $J \subseteq I$.
3. $\Pi_I(c) \sqsupseteq \pi_I(c)$.

Proof. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of \mathbf{C} and let $c, e \in \mathbf{C}$.

1. Let $S = \{d \mid c \sqsupseteq \mathbb{D}_I(d)\}$. First, assume that $c \sqsupseteq \mathbb{D}_I(e)$. Since $e \in S$, by Def. 17, $\Pi_I(c) = \bigsqcup S \sqsupseteq e$. Second, assume $\Pi_I(c) \sqsupseteq e$. Then by monotonicity, $\mathbb{D}_I(\Pi_I(c)) \sqsupseteq \mathbb{D}_I(e)$. From continuity of \mathbb{D}_I , we know that $\mathbb{D}_I(\Pi_I(c)) = \mathbb{D}_I(\bigsqcup S) = \bigsqcup \{\mathbb{D}_I(d) \mid d \in S\}$ and by definition of S , for every $d \in S$, we have $c \sqsupseteq \mathbb{D}_I(d)$, then $c \sqsupseteq \bigsqcup \{\mathbb{D}_I(d) \mid d \in S\}$. Therefore, $c \sqsupseteq \mathbb{D}_I(e)$.
2. Given that $(\mathbb{D}_I)_{I \subseteq G}$ is a gcd (see Th. 1), if $J \subseteq I$, then $\mathbb{D}_J \sqsupseteq_s \mathbb{D}_I$. Hence $\{d \mid c \sqsupseteq \mathbb{D}_J(d)\} \subseteq \{d \mid c \sqsupseteq \mathbb{D}_I(d)\}$ and thus $\Pi_I(c) \sqsupseteq \Pi_J(c)$ for every $c \in \mathbf{C}$.
3. By part (2), for every $\{i\} \subseteq I$ and every $c \in \mathbf{C}$, we have $\Pi_I(c) \sqsupseteq \Pi_{\{i\}}(c)$. It implies, $\Pi_I(c) \sqsupseteq \bigsqcup_{i \in I} \Pi_{\{i\}}(c)$, for every $c \in \mathbf{C}$. Then $\bigsqcup_{i \in I} \Pi_{\{i\}}(c) = \bigsqcup_{i \in I} \{\bigsqcup \{d \mid c \sqsupseteq \mathbb{D}_{\{i\}}(d)\}\} = \bigsqcup_{i \in I} \{\bigsqcup \{d \mid c \sqsupseteq \mathfrak{s}_i(d)\}\} = \bigsqcup_{i \in I} \{\pi_i(c)\} = \pi_I(c)$. Therefore, $\Pi_I(c) \sqsupseteq \pi_I(c)$, for every $c \in \mathbf{C}$. \square

The first property in Prop. 9, a Galois connection, states that we can conclude from c that e is in the distributed space of I exactly when e is I -group derivable from c . The second says that the bigger the group, the bigger the projection. The last property says that whatever is I -join derivable is I -group derivable, although the opposite is not true as shown in Ex. 10.

4.7. Group Compactness.

Suppose that an *infinite* group of agents I can derive e from c (i.e., $c \sqsupseteq \mathbb{D}_I(e)$). A legitimate question is whether there exists a *finite* sub-group J of agents from I that can also derive e from c . The following theorem provides a positive answer to this question given that e is a compact element (see Section 2) and I -join derivable from c .

Theorem 2 (Group Compactness). *Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $c \sqsupseteq \mathbb{D}_I(e)$. If e is compact and I -join derivable from c then there exists a finite set $J \subseteq I$ such that $c \sqsupseteq \mathbb{D}_J(e)$.*

Proof. Suppose that $c \sqsupseteq \mathbb{D}_I(e)$. If I is finite then take $J = I$. If I is not finite, since e is I -join derivable from c we have $\pi_I(c) = \bigsqcup S \sqsupseteq e$ where $S = \{\pi_i(c) \mid i \in I\}$.

Define $D_I = \{\pi_J(c) \mid J \subseteq I \text{ and } J \text{ is finite}\}$. Take any $\pi_H(c), \pi_K(c) \in D_I$. Since H and K are finite, their union $K \cup H$ must also be finite and included in I . Hence $\pi_{H \cup K}(c) \in D_I$. Therefore, D_I is a directed set.

Since $S = \{\pi_i(c) \mid i \in I\} = \{\pi_{\{i\}}(c) \mid i \in I\}$ is included in D_I , we obtain $\bigsqcup D_I \sqsupseteq \bigsqcup S \sqsupseteq e$. But e is compact and D_I directed hence there must be $\pi_J(c) \in D_I$, with J a finite set, such that $\pi_J(c) \sqsupseteq e$. From Prop.9 (3) and Prop.9 (1), we conclude $c \sqsupseteq \mathbb{D}_J(e)$ as wanted. \square

Let us illustrate Th. 2 with our recurrent example.

Example 17. Consider an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $G = \mathbb{N}$ and \mathbf{C} is a countable set. Let $d = \bigsqcup_{i \in \mathbb{N}} \mathfrak{s}_i(a_i)$ for some increasing chain $a_0 \sqsubseteq a_1 \sqsubseteq \dots$, and $b \in \mathbf{C}$ such that $b \sqsubseteq \bigsqcup_{i \in \mathbb{N}} a_i$.

Notice that $d \sqsupseteq \mathbb{D}_{\mathbb{N}}(b)$ and $\pi_{\mathbb{N}}(d) \sqsupseteq b$. Hence b is \mathbb{N} -join derivable from d . If b is compact, by Th. 2 there must be a finite subset $J \subseteq \mathbb{N}$ such that $d \sqsupseteq \mathbb{D}_J(b)$. \square

We rewrote this example.

4.8. Group-Compactness without I -join derivability

Let us assume $c \sqsupseteq \mathbb{D}_I(e)$ as in Th. 2. By Prop. 9 (1), we know that e is I -group derivable from c but not necessarily I -join derivable from c . The problem in establishing group compactness in the *absence of I -join derivability* has to do with d' in the infinite case in Ex. 10. We have $d' = \prod_{i \in \mathbb{N}} \mathfrak{s}_i(b')$. Notice that we cannot guarantee that b' is \mathbb{N} -join derivable from d' ($\pi_{\mathbb{N}}(d') \not\sqsupseteq b'$). One can verify that $d' \sqsupseteq \mathbb{D}_{\mathbb{N}}(b')$, i.e., b' resides in the space of agent i for some $i \in \mathbb{N}$. Then, b' is I -group derivable from d' ($\Pi_{\mathbb{N}}(d') \sqsupseteq b'$). Nevertheless we cannot guarantee the existence of a finite $J \subset \mathbb{N}$ such that that $d' \sqsupseteq \mathbb{D}_J(b')$. In fact, the existence of such a J cannot be guaranteed even if e (b' in Ex. 10) is compact as stated in the next theorem.

Theorem 3 (Non-Compactness). *There exists an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ with distributed spaces $(\mathbb{D}_I)_{I \subseteq G}$ such that for some $c, e \in \mathbf{C}$ and $I \subseteq G$: (1) e is compact, (2) $c \sqsupseteq \mathbb{D}_I(e)$ but (3) there is no finite subset $J \subseteq I$ with $c \sqsupseteq \mathbb{D}_J(e)$.*

Proof. Consider the scs $(\mathbf{C}, \leq, (\mathfrak{s}_n)_{n \in \mathbb{N}})$ (Fig. 5) defined by

$$\mathbf{C} = \left\{0, \frac{1}{2}\right\} \cup \left\{\frac{1}{2} + \frac{1}{2n} \mid n \geq 1\right\} \quad \text{and} \quad \mathfrak{s}_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{2n} & x \geq \frac{1}{2} \\ 0 & x < \frac{1}{2} \end{cases}$$

where \mathfrak{s}_n is a self-map on \mathbf{C} for every $n \geq 1$. For $n = 0$, $\mathfrak{s}_n(x) = 0$ for every $x \in \mathbf{C}$.

First we prove that for every $n \in \mathbb{N}$, \mathfrak{s}_n is a space function. From Def. 6 we prove:

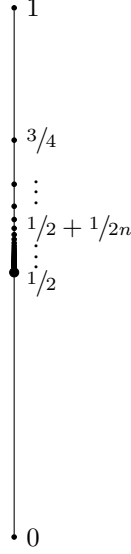


Figure 5: Constraint system (\mathbf{C}, \leq) with $\mathbf{C} = \{0, \frac{1}{2}\} \cup \{\frac{1}{2} + \frac{1}{2n} \mid n \geq 1\}$.

a. \mathfrak{s}_n preserves 0 and binary joins.

Recall that for every $n \in \mathbb{N}$, $\mathfrak{s}_n(0) = 0$. If $x < 1/2$ and $y < 1/2$, then $\mathfrak{s}_n(x \sqcup y) = 0 = \mathfrak{s}_n(x) \sqcup \mathfrak{s}_n(y)$. If either $x \geq 1/2$ or $y \geq 1/2$ hold, then $\mathfrak{s}_n(x \sqcup y) = 1/2 + 1/2n = \mathfrak{s}_n(x) \sqcup \mathfrak{s}_n(y)$.

b. \mathfrak{s}_n is continuous.

For $n = 0$, it is immediate that \mathfrak{s}_n is continuous.

Since \mathbf{C} is countable, from Prop. 1 it suffices to prove that \mathfrak{s}_n preserves the join of increasing chains.

For any $n \geq 1$, consider any increasing chain $x_1 \leq x_2 \leq \dots$ in (\mathbf{C}, \leq) . To prove $\bigsqcup_{i \geq 1} \mathfrak{s}_n(x_i) =$

$$\mathfrak{s}_n\left(\bigsqcup_{i \geq 1} x_i\right).$$

Let $J, K \subseteq \mathbb{N} \setminus \{0\}$ be two index sets such that: if $x_i < 1/2$ then $i \in J$ and if $x_i \geq 1/2$ then $i \in K$. Notice that $J \cap K = \emptyset$ and $J \cup K = \mathbb{N} \setminus \{0\}$. Then using this and part (a), we have $\mathfrak{s}_n\left(\bigsqcup_{i \geq 1} x_i\right) = \mathfrak{s}_n\left(\left(\bigsqcup_{j \in J} x_j\right) \sqcup \left(\bigsqcup_{k \in K} x_k\right)\right) = \mathfrak{s}_n\left(\bigsqcup_{j \in J} x_j\right) \sqcup \mathfrak{s}_n\left(\bigsqcup_{k \in K} x_k\right)$. Also, $\bigsqcup_{j \in J} x_j = 0$ and $\bigsqcup_{k \in K} x_k \geq 1/2$, therefore $\mathfrak{s}_n\left(\bigsqcup_{j \in J} x_j\right) \sqcup \mathfrak{s}_n\left(\bigsqcup_{k \in K} x_k\right) = 0 \sqcup (1/2 + 1/2n) = \bigsqcup_{j \in J} 0 \sqcup \bigsqcup_{k \in K} (1/2 + 1/2n) = \bigsqcup_{j \in J} \mathfrak{s}_n(x_j) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_n(x_k) = \bigsqcup_{i \geq 1} \mathfrak{s}_n(x_i)$. Then $\mathfrak{s}_n\left(\bigsqcup_{i \geq 1} x_i\right) = \bigsqcup_{i \geq 1} \mathfrak{s}_n(x_i)$ as wanted.

To complete the proof we now show that given the above scs $(\mathbf{C}, \leq, (\mathfrak{s}_n)_{n \in \mathbb{N}})$ for $I = \mathbb{N} \setminus \{0\}$ and $c = e = 1/2$: (1) e is compact, (2) $c \geq \mathbb{D}_I(e)$ but (3) $c \not\geq \mathbb{D}_N(e)$ for any finite set $N \subseteq I$.

