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► **To cite this version:**

Marco Panza, Francesca Boccuni. Frege's Theory of Real Numbers: A Consistent Rendering. *The review of symbolic logic*, 2021, 15, 2022 (3), pp.624-667. hal-03098430

HAL Id: hal-03098430

<https://hal.science/hal-03098430>

Submitted on 5 Jan 2021

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Frege’s Theory of Real Numbers: A Consistent Rendering

FRANCESCA BOCCUNI* & MARCO PANZA†

Abstract

Frege’s definition of the real numbers, as envisaged in the second volume of *Grundgesetze der Arithmetik*, is fatally flawed by the inconsistency of Frege’s ill-fated *Basic Law V*. We restate Frege’s definition in a consistent logical framework and investigate whether it can provide a logical foundation of real analysis. Our conclusion will deem it doubtful that such a foundation along the lines of Frege’s own indications is possible at all.

1 Overview

The aim of the present paper is twofold: (i) rephrasing Frege’s inconsistent definition¹ of real numbers, as envisaged in Part III of *Grundgesetze der Arithmetik* (Frege 1893-1903), in a consistent setting ruling out value-ranges, and so involving no version of the infamous *Basic Law V* (BLV); (ii) wondering whether the rephrased definition can be considered logical, and, as such, as a ground for a logicist view about real analysis.

Concerning (ii) a proviso is in order. In the debate on neologicism, a distinction has been made between logicity and analyticity, by suggesting, for instance, that, though not logical, Hume’s Principle (HP) is analytic. We are far from undermining the relevance of this distinction, but we consider unnecessary to stress it for our present purpose. There are two reasons for that. On the one side, we deem all the arguments we will advance against the logicity of the relevant principles and definitions also apt to oppose their analyticity—though some of those advanced in favor of the former are possibly only sufficient to argue for the latter. On the other side, we are interested in the epistemic attitude that a faithful Fregean (or even Frege himself) might have (had) in the face of a definition such as our own. Hence, for the sake of our discussion, we must follow Frege himself in taking a “truth” to be analytic if, in its proof, “one only runs into logical laws and definitions” (Frege 1884, § 3; Frege 1953, p. 4)², and in regarding definitions as mandatorily explicit, which suggests regarding logicity as a necessary condition for analyticity, rather than the latter as a weaker condition than the former.

Concerning (i), it is important to observe that, for Frege, real numbers had to be defined as ratios of magnitudes, and magnitudes had to belong to different domains. Hence, his definition should have included two successive steps: a definition of domains of magnitudes, and a definition of ratios on these domains. As a matter of fact, he accomplished only the former step, and merely gave some informal indications on how to accomplish the latter. Both things are done in the

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¹In what follows we will use ‘definition’ quite broadly. The meaning of its occurrences will be clarified in context.

²We slightly modify Austin’s translation.

second volume of *Grundgesetze*³. The latter step should have presumably been accomplished in a third volume that, once aware of Russell’s paradox, Frege never wrote.

Had he accomplished this step, he should have made it conform to a crucial requirement: supposing that several domains of magnitude exist, the definition of real numbers should have identified a ratio on one of these domains with the same real number as a ratio on each one of the others. Our definition actually complies with this requirement.

After a short presentation of Frege’s strategy in § 2, we will consistently rephrase his definition of domains of magnitudes in § 3⁴. To eliminate value-ranges, we will rephrase first-order formulas involving terms for them as higher-order formulas proper to a system of higher-order predicate logic as weak as possible, on which we will take stock in § 4. This seems to us the most faithful way to consistently render Frege’s original definition. Insofar as our appreciation of the logicity of our definition depends on assuming, in a genuinely Fregean vein, the logicity of higher-order logic, we contend that this appreciation *ipso facto* provides an appreciation of the logicity of Frege’s own definition that remains perfectly independent of any judgement about the logicity of (any consistent version of) BLV⁵.

In § 5, we will investigate how to define real numbers by following Frege’s indications, on the base of our definition of domains of magnitudes. In § II.164 of his treatise, Frege explicitly acknowledged that his envisaged definition of these numbers requires an existence proof of nonempty such domains. We will explain why this is so. Here, it is only in order to observe that, in this same §, he also argues that this existence depends on the existence of continuously many objects (an infinity of objects larger than “*Endloss*”, the cardinality of “finite cardinal numbers”), and sketches a plan for this proof, which, taking the existence of natural numbers for granted, aims at constructing these objects from them. He then claims that, thanks to this proof, he would have succeeded “in defining the real number purely arithmetically or logically as a ratio of magnitudes that are demonstrably there” (Frege 2013, p. 162₂).

The adverb “arithmetically” is clearly used to emphasize that the envisaged definition would have been independent of both empirical considerations and geometry. In this sense, the definition would have surely been arithmetical, and our rendering of it will be as well. But there is another sense in which, despite his appealing to natural numbers, Frege did not certainly want his definition to be arithmetical: both his criticisms to the alternative definitions depending on an extension of the domain of rationals—including Cantor’s (§§ II.68-85), Dedekind’s (§§ II.138-147), and Weierstrass’s (§§ II.148-155)—and the very purpose of identifying real numbers with ratios of magnitudes make clear he wanted these numbers to be strictly independent of natural ones, to be properly *Zahlen*, rather than *Anzahlen*. By offering our definition, we will try, among other things, to comply with this requirement.

In § 6, we will account for two distinct strategies to get the required existence proof in our setting. One of them conforms to Frege’s indications, while the other might be considered more appropriate for ensuring logicity, since, *pace* Frege, it does not require that the existence of continuously many objects be established. In § 7, we will investigate whether the resulting

³Frege (1893-1903), §§ II.165-II.245 and § II.164, respectively; §§ II.55-II.159 contain a critical discussion of alternative definitions, while §§ II.160-II.163 contain an informal introduction and a principled justification of the definition of domains of magnitudes.

⁴Knowledge of Frege’s original definition is required to appreciate its correspondence with our rephrasing. Useful accounts of it can be found in Dummett (1991, ch. 22); Schirn (2013); Simons (1987); Shapiro & Snyder (2020).

⁵Anyone supporting Quine’s view on the non-logicity of higher-order logic can take our granting it as made for the sake of the argument.

definition does comply with logicity and non-arithmeticity—in the mentioned sense. In § 8, we will provide some concluding remarks.

2 Frege’s Strategy

Frege’s strategy agrees with the “application constraint”: the requirement that a mathematical theory be shaped as to immediately account for its applications⁶. This motivates his suggestion to define real numbers as ratios of magnitudes, magnitudes as elements of distinct domains supposedly including those of geometric, mechanic and empirical ones, and ratios on these domains as measures of the relevant magnitudes. Insofar as it would be odd to require that the theory of real numbers involve these magnitudes as such, together with their respective theories, this makes providing a structural definition of domains of magnitudes mandatory: a definition that merely fixes the conditions that a certain domain of independent items has to meet in order to be recognized as a domain of magnitudes. Frege himself clearly stresses this crucial point (Frege 1893-1903, §II.161; Frege 2013, p. 158₂):

There are many different kinds of magnitudes: lengths, angles, periods of time, masses, temperatures, etc., and it will scarcely be possible to say how objects that belong to these kinds of magnitudes differ from other objects that do not belong to any kind of magnitude. Moreover, little would be gained thereby; for we still lack any way of recognizing which of these magnitudes belong to the same domain of magnitudes.

Instead of asking which properties an object must have in order to be a magnitude, one needs to ask: how must a concept be constituted in order for its extension to be a domain of magnitudes?

A natural way to render the required structural definition would have provided definitional axioms, as usually done for groups or fields. An informal conception of magnitudes recognizing the existence of “lengths, angles, periods of time, masses, temperatures, etc.” might have suggested that there are non-isomorphic models satisfying these axioms. Still, for Frege, magnitudes are just those items that real numbers are ratios of, and they all behave as lengths do, so that domains of magnitudes are all isomorphic to each other. Had he defined them through appropriate axioms, these should have then been expected to be categorical, though algebraic in nature—as it happens for the usual axioms for real numbers themselves, namely the axioms of a totally ordered and Dedekind-complete field. Moreover, insofar as magnitudes are required to add to each other but not to multiply with each other (namely to admit only a single internal composition law), what he would have needed is a categorical axiomatization for totally ordered, dense and Dedekind-complete (and, then, also Abelian and Archimedean)⁷ groups.

Frege did not straightforwardly follow this route, however. Conforming with a remark by Gauss (1931, p. 635, also in *Werke*, II, pp. 175-76; quoted in Frege, 1893-1903, § II.161) and putting it in his perspective, he conceives of magnitudes as value-ranges of permutations, and so defines their domains not as domains of items merely satisfying certain conditions, but rather as domains of extensions of appropriate first-level binary relations satisfying these conditions. This makes him able to appeal, along with his definitions, to structural properties of first-level binary

⁶See Panza & Sereni (2020) and Sereni (2019), which include a critical survey of the recent discussion on Frege’s attitude toward applications of mathematical theories.

⁷See footnotes 19 and 28 below.

relations, namely to the way they compose and are inverted, as well as to there being the identity relation among them. Accordingly, rather than listing a number of axioms, and finally getting an implicit definition, Frege explicitly defines domains of magnitudes as extensions of concepts under which extensions of certain first-level relations fall. The difficulty he tackles is then that of looking for an explicit definition of the concept of being one of these extensions (and, then, a magnitude), and falling under one of these concepts.

Since for Frege extensions are objects, this concept is first-level. In order to define it, he appeals to a special function allowing him to reduce higher-level concepts to first-level ones, so as to work in a first-order fragment of his second-order theory. This is the first-level two-arguments function $\xi \frown \zeta$, often too quickly identified with set-theoretic membership, whose definition is licensed by BLV. Once BLV is omitted, this function can no more be defined, and the reduction to first-order is no more possible—unless by a form of set theory. Hence, making Frege’s definition consistent by eliminating BLV without falling into a set-theoretical setting requires replacing Frege’s first-order definitions with higher-order ones. We will explain how this can be done by appropriately rephrasing Frege’s definitions, and in clarifying the logical nature of the (logical) system that is required for that. This is the purpose of the next two §§.

3 Frege’s Definition of Domains of Magnitudes Rephrased

3.1 Eliminating Value-Ranges

The omission of BLV is made possible by the elimination, from Frege’s language, of terms for value-ranges. Insofar as the presence of these terms in his definition of domains of magnitudes entirely depends on the function $\xi \frown \zeta$, we have to make its use pointless.

To make a long story short, this function is such that for any objects Γ and Δ , if Γ is the value-range $\dot{\varepsilon}\Phi(\varepsilon)$ of a first-level one-argument function $\Phi(\xi)$ ⁸, then $\Delta \frown \Gamma$ is $\Phi(\Delta)$, and if Γ is not such a value-range—or, better, it is not a value-range at all, since, in Frege’s formalism, any value-range reduces to the value-range of a first-level one-argument function—, then $\Delta \frown \Gamma$ is the value-range of a first-level concept under which no object falls—for example that of $\neg(\xi = \xi)$, which we could denote by ‘ \emptyset ’, for short. In other terms, $\Delta \frown \Gamma$ is the value, for Δ as argument, of the first-level one-argument function of which Γ is the value-range, if Γ is a value-range, and \emptyset , if it isn’t—whatever object Δ might be.

In a rich enough second-order predicate language including the operator ‘ $\iota z [z : \varphi]$ ’ for definite descriptions, together with a symbol for value-ranges, the individual variables ‘ x ’, ‘ y ’ and ‘ z ’, and the monadic predicate one ‘ F ’, this stipulation could be rendered as follows:

$$\forall x, y [x \frown y = \iota z [z : \exists F (y = \dot{\varepsilon}F(\varepsilon) \wedge F(x) = z)]] ,$$

provided that ‘ $\iota z [z : \varphi]$ ’ designates a well-defined object, namely \emptyset , even if there is no z such that φ . If ‘ a ’ and ‘ b ’ are terms, this makes ‘ $a \frown b$ ’ be a term in turn.

This licenses using this term to denote the ζ -argument of the same function $\xi \frown \zeta$. Taking a new term ‘ c ’ to denote the ξ -argument, one has the new term ‘ $c \frown (a \frown b)$ ’ such that

$$c \frown (a \frown b) = \iota z [\exists G (b = \dot{\alpha}G(\varepsilon, \alpha) \wedge G(c, a) = z)] .$$

⁸Our use of Greek capital letters to denote objects and functions whatsoever corresponds to Frege’s, in his “exposition” of his formal language (Frege 1893-1903, Part I, § I.1-52).

It follows that ‘ $c \frown (a \frown b)$ ’ is a term that denotes the value for c and a as arguments of the first-level two-argument function of which b is the value-range, if b is such a value-range⁹.

Hence, if $\Phi(\xi)$ and $\Psi(\xi, \zeta)$ are a first-level one-argument and a first-level two-argument function, respectively, then

$$\begin{aligned} a \frown \dot{\varepsilon}\Phi(\varepsilon) &= \Phi(a) \\ c \frown (a \frown \dot{\alpha}\dot{\varepsilon}\Psi(\varepsilon, \alpha)) &= \Psi(c, a). \end{aligned}$$

Suppose that ‘ \mathbf{P}_b ’ and ‘ \mathbf{R}_b ’ be respectively a monadic and a dyadic predicate¹⁰ appropriate for rendering, in an appropriate predicate language, two functions $\Phi(\xi)$ and $\Psi(\xi, \zeta)$ of which b is the value-range. It follows that, in order to make the use of the function $\xi \frown \zeta$ pointless, and so eliminate value-ranges while restating Frege’s definition of domains of magnitudes, it is enough to replace each term of the form ‘ $a \frown b$ ’ with the formula ‘ $\mathbf{P}_b a$ ’ and each term of the form ‘ $c \frown (a \frown b)$ ’ with the formula ‘ $c\mathbf{R}_b a$ ’, and to transform Frege’s formal system accordingly¹¹. The system so obtained will be independent of BLV, and so will any definition stated in it.

3.2 Working with Binary First-Order Relations

Informally speaking, Frege conceived of a nonempty domain of magnitudes as a totally ordered, dense and Dedekind-complete additive group of permutations. In light of his rejection of implicit definitions, defining such a group required to explicitly defining a particular function to play the role of its (additive) law of composition, which required, in turn, to have objects available, endowed with an internal structure making such a definition possible. To this purpose, he made the simplest choice possible: he took those objects to be extensions of functional first-level binary relations, and assigned the role of this law to the composition of the corresponding relations. This obviously resulted in taking the extension of the identity relation as the neutral element of the group, and the extensions of the inverse relations as its inverse elements.

This choice is easily rendered in a predicate setting lacking extensions. We merely have to fix the conditions under which a first-level binary relation is functional and results either from the inversion of another such relation, or from the composition of two other such relations. Taking ‘ R ’ and ‘ S ’ to range over first-level binary relations, this can be formally done through the following explicit definitions:

$$\begin{aligned} [\text{Functionality}] \quad & \forall R (\mathcal{S}R \Leftrightarrow \forall x, y (xRy \Rightarrow \forall z (xRz \Rightarrow y = z))), \\ [\text{Inversion}] \quad & \forall R \forall x, y (xR^{-1}y \Leftrightarrow yRx), \\ [\text{Composition}] \quad & \forall R, S \forall x, y (x[R \sqcup S]y \Leftrightarrow \exists z (xRz \wedge zSy)). \end{aligned}$$

These definitions define three third-order constants, respectively: the monadic predicate constant ‘ \mathcal{S} ’, designating a property of first-level binary relations; the monadic functional constant

⁹If b is not such a value-range, different cases are possible. It is not necessary to account for them, here. For a complete treatment, see Panza (FC1).

¹⁰Here and in what follows, we take boldface capital latin letters as dummy letters for first-level properties and relations. The same letters in italics will, instead, be used for the corresponding variables.

¹¹To be sure, this rendering of the function $\xi \frown \zeta$ in terms of predication is not fully faithful to Frege’s original view: for Frege, both ‘ $\Delta \frown \Gamma$ ’ and ‘ $_ \Delta \frown \Gamma$ ’ (see footnote (12), below) denote an object, while for us ‘ $a \frown b$ ’ is rather a formula. Nevertheless, whenever ‘ Γ ’ denotes the extension of a concept $\Phi(\xi)$, ‘ $\Delta \frown \Gamma$ ’ and ‘ $_ \Delta \frown \Gamma$ ’ are, for Frege, names of the True if and only if ‘ $\Phi(\Delta)$ ’ itself is a name of the True. Insofar as only this case is relevant here, this rendering does not alter the aspects of Frege’s definition that are of interest here.

‘ \sim ’, designating a one-argument function from and to first-level binary relations; the dyadic functional constant ‘ $\sim \sqcup \sim$ ’, designating a two-argument function from and to first-level binary relations, again. Hence, in order to be licensed, they require appropriate instances of predicative comprehension. The first requires the following instance of third-order predicative comprehension without parameters:

$$\text{(Functionality-CA)} \quad \exists \mathcal{X} \forall R (\mathcal{X} R \Leftrightarrow \forall x, y (x R y \Rightarrow \forall z (x R z \Rightarrow y = z))).$$

The other two require the following instances of second-order dyadic predicative comprehension with parameters respectively:

$$\text{(Inversion-CA)} \quad \forall R \exists S \forall x, y (x S y \Leftrightarrow y R x),$$

$$\text{(Composition-CA)} \quad \forall R, S \exists T \forall x, y (x T y \Leftrightarrow \exists z (x R z \wedge z S y)),$$

where ‘ T ’ ranges over first-level binary relations, too.