1. $1/2$ is compact. This follows directly from the definition of our constraint system (\mathbf{C}, \leq) .
2. $1/2 \geq \mathbb{D}_I(1/2)$. By definition of \mathbb{D}_I , $\mathbb{D}_I(1/2) \leq \mathfrak{s}_n(1/2)$ for every $n \in I$. Then $\mathbb{D}_I(1/2) \leq \prod_{n \geq 1} \mathfrak{s}_n(1/2)$. Since for every $n \geq 1$, $\mathfrak{s}_n(1/2) = 1/2 + 1/2n \geq 1/2$ then $\prod_{n \geq 1} \mathfrak{s}_n(1/2) = 1/2$. Thus, $1/2 \geq \mathbb{D}_I(1/2)$.
3. $1/2 \not\geq \mathbb{D}_N(1/2)$ for any finite set $N \subseteq I$. Notice that, for any finite set $N \subseteq I$, $\mathbb{D}_N(1/2) = \mathfrak{s}_m(1/2) > 1/2$, where $m = \max(N)$. Hence, $1/2 \not\geq \mathbb{D}_N(1/2)$.

□

4.9. Distributed Spaces in Completely Distributive Lattices

In this section we present another characterization of distributed spaces for distributive lattices: For a group I , $\mathbb{D}_I(c)$ can be understood as the greatest information below all possible combinations of information in the spaces of the agents in I that derive c . We also provide compositionality properties capturing the intuition that just like *distributed information* of a group I is the collective information from all its members, it is also the collective information of its subgroups. We shall argue that the following results can be used to produce algorithms to efficiently compute $\mathbb{D}_I(c)$ for finite constraint systems.

We recall J -tuples, a general form of tuples that allows for an arbitrary index set J .

Definition 18 ([30]). Let J be an index set. Given a set X , a J -tuple of elements of X is a function $\mathbf{x} : J \rightarrow X$. If $j \in J$, we denote $\mathbf{x}(j)$ by x_j and refer to it as the j -th coordinate of \mathbf{x} . The function \mathbf{x} is denoted itself by $(x_j)_{j \in J}$. We use X^J to denote the set of all J -tuples.

The next theorem is one of main results of this paper. It establishes that for completely distributed lattices, $\mathbb{D}_K(c)$ is the greatest information below all possible combinations of information in the spaces of the agents in K that derive c .

Theorem 4. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Let $\overline{\mathbb{D}}_K : \mathbf{C} \rightarrow \mathbf{C}$, with $K \subseteq G$, be the function defined as follows:

$$\overline{\mathbb{D}}_K(c) \stackrel{\text{def}}{=} \bigsqcap \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in \mathbf{C}^K \text{ and } \bigsqcup_{k \in K} a_k \sqsupseteq c \right\}.$$

Then $\mathbb{D}_K = \overline{\mathbb{D}}_K$.

The above theorem is presented in [10] for finite cs. Here we extend this for completely distributive lattices.

For the sake of the presentation, we give the proof of the above theorem in Section 6. Nevertheless, we would like to mention that the central and non-obvious property used in the proof is that of $\overline{\mathbb{D}}_K$ being a *continuous function*. The distributivity of $(\mathbf{C}, \sqsubseteq)$ is crucial for this. In fact without it the equality $\mathbb{D}_K = \overline{\mathbb{D}}_K$ does not necessarily hold as shown by the following counter-example.

Example 18. Consider the lattice \mathbf{M}_3 , which is not (completely) distributive, and the space functions \mathfrak{s}_1 and \mathfrak{s}_2 in Fig. 6b. We obtain $\overline{\mathbb{D}}_I(b \sqcup c) = \overline{\mathbb{D}}_I(e) = a$ and $\overline{\mathbb{D}}_I(b) \sqcup \overline{\mathbb{D}}_I(c) = b \sqcup a = b$. Then, $\overline{\mathbb{D}}_I(b \sqcup c) \neq \overline{\mathbb{D}}_I(b) \sqcup \overline{\mathbb{D}}_I(c)$, i.e., $\overline{\mathbb{D}}_I$ is not a space function. \square

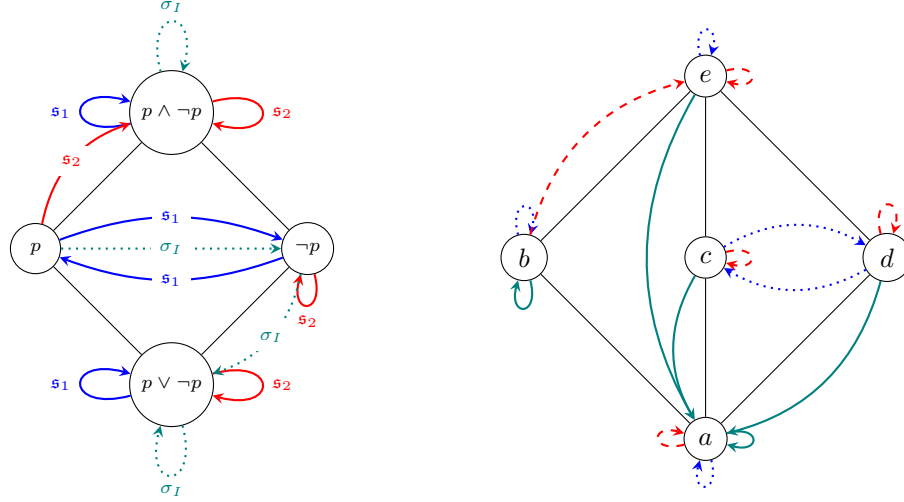
We can use Th.4 to prove the following properties characterizing the information of a group from that of its subgroups.

Theorem 5. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Let $I, J, K \subseteq G$ be such that $I = J \cup K$. Then the following equalities hold:

$$1. \mathbb{D}_I(c) = \bigsqcap \{ \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c \}. \quad (4)$$

$$2. \mathbb{D}_I(c) = \bigsqcap \{ \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C} \}. \quad (5)$$

$$3. \mathbb{D}_I(c) = \bigsqcap \{ \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C} \text{ and } a \sqsubseteq c \}. \quad (6)$$



(a) For $I = \{1, 2\}$, $\sigma_I(c) = \prod_{i \in I} s_i(c)$ is not a space function: $\sigma_I(p \sqcup \neg p) \neq \sigma_I(p) \sqcup \sigma_I(\neg p)$.
 (b) For $I = \{1, 2\}$, s_1 ($\cdots \rightarrow$) and s_2 (\dashrightarrow) are space functions. The function \mathbb{D}_S (\rightarrow) in Th. 4 is not a space function: $\mathbb{D}_I(b) \sqcup \mathbb{D}_I(c) = b \neq a = \mathbb{D}_I(b \sqcup c)$.

Figure 6: Counter-examples over the four-element boolean algebra (a) and non-distributive lattice M_3 (b).

We find it convenient to give the proof of Th. 5 in Section 6. The properties in this theorem bear witness to the inherent compositional nature of our notion of distributed space. The first property in Th. 5 essentially reformulates Th. 4 in terms of subgroups rather than agents. It can be proven by replacing $\mathbb{D}_J(a)$ and $\mathbb{D}_K(b)$ by $\overline{\mathbb{D}}_J(a)$ and $\overline{\mathbb{D}}_K(b)$, defined in Th. 4 and using distributivity of joins over meets. The second and third properties in Th. 5 are pleasant simplifications of the first one using co-Heyting subtraction. These properties realize the intuition that by joining the information a and $c \ominus a$ of their subgroups, the group I can obtain c .

In Section 6 we use Th.4 to prove Th. 5. We now conclude this section with a brief discussion on how to use Th.4 to solve a computational lattice problem.

Computing Distributed Information. Let us assume that \mathbf{C} is finite and distributive. We wish to compute \mathbb{D}_I . Notice that under this finiteness assumption, space functions are exactly those that preserve the join of finite sets, also known as *join-endomorphisms* [29]. Recall that $\mathcal{S}(\mathbf{C})$ denotes the set of space functions (join-endomorphism in this case) over \mathbf{C} .

From Remark 6, computing the distributed space \mathbb{D}_I is then equivalent to the following lattice problem: *Given a finite set $S = \{s_i \mid i \in I\}$ of join-endomorphisms over the finite distributive lattice \mathbf{C} , find its meet $\prod_{S(\mathbf{C})} S$.* Even in small lattices with four elements and two space functions, finding $\mathbb{D}_I = \prod_{S(\mathbf{C})} S$ may not be immediate, e.g., consider $S = \{s_1, s_2\}$ in Fig. 4b.

A naive approach would be to compute each $\mathbb{D}_I(c)$ by taking the point-wise meet construction $\sigma_I(c) \stackrel{\text{def}}{=} \prod \{s_i(c) \mid i \in I\}$. But this does not work in general since $\prod \{s_i \mid i \in I\}(c)$ is not necessarily equal to $\sigma_I(c)$. In fact $\sigma_I \sqsupseteq_S \mathbb{D}_I$ but σ_I may not even be a space function as shown in Fig. 6a.

A brute force solution to computing $\mathbb{D}_I(c)$ is to generate the set $\{f(c) \mid f \in \mathcal{S}(\mathbf{C}) \text{ and } f \sqsubseteq s_i \text{ for all } i \in I\}$ and then take its join (see Remark 6). This approach works since for any set S of join-endomorphisms $(\prod_{S(\mathbf{C})} S)(c) = \bigsqcup_{\mathbf{C}} \{f(c) \mid f \in S\}$. The problem, however, is that the number of join-endomorphisms over a distributive lattice can be *non-polynomial* in the size of the lattice [31].

Nevertheless we can use Th. 4 to obtain a worst-case polynomial bound for computing \mathbb{D}_I . The next proposition shows this.

Proposition 10. *Let \mathbf{C} be a distributive lattice of size n . Let $S = \{\mathfrak{s}_i \mid i \in I\}$, where $I = \{1, \dots, m\}$, be a set of join-endomorphisms over \mathbf{C} . Assuming that binary meets and joins over \mathbf{C} can be computed in $O(1)$, the meet $\mathbb{D}_I = \prod_{S(\mathbf{C})} S$ can be computed in $O(mn^3)$ worst-case time complexity.*

Proof. Let \mathbf{C} be a distributive cs (lattice) of size n and let \mathfrak{s}_j and \mathfrak{s}_k be space functions (join-endomorphisms) over \mathbf{C} . From Th. 4, the value $\mathbb{D}_K(c) = (\mathfrak{s}_j \prod_{S(\mathbf{C})} \mathfrak{s}_k)(c) = \overline{\mathbb{D}}_K(c)$, with $K = \{j, k\}$, can be computed in $O(n^2)$ by performing $O(n^2)$ joins and $O(n^2)$ meets. Hence the function \mathbb{D}_K can be computed in $O(n^3)$ whenever $|K| = 2$.

We now proceed by induction on the size of $S = \{\mathfrak{s}_i \mid i \in I\}$ where $I = \{1, \dots, m\}$. Suppose that $m = 1$. We can then compute $\mathbb{D}_{\{1\}} = \mathfrak{s}_1$ in $O(n)$ and hence in $O(n^3)$. Assume that $\mathbb{D}_{\{1, \dots, m-1\}}$ can be computed in $O((m-1)n^3)$. From the associativity of the meet operation, $\mathbb{D}_I = \mathbb{D}_{\{1, \dots, m-1\}} \prod_{S(\mathbf{C})} \mathfrak{s}_m$. Thus, we can compute first $\mathbb{D}_{\{1, \dots, m-1\}}$ in $O((m-1)n^3)$ and then \mathbb{D}_I , with $\mathfrak{s}_j = \mathbb{D}_{\{1, \dots, m-1\}}$ and $\mathfrak{s}_k = \mathfrak{s}_m$, in $O(n^3)$. The total worst-case time complexity for computing \mathbb{D}_I is then in $O(mn^3)$. \square

4.10. Summary of Section 4

In this section we presented the main technical results of this paper. We have formalized and developed the theory of the collective information of a group of agents I as the space function \mathbb{D}_I . Intuitively, the space function \mathbb{D}_I represents the smallest space that includes all the local information of the agents in I .

We first constructed the complete lattice $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$ (Lemma 1) where $\mathcal{S}(\mathbf{C})$ is the set of all space functions defined on the complete lattice $(\mathbf{C}, \sqsubseteq)$. We then defined \mathbb{D}_I as the greatest space function in $\mathcal{S}(\mathbf{C})$ below the space functions of agents in I (Def. 15) and presented some of its basic compositional properties (Prop. 6). We showed that \mathbb{D}_I could also be alternative defined as the greatest group distribution candidate (gdc) (Th. 1). We illustrated in Ex.14 that \mathbb{D}_I can be interpreted as *Distributed Knowledge* in Aumann structures, a representative model for epistemic group reasoning. **Furthermore, in Ex. 15 we showed that distributed knowledge in epistemic logic is also an instance of distributed information.**

We added the next phrase.

We also defined agent, join and group projections (Def. 12, Def 17). Group (agent) projection of a given $c \in \mathbf{C}$ represents the join of all the information that the group (agent) has in c . Join projections are the join of individual agent projections. We stated that group projections and distributed spaces form a Galois connection (Prop. 9). We then provided a *group compactness* result: Given an *infinite* group I , we identified *join-derivability* (Def. 12) as a condition under which $c \sqsubseteq \mathbb{D}_I(e)$ implies $c \sqsupseteq \mathbb{D}_J(e)$ for some finite group $J \subseteq I$ (Th. 2). We then showed that without this condition we cannot guarantee the existence of such finite set $J \subseteq I$ (Th. 3).

Finally we showed that if \mathbf{C} is completely-distributive, $\mathbb{D}_I(c)$ can be characterized as the greatest information below all possible combinations of information in the spaces of the agents in I (Th. 4) that derive c and, more succinctly, as the combination of the information of its subgroups (Th. 5) that derive c . For the finite-case we briefly explained how Th. 4 can be used to compute distributed space functions in polynomial time.

5. Applications to Minkowski Addition and Mathematical Morphology

In this section we shall show that some fundamental operations from Mathematical Morphology (MM) have a counterpart in the theory we developed in the previous sections. In particular we shall show that *distributed spaces*, the central notion of this paper, have a natural interpretation in MM. Furthermore, we shall use our results on distributed information to provide new constructions and results for MM.

5.1. Modules

We assume that the reader is familiar with basic concepts of abstract algebra [32, 33]. To present the results in this section uniformly we shall use a fundamental structure from algebra, namely, that of a *module*. Recall that a module M over a ring R is a generalization of the notion of *vector space*. In a vector space the ring R needs to be a *field*. We shall take the liberty of referring to the elements of M and R as *vectors* and *scalars*, resp. The former will be written in boldface to distinguish them from the latter.

More precisely, a *module* M over a ring R [33], also called an (left) R -*module* M , is a set that satisfies the following three conditions. It must be closed under addition and scalar multiplication: $\mathbf{u} + \mathbf{v} \in M$ and $r\mathbf{u} \in M$ whenever $\mathbf{u}, \mathbf{v} \in M$ and $r \in R$. It must also form an *abelian group* under addition: $+$ is a commutative and associative operator with $\mathbf{0}$ as additive identity and with an additive inverse $-\mathbf{u}$ for every $\mathbf{u} \in M$. Finally, it must also satisfy the following axioms for scalar multiplication: For every $\mathbf{u}, \mathbf{v} \in M$ and every $r, s \in R$, $r(s\mathbf{u}) = (rs)\mathbf{u}$, $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$, $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$, and $1\mathbf{u} = \mathbf{u}$ where 1 is the *multiplicative identity* of R .

We shall use the following basic properties of modules. The additive identity for any module M and the additive inverse for every $\mathbf{u} \in M$ are unique. If R is a field then the R -module M is a *vector space* over R . If M is a vector space over R then 1 is the only multiplicative identity for M .

The following examples of modules are fundamental in Mathematical Morphology. One of them is not a vector space; it justifies using modules rather than vector spaces as the underlying structure.

Example 19. The set R^n of all n -tuples of elements of a ring R can be made into an R -module. Given $\mathbf{u} = (p_1, \dots, p_n) \in R^n$, $\mathbf{v} = (q_1, \dots, q_n) \in R^n$ and $r \in R$, define $\mathbf{u} + \mathbf{v} = (p_1 + q_1, \dots, p_n + q_n)$ and $r \cdot \mathbf{v} = (rp_1, \dots, rp_n)$. The module additive identity $\mathbf{0}$ is $(0, \dots, 0) \in R^n$, where 0 is the ring additive identity, and the module inverse additive $-\mathbf{u}$ is $(-p_1, \dots, -p_n)$ where $-p_i$ is the ring additive inverse of p_i .

The *Euclidean n -dimensional space* \mathbb{R}^n is obtained by taking R as the set of reals numbers \mathbb{R} in the above example. Since \mathbb{R} is also a field, \mathbb{R}^n is also a vector space. The *n -dimensional grid* \mathbb{Z}^n is obtained by taking R as the set of integers \mathbb{Z} . This is an example of a module that it is not a vector space since the ring \mathbb{Z} is not a field. \square

5.2. Minkowski Addition

In geometry, vector addition is extended to addition of sets of vectors in an operation known as *Minkowski addition*. From now on we shall omit mentioning the ring of the module when it is unimportant or clear from the context.

Definition 19 (Minkowski Sum [8]). Let M be a module and $A, B \subseteq M$. The *Minkowski addition* of A and B is defined thus $A \oplus B = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in A \text{ and } \mathbf{v} \in B\}$.

It is easy to see that \oplus is associative and commutative, it has $\{\mathbf{0}\}$ and \emptyset as identity and absorbent elements, resp., and that it distributes over set union.

Proposition 11 ([8]). Let M be a module. Then $(\mathcal{P}(M), \oplus)$ is a commutative monoid with zero element \emptyset and identity $\{\mathbf{0}\}$. Furthermore, $X \oplus (A \cup B) = (X \oplus A) \cup (X \oplus B)$ for every $X, A, B \subseteq M$.

Recall that a convex set is a set of points such that, given any two points in that set, the line segment joining them lies entirely within that set. It is well-known that the distribution law $A \oplus (B \cap C) = (A \cap B) \oplus (A \cap C)$ holds for convex sets [8]. Nevertheless, in general, \oplus does not distribute over set intersection as illustrated next.

Example 20. Take the Euclidean one-dimensional vector space \mathbb{R} and let $X = \{0, 1\}$, $A = \{1\}$ and $B = \{2\}$. One can verify that $\emptyset = X \oplus (A \cap B) \neq (X \oplus A) \cap (X \oplus B) = \{2\}$. \square

However, as part of the application results in this section we shall establish a pleasant new equation for $X \oplus (A \cap B)$. Namely, for every module M and for all $X, A, B \subseteq M$ we have

$$X \oplus (A \cap B) = \bigcap_{Y \subseteq X} (Y \oplus A) \cup ((X \setminus Y) \oplus B). \quad (7)$$

The Minkowski sum has been applied in mathematical morphology as well as in collision detection, robot motion planning, aggregation theory [34, 35, 36]. In this section we focus on applications to mathematical morphology.

5.3. Mathematical Morphology

Mathematical morphology (MM) is a theory developed for the analysis of geometric structures [37]. It is founded upon, among others, set theory, lattice theory, geometry, topology and probability. Basically, this theory considers an arbitrary space M where its objects are transformed by two fundamental operations: *dilation* and *erosion*.

In [9] dilations and erosions are typically defined in terms of Minkowski additions over the modules \mathbb{R}^n or \mathbb{Z}^n given in Ex. 19. Here we generalize the definition in [9] to arbitrary modules.

Definition 20 (Dilations and Erosions in Modules). Let M be a module. A *dilation* by $S \subseteq M$ is a function $\delta_S : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ given by $\delta_S(X) = X \oplus S = \bigcup_{\mathbf{u} \in S} X \oplus \{\mathbf{u}\}$. An *erosion* by $S \subseteq M$ is a function $\varepsilon_S : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ given by $\varepsilon_S(X) = X \ominus S$ where $X \ominus S \stackrel{\text{def}}{=} \bigcap_{\mathbf{u} \in S} X \oplus \{-\mathbf{u}\}$.

In MM, a binary image X is typically represented as a subset of the module $M = \mathbb{Z}^2$ where a pixel is activated if its corresponding coordinate (or position vector) is in X . The *translation* of a vector \mathbf{u} by a vector \mathbf{v} is given by $\mathbf{u} + \mathbf{v}$. The dilation $\delta_S(X)$ describes the interaction of X with another image S referred to as a *structuring element* and typically assumed to include the center, i.e., $\mathbf{0} = (0, 0) \in S$. The dilated image $\delta_S(X)$ “inflates” the original one by including X and adding the pixels from the translation of every \mathbf{v} in X by each \mathbf{u} in S . Intuitively, $\delta_S(X)$ can be viewed as redrawing the image X with the brush S [9]. This is illustrated in Fig. 7 where an image X is dilated by the structuring element $S = \{(0, 0), (0, -1)\}$, and in Fig. 8 where an image X is dilated by two different structuring elements $A = \{(1, 1), (0, 0), (1, 0), (-1, -1), (0, -1)\}$ and $B = \{(-1, 1), (-1, 0), (0, 0), (0, -1), (1, -1)\}$.

Furthermore, the erosion $\varepsilon_S(X)$ in \mathbb{R}^n or \mathbb{Z}^n can be defined in terms of the translations of S that are contained in X . The next proposition states this result for modules.

Proposition 12. Let M be a module and $S, X \subseteq M$. Then $\varepsilon_S(X) = \{\mathbf{u} \in M \mid S \oplus \{\mathbf{u}\} \subseteq X\}$.

Proof. Let M be a module and $S, X \subseteq M$. From Def. 20, we will prove that $\bigcap_{\mathbf{u} \in S} X \oplus \{-\mathbf{u}\} = \{\mathbf{u} \in M \mid S \oplus \{\mathbf{u}\} \subseteq X\}$. Take any $\mathbf{v} \in \bigcap_{\mathbf{u} \in S} X \oplus \{-\mathbf{u}\}$, then for every $\mathbf{u} \in S$, $\mathbf{v} = \mathbf{w} + (-\mathbf{u})$ for some $\mathbf{w} \in X$. Since for every $\mathbf{u} \in S$, $\mathbf{u} + \mathbf{v} \in S \oplus \{\mathbf{v}\}$ and $\mathbf{u} + \mathbf{v} = \mathbf{w} \in X$ (see Prop. 11), we have $S \oplus \{\mathbf{v}\} \subseteq X$. Therefore, $\mathbf{v} \in \{\mathbf{u} \in M \mid S \oplus \{\mathbf{u}\} \subseteq X\}$.