One might replace, however, these explicit definitions with mere (metalinguistic) typographic stipulations:

$$\text{(Functionality')} \quad \mathcal{S}(R) := \forall x, y (x R y \Rightarrow \forall z (x R z \Rightarrow y = z)),$$

$$\text{(Inversion')} \quad R^- (xy) := y R x,$$

$$\text{(Composition')} \quad R \sqcup S (xy) := \exists z (x R z \wedge z S y).$$

Any instance of the left-hand side of these stipulations is intended to be a mere abbreviation of the corresponding instance of the right-hand side. For example, while ‘ $\mathcal{S}R$ ’ in (Functionality) is an atomic third-order (open) formula, ‘ $\mathcal{S}(R)$ ’ in (Functionality’) is an atomic symbol abbreviating the second-order (open) formula ‘ $\forall x, y (x R y \Rightarrow \forall z (x R z \Rightarrow y = z))$ ’. And analogously for ‘ $x R^- y$ ’ and ‘ $R^- (xy)$ ’ in (Inversion) and (Inversion’), respectively, and for ‘ $x [R \sqcup S] y$ ’ and ‘ $R \sqcup S (xy)$ ’ in (Composition) and (Composition’), respectively. Adopting these stipulations requires neither any instance of comprehension, nor any extension of the usual second-order language.

We will see in what follows whether these stipulations are enough for our purpose, or the corresponding explicit definitions are needed, and the instances of comprehension they require¹².

¹²In Frege’s original setting things would not be so simple. Consider only the example of (Functionality). In this setting, the role of this definition is played by the definition of the first-level concept $\mathbf{I}\xi$ (Frege, 1893-1903, § I.37). By adapting Frege’s notation to our modern one, the definition might be stated as follows:

$$[\forall x, y [_ (x \frown (y \frown a)) \Rightarrow \forall z [_ (x \frown (z \frown a)) \Rightarrow y = z]]] = \mathbf{I}a.$$

where ‘ a ’ is a term used as a parameter, and $_ \xi$ is the horizontal concept (*ibidem*, § I.8), which is such that $_ \Gamma$ is the True if Γ is also the True, and the False otherwise. It follows that $\mathbf{I}a$ is the same object as $\forall x, y [_ (x \frown (y \frown a)) \Rightarrow \forall z [_ (x \frown (z \frown a)) \Rightarrow y = z]]$, which is a truth-value. If a is not a value-range of a first-level binary relation, $_ (b \frown (c \frown a))$ is the False for whatever pair of objects b and c , and $\mathbf{I}a$ is then the True, which makes any object other than a value-range of a first-level binary relation fall under the concept $\mathbf{I}\xi$. If a is the value-range of a first-level binary relation $\Phi(\xi, \zeta)$, a falls under the concept $\mathbf{I}\xi$ if and only if either $\Phi(\xi, \zeta)$ is empty, or, for any x , there is at most one y such that $\Phi(x, y)$ is the True. Clearly, there is no way to regard this definition as a mere typographic stipulation. It rather defines a total first-level concept by introducing a functional constant to designate it. Among many others, there are two relevant differences with our case: *i*) Frege’s definition applies in general, whereas both (Functionality) and (Functionality’) only apply to first-order binary relations; *ii*) Differently from (Functionality’), Frege’s definition is licensed only via a stipulation analogous to second-order comprehension. *Mutatis mutandis*, this also applies to (Inversion) and (Composition), and to any other particular definition entering his definition of domains of magnitudes.

3.3 Domains of Classes

For Frege, a domain of magnitudes is the domain of a “positive class”, which is in turn a “positival class” of an appropriate sort. In his setting, a class is the extension of a first-level concept (Frege, 1893-1903, § II.16), and the objects falling under this concept are said to belong to the class. Positival and positive classes are, in particular, extensions of concepts under which (only) extensions of first-level binary relations fall. Defining them amounts to fixing the conditions that a concept is to meet for the objects falling under it to be just these extensions. To do this, Frege appeals to their “domains”. He has, then, to firstly define, in general, domains of classes (*ibidem*, § II.173). The definition applies to any class, but we only need to consider its application to the case of the domain of a class of extensions of first-level binary relations.

This is the extension of a concept under which fall: the extensions in the class; the extensions of the inverses of the relations whose extensions are in the class; and the extensions of the relations composed by each of the relations whose extensions are in the class and their inverses—which in case these relations are functional, as required for both positival and positive classes, all coincide with the extension of the identity relation. In our setting, we can, then, rephrase, Frege’s definition of the domain of a class of extensions of first-level binary relations as follows:

$$(3.1) \quad \forall \mathcal{X} \forall R \left(\bar{\partial} \mathcal{X} R \Leftrightarrow \left\{ \begin{array}{l} \mathcal{X} R \forall \\ \exists S \left[\mathcal{X} S \wedge \forall x, y \left[\begin{array}{l} [xRy \Leftrightarrow S^-(xy)] \vee \\ [xRy \Leftrightarrow S \sqcup S^-(xy)] \end{array} \right] \right] \right\} \right)$$

where ‘ \mathcal{X} ’ is a third-order monadic variable, and ‘ $\bar{\partial}$ ’ a functional operator applied to it. This definition makes clear that, when applied to whatever (second-level) property \mathcal{Q} of first-level binary relations¹³, $\bar{\partial}$ gives another property $\bar{\partial}\mathcal{Q}$ of these same relations.

To license this definition, we need to ensure the existence and uniqueness of a second-level property providing a putative value for $\bar{\partial}\mathcal{X}$ under the existence of a second-level property providing a value for \mathcal{X} , and this requires, in turn, third-order comprehension with parameters. But suppose we wanted to define a certain (third-level) property \mathcal{Q} ¹⁴ that a class of first-level binary relations should have in order to be positival, which is required to render Frege’s definition of positival classes. If, in defining it, we had to appeal to the domains of the classes that could have it, as is also required to render Frege’s definition, we should have recourse to a definition like this:

$$\forall \mathcal{X} [\mathcal{Q}\mathcal{X} \Leftrightarrow \phi(\bar{\partial}\mathcal{X})]$$

where ‘ $\phi(\bar{\partial}\mathcal{X})$ ’ stands for an appropriate formula involving the predicate ‘ $\bar{\partial}\mathcal{X}$ ’. Hence, insofar as, in our rendering of Frege’s definition of positival and positive classes and domains of magnitudes, this predicate would only appear in instances of formulas of the form ‘ $\bar{\partial}\mathcal{X}R$ ’, we can replace (3.1) with the following abbreviation stipulation

$$(3.1') \quad \bar{\partial}(\mathcal{X})(R) := \left\{ \begin{array}{l} \mathcal{X} R \forall \\ \exists S \left[\mathcal{X} S \wedge \forall x, y \left[\begin{array}{l} [xRy \Leftrightarrow S^-(xy)] \vee \\ [xRy \Leftrightarrow S \sqcup S^-(xy)] \end{array} \right] \right] \right\},$$

then use appropriate instances of ‘ $\bar{\partial}(\mathcal{X})(R)$ ’ instead of the corresponding instances of ‘ $\bar{\partial}\mathcal{X}R$ ’. As a matter of fact, this stipulation is all we need for our present purpose, and it must be supplied

¹³Here and in what follows we use ‘ \mathcal{Q} ’ as a dummy letter for second-level properties. Later we will also use ‘ \mathcal{A} ’, ‘ \mathcal{E} ’, ‘ \mathcal{H} ’, ‘ \mathcal{L} ’, ‘ \mathcal{M} ’ and ‘ \mathcal{P} ’ for the same purpose.

¹⁴Here we use ‘ \mathcal{Q} ’ as a dummy letter for third-level properties.

by no sort of comprehension, since it introduces no new predicate, but merely lets each instance of its left-hand side be an abbreviation of the corresponding instance of the right-hand side. For short, read both ‘ ∂XR ’ and ‘ $\partial(\mathcal{X})(R)$ ’ as ‘ R belongs to the domain of the class of first-order binary relations that have \mathcal{X} ’.

3.4 Positival Classes

We can now consider Frege’s definition of positival classes. If we had to render it through a(n explicit) definition, we should define a fourth-order monadic predicate constant designating a third-level property. This would require to quantify over second-level properties, and, then, to appeal to fourth-order comprehension. But, once again, we are not forced to do it. As above, we might recur to an abbreviation stipulation by so avoiding any sort of comprehension.

In agreement with Frege’s definition, the extension of a first-level binary relation \mathbf{R} belongs to a positival class if (and only if): both \mathbf{R} and its inverse are functional; the extension of $\mathbf{R} \sqcup \mathbf{R}^-$, i.e. the identity relation, does not belong to the class; and for any first-level binary relation \mathbf{S} , if its extension belongs to the class, then: the class of the objects that bear \mathbf{R} to some other object coincides with the class of the objects to which some object bears \mathbf{S} ¹⁵; the extension of $\mathbf{R} \sqcup \mathbf{S}$ belongs to the class; both the extension of $\mathbf{R}^- \sqcup \mathbf{S}$ and that of $\mathbf{R} \sqcup \mathbf{S}^-$ belong to the domain of the class. In our setting, this can be rendered either this way

$$(3.2) \quad \forall \mathcal{X} \left[\mathcal{L}\mathcal{X} \Leftrightarrow \forall R \left[\mathcal{X}R \Rightarrow \left[\begin{array}{l} \forall S \left[\mathcal{X}S \Rightarrow \left[\begin{array}{l} \forall x [\exists y (xRy) \Leftrightarrow \exists z (zSx)] \wedge \\ \mathcal{X}R \sqcup S \wedge \\ \partial \mathcal{X}R \sqcup S^- \wedge \\ \partial \mathcal{X}R^- \sqcup S \\ \mathcal{I}R \wedge \mathcal{I}R^- \wedge \neg \mathcal{X}R \sqcup R^- \end{array} \right] \right] \wedge \right] \right] \right],$$

or this way:

$$(3.2') \quad \mathcal{L}(\mathcal{X}) := \forall R \left[\mathcal{X}R \Rightarrow \left[\begin{array}{l} \forall S \left[\mathcal{X}S \Rightarrow \left[\begin{array}{l} \forall x [\exists y (xRy) \Leftrightarrow \exists z (zSx)] \wedge \\ \mathcal{X}R \sqcup S \wedge \\ \partial(\mathcal{X})(R \sqcup S^-) \wedge \\ \partial(\mathcal{X})(R^- \sqcup S) \\ \mathcal{I}(R) \wedge \mathcal{I}(R^-) \wedge \neg \mathcal{X}R \sqcup R^- \end{array} \right] \right] \wedge \right] \right] \right],$$

where both ‘ $\mathcal{L}\mathcal{X}$ ’ and ‘ $\mathcal{L}(\mathcal{X})$ ’ are short for ‘ \mathcal{X} is a positival class’ or, more precisely, ‘the first-level binary relations having \mathcal{X} form a positival class’.

In (3.2), ‘ \mathcal{L} ’ is a fourth-order predicate constant and ‘ $\mathcal{L}\mathcal{X}$ ’ an atomic (open) formula. This is, then, an explicit definition, which is to be licensed by an appropriate form of fourth-order comprehension. In (3.2’), ‘ $\mathcal{L}(\mathcal{X})$ ’ is, instead, an abbreviated (open) formula, and ‘ \mathcal{L} ’ is merely a symbol occurring in it. Hence (3.2’) neither requires a fourth-order language nor is to be licensed by any form of fourth-order comprehension.

¹⁵This makes the relations whose extensions belong to the class permutations on a subjacent first-order domain.

This does not mean that no form of comprehension is required to license it. Since, in its right-hand side, the signs ‘ \neg ’ and ‘ \sqcup ’ do not merely occur as parts of the abbreviated formulas ‘ $R^-(xy)$ ’ and ‘ $R \sqcup S(xy)$ ’ introduced by (Inversion’) and (Composition’), but as functional signs allowing to construe the predicate variables ‘ R^- ’ and ‘ $R \sqcup S$ ’. Their use in (3.2’) is, then, to be licensed by the explicit definitions (Inversion) and (Composition), which respectively require, in turn, (Inversion-CA) and (Composition-CA), or, more generally, the following second-order predicative comprehension axiom schema with parameters:

$$(PCA_{\Delta_1}^2) \quad \forall R \dots T \exists U \forall x, y \left[xUy \Leftrightarrow \phi_{\Delta_1}(R \dots T) \right],$$

where ‘ U ’, ‘ R ’, ‘ S ’, and ‘ T ’ range over first-level binary relations, and ‘ $\phi_{\Delta_1}(R \dots T)$ ’ stands for any second-order formula containing the parameters ‘ R ’, \dots ‘ T ’, but no higher-order quantifiers.

Before going ahead with the definition of positive classes, some remarks are in order about the informal import of the conditions characterizing a positival class. They apply, *mutatis mutandis*, both to (3.2) and to (3.2’), but, for short and simplicity, we only make them about the latter.

Let \mathcal{L} be a second-level property. Requiring that

$$\forall R [\mathcal{L}R \Rightarrow (\mathcal{I}(R) \wedge \mathcal{I}(R^-))]$$

amounts to requiring that both a binary relation that has \mathcal{L} and its inverse are functional. If this condition obtains, requiring that

$$\forall R [\mathcal{L}R \Rightarrow \neg \mathcal{L}R \sqcup R^-]$$

and that

$$\forall R, S \forall x [(\mathcal{L}R \wedge \mathcal{L}S) \Rightarrow \exists y [(xRy) \Leftrightarrow \exists z (zSx)]]$$

respectively amount to requiring that the identity relation has not \mathcal{L} , and that all the relations having \mathcal{L} are permutations¹⁶ on a subjacent unspecified set. Hence, only permutations but the identity one, have \mathcal{L} . Thus, \sqcup is an associative law of composition without neutral element on the relations having \mathcal{L} . Again, if all the above conditions obtain, requiring that

$$\forall R, S [(\mathcal{L}R \wedge \mathcal{L}S) \Rightarrow \mathcal{L}R \sqcup S]$$

amounts to requiring that the family of permutations having \mathcal{L} is closed under \sqcup . This makes: the inverse of any such permutation not have \mathcal{L} —since, if it did, the identity permutation would also have it; the family of permutations that satisfy the open formula ‘ $\bar{\partial}(\mathcal{L})(R)$ ’ be also closed under composition of the inverses of those having \mathcal{L} —since, for whatever permutations \mathbf{R} and \mathbf{S} that have \mathcal{L} , $\mathbf{R}^- \sqcup \mathbf{S}^-$ is the same permutation as $(\mathbf{S} \sqcup \mathbf{R})^-$. All this is still not enough to ensure that the family of permutations that satisfy the open formula ‘ $\bar{\partial}(\mathcal{L})(R)$ ’, if any, is closed under \sqcup , and forms, then, a(n additive) group of permutations. Also requiring that

$$\forall R, S [(\mathcal{L}R \wedge \mathcal{L}S) \Rightarrow [\bar{\partial}(\mathcal{L})(R \sqcup S^-) \wedge \bar{\partial}(\mathcal{L})(R^- \sqcup S)]]$$

just amounts to requiring it. If \mathcal{L} is a second-level (monadic) property such that $\mathcal{L}(\mathcal{L})$, the first-level binary relations satisfying the open formula ‘ $\bar{\partial}(\mathcal{L})(R)$ ’, if any, form, then, a(n additive)

¹⁶One should better say ‘correspond to permutations’, since, strictly speaking, permutations are functions, not relations. Let us adopt, however, a more straightforward, though abusive, language, for short.

group of permutations, whose internal law of composition is \sqcup , whose neutral element is the identity permutation, and whose inverse function is $R \mapsto R^-$.

This group is not necessarily Abelian, for \sqcup is not commutative on permutations. But it is endowed with a total and right-invariant order defined in terms of the composition operation. Since, if \mathbf{H} and \mathbf{K} are two permutations whatsoever that satisfy ‘ $\partial(\mathcal{L})(R)$ ’, requiring that $\mathbf{H} \sqcup \mathbf{K}^-$ have the property \mathcal{L} is equivalent to requiring that \mathbf{K} and \mathbf{H} bear a right-invariant strict-order relation, let us say $\sqsubset_{\mathcal{L}}$, on these permutations¹⁷. Hence, if this relation is conceived of as the smaller-than relation (that is, ‘ $\mathcal{L}\mathbf{H} \sqcup \mathbf{K}^-$ ’ or ‘ $\mathbf{K} \sqsubset_{\mathcal{L}} \mathbf{H}$ ’ are read as ‘ \mathbf{K} is smaller than \mathbf{H} ’), then we can take the collection of the permutations that have \mathcal{L} , if any, as the positive semi-group of the group of permutations formed by the permutations that satisfy ‘ $\partial(\mathcal{L})(R)$ ’¹⁸.

3.5 Positive Classes and Domains of Magnitudes

Informally speaking, a nonempty positive class is a positival class whose domain is a totally-ordered, dense and Dedekind-complete group of permutations, which is, by consequence, also

¹⁷The proof is simple. As it has been required that $\neg \mathcal{L}\mathbf{H} \sqcup \mathbf{H}^-$, we immediately have that $\neg \mathbf{H} \sqsubset_{\mathcal{L}} \mathbf{H}$. As $\mathbf{K} \sqcup \mathbf{H}^-$ is the same permutation as $(\mathbf{H} \sqcup \mathbf{K}^-)^-$, we have that $\mathcal{L}\mathbf{H} \sqcup \mathbf{K}^- \Rightarrow \neg \mathcal{L}\mathbf{K} \sqcup \mathbf{H}^-$, i.e. $\mathbf{K} \sqsubset_{\mathcal{L}} \mathbf{H} \Rightarrow \neg \mathbf{H} \sqsubset_{\mathcal{L}} \mathbf{K}$. Again, if \mathbf{J} is, also, a permutation that satisfies ‘ $\partial(\mathcal{L})(R)$ ’, then $\mathbf{J} \sqcup \mathbf{K}^-$ is the same permutation as $(\mathbf{J} \sqcup \mathbf{H}^-) \sqcup (\mathbf{H} \sqcup \mathbf{K}^-)$, and so we have that $(\mathcal{L}\mathbf{H} \sqcup \mathbf{K}^- \wedge \mathcal{L}\mathbf{J} \sqcup \mathbf{H}^-) \Rightarrow \mathcal{L}\mathbf{J} \sqcup \mathbf{K}^-$, i.e. $(\mathbf{K} \sqsubset_{\mathcal{L}} \mathbf{H} \wedge \mathbf{H} \sqsubset_{\mathcal{L}} \mathbf{J}) \Rightarrow \mathbf{K} \sqsubset_{\mathcal{L}} \mathbf{J}$. Finally, as $(\mathbf{H} \sqcup \mathbf{J}) \sqcup (\mathbf{K} \sqcup \mathbf{J})^-$ is the same permutation as $\mathbf{H} \sqcup \mathbf{K}^-$, we have that $\mathcal{L}\mathbf{H} \sqcup \mathbf{K}^- \Rightarrow \mathcal{L}(\mathbf{H} \sqcup \mathbf{J}) \sqcup (\mathbf{K} \sqcup \mathbf{J})^-$, i.e. $\mathbf{K} \sqsubset_{\mathcal{L}} \mathbf{H} \Rightarrow \mathbf{K} \sqcup \mathbf{J} \sqsubset_{\mathcal{L}} \mathbf{H} \sqcup \mathbf{J}$.