Now, let $\mathbf{v} \in \{\mathbf{u} \in M \mid S \oplus \{\mathbf{u}\} \subseteq X\}$, then $S \oplus \{\mathbf{v}\} \subseteq X$. Notice that for any $\mathbf{w} \in S$, $\mathbf{w} + \mathbf{v} \in S \oplus \{\mathbf{v}\} \subseteq X$ and therefore $\mathbf{v} = (\mathbf{w} + \mathbf{v}) + (-\mathbf{w}) \in X \oplus \{-\mathbf{w}\}$ for every $\mathbf{w} \in S$. Then $\mathbf{v} \in \bigcap_{\mathbf{u} \in S} X \oplus \{-\mathbf{u}\}$. \square

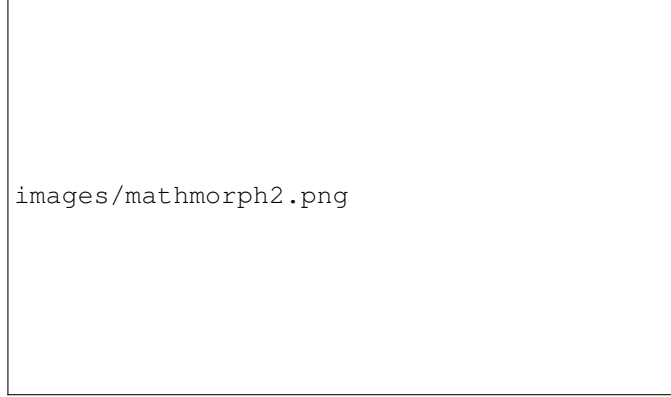


Figure 7: Example of Galois connection between dilations and erosions: $\delta_S(X) \subseteq Y$ iff $X \subseteq \varepsilon_S(Y)$. Here X and Y are images, S is the structuring element (centered at the origin), $\delta_S(X)$ is a dilation and $\varepsilon_S(Y)$ is an erosion. Pixels added/removed by dilation/erosion are depicted in dark/light gray.

Assume that S includes the center. The above proposition tells us that an erosion $\varepsilon_S(X)$ reduces the image X by erasing the pixels in X whose translation by some element of S is not within X . This can also be easily seen from the fact that erosions satisfy the equation $\varepsilon_S(X) = \{\mathbf{u} \in M \mid \text{for each } \mathbf{v} \in S : \mathbf{u} + \mathbf{v} \in X\}$ which follows directly from Prop. 12. Fig. 7 illustrates the erosion of an image Y by the structuring element S .

5.4. Dilations as Space Functions

We now state that the power set with the usual order and the set of all dilations over it form a (*completely distributive*) *spatial constraint system*; i.e., an scs whose underlying lattice is completely distributive.

Theorem 6. *Let M be a module. Then $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$ is a completely distributive scs.*

Proof. The power set of any set ordered by inclusion is a completely distributive lattice with join $\sqcup = \cup$ and meet $\sqcap = \cap$ [11]. Hence $(\mathcal{P}(M), \subseteq)$ is a completely distributive cs.

It remains to prove that for any $S \in \mathcal{P}(M)$, the dilation δ_S is a space function. From Prop. 5 part (2) it suffices to show that $\delta_S(\bigcup_i A_i) = \bigcup_i \delta_S(A_i)$ for every arbitrary union $\bigcup_i A_i \in \mathcal{P}(M)$. This follows from the following equations:

$$\begin{aligned} \delta_S \left(\bigcup_i A_i \right) &= \left\{ \mathbf{x} + \mathbf{e} \mid \mathbf{x} \in \bigcup_i A_i \text{ and } \mathbf{e} \in S \right\} \\ &= \{ \mathbf{x} + \mathbf{e} \mid \mathbf{x} \in A_i \text{ for some } i \text{ and } \mathbf{e} \in S \} \\ &= \{ \mathbf{x} + \mathbf{e} \mid \mathbf{x} + \mathbf{e} \in \delta_S(A_i) \text{ for some } i \} \\ &= \bigcup_i \delta_S(A_i) \end{aligned}$$

Thus, $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$ is a completely distributive scs. □



Figure 8: From left to right, image X , structuring elements A , B and $A \cap B$, dilations $\delta_A(X)$ and $\delta_B(X)$, and dilation $\mathbb{D}_{\{A,B\}}(X)$. Structuring elements are centered at the origin. Pixels added by dilation are depicted in gray.

The above theorem states that dilations are space functions. Erosions, on the other hand, are space projections (see Def. 12).

Proposition 13. *Let M be a module. For every $S \subseteq M$, the function ε_S is the S -projection in the scs $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$.*

Proof. Let M be a module and $S \subseteq M$. To prove that ε_S is an S -projection we show that dilations and erosions form a Galois connection, i.e., for every $X, Y \subseteq \mathcal{P}(M)$, $\delta_S(X) \subseteq Y$ iff $X \subseteq \varepsilon_S(Y)$. It is known that a Galois connection determines each function uniquely. Therefore from Prop. 9 (1) it follows that the erosion ε_S must then be a projection.

Pick arbitrary $X, Y, S \subseteq \mathcal{P}(M)$. We have $\delta_S(X) \subseteq Y$ iff for every $\mathbf{u} \in S$, $X \oplus \{\mathbf{u}\} \subseteq Y$. Furthermore, with the help of the monoid laws for \oplus (Prop. 11) we can show that for every $\mathbf{u} \in S$:

$$\begin{aligned}
& X \oplus \{\mathbf{u}\} \subseteq Y \\
& \text{iff } \langle \text{Property of } \subseteq \rangle \\
& (X \oplus \{\mathbf{u}\}) \cup Y = Y \\
& \text{iff } \langle Z = Z \oplus \{\mathbf{0}\} \text{ and } \{\mathbf{0}\} = \{-\mathbf{u}\} \oplus \{\mathbf{u}\} \rangle \\
& (X \oplus \{\mathbf{u}\}) \cup (Y \oplus \{-\mathbf{u}\} \oplus \{\mathbf{u}\}) = Y \\
& \text{iff } \langle X \oplus (A \cup B) = (X \oplus A) \cup (X \oplus B) \rangle \\
& (X \cup (Y \oplus \{-\mathbf{u}\})) \oplus \{\mathbf{u}\} = Y \\
& \text{iff } \langle \text{Adding } \{-\mathbf{u}\} \text{ by } \oplus \rangle \\
& (X \cup (Y \oplus \{-\mathbf{u}\})) \oplus \{\mathbf{u}\} \oplus \{-\mathbf{u}\} = Y \oplus \{-\mathbf{u}\} \\
& \text{iff } \langle \{-\mathbf{u}\} \oplus \{\mathbf{u}\} = \{\mathbf{0}\} \text{ and } Z \oplus \{\mathbf{0}\} = Z \rangle \\
& X \cup (Y \oplus \{-\mathbf{u}\}) = Y \oplus \{-\mathbf{u}\} \\
& \text{iff } \langle \text{Property of } \subseteq \rangle \\
& X \subseteq Y \oplus \{-\mathbf{u}\}.
\end{aligned}$$

Clearly for every $\mathbf{u} \in S$, $X \subseteq Y \oplus \{-\mathbf{u}\}$ iff $X \subseteq \bigcap_{\mathbf{v} \in S} Y \oplus \{-\mathbf{v}\} = \varepsilon_S(Y)$. We have then established that $\delta_S(X) \subseteq Y$ iff $X \subseteq \varepsilon_S(Y)$ as wanted. \square

In the proof of the above proposition, we show that dilations and erosions form a Galois connection. This is a known fact in the MM community for \mathbb{R}^n or \mathbb{Z}^n space [38]. Here we proved a more general version of it for modules.

Remark 8 (An Epistemic Interpretation of MM operations). Since we have shown that dilation is a space function, and erosion is a space projection, we can now think of these functions in terms of information available to an agent. Thus, for example in Fig. 7, when Y is the actual state, $\varepsilon_S(Y)$ shows what S perceives. The interpretation of δ_S is more complicated: in fact, we interpret X as the information available to, or perceived by, agent S , when $\delta_S(X)$ is the real situation.

We extended the following remark according to N46. Also we added Fig. 9

We notice that with the interpretation of dilations as visual perception, an agent with perfect vision has the structuring element which is a single pixel at the origin, and if structuring elements $S_1 \subseteq S_2$, then S_2 represents vision which is more blurry than the vision represented by S_1 , as seen in Fig. 9. Thus, the agent only perceives some of the real information, and some parts of the actual state of the world are hidden from the agent. Specifically, we can consider this as an agent with blurred vision who does not perceive some of the edges of objects. In the instance of Fig. 9, there are two agents, agent 1, corresponding to structuring element S_1 , with slightly blurred vision, and agent 2 with corresponding structuring element S_2 , with extremely blurred vision. Neither agent can perceive the edges of the objects. So, when $\delta_{S_1}(X)$ is the true state of affairs, S_1 perceives the information in X ; in effect, losing one pixel at the edge of each object. On the other hand, when $\delta_{S_2}(X)$ is the real situation, S_2 perceives it as X , because this agent loses two pixels from the edge of every object.

Thus, given an agent S , we may think of $\delta_S(X)$ as the situation where agent S has information X , because X is what the agent perceives when $\delta_S(X)$ is the real situation.

In contrast, in Fig. 9 we note that if X represents the real situation, then agent 1 perceives $\varepsilon_{S_1}(X)$, losing one pixel at the edge of each object. Similarly, agent 2 perceives $\varepsilon_{S_2}(X)$ losing two pixels at the edge of each object.

In Fig. 7, the agent's vision is blurred in the vertical direction: they only perceive a pixel if it also has another pixel below it, but they are unable to perceive the bottom edge of any object. \square

One may wonder if every space function over $(\mathcal{P}(M), \subseteq)$ is a dilation δ_S for some $S \subseteq M$. Prop. 15 answers this question negatively. First, we need to introduce a new family of functions over modules.

Definition 21 (Scale Function). Let M be a module over a ring R . Given $r \in R$, a scale by r is a function $s_r : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ defined as $s_r(X) = \{r\mathbf{u} \mid \mathbf{u} \in X\}$.

It is easy to see that s_r is a space function over $(\mathcal{P}(M), \subseteq)$.

Proposition 14. Let M be a module over a ring R . Then for every $r \in R$, s_r is a space function over $(\mathcal{P}(M), \subseteq)$.

Proof. Let M be a module over a ring R . From Prop. 5 (2), it suffices to show that given $r \in R$ and an arbitrary $\bigcup_i A_i \in \mathcal{P}(M)$, $s_r(\bigcup_i A_i) = \bigcup_i s_r(A_i)$. Indeed, we have $s_r(\bigcup_i A_i) = \{r\mathbf{u} \mid \mathbf{u} \in \bigcup_i A_i\} = \{r\mathbf{u} \mid \mathbf{u} \in A_i \text{ for some } i\} = \{r\mathbf{u} \mid r\mathbf{u} \in s_r(A_i) \text{ for some } i\} = \bigcup_i s_r(A_i)$ as wanted. \square

The following proposition gives us a necessary and sufficient condition for a scale function to be a dilation. In particular, it tells us that we can have infinitely many space functions that *are not dilations* by some structuring element if the underlying module is, for example, the Euclidean vector space \mathbb{R}^n or the grid \mathbb{Z}^n (see Ex. 19).