¹⁸In commenting his definition of positival classes, Frege (1893-1903, § II.175; Frege 2013, pp. 171₂-72₂) claims to have “tried [*bemüht*]” to include in it only “necessary [*nothwendigen*]” and “mutually independent [*einander unabhängig*]” conditions, though taking as unprovable his having succeeded in this. In a note added at the end of his book (*ibidem* vol. 2, p. 243, Frege 2013, p. 243₂), explicitly referred to this comment, he corrects himself by observing that a proof could have been possible by means of counterexamples, though taking it to be “doubtful [*bezweifeln*]” that these counterexamples could be given in his formal setting. Dummett (1991, p. 288) suggests that his doubt concerned the independence of the condition we expressed by ‘ $\forall R, S [(\mathcal{L}R \wedge \mathcal{L}S) \Rightarrow \partial(\mathcal{L})(R^- \sqcup S)]$ ’ from the other ones characterizing a positival class, by observing that, in his developments concerning domains of magnitudes, Frege appeals to this condition as late as possible (namely only in § II.218), after making explicit (§ II.217) the “indispensability” of this condition for the purpose for which it is used, which, in our setting, corresponds to prove that if \mathbf{H} and \mathbf{K} belong to a positive class and \mathbf{H} is smaller than \mathbf{K} over the positive semigroup involved in this class, then \mathbf{K}^- is smaller than \mathbf{H}^- over the corresponding group. Adeleke, Dummett and Neumann (1987, th. 2.1) have finally proved that this condition is actually independent of the others. When transposed in our setting, the proof goes along the following lines. Let \mathcal{L}^* be a property satisfying ‘ $\mathcal{L}(\mathcal{X})$ ’ except for the condition at issue, \mathcal{G}^* be the structure formed by the permutations that satisfy ‘ $\partial(\mathcal{L}^*)(R)$ ’, and \mathbf{H} and \mathbf{K} two binary first-level relations having \mathcal{L}^* . Insofar as $(\mathbf{K}^- \sqcup \mathbf{H})^-$ is the same permutation as $\mathbf{H}^- \sqcup \mathbf{K}$, not ensuring that $\mathbf{K}^- \sqcup \mathbf{H}$ satisfies ‘ $\partial(\mathcal{L}^*)(R)$ ’ is the same as not ensuring that the disjunction

$$\mathcal{L}\mathbf{K}^- \sqcup \mathbf{H} \vee \forall x, y [x\mathbf{H}^-y \Leftrightarrow x\mathbf{K}^-y] \vee \mathcal{L}\mathbf{H}^- \sqcup \mathbf{K} \quad \text{i.e.} \quad \mathbf{H}^- \sqsubset_{\mathcal{L}} \mathbf{K}^- \vee \mathbf{H}^- =_{\mathcal{L}} \mathbf{K}^- \vee \mathbf{K}^- \sqsubset_{\mathcal{L}} \mathbf{H}^-$$

holds, namely that \mathbf{H}^- and \mathbf{K}^- are comparable according to the order over \mathcal{G}^* . It would follow that, besides of not being a group, this last structure is not endowed with a total order, but only with a partial one. It can be proved (*ibidem*, Lemma 1.2) that this partial order is an “upper semilinear order”—that is, a strict partial order “such that the elements greater than any given one are comparable, and that, for any two incomparable elements, there is a third greater than both of them”, or, more simply, a strict partial order that “may branch downwards, but cannot branch upwards” (Dummett 1991, p. 288). But \mathcal{G}^* is a sub-structure of the group \mathcal{G} formed by the permutations that satisfy ‘ $\partial(\mathcal{L})(R)$ ’, where \mathcal{L} satisfies ‘ $\mathcal{L}(\mathcal{X})$ ’ as a whole. Hence, the condition at issue follows from the others if and only if \mathcal{G}^* can be extended in no group other than \mathcal{G} . To prove the independence of this condition it is, then, enough to prove that there is a group including \mathcal{G}^* other than \mathcal{G} . By Cayley’s theorem, any group is isomorphic to a group of permutations. It is, then, enough to prove that there is a partially ordered group whatsoever not isomorphic to \mathcal{G} (that is, not totally ordered) that includes a sub-structure isomorphic to \mathcal{G}^* . This is just what Adeleke, Dummett and Neumann do.

Archimedean and Abelian¹⁹. By having a strict order available, the density condition can be stated easily. For stating the Dedekind-completeness one, further means are required.

To this purpose, Frege defines the upper rims over a positival class. Let \mathcal{L} be such that $\mathcal{L}(\mathcal{L})$, and \mathcal{A} a sub-property of it. In our setting, an upper rim \mathbf{U} of the collection of permutations that have \mathcal{A} over the collection of those that have \mathcal{L} is a relation having \mathcal{L} , such that any other relation that has \mathcal{L} and is smaller than \mathbf{U} over the former collection has \mathcal{A} . To define it, Frege begins with a general definition, then applies it to positival classes. In the general case, both the informal notion of an upper rim and the subsequent one of an upper limit become nonsensical. Formally speaking, this is immaterial, however, since the following definition of positive classes excludes that the deviant cases obtain in the case of such a class²⁰.

Once again, there are two ways to render the general definition: either as

$$(3.3) \quad \forall \mathcal{X}, \mathcal{Y} \forall R [\mathcal{X} \triangleleft_R \mathcal{Y} \Leftrightarrow \forall S [(\mathcal{X} S \wedge \mathcal{X} R \sqcup S^-) \Rightarrow \mathcal{Y} S]],$$

or as

$$(3.3') \quad [(\mathcal{X}) \triangleleft (\mathcal{Y})](R) := \forall S [(\mathcal{X} S \wedge \mathcal{X}(R \sqcup S^-)) \Rightarrow \mathcal{Y} S].$$

The upper limit of a sub-class of a positival class is the greatest of all the upper rims of the former over the latter, if there is one. Anew, Frege's definition can be rendered in two ways: either as

$$(3.4) \quad \forall \mathcal{X}, \mathcal{Y} \forall R [\mathcal{X} \uparrow_R \mathcal{Y} \Leftrightarrow \{\mathcal{L}\mathcal{X} \wedge \mathcal{X} R \wedge \mathcal{X} \triangleleft_R \mathcal{Y} \wedge \neg \exists S [\mathcal{X} S \wedge \mathcal{X} S \sqcup R^- \wedge \mathcal{X} \triangleleft_S \mathcal{Y}]\}],$$

or as:

$$(3.4') \quad [(\mathcal{X}) \uparrow (\mathcal{Y})](R) := \left\{ \mathcal{L}(\mathcal{X}) \wedge \mathcal{X} R \wedge [(\mathcal{X}) \triangleleft (\mathcal{Y})](R) \wedge \neg \exists S \left[\begin{array}{l} \mathcal{X} S \wedge \mathcal{X}(S \sqcup R^-) \wedge \\ [(\mathcal{X}) \triangleleft (\mathcal{Y})](S) \end{array} \right] \right\}.$$

For short, read both ' $\mathcal{X} \triangleleft_R \mathcal{Y}$ ' and ' $[(\mathcal{X}) \triangleleft (\mathcal{Y})](R)$ ' as ' R is an upper rim of \mathcal{Y} over \mathcal{X} ', and both ' $\mathcal{X} \uparrow_R \mathcal{Y}$ ' and ' $[(\mathcal{X}) \uparrow (\mathcal{Y})](R)$ ' as ' R is the upper limit of \mathcal{Y} over \mathcal{X} '.

Let \mathcal{P} be a third-level monadic property of first-order binary relations. Informally speaking, the relations that have it form a positive class if they form a positival one, and are such that: for any relation R which has \mathcal{P} , there is another relation S smaller than it over \mathcal{P} (density); any proper subclass \mathcal{Y} of \mathcal{P} which has an upper rim over \mathcal{P} also has an upper limit over \mathcal{P} (Dedekind-completeness). These conditions can be rendered in two ways: either as

$$(3.5) \quad \forall \mathcal{X} \left[\mathcal{P}\mathcal{X} \Leftrightarrow \left[\begin{array}{l} \mathcal{L}\mathcal{X} \wedge \\ \forall R [\mathcal{X} R \Rightarrow \exists S [\mathcal{X} S \wedge \mathcal{X} R \sqcup S^-]] \wedge \\ \forall \mathcal{Y} \left[\left[\begin{array}{l} \exists R [\mathcal{X} \triangleleft_R \mathcal{Y} \wedge \mathcal{X} R] \wedge \\ \exists S [\mathcal{X} S \wedge \neg \mathcal{Y} S] \end{array} \right] \Rightarrow \exists T [\mathcal{X} \uparrow_T \mathcal{Y}] \right] \end{array} \right] \right],$$

¹⁹That a totally-ordered, dense and Dedekind-complete group (of permutations) is also Archimedean and Abelian is in fact proved by Frege himself. He proves that a Dedekind-complete positival class is Archimedean (Frege 1893-1903, th. 635, § II. 214), and that the domain of a positive class is Abelian (*ibidem*, th. 689, § II. 244). This is the last theorem he proves. Insofar as the proof of the former theorem does not appeal to the condition considered in footnote (18) above, Adeleke, Dummett and Neumann (1987, p. 516) restate these theorems as follows: a Dedekind-complete upper semilinear order is Archimedean—which, of course makes it also a Dedekind-complete total order; if the order of a group is dense, Archimedean and total, then the group is Abelian.

²⁰See footnote 21 below.

or as:

$$(3.5') \quad \mathcal{P}(\mathcal{X}) := \left\{ \begin{array}{l} \mathcal{L}(\mathcal{X}) \wedge \\ \forall R[\mathcal{X}R \Rightarrow \exists S[\mathcal{X}S \wedge \mathcal{X}R \sqcup S^-]] \wedge \\ \forall \mathcal{Y} \left[\left[\begin{array}{l} \exists R[(\mathcal{X})\downarrow(\mathcal{Y})](R) \wedge \mathcal{X}R \wedge \\ \exists S(\mathcal{X}S \wedge \neg \mathcal{Y}S) \end{array} \right] \Rightarrow \exists T[(\mathcal{X})\downarrow(\mathcal{Y})](T) \right] \end{array} \right\}.$$

For short, read both ‘ $\mathcal{P}\mathcal{X}$ ’ and ‘ $\mathcal{P}(\mathcal{X})$ ’ as ‘ \mathcal{X} is a positive class’ or, more precisely, ‘the first-level binary relations having \mathcal{X} form a positive class’.

While (3.3), (3.4) and (3.5) are explicit definitions, and have to be licensed by some form of fourth-order comprehension, (3.3’), (3.4’) and (3.5’) are abbreviation stipulations, and require no form of comprehension stronger than $(\text{PCA}_{\Delta_0^2}^2)^{21}$.

From the previous remarks, it should be clear that if the second-level monadic property \mathcal{P} is such that $\mathcal{P}(\mathcal{P})$, then the permutations that respectively satisfy ‘ $\mathfrak{D}\mathcal{P}R$ ’ or ‘ $\mathfrak{D}(\mathcal{P})(R)$ ’, if any, form a totally-ordered, dense and Dedekind-complete group of permutations. This is just what a

²¹Comparing (3.5) and (3.5’), on the one side, with (3.3-3.4) and (3.3’-3.4’), on the other, allows us to see why the deviant cases pertaining to the definition of an upper rim and an upper limit become immaterial by passing to the definition of a positive class. For short and simplicity, we only consider (3.3’-3.5’). The right-hand side of (3.3’) fails, as such, in rendering the informal notion of an upper rim since it does not express the two crucial conditions that \mathcal{Y} be a sub-class of \mathcal{X} , and that \mathcal{X} be a positival class and R belong to it. This makes ‘ $\mathcal{X}R \sqcup S^-$ ’ do not render the condition that S be smaller than R over \mathcal{X} . Hence, according to (3.3’), it could happen that $[(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$ even if \mathcal{H} is not a sub-class of \mathcal{L} or \mathbf{R} it is not such that any relation smaller than it over \mathcal{L} has \mathcal{H} . It follows that the conjunction

$$\mathcal{L}(\mathcal{L}) \wedge \mathcal{L}\mathbf{R} \wedge [(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$$

renders the informal condition that \mathbf{R} be an upper rim of \mathcal{H} over \mathcal{L} except for the requirement that \mathcal{H} be a sub-class of \mathcal{L} . What are the consequences of missing this requirement? To see it, let us write the implication

$$\forall S[(\mathcal{X}S \wedge \mathcal{X}\mathbf{R} \sqcup S^-) \Rightarrow \mathcal{Y}S] \quad \text{as} \quad \neg \exists S[\mathcal{X}S \wedge \mathcal{X}\mathbf{R} \sqcup S^- \wedge \neg \mathcal{Y}S].$$

It is clear that this formula can be satisfied by \mathcal{L} (as value of \mathcal{X}) and \mathcal{H} (as value of \mathcal{Y}) even if \mathcal{H} is not a sub-class of \mathcal{L} . For instance, this is just what happens, whatever first-level binary relation R might be, if \mathcal{L} is a sub-class of \mathcal{H} . Hence, missing the mentioned requirement results in admitting that, for any R , if \mathcal{L} is a sub-class of \mathcal{H} , then $[(\mathcal{L})\downarrow(\mathcal{H})](R)$. But suppose that the first-level binary relations that have \mathcal{L} form a positival class and that \mathbf{R} be one of them, that is, that $\mathcal{L}(\mathcal{L}) \wedge \mathcal{L}\mathbf{R}$. If \mathbf{R} is not the smallest relation that has \mathcal{L} , there is certainly another relation S such that $\mathcal{L}S \wedge \mathcal{L}\mathbf{R} \sqcup S^-$. Hence, for it to hold that

$$\neg \exists S[\mathcal{L}S \wedge \mathcal{L}\mathbf{R} \sqcup S^- \wedge \neg \mathcal{H}S] \quad \text{and} \quad [(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R}),$$

it is necessary that any such S have \mathcal{H} . But if this is so, then \mathcal{L} and \mathcal{H} are not disjoint. This having been established, rewrite the right-hand side of (3.4’) in agreement with (3.3’), i.e. as follows

$$\mathcal{L}(\mathcal{X}) \wedge \mathcal{X}\mathbf{R} \wedge \neg \exists S[\mathcal{X}S \wedge \mathcal{X}\mathbf{R} \sqcup S^- \wedge \neg \mathcal{Y}S] \wedge \neg \exists T[\mathcal{X}T \wedge \mathcal{X}T \sqcup \mathbf{R}^- \wedge \neg \exists W(\mathcal{X}W \wedge \mathcal{X}T \sqcup W^- \wedge \neg \mathcal{Y}W)].$$

For this conjunction to hold, it has to exist a first-order binary relation that has \mathcal{X} but not \mathcal{Y} . The case where \mathcal{L} is a sub-class of \mathcal{H} is then expunged from those in which it can happen that $[(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$ for some \mathbf{R} . Insofar as (3.4’) implies that $[(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$ only if \mathcal{L} is positival, \mathbf{R} has it, and $[(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$, it follows that, provided that \mathbf{R} be not the smallest relation having \mathcal{L} , it can happen that $[(\mathcal{L})\downarrow(\mathcal{H})](\mathbf{R})$ only if \mathcal{H} is a sub-class of \mathcal{L} or, at least, \mathcal{L} and \mathcal{H} are not disjoint, but \mathcal{L} is not a sub-class of \mathcal{H} . Let now \mathcal{P} be a property of first-level binary relations satisfying the right-hand side of (3.5’), and, then, such that $\mathcal{P}(\mathcal{P})$. The sub-group formed by the relations having it is, then, dense. If \mathbf{R} has \mathcal{P} , it cannot happen that it be the smallest relation having it. Hence, it can happen that $[(\mathcal{P})\downarrow(\mathcal{H})](\mathbf{R})$, only if \mathcal{H} is either a subclass of \mathcal{P} , or \mathcal{P} and \mathcal{H} are not disjoint, but \mathcal{P} is not a subclass of \mathcal{H} . Hence \mathcal{P} and \mathcal{H} are not disjoint, that is, some relation having \mathcal{H} has also \mathcal{P} . Thus, even if \mathcal{H} is not a sub-class of \mathcal{P} , it can happen that it has both some upper rims and an upper limit over \mathcal{P} , which is just what is relevant for both (3.4’) and (3.5’) to comply with the informal explanations given above.

domain of magnitudes is in our setting. Indeed, for Frege, domains of magnitudes are nothing but domains of positive classes. This suggests either the following explicit definition licensed, again, only by an appropriate form of fourth-order comprehension,

$$(3.6) \quad \forall \mathcal{X} [\mathcal{M}\mathcal{X} \Leftrightarrow \exists \mathcal{Y} [\mathcal{P}\mathcal{Y} \wedge \forall R [\mathcal{X}R \Leftrightarrow \bar{\delta}\mathcal{Y}R]]],$$

or the following abbreviation stipulation requiring no comprehension stronger than $(\text{PCA}_{\Delta_0}^2)$,

$$(3.6') \quad \mathcal{M}(\mathcal{X}) := \exists \mathcal{Y} [\mathcal{P}(\mathcal{Y}) \wedge \forall R [\mathcal{X}R \Leftrightarrow \bar{\delta}(\mathcal{Y})(R)]].$$

where both ‘ $\mathcal{M}\mathcal{X}$ ’ and ‘ $\mathcal{M}(\mathcal{X})$ ’ are to be read as ‘ \mathcal{X} is a domain of magnitudes’ or, more precisely, ‘the first-level binary relations having \mathcal{X} form a domain of magnitudes’.