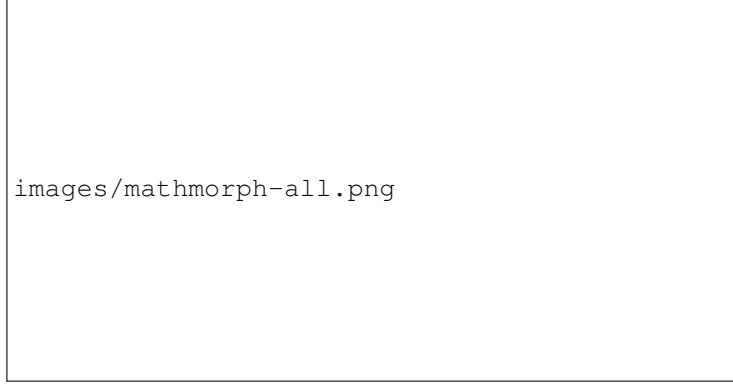


Figure 9: From left to right, image X , structuring elements S_1 and S_2 , dilations $\delta_{S_1}(X)$ and $\delta_{S_2}(X)$, and erosions $\varepsilon_{S_1}(X)$ and $\varepsilon_{S_2}(X)$. Structuring elements are centered at the origin. Pixels added/removed by dilation/erosion are depicted in dark/light grey.

Proposition 15. *Let M be a module over a ring R . Then for each $r \in R$, $s_r = \delta_S$ for some $S \subseteq M$ if and only if r is a multiplicative identity for M .*

Proof. Suppose that r is a multiplicative identity for M . Take $S = \{\mathbf{0}\}$. Clearly $s_r = \delta_{\{\mathbf{0}\}}$. For the other direction we proceed by contradiction. Let us suppose that r is not a multiplicative identity for M but that there exists S such that $s_r = \delta_S$. By applying both space functions to $\{\mathbf{0}\}$, we obtain $s_r(\{\mathbf{0}\}) = \{r\mathbf{0}\} = \{\mathbf{0}\} = \delta_S(\{\mathbf{0}\})$. Then for every $\mathbf{v} \in S$, $\mathbf{0} + \mathbf{v} = \mathbf{0}$, hence $\mathbf{v} = \mathbf{0}$. Thus $S = \{\mathbf{0}\}$. It follows that for every $\mathbf{u} \in M$, $s_r(\{\mathbf{u}\}) = \{r\mathbf{u}\} = \delta_{\{\mathbf{0}\}}(\{\mathbf{u}\}) = \{\mathbf{u}\}$. This implies that for every $\mathbf{u} \in M$, $r\mathbf{u} = \mathbf{u}$, thus r is a multiplicative identity for M , a contradiction. □

5.5. The Distributed Spaces and Dilations

We have shown that dilations are space functions while erosions are space projections. However, the main construction of this paper is that of *distributed spaces*: The greatest space function below a given set of space functions. The problem we shall address is the following: Given two dilations δ_A and δ_B , find the greatest space function $\mathbb{D}_{\{A,B\}}$ below them. Let us consider some issues regarding this question.

Recall that join and meet operations of the power set $(\mathcal{P}(M), \subseteq)$ are set union and intersection. Notice that simply taking the point-wise greatest lower bound does not work, i.e., in general, the equation $\mathbb{D}_{\{A,B\}}(X) = \delta_A(X) \cap \delta_B(X)$ does not hold.

Example 21. Consider Ex. 20 with the Euclidean one-dimensional vector space $M = \mathbb{R}$, $X = \{0, 1\}$, $A = \{1\}$ and $B = \{2\}$. Let $f(Y) = \delta_A(Y) \cap \delta_B(Y)$ for every $Y \subseteq M$. One can verify that $f(\{0\} \cup \{1\}) = \{2\} \neq \emptyset = f(\{0\}) \cup f(\{1\})$, hence f is not even a space function. □

Furthermore, notice Prop. 15 tells us that there are space functions over $(\mathcal{P}(M), \subseteq)$ that are not dilations. Thus, in principle it is not clear if $\mathbb{D}_{\{A,B\}}$ is itself a dilation and if it is, we would like to identify what its structuring element should be.

The main result of this section, given next, addresses the above issues.

Theorem 7 (Distributed Spaces as Dilations). *Suppose that M is a module. Let $(\mathbb{D}_S)_{S \subseteq M}$ be the distributed spaces of the scs $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$. Then $\mathbb{D}_{\{A,B\}} = \delta_{A \cap B}$ for every $A, B \subseteq M$.*

Therefore, the greatest space function below two dilations is a dilation by the intersection of their structuring elements. According to our intuition about dilations, the theorem tells us that given δ_A and δ_B , the dilation $\mathbb{D}_{\{A,B\}}$ applied to an image X can be intuitively described as the image obtained by re-drawing X with (the brush) $A \cap B$. This is illustrated in Fig. 8 by showing $\mathbb{D}_{\{A,B\}}$ applied to an image X ⁷.

We now devote the final part of this section to illustrate the application of the theory developed in previous sections to prove the above result.

5.6. Application: Proof of Theorem 7

We wish to prove that $\mathbb{D}_{\{A,B\}} = \delta_{A \cap B}$ for $A, B \subseteq M$. From Def. 20, we have $\delta_{A \cap B}(X) = X \oplus (A \cap B)$ and, from Th. 5, we have $\mathbb{D}_{\{A,B\}}(X) = \bigcap_{Y \subseteq X} (Y \oplus A) \cup ((X \setminus Y) \oplus B)$. Therefore, it suffices to show that

$$X \oplus (A \cap B) = \bigcap_{Y \subseteq X} (Y \oplus A) \cup ((X \setminus Y) \oplus B)$$

for all $X \subseteq M$. Recall that the above equality is the distributivity equation for the Minkowski addition discussed in Eq. 7. It is important also to recall the equation $X \oplus (A \cap B) = (X \oplus A) \cap (X \oplus B)$ *does not hold* in general. Nevertheless, it does if X is a singleton set as shown next.

Let us consider the *singleton case*: Suppose that X is an arbitrary set of the form $\{\mathbf{v}\}$. We obtain the following equations:

$$\begin{aligned} \mathbb{D}_{\{A,B\}}(\{\mathbf{v}\}) &= \bigcap_{Y \subseteq X} (Y \oplus A) \cup ((X \setminus Y) \oplus B) \\ &= ((\emptyset \oplus A) \cup (\{\mathbf{v}\} \oplus B)) \cap ((\{\mathbf{v}\} \oplus A) \cup (\emptyset \oplus B)) \\ &= (\{\mathbf{v}\} \oplus B) \cap (\{\mathbf{v}\} \oplus A) \\ &= \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in B\} \cap \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in A\} \\ &= \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in A \cap B\} \\ &= \delta_{A \cap B}(\{\mathbf{v}\}). \end{aligned}$$

Now for the general case we will use the *continuity of space functions*. Suppose that X is an arbitrary set. From Def. 15 and Th. 6 we know that $\mathbb{D}_{\{A,B\}}$ and $\delta_{A \cap B}$ are *space functions*. Furthermore from Prop. 5 it follows that space functions preserve arbitrary joins. Thus, with the help of this preservation of arbitrary joins (unions) and the singleton case above, we can obtain the desired result in a simple way; namely $\mathbb{D}_{\{A,B\}}(X) =$

$$\mathbb{D}_{\{A,B\}}\left(\bigcup_{\mathbf{v} \in X} \{\mathbf{v}\}\right) = \bigcup_{\mathbf{v} \in X} \mathbb{D}_{\{A,B\}}(\{\mathbf{v}\}) = \bigcup_{\mathbf{v} \in X} \delta_{A \cap B}(\{\mathbf{v}\}) = \delta_{A \cap B}\left(\bigcup_{\mathbf{v} \in X} \{\mathbf{v}\}\right) = \delta_{A \cap B}(X).$$

□

5.7. Summary of Section 5

In this section we provided an application of the theory developed in Section 4 to geometry and Mathematical Morphology. First, we recalled the notion of Minkowski addition of sets in vector spaces as an

⁷The result of $\mathbb{D}_{\{A,B\}}(X)$ was computed using the $O(n^2)$ procedure mentioned in the beginning of the proof of Prop. 10 from Section 4.9.

operation in the algebraic structure of modules (Section 5.1) and showed some of its basic properties (Sections 5.2 and 5.3). We then proved that given a module M , the structure $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$ is an scs where dilations are space functions (Section 5.4). Furthermore, we showed that erosions are space projections (Prop.13). Since space functions and projections can also be viewed as the information a given agent sees, we then gave a natural epistemic interpretation of these MM operations as an agent's perception of a given image (Remark 8).

In Sec. 5.5 we proved that given two dilations δ_A and δ_B , the *distributed space function* of the group $\{A, B\}$ (i.e., $\mathbb{D}_{\{A, B\}}$) corresponds to the dilation $\delta_{A \cap B}$. Finally, in Section 5.6 we used the theory developed in previous sections to prove Th. 7 and a novel law for $X \oplus (A \cap B)$: i.e., $X \oplus (A \cap B) = \bigcap_{Y \subseteq X} (Y \oplus A) \cup ((X \setminus Y) \oplus B)$.

6. Proofs of Section 4

In this section we present proofs of some results from Sec. 4.

6.1. Proof of Proposition 7

Let $\mathcal{A} = (S, \mathcal{P}_1, \dots, \mathcal{P}_n)$ be an Aumann structure and let $\mathbf{C}(\mathcal{A})$ be its induced scs defined in Ex.14. Then $\mathbb{D}_I = D_I$ for every $I \subseteq G = \{1, \dots, n\}$.

The next proof is new, associated to N43.

Proof. Let $I \subseteq G$. We shall prove that (i) $D_I \in \mathcal{S}(\mathbf{C})$, (ii) $\mathbb{D}_I(c) \supseteq D_I(c)$ and, (iii) $\mathbb{D}_I(c) \subseteq D_I(c)$.

- (i) Recall that in $\mathbf{C}(\mathcal{A})$ joins are intersections (Sec. 4.4). From Prop. 5 (2), it suffices to show that given an arbitrary $\bigcap_j A_j \in \mathbf{C}$, $D_I(\bigcap_j A_j) = \bigcap_j D_I(A_j)$. Indeed, we have $D_I(\bigcap_j A_j) = \{s \mid \bigcap_{i \in I} \mathcal{P}_i(s) \subseteq \bigcap_j A_j\} = \{s \mid \bigcap_{i \in I} \mathcal{P}_i(s) \subseteq A_j \text{ for every } j\} = \bigcap_j \{s \mid \bigcap_{i \in I} \mathcal{P}_i(s) \subseteq A_j\} = \bigcap_j D_I(A_j)$ as wanted.
- (ii) By definition of K_i and D_I (see Sec. 4.4), we know that for every $c \in \mathbf{C}$, $K_i(c) \subseteq D_I(c)$. This implies $D_I(c) \subseteq K_i(c)$. Then $D_I(c)$ is a lower bound in $\mathcal{S}(\mathbf{C})$ of the set of all the $K_i(c)$. Thus, by Remark 6 $\mathbb{D}_I(c) \supseteq D_I(c)$ for every $c \in \mathbf{C}$.
- (iii) Now, let $t \in D_I(c)$. Then, by definition of D_I , we have $\bigcap_{i \in I} \mathcal{P}_i(t) \subseteq c$. From Th. 4,

$$\mathbb{D}_I(c) = \bigcup \left\{ \bigcap_{i \in I} K_i(e_i) \mid \bigcap_{i \in I} e_i \subseteq c \right\}.$$

Take $e_i = \mathcal{P}_i(t)$ for every $i \in I$. By assumption, $\bigcap_{i \in I} e_i \subseteq c$ and, by definition of K_i , we know that $t \in K_i(e_i)$. Therefore, $t \in \bigcap_{i \in I} K_i(e_i)$, i.e., $t \in \mathbb{D}_I(c)$ for every $c \in \mathbf{C}$. Thus $\mathbb{D}_I(c) \subseteq D_I(c)$ for every $c \in \mathbf{C}$, as wanted.