4 Which Definition, in which System?

As a matter of fact, (3.6) and (3.6’) provide two different definitions of domains of magnitudes. The former results from the explicit definition of the third-order predicate constant ‘ \mathcal{M} ’. The latter merely exhibits the third-order open formula briefly designated by ‘ $\mathcal{M}(\mathcal{X})$ ’, and involves no explicit definition other than (Inversion) and (Composition). Both render Frege’s definition, but require different logical resources, and play distinct roles in our setting.

Let us begin with the logical resources they require. A first difference is manifest: while (3.6) requires a fourth-order system, a third-order system is enough for (3.6’). Though both systems encompass no proper axioms, the former is, by far, more entangled than the latter. This is not only because of its higher order, but also because of the forms of comprehension it has to incorporate, in order to license (3.6). Besides $(\text{PCA}_{\Delta_0}^2)$ —or its instances (Composition-CA) and (Inversion-CA)—, required to license (Composition) and (Inversion), it also calls for other comprehension axioms, respectively required to license (Functionality) and (3.1-3.6)²². The latter system only needs to involve $(\text{PCA}_{\Delta_0}^2)$, or merely (Composition-CA) and (Inversion-CA), instead.

First-order variables (and the usual logical constants) apart, the language of this latter system has only to include dyadic second-order and monadic third-order variables, together with the

²²Namely:

$$(\text{CA}_{\Delta_0}^3) \quad \exists \mathcal{X} \forall R [\mathcal{X}R \Leftrightarrow \phi_{\Delta_0}^1] \quad [\text{where ‘}\phi_{\Delta_0}^1\text{’ stands for a second-order predicative formula}],$$

required to license (Functionality);

$$(\text{PCA}_{\Sigma_1(1)}^3) \quad \forall \mathcal{X} \exists \mathcal{Y} \forall R \left[\mathcal{Y}R \Leftrightarrow \phi_{\Sigma_1(1)}^2(\mathcal{X}) \right] \quad \left[\begin{array}{l} \text{where ‘}\phi_{\Sigma_1(1)}^2(\mathcal{X})\text{’ stands for a third-order formula} \\ \text{involving a second-order existential quantifier} \\ \text{and the parameter ‘}\mathcal{X}\text{’} \end{array} \right],$$

required to license (3.1);

$$(\text{CA}_{\Pi_1(1)}^4) \quad \exists \mathcal{X} \forall \mathcal{X} \left[\mathcal{X}\mathcal{X} \Leftrightarrow \phi_{\Pi_1(1)}^2 \right] \quad \left[\begin{array}{l} \text{where ‘}\phi_{\Pi_1(1)}^2\text{’ stands for a third-order formula} \\ \text{involving a second-order universal quantifier} \end{array} \right],$$

required to license (3.2);

$$(\text{CA}_{\Pi_1(1)}^4) \quad \exists \mathcal{V} \forall \mathcal{X}, \mathcal{Y} \forall R \left[\mathcal{X}\mathcal{V}_R\mathcal{Y} \Leftrightarrow \phi_{\Pi_1(1)}^2 \right] \quad [\text{where ‘}\phi_{\Pi_1(1)}^2\text{’ is as in } (\text{CA}_{\Pi_1(1)}^4)],$$

required to license (3.3);

corresponding quantifiers²³, and the functional constants ‘-’ and ‘ \sqcup ’. Though third-order, this system is, then, quite weak. As a matter of fact, we have nevertheless shown that Frege’s definition of domains of magnitudes can be consistently rephrased in such a weak system, and is, then, so rephrased, equiconsistent with it²⁴. For future reference, call this system ‘ $L_2PCA_{\Delta_0}^2$ ’.

It remains, however, that (3.6’), and, then, this very system, are suitable for our present purpose only if we are content with admitting that a second-level property \mathcal{M} is a domain of magnitudes (or that the first-level binary relations that have it form such a domain) if and only if it satisfies the right-hand side of (3.6’). Were we, instead, in need of defining a (third-level) property that \mathcal{M} has if and only if it is so, (3.6’) would no more be suitable, and we would have to recur to (3.6) and the corresponding fourth-order and much stronger system. Provided that the definition of positival and positive classes is, in the present setting, merely instrumental to that of domains of magnitudes, the former attitude might be easily admitted for the corresponding properties. But one might consider that the same attitude is not admissible in the case of these very domains, whose definition is the final outcome of Frege’s work, on pain of missing a genuine entity counting as such a domain, and, then, the definition itself.

Still, even if the definition of domains of magnitudes is the last step Frege reached in his formalization of real analysis²⁵, it is in no way the final step such a formalization should have reached: this should have rather been an explicit definition of real numbers, and, possibly, of the operations and relations making them form a totally ordered and Dedekind-complete field. Hence, if Frege’s informal indications for reaching this final aim may be rendered in our setting without defining the predicate constant ‘ \mathcal{M} ’, there is no stringent reason for accepting the foregoing argument, so that (3.6’) and $L_2PCA_{\Delta_0}^2$ may be considered enough for the purpose of rendering the result he was envisaging. In § 5, we will show that this can be actually done. We can, then, conclude that, whereas (3.6), and the fourth-order system it requires, are suitable for the purpose of defining domains of magnitude, as such, (3.6’) and $L_2PCA_{\Delta_0}^2$ are so for the purpose of defining real numbers as ratios on such domains. As this is our goal, we’ll go, then, for the latter option.

This is all the more justified because no form of comprehension can guarantee the existence

$$(CA_{\Pi_1(1)}^4) \quad \exists \forall \mathcal{X}, \mathcal{Y} \forall R \left[\mathcal{X} \mathcal{V}_R \mathcal{Y} \Leftrightarrow \phi_{\Sigma_1(1)}^3 \right] \quad \left[\begin{array}{l} \text{where ‘}\phi_{\Sigma_1(1)}^3\text{’ stands for a fourth-order formula} \\ \text{involving a second-order existential quantifier} \end{array} \right],$$

required to license (3.4);

$$(CA_{\Pi_1(2)}^4) \quad \exists \mathcal{X} \forall \mathcal{X} \left[\mathcal{X} \mathcal{X} \Leftrightarrow \phi_{\Pi_1(2)}^3 \right] \quad \left[\begin{array}{l} \text{where ‘}\phi_{\Pi_1(2)}^3\text{’ stands for a fourth-order formula} \\ \text{involving a third-order universal quantifier} \end{array} \right],$$

required to license (3.5);

$$(CA_{\Pi_1(2)}^4) \quad \exists \mathcal{X} \forall \mathcal{X} \left[\mathcal{X} \mathcal{X} \Leftrightarrow \phi_{\Sigma_1(2)}^3 \right] \quad \left[\begin{array}{l} \text{where ‘}\phi_{\Pi_1(2)}^3\text{’ stands for a fourth-order formula} \\ \text{involving a third-order existential quantifier} \end{array} \right],$$

required to license (3.6).

²³Notice that a third-order quantifier only occurs once: in the right-hand side of (3.5’). This single occurrence is however essential for the definition of domains of magnitude to succeed.

²⁴A note of caution. The fact that, when rephrased as suggested, Frege’s definition is equiconsistent with this system does not entail at all that the original definition requires no impredicative comprehension, and is, then, consistent in itself. It crucially involves the function $\xi \frown \zeta$, which cannot be defined without impredicative (second-order) comprehension.

²⁵To be more precise, Frege offers no explicit formal definition of domains of magnitudes, and rather is content with informally claiming that a domain of magnitudes is the domain of a positive class.

of positival and positive classes and domains thereof. Surely, by the standard interpretation of higher-order logic, the stipulations (3.2') and (3.5'-3.6') being given, the following instances of third-order impredicative comprehension

$$\begin{aligned} \exists \mathcal{X} \forall R [\mathcal{X} R \Leftrightarrow \exists \mathcal{Y} (\mathcal{L}(\mathcal{Y}) \wedge \mathcal{Y} R)], \\ \exists \mathcal{X} \forall R [\mathcal{X} R \Leftrightarrow \exists \mathcal{Y} (\mathcal{P}(\mathcal{Y}) \wedge \mathcal{Y} R)], \\ \exists \mathcal{X} \forall R [\mathcal{X} R \Leftrightarrow \exists \mathcal{Y} (\mathcal{M}(\mathcal{Y}) \wedge \mathcal{Y} R)] \end{aligned}$$

are enough to ensure the existence of the second-level properties that a first-level binary relation has to have for being respectively included in a positival and a positive class and in a domain of magnitudes. Again, the following instances of fourth-order predicative comprehension

$$\begin{aligned} \exists \mathcal{X} \forall \mathcal{X}' [\mathcal{X} \mathcal{X}' \Leftrightarrow \mathcal{L}(\mathcal{X}')], \\ \exists \mathcal{X} \forall \mathcal{X}' [\mathcal{X} \mathcal{X}' \Leftrightarrow \mathcal{P}(\mathcal{X}')], \\ \exists \mathcal{X} \forall \mathcal{X}' [\mathcal{X} \mathcal{X}' \Leftrightarrow \mathcal{M}(\mathcal{X}')] \end{aligned}$$

are enough to ensure the existence of the third-level properties of being a positival and a positive class and a domain of magnitudes. Still, securing this existence is no substantial achievement, since these properties would exist even if they were empty, or there were not enough relations satisfying them, to make them play the required role in a definition of the reals.

Even more so, if we grant the extensional conception of properties and relations, we also have to grant that, no matter how the first-order domain might be, both the empty first-level binary relation and the empty second-level property exist²⁶, which alone is enough to ensure that all six foregoing second- and third-level properties exist, in turn, and are nonempty, since, according to (3.2') and (3.5'-3.6'), the empty second-level property is at the same time a positival class, a positive one, and a domain of magnitudes²⁷.

²⁶Frege is not clear about what he takes to make a property or a relation exist, if at all. It is even plausible to ascribe him an intensional conception, which makes the talk of existence of properties and relations inappropriate (Panza 2015). What is unquestionable is that he does not explicitly endorse any sort of comprehension axiom, by rather pervasively admitting of a substitution policy which we could consider as equivalent to full second-order comprehension. In the light of BLV, our replacing value-ranges of binary first-level relations with these very relations seems, however, in line with granting the extensional conception of properties and relations while doing semantic considerations about $L_2PCA_{\Delta_0}^2$. This means informally taking a $n + 1$ -order m -adic predicate variable to range on all the subsets of ordered m -tuples of the elements of the n -order domain ($n, m = 1, 2, \dots$), and a predicate constant to designate one of these subsets.

²⁷Let \mathcal{E}_0 be the second-level empty property. From (3.2') it follows that $\mathcal{L}(\mathcal{E}_0)$, since $\forall R \neg [\mathcal{E}_0 R]$. Let \mathbf{V} be the empty first-level binary relation, and \mathcal{E}_1 the second-level property of being this property. If \mathbf{R} is a first-level binary relation (extensionally) distinct from \mathbf{V} , then, we have that both $\mathcal{E}_1 \mathbf{V}$ and $\neg \mathcal{E}_1 \mathbf{R}$. Moreover, from (Functionality), (Inversion') and (Composition'), it follows that $\mathcal{S}(\mathbf{V})$, and that \mathbf{V} extensionally coincides both with \mathbf{V}^- and with $\mathbf{V} \sqcup \mathbf{V}^-$, and, then, also with $\mathbf{V}^- \sqcup \mathbf{V}$ (see Frege, 1893-1903, § II.164). This is enough to show that \mathbf{V} does belong to no positival (and, then, positive) class, and that both $\neg \mathcal{L}(\mathcal{E}_1)$ and $\mathcal{L}^*(\mathcal{E}_1)$, where ' $\mathcal{L}^*(\mathcal{X})$ ' is the abbreviation of the formula resulting from the right-hand side of (3.2'), by skipping the conjunct ' $\neg \mathcal{X} R \sqcup R^-$ '. Then, \mathcal{E}_1 is positival, except for admitting the possibility that the identity relation be included in it. Thus, both the second-level properties $[R : \exists \mathcal{Y} (\mathcal{L}(\mathcal{Y}) \wedge \mathcal{Y} R)]$ and $[R : \exists \mathcal{Y} (\mathcal{L}^*(\mathcal{Y}) \wedge \mathcal{Y} R)]$ and the third level ones $[\mathcal{X} : \mathcal{L}(\mathcal{X})]$ and $[\mathcal{X} : \mathcal{L}^*(\mathcal{X})]$ not only exist by appropriate forms of comprehension, but are also nonempty. Consider now (3.5'). It is enough to observe that $\mathcal{L}(\mathcal{E}_0)$ and $\neg \exists R [\mathcal{E}_0 R]$ to conclude that $\mathcal{P}(\mathcal{E}_0)$. Look, then, at \mathcal{E}_1 . Insofar as $\mathbf{R} \sqcup \mathbf{V}^-$ coincides with \mathbf{V} , whatever first-level binary relation \mathbf{R} might be, from (3.3') it follows that $[(\mathcal{E}_1) \exists (\mathcal{Y})] (\mathbf{R})$ if and only if \mathcal{Y} is \mathcal{E}_1 itself. Insofar as $\forall S \neg [\mathcal{E}_1 S \wedge \neg \mathcal{E}_1 S]$, from (3.5') it also

Clearly though, if a positive class exists that includes at least a first-level binary relation, then it necessarily includes infinitely many (extensionally) distinct relations, which are continuously many if the class is positive²⁸. And, if this holds for a positive class, it also holds for its domain. It follows that a positive class and a domain of magnitudes exist, which include at least a first-level binary relation, if and only if there exists one including continuously many of them. No semantic consideration is, however, apt to prove the existence of nonempty positive classes and domains thereof. To prove it, it is necessary to prove that there are enough objects, for defining on them continuously many appropriate relations.

This was the crucial challenge for Frege’s definition of the real numbers, and it is also ours. Before tackling it, it is, however, in order to see what makes this proof indispensable both for Frege’s and for our purposes. This requires looking at how Frege’s original theory of the reals as objects can be revived in our setting, on the basis of (3.5’).

5 Real Numbers as Objects

Apart from some generalities, most of which we already discussed above, and the sketchy outline of a possible existence proof for nonempty domains of magnitudes, which we will consider in § 6.1, the *pars construens* of part III of *Grundgesetze* only contains the definition of these domains. No precise indication of how to define the real numbers is available. The only thing that is clear is that these numbers should be defined as ratios of magnitudes, and that these ratios have to be

follows that $\mathcal{P}^*(\mathcal{E}_1)$, where ‘ $\mathcal{P}^*(\mathcal{X})$ ’ is the abbreviation of the formula resulting from the right-hand member of (3.5’) by replacing ‘ $\mathcal{L}(\mathcal{X})$ ’ with ‘ $\mathcal{L}^*(\mathcal{X})$ ’. So, the properties $[R : \exists \mathcal{Y} (\mathcal{P}(\mathcal{Y}) \wedge \mathcal{Y}R)]$, $[R : \exists \mathcal{Y} (\mathcal{P}^*(\mathcal{Y}) \wedge \mathcal{Y}R)]$, $[\mathcal{X} : \mathcal{P}(\mathcal{X})]$ and $[\mathcal{X} : \mathcal{P}^*(\mathcal{X})]$ not only exist by appropriate forms of comprehension, but they are also nonempty. Finally, consider (3.6’). Provided that ‘ $\mathcal{M}^*(\mathcal{X})$ ’ be the abbreviation of the formula resulting from the right-hand side of (3.6’) by replacing ‘ $\mathcal{P}(\mathcal{X})$ ’ with ‘ $\mathcal{P}^*(\mathcal{X})$ ’, it is immediate that both $\mathcal{M}(\mathcal{E}_0)$ and $\mathcal{M}^*(\mathcal{E}_1)$, just because $\mathcal{P}(\mathcal{E}_0)$ and $\mathcal{P}^*(\mathcal{E}_1)$ and the domains of \mathcal{E}_0 and \mathcal{E}_1 respectively coincide with \mathcal{E}_0 and \mathcal{E}_1 themselves. Hence, the properties $[R : \exists \mathcal{Y} (\mathcal{P}(\mathcal{Y}) \wedge \mathcal{Y}R)]$, $[R : \exists \mathcal{Y} (\mathcal{P}^*(\mathcal{Y}) \wedge \mathcal{Y}R)]$, $[\mathcal{X} : \mathcal{M}(\mathcal{X})]$ and $[\mathcal{X} : \mathcal{M}^*(\mathcal{X})]$, too, not only exist by appropriate forms of comprehension, but are also nonempty. Notice that \mathcal{M}^* does not (extensionally) coincide with \mathcal{M} , since $\mathcal{M}^*(\mathcal{E}_1)$, but not $\mathcal{M}(\mathcal{E}_1)$.