□

6.2. Proof of Proposition 8

Let $\mathbf{K}(\mathcal{S}_n(\Phi)) = (\mathbf{C}, \subseteq, K_1, \dots, K_n)$ be an scs where $\mathcal{S}_n(\Phi)$ is a non-empty set of n -agent Kripke structures over Φ as in Def. 9. Then $\mathbb{D}_I = \mathcal{D}_I$ for every $I \subseteq G = \{1, \dots, n\}$.

This proof is new.

Proof. Let $I \subseteq G$. We shall prove that (i) $\mathcal{D}_I \in \mathcal{S}(\mathbf{C})$, (ii) $\mathbb{D}_I(c) \supseteq \mathcal{D}_I(c)$ and, (iii) $\mathbb{D}_I(c) \subseteq \mathcal{D}_I(c)$.

- (i) Recall that in $\mathbf{K}(\mathcal{S}_n(\Phi))$ joins are intersections (Def. 9). From Prop. 5 (2), it suffices to show that given an arbitrary $\bigcap_j A_j \in \mathbf{C}$, $\mathcal{D}_I\left(\bigcap_j A_j\right) = \bigcap_j \mathcal{D}_I(A_j)$.

Indeed, we have $\mathcal{D}_I\left(\bigcap_j A_j\right) = \{(M, s) \in \Delta \mid \forall t : (s, t) \in \bigcap_{i \in I} \mathcal{R}_i \text{ implies } (M, t) \in \bigcap_j A_j\} = \{(M, s) \in \Delta \mid \forall t : (s, t) \in \bigcap_{i \in I} \mathcal{R}_i \text{ implies } (M, t) \in A_j \text{ for every } j\} = \bigcap_j \{(M, s) \in \Delta \mid \forall t : (s, t) \in \bigcap_{i \in I} \mathcal{R}_i \text{ implies } (M, t) \in A_j\} = \bigcap_j \mathcal{D}_I(A_j)$ as wanted.

- (ii) Recall that in $\mathbf{K}(\mathcal{S}_n(\Phi))$, for every $i \in \{1, \dots, n\}$, $\mathfrak{s}_i = K_i$ (Prop. 3). By definition of K_i and \mathcal{D}_I (see Def. 9 and Ex. 15), we know that for every $X \in \mathbf{C}$, $K_i(X) \subseteq \mathcal{D}_I(X)$. This implies $\mathcal{D}_I(X) \sqsubseteq K_i(X)$. Then $\mathcal{D}_I(X)$ is a lower bound in $\mathcal{S}(\mathbf{C})$ of the set of all the $K_i(X)$. Thus, by Remark 6 $\mathbb{D}_I(X) \sqsupseteq \mathcal{D}_I(X)$ for every $X \in \mathbf{C}$.
- (iii) Now, let $(M, s) \in \mathcal{D}_I(X)$. Then, by definition of \mathcal{D}_I , we have for all t , $(s, t) \in \bigcap_{i \in I} \mathcal{R}_i$ implies $(M, t) \in X$. From Th. 4,

$$\mathbb{D}_I(X) = \bigcup \left\{ \bigcap_{i \in I} K_i(X_i) \mid \bigcap_{i \in I} X_i \subseteq X \right\}.$$

Take $X_i = \{(M, t) \mid (s, t) \in \bigcap_{i \in I} \mathcal{R}_i\}$ for every $i \in I$. It is clear that $\bigcap_{i \in I} X_i \subseteq X$. Notice that by definition of K_i , we know that $(M, s) \in K_i(X_i)$. Therefore, $(M, s) \in \bigcap_{i \in I} K_i(X_i)$, i.e., $(M, s) \in \mathbb{D}_I(X)$ for every $X \in \mathbf{C}$. Thus $\mathbb{D}_I(X) \sqsubseteq \mathcal{D}_I(X)$ for every $X \in \mathbf{C}$, as wanted. □

6.3. Proofs of Section 4.9

This section is devoted to the proofs of the compositionality properties of distributed spaces given in Section 4.9. To simplify our notation, we define sets of J -tuples whose join derive a given constraint c .

Definition 22. Let $(\mathbf{C}, \sqsubseteq)$ be a cs and J some index set. For every $c \in \mathbf{C}$, let $T_c^J = \{(a_j)_{j \in J} \in \mathbf{C}^J \mid \bigsqcup_{j \in J} a_j \sqsupseteq c\}$. For simplicity, we use T_c instead of T_c^J when no confusion arises.

6.4. Proof of Theorem 4

The function $\overline{\mathbb{D}}_K$ is defined in Th. 4. Here we find it convenient to use the following simplified version.

Proposition 16. $\overline{\mathbb{D}}_K(c) = \prod \{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_c\}$.

The following is an immediate consequence of the above definition.

Proposition 17. *The function $\overline{\mathbb{D}}_K$ is monotonic.*

Proof. Let $c \sqsupseteq d$. We have $\{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_c\} \subseteq \{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_d\}$. Thus $\overline{\mathbb{D}}_K(c) \sqsupseteq \overline{\mathbb{D}}_K(d)$. □

Next lemma states that $\overline{\mathbb{D}}_K$ is a space function. As pointed out in Section 4.9, the proof of continuity of $\overline{\mathbb{D}}_K$ uses the assumption of $(\mathbf{C}, \sqsubseteq)$ being completely distributive.

Lemma 2. *Let $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs. Suppose that $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Then, for any $K \subseteq G$, $\overline{\mathbb{D}}_K$ is a space function.*

Proof. Let $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ be an scs and assume $(\mathbf{C}, \sqsubseteq)$ to be completely distributive. To show that $\overline{\mathbb{D}}_K$ is a space function we prove: (i) it satisfies S.1 and S.2 in Def. 6, and (ii) it is continuous.

(i) $\overline{\mathbb{D}}_K$ satisfies S.1 and S.2.

For S.1, one can verify that $\overline{\mathbb{D}}_K(\text{true}) = \text{true}$.

To prove S.2, it suffices to show that $\overline{\mathbb{D}}_K(c \sqcup d) \sqsubseteq \overline{\mathbb{D}}_K(c) \sqcup \overline{\mathbb{D}}_K(d)$. The other direction follows by monotonicity (Prop. 17). Consider the following derivation:

$$\begin{aligned}
& \overline{\mathbb{D}}_K(c) \sqcup \overline{\mathbb{D}}_K(d) \\
= & \langle \text{Definition of } \overline{\mathbb{D}}_K \rangle \\
& \overline{\mathbb{D}}_K(c) \sqcup \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \sqcup \text{ distributes over } \prod \rangle \\
& \prod \left\{ \overline{\mathbb{D}}_K(c) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \text{Definition of } \overline{\mathbb{D}}_K \rangle \\
& \prod \left\{ \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_c \right\} \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \sqcup \text{ distributes over } \prod \rangle \\
& \prod \left\{ \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (a_k)_{k \in K} \in T_c \right\} \mid (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \text{Associativity of } \prod \rangle \\
& \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (a_k)_{k \in K} \in T_c \text{ and } (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \text{Associativity of } \sqcup \rangle \\
& \prod \left\{ \bigsqcup_{k \in K} (\mathfrak{s}_k(a_k) \sqcup \mathfrak{s}_k(b_k)) \mid (a_k)_{k \in K} \in T_c \text{ and } (b_k)_{k \in K} \in T_d \right\} \\
= & \langle \mathfrak{s}_k \text{ is a space function} \rangle \\
& \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k \sqcup b_k) \mid (a_k)_{k \in K} \in T_c \text{ and } (b_k)_{k \in K} \in T_d \right\} \\
\sqsupseteq & \langle c_k = a_k \sqcup b_k; (\bigsqcup_{k \in K} c_k = \bigsqcup_{k \in K} (a_k \sqcup b_k)) \sqsupseteq c \sqcup d \text{ implies } (c_k)_{k \in K} \in T_{c \sqcup d} \rangle \\
& \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(c_k) \mid (c_k)_{k \in K} \in T_{c \sqcup d} \right\} \\
= & \langle \text{Definition of } \overline{\mathbb{D}}_K \rangle \\
& \overline{\mathbb{D}}_K(c \sqcup d)
\end{aligned}$$

(ii) $\overline{\mathbb{D}}_K$ is continuous.

Let D be any directed set on \mathbf{C} , we will prove $\overline{\mathbb{D}}_K(\bigsqcup D) = \bigsqcup\{\overline{\mathbb{D}}_K(d) \mid d \in D\}$. We proceed with $\overline{\mathbb{D}}_K(\bigsqcup D) \sqsubseteq \bigsqcup\{\overline{\mathbb{D}}_K(d) \mid d \in D\}$. The other direction follows by monotonicity.

By definition of $\overline{\mathbb{D}}_K$, $\bigsqcup\{\overline{\mathbb{D}}_K(d) \mid d \in D\} = \bigsqcup\{\bigsqcup\{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_d\} \mid d \in D\}$. Since $(\mathbf{C}, \sqsubseteq)$ is completely distributive (see Def. 1), for the subset $\{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k)\}_{d \in D, (a_k)_{k \in K} \in T_d}$ of \mathbf{C} , we have

$$\bigsqcup_{d \in D} \left\{ \bigsqcup_{(a_k)_{k \in K} \in T_d} \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \right\} \right\} = \bigsqcup_{f \in F} \left\{ \bigsqcup_{d \in D} \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(f_k(d)) \right\} \right\}$$

where F is the class of choice functions f choosing for each $d \in D$ some index $f(d) \in T_d$. Recall that $f_k(d)$ is the k -th element of K -tuple $f(d)$. We can rewrite the right-hand side of the above equality using \sqcup properties and the fact that \mathfrak{s}_k preserves arbitrary joins (see Prop. 5). Then we obtain

$$\bigsqcup_{d \in D} \left\{ \bigsqcup_{(a_k)_{k \in K} \in T_d} \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \right\} \right\} = \bigsqcup_{f \in F} \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k \left(\bigsqcup_{d \in D} f_k(d) \right) \right\}.$$

We now show that for every $f \in F$, $(\bigsqcup_{d \in D} f_k(d))_{k \in K} \in T_{\bigsqcup D}$. Notice that for every $d \in D$, $\bigsqcup_{k \in K} f_k(d) \sqsupseteq d$. We have

$$\bigsqcup_{k \in K} \left(\bigsqcup_{d \in D} f_k(d) \right) = \bigsqcup_{d \in D} \left(\bigsqcup_{k \in K} f_k(d) \right) \sqsupseteq \bigsqcup_{d \in D} d = \bigsqcup D.$$

Therefore $(\bigsqcup_{d \in D} f_k(d))_{k \in K} \in T_{\bigsqcup D}$ (see Def. 22).

Then, for every $f \in F$, the element $\bigsqcup_{k \in K} \mathfrak{s}_k(\bigsqcup_{d \in D} f_k(d)) \in \{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_{\bigsqcup D}\}$, this implies

$$\left\{ \bigsqcup_{k \in K} \mathfrak{s}_k \left(\bigsqcup_{d \in D} f_k(d) \right) \mid f \in F \right\} \subseteq \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_{\bigsqcup D} \right\}.$$

Consequently,

$$\bigsqcup_{f \in F} \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k \left(\bigsqcup_{d \in D} f_k(d) \right) \right\} \sqsupseteq \bigsqcup_{k \in K} \left\{ \bigsqcup_{(a_k)_{k \in K} \in T_{\bigsqcup D}} \mathfrak{s}_k(a_k) \right\} = \overline{\mathbb{D}}_K(\bigsqcup D).$$

Thus, we conclude $\overline{\mathbb{D}}_K$ is continuous. □

Finally we prove Th. 4. Its statement now is simplified due to Prop. 16.

Theorem (4). Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $(\mathbf{C}, \sqsubseteq)$ is a completely distributive lattice. Then $\mathbb{D}_K = \overline{\mathbb{D}}_K$.

Proof. Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ and assume $(\mathbf{C}, \sqsubseteq)$ to be completely distributive.

We divide the proof in two parts: I. $\overline{\mathbb{D}}_K \sqsubseteq_s \mathbb{D}_K$ and II. $\mathbb{D}_K \sqsubseteq_s \overline{\mathbb{D}}_K$.

I. $\overline{\mathbb{D}}_K \sqsubseteq_s \mathbb{D}_K$.

Recall that from Def. 15, $\mathbb{D}_K = \max\{h \in \mathcal{S}(\mathbf{C}) \mid h \sqsubseteq_s \mathfrak{s}_k \text{ for all } k \in K\}$. Thus we have to prove: (i) $\overline{\mathbb{D}}_K \in \mathcal{S}(\mathbf{C})$ and (ii) $\overline{\mathbb{D}}_K \sqsubseteq_s \mathfrak{s}_k$ for every $k \in K$.

From Lemma 2 we have (i). For part (ii), let $c \in \mathbf{C}$ and $S = \{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_c\}$. From definition of $\overline{\mathbb{D}}_K$, for every $k \in K$, the element $\mathfrak{s}_k(c) = \mathfrak{s}_k(c) \sqcup \bigsqcup_{j \in K \setminus \{k\}} \mathfrak{s}_j(\text{true}) \in S$. Then for every $c \in \mathbf{C}$, $\overline{\mathbb{D}}_K(c) = \prod S \sqsubseteq \mathfrak{s}_k(c)$. Therefore for every $k \in K$, $\overline{\mathbb{D}}_K \sqsubseteq_s \mathfrak{s}_k$. Then, $\overline{\mathbb{D}}_K \sqsubseteq_s \mathbb{D}_K$ holds.

II. $\mathbb{D}_K \sqsubseteq_s \overline{\mathbb{D}}_K$.

Let $c \in \mathbf{C}$ and $S = \{\bigsqcup_{k \in K} \mathfrak{s}_k(a_k) \mid (a_k)_{k \in K} \in T_c\}$. Notice that, for any $(a_k)_{k \in K} \in T_c$, $\bigsqcup_{k \in K} \mathbb{D}_K(a_k) \sqsubseteq \bigsqcup_{k \in K} \mathfrak{s}_k(a_k)$. Since \mathbb{D}_K is a space function and monotonic, we know that

$$\bigsqcup_{k \in K} \mathbb{D}_K(a_k) = \mathbb{D}_K\left(\bigsqcup_{k \in K} a_k\right) \sqsupseteq \mathbb{D}_K(c).$$

Thus, for every $(a_k)_{k \in K} \in T_c$, we have $\mathbb{D}_K(c) \sqsubseteq \bigsqcup_{k \in K} \mathfrak{s}_k(a_k)$, i.e., $\mathbb{D}_K(c)$ is a lower bound of S . Then for every $c \in \mathbf{C}$, $\mathbb{D}_K(c) \sqsubseteq \prod S = \overline{\mathbb{D}}_K(c)$. Therefore $\mathbb{D}_K \sqsubseteq_s \overline{\mathbb{D}}_K$ as wanted.

Thus, from (I) and (II), we conclude Th. 4, i.e., $\mathbb{D}_K = \overline{\mathbb{D}}_K$. \square

6.5. Proof of Theorem 5

We now prove the compositional properties of distributed spaces introduced in Th. 5.

Theorem (5). *Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$. Suppose that $(\mathbf{C}, \sqsubseteq)$ is completely distributive and let $I, J, K \subseteq G$ be such that $I = J \cup K$. Then*

1. $\mathbb{D}_I(c) = \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c\}$.
2. $\mathbb{D}_I(c) = \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C}\}$.
3. $\mathbb{D}_I(c) = \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C} \text{ and } a \sqsubseteq c\}$.

We present the proof of each item separately.

PROOF OF THEOREM (5 (1)). Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Let $I, J, K \subseteq G$ be such that $I = J \cup K$.

- $\mathbb{D}_I(c) \sqsubseteq \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c\}$.

Let $a, b, c \in \mathbf{C}$ such that $a \sqcup b \sqsupseteq c$. Since $(\mathbb{D}_I)_{I \subseteq G}$ is a gcd (see Th. 1) and, both $J \subseteq I$ and $K \subseteq I$ hold, for every $e \in \mathbf{C}$, we have $\mathbb{D}_I(e) \sqsubseteq \mathbb{D}_J(e)$ and $\mathbb{D}_I(e) \sqsubseteq \mathbb{D}_K(e)$. Then $\mathbb{D}_I(a) \sqcup \mathbb{D}_I(b) \sqsubseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b)$. From the fact that \mathbb{D}_I is a space function and hence monotonic we conclude $\mathbb{D}_I(c) \sqsubseteq \mathbb{D}_I(a) \sqcup \mathbb{D}_I(b)$. Therefore $\mathbb{D}_I(c) \sqsubseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b)$. Thus, $\mathbb{D}_I(c) \sqsubseteq \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c\}$.

- $\mathbb{D}_I(c) \sqsupseteq \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c\}$.

Let $c \in \mathbf{C}$ and $S = \{\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \mid (c_i)_{i \in I} \in T_c^I\}$. We first show the following claim:

Claim. *For every $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \in S$, there are some $a, b \in \mathbf{C}$ such that (i) $a \sqcup b \sqsupseteq c$ and (ii) $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \sqsupseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b)$.*

Let $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \in S$. Since $I = J \cup K$, we have $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) = \bigsqcup_{j \in J} \mathfrak{s}_j(c_j) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(c_k)$. Let $a = \bigsqcup_{j \in J} c_j$ and $b = \bigsqcup_{k \in K} c_k$. From Th. 4, we have

$$\mathbb{D}_J(a) = \prod \left\{ \bigsqcup_{j \in J} \mathfrak{s}_j(a_j) \mid (a_j)_{j \in J} \in T_a^J \right\} \text{ and } \mathbb{D}_K(b) = \prod \left\{ \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (b_k)_{k \in K} \in T_b^K \right\}.$$

Given that $(\mathbf{C}, \sqsubseteq)$ is completely distributive and by associativity of \prod , $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) = \prod R$ where $R = \left\{ \bigsqcup_{j \in J} \mathfrak{s}_j(a_j) \sqcup \bigsqcup_{k \in K} \mathfrak{s}_k(b_k) \mid (a_j)_{j \in J} \in T_a^J \text{ and } (b_k)_{k \in K} \in T_b^K \right\}$. Clearly $a \sqcup b \sqsupseteq c$ and $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \in R$. Then $\bigsqcup_{i \in I} \mathfrak{s}_i(c_i) \sqsupseteq \prod R = \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b)$. This shows (i) and (ii).

From the above claim and Th. 4, we obtain $\mathbb{D}_I(c) = \prod S \sqsupseteq \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a, b \in \mathbf{C} \text{ and } a \sqcup b \sqsupseteq c\}$ as wanted. \square

PROOF OF THEOREM (5 (2)). Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Let $I, J, K \subseteq G$ be such that $I = J \cup K$.

Let $a, c \in \mathbf{C}$. Recall that $c \ominus a$ represents the least element $e \in \mathbf{C}$ such that $a \sqcup e \sqsupseteq c$. Take any $b \in \mathbf{C}$ such that $a \sqcup b \sqsupseteq c$. Then $b \sqsupseteq c \ominus a$ and since space functions are monotonic $\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \sqsupseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)$. From this it follows that $\prod (S \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a), \mathbb{D}_J(a) \sqcup \mathbb{D}_K(b)\}) = \prod (S \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)\})$ for any $S \subseteq \mathbf{C}$.

From Th. 5 (1) and the above argument, we have

$$\begin{aligned} \mathbb{D}_I(c) &= \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a \sqcup b \sqsupseteq c\} \\ &= \prod (\{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(b) \mid a \sqcup b \sqsupseteq c\} \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C}\}) \\ &= \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C}\}. \end{aligned}$$

Thus $\mathbb{D}_I(c) = \prod \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C}\}$. \square

PROOF OF THEOREM (5 (3)). Let $(\mathbb{D}_I)_{I \subseteq G}$ be the distributed spaces of an scs $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$ where $(\mathbf{C}, \sqsubseteq)$ is completely distributive. Let $I, J, K \subseteq G$ be such that $I = J \cup K$.

Let $c \in \mathbf{C}$ and take any $a' \not\sqsupseteq c$. It suffices to find $a \in \mathbf{C}$ such that $a \sqsubseteq c$ and $\mathbb{D}_J(a') \sqcup \mathbb{D}_K(c \ominus a') \sqsupseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)$ since then $\prod (S \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a), \mathbb{D}_J(a') \sqcup \mathbb{D}_K(c \ominus a')\}) = \prod (S \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)\})$ for any $S \subseteq \mathbf{C}$.

Given $a' \not\sqsupseteq c$ either (a) $a' \sqsupset c$ or (b) a' and c are incomparable w.r.t \sqsubseteq , written $a' \parallel c$.

- Suppose (a) holds. Then take $a = c$ thus $c \ominus a = \text{true}$. By monotonicity we have $\mathbb{D}_J(a') \sqcup \mathbb{D}_K(c \ominus a') \sqsupseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)$ as wanted.
- Suppose (b) holds, i.e., $a' \parallel c$. Notice that $c \ominus a' \sqsubseteq c$. By cases, assume $c \ominus a' = c$. Then we can take $a = \text{true}$, and thus $c \ominus a = c = c \ominus a'$. By monotonicity we have $\mathbb{D}_J(a') \sqcup \mathbb{D}_K(c \ominus a') \sqsupseteq \mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a)$ as wanted. Now suppose $c \ominus a' \sqsubset c$ holds. We can build a poset $P = (\{a' \sqcup c, a', c, c \ominus a', a' \sqcap (c \ominus a')\}, \sqsubseteq)$ which is a non-distributive sub-lattice of $(\mathbf{C}, \sqsubseteq)$, isomorphic to a lattice known as \mathbf{N}_5 (see Fig. 10). But this contradicts $(\mathbf{C}, \sqsubseteq)$ to be distributive (see [11]).

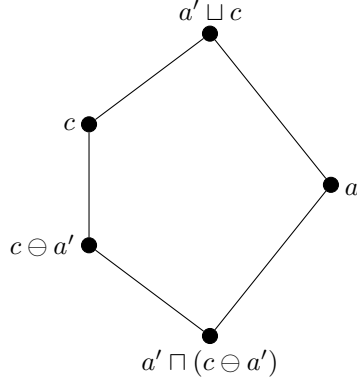


Figure 10: Poset $(\{a' \sqcup c, a', c, c \ominus a', a' \sqcap (c \ominus a')\}, \sqsubseteq)$ isomorphic to lattice \mathbf{N}_5 .

From Th. 5 (2) and the above argument, we have

$$\begin{aligned}
\mathbb{D}_I(c) &= \bigsqcap \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \in \mathbf{C}\} \\
&= \bigsqcap (\{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \sqsubseteq c\} \cup \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \not\sqsubseteq c\}) \\
&= \bigsqcap \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \sqsubseteq c\}.
\end{aligned}$$

Thus, $\mathbb{D}_I(c) = \bigsqcap \{\mathbb{D}_J(a) \sqcup \mathbb{D}_K(c \ominus a) \mid a \sqsubseteq c\}$. □

7. Conclusions and Related Work

We have introduced an algebraic theory for reasoning about possibly infinite groups of agents. We have also shown that the theory can be applied to other domains such as geometry and mathematical morphology.