²⁸More precisely, this is with respect to the full model of $L_2PCA_{\Delta_0}^2$, which we also take as its intended one, where the first-order variables, x, y, z, \dots vary over a large enough domain \mathbf{A} of objects, the second-order dyadic variables R, S, T, \dots vary over the full power set of $\mathbf{A} \times \mathbf{A}$, and the third-order monadic variables $\mathcal{X}, \mathcal{Y}, \dots$ vary over the full power set of the power set of \mathbf{A} . It is, indeed, easy to see that $L_2PCA_{\Delta_0}^2$ is far from categorical. A simple way to see it (thanks to Andrew Moshier for suggesting it to us) is to observe that the lack of third-order comprehension makes this system have a model where the third-order variables vary over the empty set. In this model, all closed formulas beginning by a third-order universal quantifier, as the third conjunct in the right-hand side of (3.5’), are vacuously true. This makes ‘ $\mathcal{P}(\mathcal{X})$ ’ trivially satisfied by countably many (appropriate) binary relations (whereas $PCA_{\Delta_0}^2$ makes any model of $L_2PCA_{\Delta_0}^2$ include countably many first-level binary relations). One might be surprised we take the full model to be the intended model, rather than an appropriate Henkin one. Still, we think a restriction on comprehension, or even the lack of it, is no reason for imposing a restricted semantics: one thing is the logical resources, in particular the instances of comprehension, required for a definition; another the selection of the intended model. The former are deductive, syntactical tools required to formulate definitions; the latter depends on semantic considerations relative to the informal piece of knowledge that the relevant system is expected to render—which, in this case, involves the idea of a positive class as a complete semi-group of permutations. We are not going to take a stand on this matter, here, but costs and benefits are worth mentioning. Should our definition be required to recover positive classes including exactly 2^{\aleph_0} permutations, then the intended semantic is to be clearly full. If impredicativity were deemed a price worth paying for this purpose, one should also be aware of the heavy-duty logical resources called for it.

defined as objects, i.e., first-order items, logically speaking²⁹.

5.1 Euclid’s Principle with Natural Numbers

A simple way to accomplish this plan is by an abstraction principle governing an operator taking pairs of relations (i.e. magnitudes) from a domain of magnitudes as arguments and having objects (i.e. ratios) as values. As suggested by Simons (1987, pp. 39–40) and Dummett (1991, pp. 290–91), this can be done by rephrasing definition V.5 of Euclid’s *Elements*, and defining the relation of proportionality between four magnitudes taken two by two³⁰

This raises a technical difficulty, however. Whereas this definition applies only if the magnitudes composing each pair are such that it cannot be the case that any multiple of one of them be smaller, equal, or greater than any multiple of the other, this condition is not met by pairs of magnitudes of the same domain, since, differently from what happened for Euclid’s ones, these are intended to be either positive, negative, or null. A way to solve the difficulty is by appropriately modifying the very structure of Euclid’s definition, in order to get the following abstraction principle, which deserves, nevertheless, the name of ‘Euclid’s Principle’ (or ‘EP’, from now on),

$$(EP) \quad \left\{ \begin{array}{l} \forall_{(\mathcal{P})} \mathcal{X}, \mathcal{X}' \\ \forall_{(\mathfrak{R}(\mathcal{X}))} R \forall_{\mathcal{X}} S \\ \forall_{(\mathfrak{R}(\mathcal{X}'))} R' \forall_{\mathcal{X}'} S' \end{array} \right\} \left[\begin{array}{l} \mathfrak{R}[R, S] = \mathfrak{R}[R', S'] \Leftrightarrow \\ \left[\begin{array}{l} \mathcal{X} R \wedge \mathcal{X}' R' \wedge \\ \forall_{\mathbf{N}} x, y (xR \sqsubset_{\mathcal{X}} yS \Leftrightarrow xR' \sqsubset_{\mathcal{X}'} yS') \end{array} \right] \vee \\ \left[\begin{array}{l} \mathcal{X} R^- \wedge \mathcal{X}' R'^- \wedge \\ \forall_{\mathbf{N}} x, y (xR^- \sqsubset_{\mathcal{X}} yS \Leftrightarrow xR'^- \sqsubset_{\mathcal{X}'} yS') \end{array} \right] \vee \\ \left[\forall z, w \left[\begin{array}{l} (zRw \Leftrightarrow z[S \sqcup S^-]w) \wedge \\ (zR'w \Leftrightarrow z[S' \sqcup S'^-]w) \end{array} \right] \right] \end{array} \right],$$

where:

- ‘ \mathfrak{R} ’ is the relevant abstraction operator;
- ‘ $\forall_{(\mathcal{P})} \mathcal{X} [\varphi]$ ’ abbreviates ‘ $\forall \mathcal{X} [\mathcal{P}(\mathcal{X}) \Rightarrow \varphi]$ ’, and the same for ‘ \mathcal{X}' ’;

²⁹From what he writes in the very § of his treatise, it seems that Frege was also requiring that real numbers form themselves a domain of magnitudes (Frege 1893-1903, § II.245; Frege 2013, p. 243₂):

The commutative law for the domain of a positive class is thus proven. The next task is now to show that there is a positive class, as indicated in § 164 [see p. 2, above]. This opens the possibility of defining a real number as a ratio of magnitudes of a domain that belongs to a positive class. Moreover, we will then be able to prove that the real numbers themselves belong as magnitudes to the domain of a positive class.

This further requirement would have not only uselessly entangled Frege’s own first-order definition, if he completed it (Dummett 1991, pp. 190-91), but it is also logically incompatible, in our predicate setting, with the requirement that real numbers be objects. This is why we will set it aside in our reconstruction.

³⁰As observed by Dummett (1991, pp. 282–83), Euclid’s definition, probably tracing back to Eudoxus, in fact, had been explicitly appealed to by Hölder in his paper on the “axioms of quantity” (Hölder 1901), which appeared two years before the second volume of the *Grundgesetze*. But, apparently, Frege’s was not aware of this. On Hölder (1901), cf. Błaszczyk (2013), which also sums up how the notion of magnitude was investigated around the end of 19th century by several mathematicians, including Du Bois-Reymond, Stolz, and Weber, by explicitly referring to Euclid’s theory, and achieving results mathematically equivalent to Frege’s.

- ‘ $\forall_{(\delta(\mathcal{X}))}R[\varphi]$ ’ abbreviates ‘ $\forall R[\delta(\mathcal{X})(R) \Rightarrow \varphi]$ ’, and the same for ‘ \mathcal{X}' ’ and ‘ R' ’;
- ‘ $\forall_{\mathcal{X}}S[\varphi]$ ’ abbreviates ‘ $\forall S[\mathcal{X}R \Rightarrow \varphi]$ ’, and the same for ‘ \mathcal{X}' ’ and ‘ S' ’;
- ‘ $\forall_{\mathbf{N}}x[\varphi]$ ’ abbreviates ‘ $\forall x[\mathbf{N}x \Rightarrow \varphi]$ ’, and the same for ‘ y ’;
- ‘ \mathbf{N} ’ denotes the property of being a natural number;
- ‘ xR ’ abbreviates ‘ $\underbrace{R \sqcup R \sqcup \dots \sqcup R}_x$ ’, and the same for ‘ y ’, ‘ S ’, ‘ S' ’, ‘ R' ’, ‘ R^- ’, ‘ R'^- ’;
- ‘ $xR \sqsubset_{\mathcal{X}} yS$ ’ abbreviates ‘ $\mathcal{X}yS \sqcup (xR)^-$ ’, and the same for ‘ \mathcal{X}' ’, ‘ R' ’, ‘ S' ’, ‘ R^- ’, ‘ R'^- ’.

Informally speaking, EP states that for whatever pairs of domains of magnitudes, issued by two positive classes \mathcal{P} and \mathcal{P}' ³¹, and whatever two ordered pairs of permutations \mathbf{R}, \mathbf{S} and \mathbf{R}', \mathbf{S}' , such that \mathbf{R} , and \mathbf{R}' respectively belong to the domains of these classes, while \mathbf{S} and \mathbf{S}' belong to the classes themselves (and are, then, intended as positive), the ratio $\mathfrak{R}[\mathbf{R}, \mathbf{S}]$ of the elements of the first pair is the same as the ratio $\mathfrak{R}[\mathbf{R}', \mathbf{S}']$ of the elements of the second pair if and only if:

- either both the first elements \mathbf{R} and \mathbf{R}' of these pairs belong to the respective positive classes \mathcal{P} and \mathcal{P}' (and are, then, intended as positive), and their equimultiples are always smaller than the equimultiples of the second elements³²;
- or both the inverses \mathbf{R}^- and \mathbf{R}'^- of the first elements \mathbf{R} and \mathbf{R}' of these pairs belong to the respective positive classes \mathcal{P} and \mathcal{P}' (so that \mathbf{R} and \mathbf{R}' are intended as negative), and their equimultiples are always smaller than the equimultiples of the second elements;
- or both the first elements \mathbf{R} and \mathbf{R}' of these pairs are the identity relation (and are, then, intended as null).

So rephrasing Euclid’s definition surely solves the technical difficulty, but it does not solve all problems: though EP involves neither a predicate constant ‘ \mathcal{P} ’ for the third-level property of being a positive class, nor a functional constant ‘ δ ’ for the domain operator, but merely depends on the stipulations (3.5’) and (3.1’), it cannot, as such, be added to $L_2PCA_{\Delta_0}^2$ as a new axiom, so as to get an extended system in which real numbers are to be defined. There are two reasons for that. First of all, EP involves the predicate constant ‘ \mathbf{N} ’ for the first-level property of being a natural number, which is not and cannot be defined within $L_2PCA_{\Delta_0}^2$. Secondly, it involves the symbol ‘ xR ’ (or ‘ $\underbrace{R \sqcup R \sqcup \dots \sqcup R}_x$ ’) (where ‘ x ’ is a variable ranging over the natural numbers)

whose use in a formal system is licensed only if this latter contains a device to count the iterated applications of the functional constant ‘ \sqcup ’, which is not and cannot be provided within $L_2PCA_{\Delta_0}^2$.

A way to overcome the first issue is to extend $L_2PCA_{\Delta_0}^2$ to a stronger system, in which the property of being a natural number can somehow be defined, e.g. by adding HP as a new axiom

³¹Notice that EP does not involve domains of magnitudes as such, but rather positive classes and their domains. This is perfectly in line with Frege’s missing a formal definition of these domains: see footnote (25), above.

³²Notice also that, whereas Euclid’s definition requires that the equimultiples of the first and the third, among the four relevant magnitudes, “alike exceed, are alike equal to, or alike fall short of [$\acute{\alpha}\mu\alpha \acute{\upsilon}\pi\epsilon\rho\acute{\epsilon}\chi\eta\ \eta\ \acute{\alpha}\mu\alpha \acute{\iota}\sigma\alpha \eta\ \eta\ \acute{\alpha}\mu\alpha \acute{\epsilon}\lambda\lambda\epsilon\acute{\iota}\pi\eta$]” (Euclid EH, vol. II, p. 114) the equimultiples of the second and the fourth, we can just require that the equimultiples of the first and the third magnitudes all be smaller than those of the second and the fourth, since, as noticed by Scott (1958-59), in the case of Archimedean magnitudes, the latter condition entails the former.

and appropriately extending its language by monadic first-order predicates, to make it include Frege Arithmetic (FA). This would be, however, a quite radical move, which would also openly conflict with Frege's requirement of non-arithmeticity for his definition of real numbers. Even more so, it would not overcome the second issue, unless the new system were supplied by some ingenious device not usually available in (the current versions of) FA.

A much less radical and costly, though a bit laborious, way to overcome both issues at once is available. It is in fact suggested by a trick Frege appeals to in proving the Archimedeanity of positive classes (Frege, 1893-1903, §§ II.199-214). It consists in amending EP with the help of some new abbreviation stipulations, which merely require adding new third-order binary variables.

5.2 Euclid's Principle without Natural Numbers

Let us begin by adopting the following new abbreviation stipulation:

$$\mathcal{D}_{(T)}(R, S) := \forall x, y (xSy \Leftrightarrow x[T \sqcup R]y).$$

Let \mathbf{R} , \mathbf{S} , and \mathbf{T} , be whatever first-level binary relations. According to this stipulation, the formula ' $\mathcal{D}_{(T)}(\mathbf{R}, \mathbf{S})$ ' asserts that \mathbf{S} results from, or extensionally coincide with, the composition of \mathbf{T} and \mathbf{R} . Hence, ' $\mathcal{D}_{(\mathbf{R})}(\mathbf{R}, \mathbf{S})$ ' asserts that \mathbf{S} results from the composition of \mathbf{R} with itself. In the notation employed in stating EP, this means that \mathbf{S} coincides with $2\mathbf{R}$.

This allows to simulate the usual definition of the weak ancestral of a binary relation:

$$\mathcal{D}_{(\mathbf{R})}^{\text{H}}(S) := \forall \mathcal{X} \left[\left[\begin{array}{l} \mathcal{X}R \wedge \\ \forall R', S' \left[\left[\begin{array}{l} \mathcal{X}R' \wedge \\ \mathcal{D}_{(\mathbf{R}')} (R', S') \end{array} \right] \Rightarrow \mathcal{X}S' \right] \end{array} \right] \Rightarrow \mathcal{X}S \right].$$

This makes the formula ' $\mathcal{D}_{(\mathbf{R})}^{\text{H}}(\mathbf{S})$ ' assert that \mathbf{S} extensionally coincides with \mathbf{R} or with the relation resulting from a iterated composition of \mathbf{R} with itself, and is, then, a multiple of \mathbf{R} itself. In the notation employed in stating EP, this means that \mathbf{S} coincides with $n\mathbf{R}$, for some natural number n . Let, now, \mathcal{P} be a positive class, and \mathbf{R} a relation in it. It is clear that if $\mathcal{D}_{(\mathbf{R})}^{\text{H}}(\mathbf{T})$ and $\mathcal{D}_{(\mathbf{R})}^{\text{H}}(\mathbf{S})$, then both \mathbf{T} and \mathbf{S} belong to \mathcal{P} . Suppose it is so, and that $\mathcal{P}\mathbf{T} \sqcup \mathbf{S}^-$. We can, then, take \mathbf{S} to be smaller than \mathbf{T} over \mathcal{P} . Hence, if also \mathcal{P}' is a positive class (either distinct from \mathcal{P} or not), \mathbf{R}' is a first-level binary relation that belongs to it, and it is also the case that $\mathcal{D}_{(\mathbf{R}')}^{\text{H}}(\mathbf{T}')$ for some first-level binary relation \mathbf{T}' , then \mathbf{T} is the same multiple of \mathbf{R} over \mathcal{P} as \mathbf{T}' of \mathbf{R}' over \mathcal{P}' if and only if there are as many first-level binary relations that satisfy the open formula ' $\mathcal{D}_{(\mathbf{R})}^{\text{H}}(S) \wedge \mathcal{P}\mathbf{T} \sqcup \mathbf{S}^-$ ' as those that satisfy the other open formula ' $\mathcal{D}_{(\mathbf{R}')}^{\text{H}}(S') \wedge \mathcal{P}'\mathbf{T}' \sqcup \mathbf{S}'^-$ '.

This suggests enriching the language of $L_2\text{PCA}_{\Delta_0}^2$ by introducing third-order binary variables, ranging over second-level binary relations between first-level such relations, and adopting the

following further abbreviation stipulation

$$(5.1) \quad (\mathcal{X}, \mathcal{X}') \mathcal{E}(R, T, R', T') := \left\{ \begin{array}{l} \mathcal{D}_{(R)}^{\mathbb{H}}(T) \wedge \mathcal{D}_{(R')}^{\mathbb{H}}(T') \wedge \\ \exists \mathcal{R} \left[\begin{array}{l} \forall S \left[\begin{array}{l} \left(\mathcal{D}_{(R)}^{\mathbb{H}}(S) \wedge S \sqsubseteq_{\mathcal{X}} T \right) \Rightarrow \\ \exists ! S' \left[S \mathcal{R} S' \wedge \mathcal{D}_{(R)}^{\mathbb{H}}(S') \wedge S' \sqsubseteq_{\mathcal{X}'} T' \right] \end{array} \right] \wedge \\ \forall S' \left[\begin{array}{l} \left(\mathcal{D}_{(R')}^{\mathbb{H}}(S') \wedge S' \sqsubseteq_{\mathcal{X}'} T' \right) \Rightarrow \\ \exists ! S \left[S \mathcal{R} S' \wedge \mathcal{D}_{(R)}^{\mathbb{H}}(S) \wedge S \sqsubseteq_{\mathcal{X}} T \right] \end{array} \right] \end{array} \right] \end{array} \right\},$$

where ‘ \mathcal{R} ’ is such a variable, and ‘ $S \sqsubseteq_{\mathcal{X}} T$ ’ abbreviates ‘ $\mathcal{X}T \sqcup S^- \vee \forall x, y [xSy \Leftrightarrow xTy]$ ’, and the same as for ‘ \mathcal{X}' ’, ‘ T' ’ and ‘ S' ’. Thus, if \mathcal{P} , \mathcal{P}' , \mathbf{R} , \mathbf{R}' , \mathbf{T} , and \mathbf{T}' are as above, then the formula ‘ $(\mathcal{P}, \mathcal{P}') \mathcal{E}(\mathbf{R}, \mathbf{T}, \mathbf{R}', \mathbf{T}')$ ’ asserts that \mathbf{T} is the same multiple of \mathbf{R} over \mathcal{P} as \mathbf{T}' of \mathbf{R}' over \mathcal{P}' .

For short, let us, now, adopt this other abbreviation stipulation:

$$(\mathcal{X}, \mathcal{X}') \mathcal{E}_{(S, U, S', U')}^{(R, T, R', T')} := (\mathcal{X}, \mathcal{X}') \mathcal{E}(R, T, R', T') \wedge (\mathcal{X}, \mathcal{X}') \mathcal{E}(S, U, S', U').$$

EP can, then, be restated as follows:

$$(EP^*) \quad \left\{ \begin{array}{l} \forall_{(\mathcal{P})} \mathcal{X}, \mathcal{X}' \\ \forall_{(\mathfrak{B}(\mathcal{X}))} R \forall_{\mathcal{X}} S \\ \forall_{(\mathfrak{B}(\mathcal{X}'))} R' \forall_{\mathcal{X}'} S' \end{array} \right\} \left[\begin{array}{l} \mathfrak{R}[R, S] = \mathfrak{R}[R', S'] \Leftrightarrow \\ \left[\begin{array}{l} \mathcal{X}R \wedge \mathcal{X}'R' \wedge \\ \forall T, U, T', U' \left[\begin{array}{l} (\mathcal{X}, \mathcal{X}') \mathcal{E}_{(S, U, S', U')}^{(R, T, R', T')} \Rightarrow \\ (T \sqsubseteq_{\mathcal{X}} U \Leftrightarrow T' \sqsubseteq_{\mathcal{X}'} U') \end{array} \right] \end{array} \right] \vee \\ \left[\begin{array}{l} \mathcal{X}R^- \wedge \mathcal{X}'R'^- \wedge \\ \forall T, U, T', U' \left[\begin{array}{l} (\mathcal{X}, \mathcal{X}') \mathcal{E}_{(S, U, S', U')}^{(R^-, T, R'^-, T')} \Rightarrow \\ (T \sqsubseteq_{\mathcal{X}} U \Leftrightarrow T' \sqsubseteq_{\mathcal{X}'} U') \end{array} \right] \end{array} \right] \vee \\ \left[\begin{array}{l} \forall zw [zRw \Leftrightarrow z[S \sqcup S^-]w] \wedge \\ \forall zw [zR'w \Leftrightarrow z[S' \sqcup S'^-]w] \end{array} \right] \end{array} \right].$$

It should be easy to verify that, informally speaking, EP* has the same content as EP. But it expresses this content in the language of $L_2\text{PCA}_{\Delta_0}^2$, merely enriched by the addition of third-order binary variables as ‘ \mathcal{R} ’. This addition being admitted, EP* can, then, be added to this system as a supplementary axiom. Since EP* is an abstraction principle, its left-hand side is a first-order identity (i.e. ‘ $\mathfrak{R}[R, S]$ ’ and ‘ $\mathfrak{R}[R', S']$ ’ are singular terms). Moreover, its right-hand side involves no constant other than ‘ $-$ ’ and ‘ \sqcup ’. Hence, adding it to $L_2\text{PCA}_{\Delta_0}^2$ requires no further comprehension axiom. The theory obtained is, then, a third-order one, including first-order, second-order binary, and third-order monadic and binary variables, but only admitting predicative second-order comprehension.

5.3 Real Numbers

Though EP* supplies the required grounds for defining real numbers as objects, this theory falls short of achieving it. All that one can do, in the light of it, is informally (and meta-theoretically) identify these numbers with ratios like $\mathfrak{R}[\mathbf{R}, \mathbf{S}]$. If a predicate constant designating the first-level property of being a real number is to be available, one also has to admit a new form of comprehension, for licensing the following explicit definition:

$$(5.2) \quad \forall x [\mathbf{R}x \leftrightarrow \exists_{(\mathcal{P})} \mathcal{X} \exists_{(\mathfrak{B}(\mathcal{X}))} R \exists_{\mathcal{X}} S [x = \mathfrak{R}[R, S]]].$$

What we need, then, is the following second-order third-orderly impredicative axiom-scheme:

$$(CA_{\Sigma_1^2}^2) \quad \exists X \forall x [Xx \leftrightarrow \phi_{\Sigma_1^2}],$$

where ‘ $\phi_{\Sigma_1^2}$ ’ stands for a third-order formula involving a third-order existential quantifier— together with a second-order one.

It is then only in such a (highly) impredicative third-order theory obtained from $L_2PCA_{\Delta_0}^2$ by adding to it both the proper axiom EP* and the comprehension axiom-schema $(CA_{\Sigma_1^2}^2)$, that the property of being a real number can be properly defined in our predicate setting. For short, let us call this theory ‘FMR’ (for ‘Frege’s (theory of) magnitudes (and) real (numbers)’). If such an impredicative theory were to be avoided, definition (5.2) should be omitted. At most, one could recur to a new abbreviation stipulation as

$$(5.2') \quad \mathbf{R}(x) := \exists_{(\mathcal{P})} \mathcal{X} \exists_{(\mathfrak{B}(\mathcal{X}))} R \exists_{\mathcal{X}} S [x = \mathfrak{R}[R, S]],$$

by then admitting that a real number is an object that satisfies the open formula ‘ $\mathbf{R}(x)$ ’. Call ‘FMR’ the theory got from $L_2PCA_{\Delta_0}^2$ by merely adding EP*, and replacing (5.2) with (5.2’). The same argument used above to prefer (3.6’) over (3.6) does not apply here, since the definition of real numbers is the final purpose to be reached to revive Frege’s program. Hence, if one considers that this aim is reached only if a property, counting as the property of being a real number, is directly expressed as such, in the relevant formal setting, on pain of missing the definition itself, there is no other option than going for FMR.

According both to (5.2) and (5.2’), a real number is a ratio over some domain of magnitudes. This might appear odd at first glance, since, given different such domains, this might seem to entail that different sorts of real numbers arise, according to the domain of magnitudes they are defined on. However, from EP*, it easily follows that, if there are several domains of magnitudes, for any ratio (or *\mathfrak{R} -abstractum*) on one of them, there is just another ratio (or *\mathfrak{R} -abstractum*) on each other of them that is the same object as the former—i.e. that the ratio of two first-level binary relations having a certain property \mathcal{M} such that $\mathcal{M}(\mathcal{M})$ is the same object as the ratio of two first-level binary relations having another property \mathcal{M}' such that $\mathcal{M}(\mathcal{M}')$.

Hence, once real numbers are defined, either in FMR or in FMR’, as ratios of magnitudes, one can define the usual properties, relations and functions on them, making the development of real analysis possible, within these systems—or some appropriate extensions of them, if needed. We stop here, however, and rather tackle some meta-theoretical issues concerning these systems, and the corresponding definitions.

6 Existence Proofs

It is easy to see that EP* implicitly defines continuously many objects to be identified, either through (5.2) or through (5.2'), with the real numbers, only in the presence of nonempty positive classes. If there were no first-level binary relations \mathbf{R} such that $\exists \mathcal{X} [\mathcal{P}(\mathcal{X}) \wedge \mathcal{X}\mathbf{R}]$, its second-order universal quantifier would range on an empty domain, and this would render the right-hand side of EP* nonsensical, as well as, then, both (5.2) and (5.2'). Still, a nonempty positive class exists if and only if this is so for a nonempty domain of magnitudes. Hence, an existence proof of such a domain (or of a positive class) is an indispensable supplement to our definition of real numbers: it is required to make it sensible.

Of course, no form of comprehension might be appealed to in order to deliver this proof, since no comprehension axiom can secure the existence of an R such that $\exists \mathcal{X} [\mathcal{M}(\mathcal{X}) \wedge \mathcal{X}R]$. Moreover, it would not be enough to observe that the empty first-level binary relation exists no matter what the first-order domain looks like, since, as observed in footnote (27), this relation neither forms nor belongs to a positival (and, then, positive) class. What is to be proved, then, is that there are enough appropriate (or appropriately related) objects for defining on them continuously many (extensionally) distinct first-level binary relations forming a nonempty domain of magnitudes³³.

This cannot be accomplished by a proof following a similar pattern as the one that allows neologicists to prove the existence of natural numbers within the very theory in which they define them, namely FA. This proof goes as follows:

- The concept $[x : x \neq x]$ exists by predicative comprehension;
- Then, HP allows to define 0 as $\#[x : x \neq x]$;
- By logic, $[x : x \neq x] \approx [x : x \neq x]$;
- Hence, by HP, $0 = 0$, from which it follows that 0 exists³⁴;
- Since HP allows to define the successor relation on the whole first-order domain, natural numbers can be defined as the objects that bear the weak ancestral of this relation to 0;

³³This is just what Frege seems to signal at the beginning of § II.164, in the passage we have quickly referred to in footnote (27) above (Frege 1893-1903, § II.164; Frege 2013, pp. 1602-612; notice that the English term ‘Relation’ with capital ‘R’ is used here to translate the German term ‘Relation’, which Frege uses, as opposed to ‘Beziehung’, translated instead as ‘relation’, to name value-ranges of relations):

We can now approach the question raised earlier (§ 159) from where we obtain the magnitudes whose ratios are real numbers. They will be Relations; and they must not be empty, i.e., they must not be extensions of relations in which no objects stand to each other. For such relations have the same extension; there is only one empty Relation. We could not define any real number with it. If q is the empty Relation, then $[\dots][q^-]$ is the same; likewise $[\dots][q \sqcup q^-]$. Also the composition of Relations on our domain of magnitudes cannot result in the empty Relation; but that would happen if there were no object to which some object stood in the first Relation and which also stood to some object in the second Relation. We thus require a class of objects that stand to each other in the Relations of our domain of magnitudes, and, in particular, this class has to contain infinitely many objects.

³⁴Notice that, since HP licenses the formation of the term ‘ $\#[x : x \neq x]$ ’, the identity ‘ $0 = 0$ ’ might be derived, in classical logic, as an immediate consequence of the theorem ‘ $\forall x [x = x]$ ’. Still, if such a proof of the existence of 0 were admitted, HP would inevitably be endowed with an existential import that would be incompatible with its alleged analyticity. This is one of the reasons why it is often advanced that the subjacent logic to FA should be free: the matter has been firstly tackled in Shapiro & Weir (2000), §§ IV-V; but see also Payne (2013).

- Proving—from HP plus (impredicative) comprehension—that any such object has a (unique) successor is, then, enough to prove, by countable induction, that all the natural numbers exist.

This pattern only allows to prove that there are objects falling under a first-level concept, given both a way to identify these objects collectively, as values of a particular function such as $\#$, and a way to identify them individually, as values of this function for particular concepts as arguments. In our case, one should, instead, prove that there are enough objects on which one can define binary relations falling under some second-level concept complying with a certain structural condition, where no particular way is given to identify both these objects and these relations either individually (except for the identity relation) or collectively. Moreover, by appealing to countable induction, this pattern can, at best, be suitable for proving the existence of countably many objects, and—even if it were possible to show that such objects allow to define on them the required binary relations³⁵—the main task of the proof would just be to prove that, which is certainly not something that might be done by following this pattern. Hence, though required for making the very definition of real numbers sensible, the existence proof of nonempty domains of magnitudes cannot be carried out in the theory FMR itself, and, *a fortiori*, in FMR', in which that definition is stated.

Two alternative strategies seem possible to deliver it. The first is in line with Frege's perspective and looks for an alternative way to prove the existence of continuously many objects on which continuously many permutations, forming a domain of magnitudes, can be defined. The second departs from this perspective, and uses mathematical results unavailable to Frege. It might be appealed to, as a sort of unhoped lifeline for Frege's purpose, in order to avoid some problems the former strategy suffers from. It consists of inquiring whether continuously many permutations forming such a domain can be obtained from countably many objects, whose existence might, if needed, be proved by applying the previous proof-pattern. Let us call the first strategy 'inflationary' and the second 'non-inflationary'.

6.1 The Inflationary Strategy: Bicimal Pairs

The inflationary strategy can be implemented in at least two slightly different ways, in our setting. One follows Frege's own plan sketched in § II.164, and takes the existence of natural numbers for granted. The other appeals, instead, to a restricted version of BLV, to get an ω -sequence of objects other than Frege's natural numbers. The structural similarity of these approaches makes them suffer from the same difficulties. We merely consider the former. The reader might get an idea of the latter from the way we deal with a restricted version of BLV at the beginning of § 6.2.

Taking the existence of natural numbers for granted, Frege considers the pairs $\langle n, \{m_i\}_{i=0}^{\infty} \rangle$ composed by a natural number and an infinite sequence of positive natural numbers. These pairs are apt to code Cauchy series like

$$(6.1) \quad n + \sum_{j=1}^{\infty} \lambda_j \frac{1}{2^j} \quad (\lambda_j \in \{0, 1\}),$$

under the condition that m_i is the i -th value of j such that $\lambda_j = 1$ and the λ_j are not all 0 after a certain range. It follows that, once addition is appropriately defined on these pairs, one can

³⁵We shall hark back on this matter in § 6.2 below.

associate to each of them, let us say α , a binary (first-level) relation \mathbf{R}_α such that, for any pair of these same pairs x and y , $x\mathbf{R}_\alpha y$ if and only if $x + \alpha = y$. It is easy to see that this allows to define as many relations as pairs like $\langle n, \{m_i\}_{i=0}^\infty \rangle$, namely continuously many ones, and that these relations are such that:

- both they and their inverses are functional, since, for any such pairs x , y and z , $x + \alpha = y \wedge x + \alpha = z$ and $y + \alpha = x \wedge z + \alpha = x$ each entails that $y = z$;
- their composition mimics an addition on the pairs they are defined on, since, if β is also such a pair, $\mathbf{R}_\alpha \sqcup \mathbf{R}_\beta$ extensionally coincides with $\mathbf{R}_{\alpha+\beta}$, which is proved by observing that, for any such pairs x and y , $x + (\alpha + \beta) = y$ if and only if there is such a pair z such that $x + \alpha = z \wedge z + \beta = y$;
- the identity relation is not one of them, since no Cauchy series like (6.1) is equal to zero.

It is, then, easy to verify that these relations form a positive class, from which a domain of magnitudes is obtained by merely adding their inverses and the identity relation.

Objects rendering these pairs in a formal setting are quite easy to define in any second-order version of arithmetic. A simple way to do it (Panza, 2016 and Panza, FC3) is by adding a new axiom, under the form of the following abstraction principle:

$$(FP) \quad \forall_{\mathbf{N}} XY \forall_{\mathbf{N}} xy [\langle x, X \rangle = \langle y, Y \rangle \Leftrightarrow (x = y \wedge \forall z (Xz \Leftrightarrow Yz))],$$

where the index ‘ \mathbf{N} ’ signals that the universal quantifiers are restricted to properties of natural numbers and to these very numbers respectively—the acronym ‘FP’ stands for ‘Frege’s Principle’, and emphasizes the fact that this principle is a restricted adapted form of BLV.

Of course, to go ahead, we have to prove that FP is consistent. Assuming the consistency of second-order arithmetic, to this purpose, it is, however, enough to observe that FP has a model in the $V_{\omega+1}$ stage of ZF’s hierarchy. This is because second-order arithmetic has a model in the V_ω segment of ZF, and consequently the set \wp^ω of all subsets of the set of natural numbers is at stage $V_{\omega+1}$, and provides the required model.

This having been established, we can look at the pairs like $\langle n, \mathbf{P} \rangle$, implicitly defined by FP, and formed by a natural number n and a property \mathbf{P} of natural numbers, and select among them those whose second element \mathbf{P} is an infinite property of natural numbers—i.e. it is such that $\forall_{\mathbf{N}} x [\mathbf{P}x \Rightarrow \exists_{\mathbf{N}} y [x < y \wedge \mathbf{P}y]]$. For short, call them ‘bicimal pairs’. Clearly, FP allows to distinguish continuously many such pairs.

They can be arranged into two partitions, such that any bicimal pair $\langle n, \mathbf{P} \rangle$ belongs to one partition if $\mathbf{P}0$, and to the other if $\neg\mathbf{P}0$. A total order can, moreover, be defined on these pairs, in such a way that the pairs in the former partition count as positive, the pair $\langle 0, \mathbf{N}^+ \rangle$ (where ‘ \mathbf{N}^+ ’ designates the property of being a positive natural number) counts as the zero pair, and the other pairs in the latter partition count as negative (more details are given in Panza, 2016, p. 417; others will be found in Panza, FC3). This makes the bicimal pairs form an additive Abelian group, that can be proved to be dense, totally ordered and Dedekind-complete (and, then, Archimedean), and can also be extended to a field by an appropriate definition of multiplication (details are, again, to be found in Panza, FC3). It would, then, be not only very tempting, but also rather natural to code the real numbers by bicimal pairs, so as to avoid the very definition of domains of magnitudes and of ratios thereof as perfectly useless.

Still, this is certainly not what Frege’s strategy should lead us to. In order to follow his indications, one should rather define appropriate permutations on bicimal pairs and show that they form a domain of magnitudes. This can easily be done by associating to any such pair $\langle n, \mathbf{P} \rangle$ the binary relation $\mathbf{R}_{\langle n, \mathbf{P} \rangle}$ such that, for any two such pairs $\langle y, Y \rangle$ and $\langle z, Z \rangle$, $\langle y, Y \rangle \mathbf{R}_{\langle n, \mathbf{P} \rangle} \langle z, Z \rangle$ if and only if $\langle y, Y \rangle + \langle n, \mathbf{P} \rangle = \langle z, Z \rangle$. It would, then, be easy to verify that the relations associated with positive bicimal pairs just behave as those Frege suggests to associate to his pairs, and form, then, a positive class, which is easy to extend to a domain of magnitudes.

Following this path leads, then, to an arithmetical copy of the additive (ordered) group of the real numbers, as an intermediate step in a much more complex, supposedly non-arithmetical definition of them. Hence, real numbers might be *ipso facto* identified with bicimal pairs, by so dramatically departing, however, from Frege’s purpose. The same happens for any other way of pursuing the inflationary strategy: it inevitably leads either to encode the real numbers by objects other than ratios of magnitudes and thus depart from Frege, or to define real numbers twice, structurally speaking, by accepting the idea that the second definition requires a supplementary axiom which is not at all required by the first, namely EP*.