This paper is an extended version of our CONCUR'19 paper [10]. With respect to that work, we have provided significant advances in both the theoretical and practical implications of our notion of distributed spaces in scs. On the theoretical side, we characterized spatial functions as maps preserving arbitrary joins (Prop. 5). This allowed us to delve into the interpretation of space functions of distributed spaces as meaningful operations in the field of mathematical morphology. We also pursued the study of conditions under which a piece of information derived by the combined local information of an infinite group of agents could be derived by some finite subgroup of those agents. We showed that this is the case when the information inferred by the group from a given supplied piece of information, is itself compact, and derivable from the combination of what each agent can infer from it in the local space (Th. 2). We further showed, however, that compactness does not hold in general without those conditions (Th. 3).

Furthermore, we showed a fundamental result that provides a way to compute, for completely distributive lattices, the greatest information that can infer some other given piece of information, and is below all possible combinations of local informations deriving that piece in the spaces of some given group of agents (Th. 4). This result is presented in [10] for finite cs; in this work we extended it for completely distributive lattices. In [10] we had also stated some properties relating the information of a group of agents w.r.t the information of subgroups of those agents but only for finite cs. In this paper we generalized these for completely distributive lattices (Th. 5).

Finally, we used the developed theory to investigate applications in mathematical morphology (MM). In this domain, two fundamental operations, dilation and erosion, provide ways to perform geometric transformations, in particular within the realm of image processing based on so-called structuring elements. We considered these MM operations, generalized with Minkowski addition over modules, and used our theory to derive some interesting distribution properties. We also gave the interpretation of maps in group spatial constraint systems as MM operations over structuring elements and showed that the maximum map under two given group distribution maps (seen as dilations) corresponds to the dilation over the intersection of their structuring elements. In so doing we provided a proof that erosion and dilations for structuring elements that are modules form a Galois connection (Prop. 13). This allowed us to prove that the operation of erosion corresponds to our defined operation of projection of information into the spaces of the structural element (Def. 17). We also discussed an interpretation of dilations and erosions as epistemic as an agent's perception of a given image (Remark 8).

7.1. Related Work

Below we discuss closely related work divided into four categories.

Algebraic Epistemic/Spatial Reasoning. There are two major branches of approaches to modal logic: Kripke semantics and Algebraic semantics [39]. The majority of modern work on epistemic logic has taken the Kripke semantics approach, whereas algebraic semantics are more common in the general study of modal logic. This is probably explained by the fact that the study of epistemic logic is most common in philosophy and computer science, with specific applications in both fields, and Kripke semantics are often the most practical approach for these applications. The Handbook of Epistemic Logic, a good survey of the recent state of work on epistemic logic, does not include algebraic semantics [40]. The correspondence between modal logic and boolean algebras with operators, and lattice theory more generally, however, is well known [41, 7, 39]. Since concurrent constraint programming (ccp) uses lattices as the underlying structure [16], in our first work adding epistemic information to ccp [6], it was natural to take the algebraic approach to epistemic logic, rather than the Kripke structure approach which is much more common in computer science applications of epistemic logic and modal logic in general. We continued to develop our theory of modal constraint systems [42, 43, 44, 18, 45, 10, 31], and besides its applicability to ccp, it has the added advantage of providing a more natural method than Kripke semantics for talking about infinite groups of agents, which is a focus of the present paper.

Since we take the algebraic approach which is less common in epistemic logic, there is not a great deal of recent work in epistemic logic which is closely related to the present paper. As already mentioned, spatial constraint systems (scs) were developed in [6] and are dual to (poly)modal algebras [21]. These algebras are boolean algebras that preserve meets and top and, they characterize the minimal (multi)modal logic K_n [46]. McKinsey and Tarski extended the Stone representation theorem for Boolean algebras to modal algebras whose operators are closure operators to give topological semantics for the epistemic modal logic $S4$ [22]. Epistemic constraint systems (ecs) are scs where the space functions are closure operators and thus they are dual to closure algebras [47]. There is more recent work on the relation of several modal logics and topology, among many others $S5$, $KD45$, and first-order extensions of modal logic (see [48]).

The work in [49] generalizes the epistemic update from the Logic of Epistemic Actions and Knowledge [50], to a general class of algebras. In the future, it will be interesting to study whether these epistemic updates on algebras apply to our systems. There is also work on proof theory for versions of epistemic logic, taking the algebraic approach [51].

Nevertheless none of the above-mentioned works deal with an algebraic characterization of distributed knowledge/information/space. To our knowledge this paper provides the first-algebraic characterization for distributed information of infinitely many agents.

We added related work as suggested by the reviewers (N4, N8, N11, N47). Also, we reorganized the related work that we had before.

Constraint Systems. The closest related work is that of [6] (and its extended version [52]) which introduces spatial constraint systems (scs) for the semantics of a spatial ccp language. Their work is confined to a finite number of agents and to reasoning about agents individually rather than as groups. We added the continuity requirement to the space functions of [6] to be able to reason about possibly infinite groups. In [18, 44, 43, 53] scs are used to reason about beliefs, lies and other epistemic utterances but also restricted to a finite number of agents and individual, rather than group, behaviour of agents.

Distributed Knowledge. Our work is inspired by the epistemic concept of distributed knowledge [24]. Knowledge in distributed systems was discussed in [54], based on interpreting distributed systems using Hintikka’s notion of possible worlds. In this definition of distributed knowledge, the system designer ascribes knowledge to processors (agents) in each global state (a processor’s local state). In [1] the authors present a general framework to formalize the knowledge of a group of agents, in particular the notion of distributed knowledge. The authors consider distributed knowledge as knowledge that is distributed among the agents belonging to a given group, without any individual agent necessarily having this knowledge.

Infinitely many agents. In [3] the authors study knowledge and common knowledge in situations with infinitely many agents. The authors highlight the importance of reasoning about infinitely many agents in situations where the number of agents is not known in advance. Their work does not address distributed knowledge but points out potential technical difficulties in their future work. In the realm of Economics and game theory, models of infinitely many agents are used to discover mass phenomena that do not necessarily occur in the case of a fixed finite number of agents [55, 56, 57, 58]. Also infinite sets of agents are used in game theory [59, 60]. For example for games played with two teams, we may want to specify that everyone in a team knows that everyone in the other team knows a given proposition, regardless of the team size. This could be naturally specified with infinitely many agents.

Mathematical Morphology and Topology. Complete lattices have been used as a framework to define morphological operators specifically to study grey-level images [38]. In this context, dilations and erosions are defined as operators that preserve arbitrary suprema and infima, resp. This proposal is a generalization of what we studied in Section 5 where we present dilations and erosions by some *structuring element*. As a novelty, we proposed the scs $(\mathcal{P}(M), \subseteq, (\delta_S)_{S \subseteq M})$ where M is a module and $(\delta_S)_{S \subseteq M}$ are dilations defined on the cs $(\mathcal{P}(M), \subseteq)$. It allowed us to apply our theoretical results to prove MM properties, e.g., that dilations and erosions form a Galois connection (Prop. 13). Also, we provided the interpretation of distributed information for images. Namely, we showed that given two dilations δ_A and δ_B , the greatest dilation below them is exactly $\mathbb{D}_{\{A,B\}}$ which in turn equals $\delta_{A \cap B}$ (Th. 7). As a future work, we plan to explore these results for grey-scale images.

We note that the δ_S function bears a superficial resemblance to the topological notion of closure, and ε_S resembles topological interior. Topology has important connections with modal logic and S4 type epistemic logic in particular [48, 61]. However, in the case of mathematical morphology, the transformations are different from topological operations, most importantly because the operators are not necessarily idempotent, whereas closure and interior operators must be idempotent.

Other related work. In [31] the authors investigate the cardinality of the set $\mathcal{E}(L)$ of all join-endomorphisms of a given lattice L . (A join-endomorphism is a self-map that preserves finite joins, hence it is a space function without the continuity requirement.) The authors also provide efficient algorithms to compute the meet of a given set of join-endomorphisms. In this paper, we briefly illustrated the use of Th.4 to derive a polynomial complexity bound for computing this meet.

There are other group phenomena that are closely related to the group phenomena here studied. In particular, group polarization [62, 63], from social sciences, and group improvisation [64], from computer music. Group polarization refers to the natural tendency of a group to make more extreme decisions than their individuals. Group improvisation involves constraining musical pattern variation choices of a participant according to choices made by others in the group. We plan to study these phenomena in future work by building upon the present work.

7.2. Future Work

Some of the proofs of the main results of this work rely on the completeness of the lattice of the underlying constraint system (Th. 4). As future work we would like to generalise our results to the directed-complete partial orders, the central semantic structure of domain theory [13]. In fact, Scott-continuity is a central concept of our theory, hence this research direction seem promising.

We plan to extend our work with a process calculus with dynamic creation/removal of agents. The idea is to investigate situations where we could prove that it is not possible to reach a state where $\mathbb{D}_A(e)$ holds, with A as the set of all possible agents and e is some sensible information. This could be interpreted as saying that the agents will never be able to derive s by pooling or joining their own information. Along the same lines, for meaningful scs such as Kripke scs with infinitely many agents, we would like to study, the decision and complexity problem of whether $c \sqsupseteq \mathbb{D}_I(e)$ given c, e and infinite set I under the conditions of our compactness result (Th.2). Notice that our compactness result, does not provide us with a bound on the size of the finite subset $J \subset I$ such that $d \sqsupseteq \mathbb{D}_J(e)$.

We added future work according to N7, N17, N45, N50, N57.

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 (P, \sqsubseteq) , poset, 5
 $(\mathcal{S}(\mathbf{C}), \sqsubseteq_s)$, set of space functions ordered by \sqsubseteq_s , 18
 $(x_j)_{j \in J}$, J -tuple of elements $x_j \in X$ for each $j \in J$, 26
 $(\mathbf{C}, \sqsubseteq, (\mathfrak{s}_i)_{i \in G})$, spatial constraint system, scs, 11
 R -module M , module M over a ring R , 29
 \mathbf{C} , $(\mathbf{C}, \sqsubseteq)$, Constraint system, cs, 6
 Δ , set of pointed Kripke structures, 14
 $\mathbb{D} = (\mathbb{D}_I)_{I \subseteq G}$, distributed spaces, 19
 \mathbb{D}_I , distributed space of the group I , 19
 $D_I(e)$, distributed knowledge of e in group I , 20
 $K_i(e)$, agent i knows e , 20
 \mathbf{M}_3 , non-distributive lattice, 26
 $\Pi_I(c)$, I -group projection of c , 23
 $\mathcal{A} = (S, \mathcal{P}_1, \dots, \mathcal{P}_n)$, Aumann structure, 20
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 \sqcup_P , join in poset P , 5
 \sqcap, \sqcap , infimum, meet, glb, 5
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 $\mathcal{C}[\![\phi]\!]$, Kripke scs interpretation of ϕ , 15
 \sqsubseteq , entailment, 5, 6
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 \xrightarrow{i}_M , accessibility relation, 14
 λ_{\perp} , bottom space, 18
 λ_{\top} , top space, 18
 $\mathcal{L}_n(\cdot)$, modal language, 15
 \mathbf{N}_5 , non-modular lattice, 42
 \max , maximum operator, 5
 \ominus , co-Heyting subtraction, 9
 \oplus , Minkowski addition, 29
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