Though mathematically quite shocking, the former option poses no problem from a realist perspective such as Frege’s, since the realist may argue that bicimal pairs are intrinsically not real numbers, though they behave like them. We do not want to dig into this possibly rather odd attitude. We merely observe that, in this perspective, the ratios on the appropriate permutations defined on these pairs could not but be taken to be real numbers. Thus, the only way to avoid concluding that, *pace* Frege, real numbers are intrinsically arithmetical objects would be to prove that there are non-arithmetical nonempty domains of magnitudes. Insofar as EP* identifies the ratios on any domain of magnitudes with the ratios on any other such domain, this would leave room for arguing that being a real number is not intrinsically the same as being a ratio on permutations defined on bicimal pairs, since such a ratio would merely provide one among other possible and essentially distinct modes of presentation of such a number. But, then, how to prove the existence of other, non-arithmetical nonempty domains of magnitudes?

6.2 The Non-Inflationary Strategy

As³⁶ a matter of fact, also the non-inflationary strategy might be grounded on the assumption of the existence of natural numbers. Strictly speaking, this is not necessary, however: it might also be grounded on a consistently restricted version of BLV.

Let ‘ $\mathcal{F}(\mathbf{X})$ ’ be the abbreviation of a logical second-order formula stating that \mathbf{X} is a property satisfied at most by finitely many objects. A possibility is appealing to Dedekind-finiteness:

$$\mathcal{F}(X) := \forall Y \left[\begin{array}{l} \forall x [[Yx \Rightarrow Xx] \wedge \exists y [Xy \wedge \neg Yy]] \Rightarrow \\ \neg \exists R \left[\begin{array}{l} \forall z [Xz \Rightarrow \exists! z' (zRz' \wedge Yz')] \wedge \\ \forall z' [Yz' \Rightarrow \exists! z (zRz' \wedge Xz)] \end{array} \right] \end{array} \right].$$

The relevant restricted version of BLV, call it ‘FinBLV’, is, then, this:

$$(\text{FinBLV}) \quad \forall_{(\mathcal{F})} XY [\varepsilon X = \varepsilon Y \Leftrightarrow \forall x (Xx \Leftrightarrow Yx)],$$

³⁶Both for this § and the following one, we are very much indebted to Mirna Džamonja, Andrew Moshier and, overall, Alain Genestier who guided us in the understanding of Karrass and Solitar’s proof and annexed topics.

where ‘ εX ’ and ‘ εY ’ denote the extensions of X and Y , and the index ‘ \mathcal{F} ’ signals that the universal quantifier is restricted to the (first-level) properties satisfying the formula ‘ $\mathcal{F}(X)$ ’³⁷.

FinBLV implicitly defines a countable infinity of extensions. To see it, notice that $[x : x \neq x]$ is a finite property, whose existence is warranted by predicative second-order comprehension. So FinBLV applies to it. Let \circlearrowleft be its extension. Since it is a theorem of second-order (free)³⁸ logic that $\forall x([x : x \neq x] \Leftrightarrow [x : x \neq x])$, from FinBLV it follows that $\circlearrowleft = \circlearrowleft$, which entails that $\exists x(x = \circlearrowleft)$. One can, then, firstly appeal to second-order impredicative comprehension to define a functional first-level binary relation on the whole first-order domain, by stating that

$$\forall x, y [xSy \Leftrightarrow [\exists X_{\mathcal{F}} [x = \varepsilon X] \wedge y = \varepsilon [z : z = x]]],$$

then define the weak ancestral of S , i.e. $S^{*=}$, and finally appeal to second-order predicative comprehension to define the property E_{\circlearrowleft} of being an extension belonging to the S -succession starting from \circlearrowleft :

$$\forall x [E_{\circlearrowleft} x \Leftrightarrow \circlearrowleft S^{*=} x].$$

This allows to accomplish the task by repeating, *mutatis mutandis*, the neologist recursive proof of the existence of natural numbers³⁹.

Now, consider the symmetric group $\Sigma_{\mathbb{N}}$ on the natural numbers, i.e. the (additive) group of all permutations on \mathbb{N} . We know that $\Sigma_{\mathbb{N}}$ has cardinality 2^{\aleph_0} . But we also know that it contains

³⁷This principle is freely inspired by two different suggestions respectively advanced by Antonelli & May (2005, p. 11), and by Boolos (1998a, p. 99; 1998b, p. 178), in particular by Boolos’ *New V*, i.e. $\forall F \forall G (\varepsilon F = \varepsilon G \Leftrightarrow (Small(F) \vee Small(G) \rightarrow F \equiv G))$, where ‘*Small*’ means ‘not equinumerous with the universal concept $[x : x = x]$ ’. By remaining faithful to this suggestion, we should replace ‘*Small*’ by ‘*Fin*’ (or take the former to mean ‘finite’), and FinBLV by ‘ $\forall F \forall G (\varepsilon F = \varepsilon G \Leftrightarrow (\mathcal{F}(F) \vee \mathcal{F}(G) \rightarrow F \equiv G))$ ’. Though this latter principle would not be equivalent to FinBLV, we cannot see any relevant difference between them with respect to our purpose. We prefer FinBLV simply because it makes immediately clear that infinite concepts do not matter, here, by ascribing to them no extension, rather than ascribing to all of them the same extension.

³⁸See footnote (34), above.

³⁹The proof depends on the lemma that $\forall x \exists y [y = \varepsilon [z : z = x]]$. Here is how this can be proved. FinBLV and $\forall X \forall x [Xx \Leftrightarrow Xx]$ imply that $\forall_{(\mathcal{F})} X [\varepsilon X = \varepsilon X]$, and, then, $\forall X_{(\mathcal{F})} \exists y [y = \varepsilon X]$. What is required is, then, to prove that ‘ $[z : z = x]$ ’ denotes a (finite) property, which is ensured by predicative comprehension with parameters, since it entails that $\forall x \exists X \forall z [Xz \Leftrightarrow z = x]$. Notice that this proof also holds in (negative) free logic: thanks to Ludovica Conti for drawing our attention to this; see also Conti (2019, p. 145, fn. 426, and pp. 151-152). This lemma implies, *a fortiori*, that $\forall_{E_{\circlearrowleft}} x \exists y [y = \varepsilon [z : z = x]]$. The very definition of the weak ancestral of S allows, then, to prove quite easily the principle of induction for the FinBLV-*abstracta* having the property E_{\circlearrowleft} (or E_{\circlearrowleft} -*abstracta*, from now on)—namely ‘ $\forall X \left[\left(X \circlearrowleft \wedge X \xrightarrow[S]{E_{\circlearrowleft}} X \right) \Rightarrow \forall_{E_{\circlearrowleft}} x [Xx] \right]$ ’, where ‘ $X \xrightarrow[S]{E_{\circlearrowleft}} X$ ’ abbreviates ‘ $\forall_{E_{\circlearrowleft}} x \forall y [(Xx \wedge xSy) \Rightarrow Xy]$ ’.

With this principle at hand, it is, then, equally easy to prove that $\forall_{E_{\circlearrowleft}} x \exists X [x = \varepsilon X]$, as a consequence of the stipulation that $\circlearrowleft = \varepsilon [x : x \neq x]$, from which it immediately follows that $\exists X [\circlearrowleft = \varepsilon X]$. By appealing to $\forall_{E_{\circlearrowleft}} x \exists y [y = \varepsilon [z : z = x]]$, one gets that $\forall_{E_{\circlearrowleft}} x \exists y [\exists X [x = \varepsilon X] \wedge y = \varepsilon [z : z = x]]$, that is, $\forall_{E_{\circlearrowleft}} x \exists y [xSy]$. Next, the properties of the ancestral of a binary relation allow to prove that $\forall_{E_{\circlearrowleft}} x \forall y [xSy \Rightarrow E_{\circlearrowleft} y]$, and so to conclude that $\forall_{E_{\circlearrowleft}} x \exists_{E_{\circlearrowleft}} y [xSy]$. The existence of countably many E_{\circlearrowleft} -*abstracta* finally follows from proving by induction that $\forall_{E_{\circlearrowleft}} x [\neg x S^{*\neq} x]$, where $S^{*\neq}$ is the strong ancestral of S . This last part of the proof is a little bit harder than the previous ones, but requires no specific skills: it just parallels the analogous proof of FA, by exploiting, like this (together with the principle of induction and some properties of the strong ancestral, also) the obvious facts that $\neg \circlearrowleft S^{*\neq} \circlearrowleft$ and $\forall x, y, z [(xSy \wedge zSy) \Rightarrow x = z]$. Though quite simple, this proof of existence of countably many E_{\circlearrowleft} -*abstracta* directly involves most of Peano’s second-order axioms as theorems about them. The axioms that do not enter it, i.e. ‘ $\neg \exists_{E_{\circlearrowleft}} x [0Sx]$ ’ and ‘ $\forall x, y, z [(xSy \wedge xSz) \Rightarrow y = z]$ ’ (the second of which would allow replacing the relation S by the successor function), are, moreover, even easier to prove. Hence, if impredicative comprehension is accepted, Peano’s second-order arithmetic is interpretable on the mentioned extensions—with no appeal to set-theoretical assumptions on them. These extensions do not comply, however, with HP, which makes them crucially differ from natural numbers as defined in FA.

torsion elements⁴⁰, which prevents both from defining on $\Sigma_{\mathbb{N}}$ a total order compatible with the group structure, and from making an injective map from a (nonempty) domain of magnitudes (if any) into it surjective. It follows that $\Sigma_{\mathbb{N}}$ does not provide a nonempty model for our Fregean consistent definition of such a domain. Nevertheless, by a theorem by Karrass and Solitar (1956, p. 65), $\Sigma_{\mathbb{N}}$ provably “contains a copy of the additive group of the reals”. In other terms, there is a subgroup of $\Sigma_{\mathbb{N}}$ which is isomorphic to $(\mathbb{R}, +)$, and is, then, not only totally ordered, but also Abelian, dense, Archimedean and Dedekind-complete. Since any totally ordered, dense and Dedekind-complete group of permutations is a model of our definition of domains of magnitudes⁴¹, $\Sigma_{\mathbb{N}}$ contains such a model. Insofar as $\Sigma_{\mathbb{N}}$ is isomorphic to the symmetric group on whatever infinite countable set, it is also so to the symmetric group $\Sigma_{E_{\varnothing}}$ formed by the E_{\varnothing} -abstracta. Hence, Karras and Solitar’s theorem entails that admitting the existence of this latter symmetric group ensures the existence of a nonempty domain of magnitudes. It would, then, be enough to admit that it makes sense to speak of all permutations on an infinite countable domain \mathbf{D} of objects⁴², and that the existence of (the elements of) \mathbf{D} *ipso facto* entail the existence of the group formed by these permutations, to conclude that the existence of a nonempty domain of magnitudes is ensured by the existence of the natural numbers or of the E_{\varnothing} -abstracta, because of an immediate corollary of Karrass and Solitar’s result⁴³.

Karrass and Solitar’s proof could certainly not have been within Frege’s reach. But it is not very entangled, as such, and, under the mentioned admission, most of it can be conducted constructively, which is something Frege seems to require for his proof.

Let \mathbf{I} be an infinite countable set, for example an infinite subset of \mathbb{N} . Let $\sigma = \coprod_{i \in \mathbf{I}} \mathbf{C}_i$ be a partition of \mathbb{N} in infinite (countable) subsets \mathbf{C}_i ($i \in \mathbf{I}$), which makes $\bigcup_{i \in \mathbf{I}} \mathbf{C}_i = \mathbb{N}$ and $\mathbf{C}_h \cap \mathbf{C}_k = \emptyset$,

for any distinct h, k in \mathbf{I} . Let $\varrho : \mathbb{N} \rightarrow \mathbf{I}$ be the (surjective) application defined by this partition, associating to any n in \mathbb{N} , the single element $i = \varrho(n)$ of \mathbf{I} such that $n \in \mathbf{C}_i$. Such a partition, and therefore such an application, can easily be defined constructively. To make a simple example, take \mathbf{I} to be the set of all prime numbers plus 0, \mathbf{C}_0 the set of all natural numbers that are not (positive) powers of a prime number, namely $\{0, 1, 6, 10, \dots\}$, and, for any prime number p , \mathbf{C}_p the set of all (positive) powers of p . Though we would not be (presently) able to write a general formula providing, for any natural number n , the value $\varrho(n)$ of i , such that $n \in \mathbf{C}_i$, there is a finite algorithm allowing us to calculate such a value $\varrho(\mathbf{n})$ for whatever given natural number \mathbf{n} .

For any i in \mathbf{I} , let π_i be a permutation on \mathbf{C}_i . Define $\pi : \mathbb{N} \rightarrow \mathbb{N}$ by establishing that for any n in \mathbb{N} , $\pi(n) = \pi_{\varrho(n)}(n)$. This is clearly a permutation on \mathbb{N} , so that $\pi \in \Sigma_{\mathbb{N}}$. If $\coprod_{i \in \mathbf{I}} \mathbf{S}_i$ and ϱ have been defined constructively, any π_i , and therefore π , can also be so defined.

Supposing ϱ be defined as in the previous example, we might, for example, take π_i to be the permutation exchanging the $(2j - 1)$ -th element of any set \mathbf{C}_i , according to the usual order on \mathbb{N} , with the $2j$ -th one, and vice versa ($j = 1, 2, \dots$). Then π would permute any natural number with another natural number following or preceding it in this subset, according to this order:

⁴⁰A torsion element of a group \mathbf{G} is an element g of \mathbf{G} such that $g^n = e$ for some natural number n , where e is the identity element of \mathbf{G} .

⁴¹More precisely, the second-order property of being a permutation belonging to any such group satisfies the right-hand side of (3.6’).

⁴²As Frege himself should have admitted, in order to make his own definition of domains of magnitudes sensible.

⁴³Notice that though the definition of $\mathbf{S}^{*=}$ allows proving that the E_{\varnothing} -abstracta form an ω -sequence, as showed in footnote (39) above, all that is relevant here is the cardinality of set formed by these abstracta, namely the fact that this set is countably infinite.

$\pi(0) = 1, \pi(1) = 0, \pi(2) = 4, \dots$ Insofar as the same can be done for any permutation π_i on any \mathbf{C}_i , the application $\pi : \mathbb{N} \rightarrow \mathbb{N}$ defined by stating that $\pi(n) = \pi_{\varrho(n)}(n)$ provides a group monomorphism⁴⁴ $\prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i} \rightarrow \Sigma_{\mathbb{N}}$, where $\prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i}$ is the product of the symmetric groups on all sets \mathbf{C}_i —since, if $\{\pi_i\}_{i \in \mathbf{I}}$ and $\{\pi'_i\}_{i \in \mathbf{I}}$ are two families of permutations on all the sets \mathbf{C}_i , then $\{\pi_i\}_{i \in \mathbf{I}} \circ \{\pi'_i\}_{i \in \mathbf{I}} = \{\pi_i \circ \pi'_i\}_{i \in \mathbf{I}}$. Under the condition mentioned above, and provided all sets \mathbf{C}_i be (constructively) given, this further step of the proof is, also, constructively licensed.

By Cayley's theorem, any group \mathbf{G} is isomorphic to a subgroup of the symmetric group $\Sigma_{\mathbf{G}}$ on \mathbf{G} itself. Thus, there is a group monomorphism $(\mathbb{Q}, +) \rightarrow \Sigma_{\mathbb{Q}}$ from the additive group of the rational numbers $(\mathbb{Q}, +)$ into the symmetric group $\Sigma_{\mathbb{Q}}$ on \mathbb{Q} . Though quite general, Cayley's theorem can easily be proved constructively, which also makes this new step of Karrass and Solitar's proof admissible from Frege's perspective. A quite usual way to prove it is, for example, by considering the translation $\tau_y : x \mapsto y * x$ on the domain of \mathbf{G} (where $*$ is the composition law of this group), since it is easy to see that $\tau_{(a*b)} = \tau_a \circ \tau_b$, for any a, b in such a domain. This proof directly applies to the present case, for \mathbf{G} is nothing but $(\mathbb{Q}, +)$. To this purpose, we can take \mathbb{Q} to play the role of the domain of \mathbf{G} and $+$ that of $*$, and observe that $\tau_{(q+r)} = \tau_q \circ \tau_r$, for any q, r in \mathbb{Q} . Notice, moreover, that this application is perfectly akin to Frege's suggested definition of permutations on the pairs $\langle n, \{m_i\}_{i=0}^{\infty} \rangle$ in the outline of his existence proof.⁴⁵

For any i in \mathbf{I} , let us choose a bijection $\vartheta_i : \mathbb{Q} \rightarrow \mathbf{C}_i$ from the set \mathbb{Q} to the set \mathbf{C}_i . Because of the monomorphism $(\mathbb{Q}, +) \rightarrow \Sigma_{\mathbb{Q}}$, this engenders, for any such i , a new group monomorphism $(\mathbb{Q}, +) \rightarrow \Sigma_{\mathbf{C}_i}$ from $(\mathbb{Q}, +)$ into the symmetric group $\Sigma_{\mathbf{C}_i}$ on any \mathbf{C}_i , and, by composition, a further group monomorphism $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +) \rightarrow \prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i}$ from the product $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ of countably many copies of the additive group $(\mathbb{Q}, +)$ into the product $\prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i}$. Again, if the partition π and the application ϱ are defined constructively, making the choice of the bijections ϑ_i and so getting these two monomorphisms require no form of the Axiom of Choice, and, under the condition mentioned above, is unquestionably constructive. By combining the monomorphisms $\prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i} \rightarrow \Sigma_{\mathbb{N}}$ and $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +) \rightarrow \prod_{i \in \mathbf{I}} \Sigma_{\mathbf{C}_i}$, we finally get, constructively again, a final monomorphism

$$(6.2) \quad \prod_{i \in \mathbf{I}} (\mathbb{Q}, +) \rightarrow \Sigma_{\mathbb{N}}.$$

This provides a constructive ground for Karrass and Solitar's proof. But, for its completion, a last step is needed, which, instead, requires non-constructive means and makes, then, the whole proof non-constructive. Both additive groups $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$, enriched with the usual multiplication by a rational number, can be regarded as vector spaces over the field $(\mathbb{Q}, +, \cdot)$, and this is also the case for the product $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$. When $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are so regarded, it is however not plain that they have a basis, unless Zorn's lemma is appealed to, since this lemma allows to prove that every vectorial space has a basis⁴⁶. The non-constructive step of the proof

⁴⁴A group monomorphism is an injective group homomorphism, i.e. an injective map μ from a group $(\mathbf{G}, *)$ to another group (\mathbf{H}, \star) , such that $\mu(x * y) = \mu(x) \star \mu(y)$ for any x, y in \mathbf{G} .

⁴⁵See § 6.1, above.

⁴⁶By speaking of basis of a vector space, we more precisely mean, here, a Hamel basis. Let \mathbf{V} be a vector space on a field \mathbf{F} . An Hamel basis of \mathbf{V} is a set $\mathbf{B}_{\mathbf{V}}$ of linearly independent vectors in \mathbf{V} , such that for any vector v in \mathbf{V} there is a (unique) finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of $\mathbf{B}_{\mathbf{V}}$ such that $v = \mathbf{a}_1 \mathbf{v}_1 + \mathbf{a}_2 \mathbf{v}_2 + \dots + \mathbf{a}_k \mathbf{v}_k$, where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are elements of \mathbf{F} .

just consists, then, in appealing to this lemma for proving that $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ have a basis.

This makes it possible to appeal to a new theorem ensuring that, if a vector space has several distinct bases, all of them have the same cardinality—which is, then, to be regarded as the dimension of this space. For vector spaces with finite bases, this is quite easy to prove⁴⁷. For vector spaces whose generating sets are infinite, as $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ and $(\mathbb{R}, +)$, the proof is more entangled, but can still be given constructively. In this case, the theorem can indeed be viewed as an immediate corollary of another theorem asserting that the cardinality of any generating set of a vector space \mathbf{V} that can be regarded as the direct sum of an infinite family $\{\mathbf{V}_{\lambda \in \mathbf{\Lambda}}\}$ of non-zero vectorial sub-spaces is greater or equal to the cardinality of the set of indices $\mathbf{\Lambda}$ (Bourbaki, Algebra I, ch. II, prop. 23, cor. 2).

Provided that two vector spaces on the same field (both having bases) are isomorphic if (and only if) they have the same dimension, $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ and $(\mathbb{R}, +)$, when regarded as vector spaces over $(\mathbb{Q}, +, \cdot)$, have the same dimension, and thus are isomorphic. This obviously entails that the group monomorphism (6.2) results in a new group monomorphism

$$(\mathbb{R}, +) \longrightarrow \Sigma_{\mathbb{N}},$$

which makes $\Sigma_{\mathbb{N}}$ contain a copy of $(\mathbb{R}, +)$, as was required to be proved.

If this proof is granted, together with the existence both of an infinite countable set \mathbf{D} —as \mathbb{N} or the set formed by the $E_{\mathcal{O}}$ -abstracta—and of the symmetric group $\Sigma_{\mathbf{D}}$ on this set, the conclusion follows that there is (at least) a totally ordered, dense and Dedekind-complete subgroup of $\Sigma_{\mathbf{D}}$. Let $(\mathbf{M}_{\mathbf{D}}, \circ, <)$ be such a subgroup of $\Sigma_{\mathbf{D}}$. Claiming that $(\mathbf{M}_{\mathbf{D}}, \circ, <)$ is a domain of magnitudes in agreement with definition (3.6') is the same as claiming that the triple $(\mathcal{M}_{\mathbf{D}}, \sqcup, \mathcal{P}_{\mathbf{D}})$ satisfies the open formula $[\mathcal{P}(\mathcal{Y}) \wedge \forall R[\mathcal{X}R \Leftrightarrow \mathfrak{d}(\mathcal{Y})(R)]]$ entering the right-hand side of this definition, with ' $\mathcal{M}_{\mathbf{D}}$ ' as a value of ' \mathcal{X} ' and ' $\mathcal{P}_{\mathbf{D}}$ ' as a value of ' \mathcal{Y} ', provided that: any binary relation R has the property $\mathcal{M}_{\mathbf{D}}$ if and only if it is in $\mathbf{M}_{\mathbf{D}}$, and the property $\mathcal{P}_{\mathbf{D}}$ only if it has the property $\mathcal{M}_{\mathbf{D}}$; ' \sqcup ' denotes the same operation on the binary relations that are in $\mathbf{M}_{\mathbf{D}}$ as that denoted by ' \circ '; and, for any R, S in $\mathbf{M}_{\mathbf{D}}$, $\mathcal{P}_{\mathbf{D}}R \sqcup S^-$ if and only if $S < R$, so that $\mathcal{P}_{\mathbf{D}}R$ if and only if $0_{\mathbf{M}_{\mathbf{D}}} < R$ —where $0_{\mathbf{M}_{\mathbf{D}}}$ is, of course, the neutral element of $(\mathbf{M}_{\mathbf{D}}, \circ, <)$, namely the identity relation. To make this claim sensible, we have, of course, to grant that these conditions ensure the existence of the two second-level properties $\mathcal{M}_{\mathbf{D}}$ and $\mathcal{P}_{\mathbf{D}}$, which requires third-order comprehension. Supposing it is admitted, Karrass and Solitar's result provably establishes that, under the admission of the existence of \mathbf{D} and $\Sigma_{\mathbf{D}}$, there is a nonempty domain of magnitudes, namely the ordered group defined by the triple $(\mathcal{M}_{\mathbf{D}}, \sqcup, \mathcal{P}_{\mathbf{D}})$, since it entails that the properties $\mathcal{P}_{\mathbf{D}}$ and $\mathcal{M}_{\mathbf{D}}$ are such that $\mathcal{P}(\mathcal{P}_{\mathbf{D}})$ and $\forall R[\mathcal{M}_{\mathbf{D}}R \Leftrightarrow \mathfrak{d}(\mathcal{P}_{\mathbf{D}})(R)]$, so that $\mathcal{M}(\mathcal{M}_{\mathbf{D}})$.

This being granted, it is, then, easy to prove that there are as many distinct binary relations in such a domain as distinct ratios defined on it according to EP*, namely that these ratios are continuously many. In other terms, the ordered pairs $[R, S]$ of binary relations, the first of which has $\mathcal{M}_{\mathbf{D}}$ and the second $\mathcal{P}_{\mathbf{D}}$, form continuously many distinct equivalence classes according to the equivalence relation on the right-hand side of EP*, under the replacement of both ' \mathcal{X} ' and

⁴⁷A simple combinatorial argument allows to prove that the cardinality of any set of linearly independent vectors is smaller or equal to the cardinality of whatever generating set. Insofar as a basis of a vector space is a set of linearly independent vectors that generates the whole space, from this it immediately follows that two bases cannot have different cardinality, since, if they did, there would be a set of linearly independent vectors whose cardinality is greater than that of a generating set.

‘ \mathcal{Y} ’ with ‘ $\mathcal{P}_{\mathbf{D}}$ ’⁴⁸. For shortness and simplicity, let us sketch this proof with reference to EP. Its rephrasing with respect to EP* is only a matter of laborious routine.

Consider the first of the three disjoints forming the right-hand side of EP. Suppose that there be three binary relations \mathbf{R} , \mathbf{S} , and \mathbf{S}' that have $\mathcal{P}_{\mathbf{D}}$ and are such that

$$\begin{aligned} \mathbf{S} &\sqsubset_{\mathcal{P}_{\mathbf{D}}} \mathbf{S}', \\ \forall_{\mathbf{N}} x, y [x\mathbf{R} \sqsubset_{\mathcal{P}_{\mathbf{D}}} y\mathbf{S} &\Leftrightarrow x\mathbf{R} \sqsubset_{\mathcal{P}_{\mathbf{D}}} y\mathbf{S}']. \end{aligned}$$

If there were, then, another binary relation \mathbf{T} such that \mathbf{S}' coincided with $\mathbf{S} \sqcup \mathbf{T}$, it would follow that

$$\forall_{\mathbf{N}} x, y [x\mathbf{R} \sqsubset_{\mathcal{P}_{\mathbf{D}}} y\mathbf{S} \Leftrightarrow x\mathbf{R} \sqsubset_{\mathcal{P}_{\mathbf{D}}} y(\mathbf{S} \sqcup \mathbf{T})],$$

which is impossible because of the Archimedeanity of $(\mathbf{M}_{\mathbf{D}}, \circ, <)$. This proves that, for whatever binary relation \mathbf{R} that has $\mathcal{P}_{\mathbf{D}}$, there are as many distinct ratios $\mathfrak{R}[\mathbf{R}, S]$ (where S is a binary relation that has $\mathcal{P}_{\mathbf{D}}$) as binary relations that have $\mathcal{P}_{\mathbf{D}}$, namely continuously many such ratios. Consider a binary relation \mathbf{R}' that has $\mathcal{P}_{\mathbf{D}}$ and is distinct from \mathbf{R} . Because of the completeness of positive classes, there is one and only one binary relation S that also has $\mathcal{P}_{\mathbf{D}}$ such that

$$\forall_{\mathbf{N}} x, y [x\mathbf{R} \sqsubset_{\mathcal{P}_{\mathbf{D}}} y\mathbf{S} \Leftrightarrow x\mathbf{R}' \sqsubset_{\mathcal{P}_{\mathbf{D}}} yS],$$

which shows that there as many pairs of binary relations R and S that have $\mathcal{P}_{\mathbf{D}}$, such that the ratio $\mathfrak{R}[R, S]$ is the same as $\mathfrak{R}[\mathbf{R}, S]$, as binary relations that have $\mathcal{P}_{\mathbf{D}}$, and there is no pair of binary relations R' and S' that have $\mathcal{P}_{\mathbf{D}}$ such that the ratio $\mathfrak{R}[R', S']$ is distinct from all ratios $\mathfrak{R}[\mathbf{R}, S]$, where S is a binary relation that has $\mathcal{P}_{\mathbf{D}}$.

Insofar as an analogous argument also applies, *mutatis mutandis*, to the second of the three disjoints forming the right-hand side of EP, and the third of these disjoints only concerns ordered pairs of binary relations whose first element is the identity relation and makes all ratios whose denominator is such relation identical, this is enough to conclude that the cardinality of the set $\{\mathfrak{R}[R, S] : \mathcal{M}_{\mathbf{D}}R \wedge \mathcal{P}_{\mathbf{D}}S\}$ is the same as that of $\{R : \mathcal{M}_{\mathbf{D}}R\}$, namely 2^{\aleph_0} , as was to be proved.

Hence, defining real numbers as \mathfrak{R} -abstracta over a domain of magnitudes entails the existence of at least 2^{\aleph_0} such numbers, since there is just one real number for any equivalence class induced by the right-hand side of EP or EP* on a single domain of magnitudes. To prove that there are just 2^{\aleph_0} , recall that, as observed in § 5.3 above, from EP or EP* it follows that, if there were several such domains, for any \mathfrak{R} -abstractum on one of those domains, there would just be one \mathfrak{R} -abstractum on the others that is identical with it, which entails that, if there were several domains of magnitudes, the \mathfrak{R} -abstracta on one of them would be just the same objects as the \mathfrak{R} -abstracta on the other.

7 Logicality and Arithmeticity

Given all the previous considerations, we can finally tackle two major questions concerning the definition of reals in FMR or FMR': *i*) Is there a strong enough sense in which this definition is logical? *ii*) Is this definition independent of natural numbers and their theory? Insofar as it seems difficult to imagine a consistent definition which is closest to Frege's envisaged one, the answer to these questions is relevant to assess Frege's achievements as well: Was Frege's plan for defining real

⁴⁸Notice that from the conditions above, it immediately follows that $\mathfrak{R}(\mathcal{P}_{\mathbf{D}})(R)$ if and only if $\mathcal{M}_{\mathbf{D}}R$.

numbers as ratios of magnitudes compatible with a logicist program, inconsistency apart? Was it in line with the basic idea that real and natural numbers are essentially independent objects? The question is not only a historical one, it has also a contemporary philosophical relevance: Is a neologicist program concerning real analysis, making it both logical and independent of the arithmetic of natural numbers, envisageable along Frege’s original indications? In what follows we will suggest a negative answer to all these questions⁴⁹.

All of them boil down to two issues: whether FMR or FMR’ can be taken to be logical systems, independent of a previous definition of natural numbers (likely got through FA); whether an existence proof of nonempty domains of magnitudes and of real numbers as defined in FMR or FMR’ is compatible with the logicity and arithmeticity of these definitions.

7.1 About the Definition of Domains of Magnitudes

Insofar as FMR and FMR’ are obtained by adding some new axioms to $L_2PCA_{\Delta_0}^2$, we will begin by investigating whether this latter system is genuinely logical and independent from the natural numbers. Both issues also apply to our definition of domains of magnitudes within it.

Likely, no one would question its independence from the natural numbers. The considerations advanced in § 4 seem, moreover, to support its logicity⁵⁰. Still, admitting that $L_2PCA_{\Delta_0}^2$ indeed has these features is not enough for concluding that our definition of domains of magnitudes is, in turn, independent of natural numbers and logical in some more significant sense than the simple and quite weak one of being formulated within a logical system. There are two concerns, here.

The first is that, even in a logical system, it seems possible to define items whose logical nature is suspect. Panza (2018) and Panza (FC2) already raised the question in relation both to natural numbers and magnitudes, as originally defined by Frege as appropriate extensions. Surely, according to our reformulation of Frege’s definition, magnitudes are no more extensions, but rather binary first-level relations. Still, apart from the identity relation, the relations forming such domains are not identified as particular relations somehow precisely defined; they are rather characterized as possible places in whatever system exemplifying a certain structure. This makes this definition define domains of magnitudes, but not magnitudes as such, which is perfectly in line with Frege’s remark quoted in § 2. Hence, all that the definition fixes is the structure of a domain of magnitudes, not its content, which is to be given independently of it.

As such, this might even be taken as an argument for its logicity, if, *contra* Frege, it is admitted that logic has no content. But it makes the second concern crucial. As already claimed, whereas an existence proof of nonempty domains of magnitudes cannot be provided within $L_2PCA_{\Delta_0}^2$ (as well as within FMR and, *a fortiori*, FMR’), it is indispensable for making our definition of real numbers sensible. So, proving, necessarily outside these systems, the existence of a nonempty domain of magnitudes is an essential part of this very definition (even if this is not required to formulate the definition of domains of magnitudes themselves), not only of a model-theoretical enquiry on it. This makes both the logicity and the arithmeticity of the

⁴⁹Simons (1987, § 7) has stressed the crucial differences between Frege’s logicism for natural numbers and his views on real ones. Without undermining his arguments—which take however for granted the usual reading of Frege’s logicism for natural numbers, which we rather take as questionable under many respects: See Panza (2018) and Panza (FC2)—we follow another strategy here: we frontally attack the idea that Frege’s envisaged definition of real numbers might be taken as logical in any substantial sense.

⁵⁰But see footnote (5), on this matter.

definition crucially and questionably depend on the means, external to FMR and FMR', needed for conducting such an existence proof. Two major issues arise.

The first is that it seems plausible to require that a genuinely logical definition not be in need of an external existence proof of the items it defines—or of other associated ones. Since such a definition should purportedly ensure the existence of these items by merely showing that, if there were none, some logical, or innocent enough, truths would not be true after all. This is indeed allegedly the case of the neologicist existence proof of natural numbers⁵¹—which we mimicked in the existence proof of the $E_{\mathcal{O}}$ -abstracta, in § 6.2 above. One might argue that this is too demanding. However, a distinction should be drawn between definitions that are logical in this demanding sense, and others that are not or cannot be so. This would be enough for concluding that neither our definition of real numbers in FMR or FMR', nor that of domains of magnitudes in $L_2PCA_{\Delta_0}^2$, can pretend to be logical in the same sense in which neologicists claim their definition of natural numbers is⁵².

This leads to the second issue: once admitted that the neologicist's proof-pattern does not apply, and that this prevents our Fregean definitions of domains of magnitudes and real numbers to be logical in the above demanding sense, the question arises whether these definitions might nevertheless be deemed logical in some less demanding sense⁵³. The question seems to have different answers according to whether it concerns the former definition or the latter, and whether domains of magnitudes are regarded as such or as tools for defining real numbers. If we look at the definition of domains of magnitudes as such, and admit that $L_2PCA_{\Delta_0}^2$ be a genuine logical system, it is hard to find any other reason than that raised above to deny its logicity. But if we look at domains of magnitudes as tools for defining real numbers, the situation changes. Insofar as proving the existence of nonempty such domains is essential for enabling them to play this role, the question becomes whether the proof can be so shaped as to make it logical, and, then, part of a logical definition of these numbers. This is, then, the question we have to tackle, now.

Above we explored two different strategies for conducting this proof: an inflationary and a non-inflationary one. In what follows, we will expand on them by considering how they score with respect to the issue of logicity. Insofar as the question of the logicity of our definition of real numbers has multiple interconnections with that of its arithmeticity, we will also consider in the

⁵¹To see it, consider the argument proving the existence of 0 that we have detailed at the beginning of § 6 above. The concept $[x : x \neq x]$ exists not only by predicative, but also by logical comprehension, as well as logic is enough to get that $[x : x \neq x] \approx [x : x \neq x]$, and HP is so for getting that $0 = 0$, which could not be true if 0 did not exist. The same pattern allows to prove the existence of each natural number, provided that comprehension be extended to formulas involving the operator '#'. So, 1 is proved to exist, for example, since, if it did not, it would be false that $1 = 1$, which follows, in agreement with HP, from $[x : x = 0] \approx [x : x = 0]$, which follows, in turn, by logic, from the existence of the concept $[x : x = 0]$, ensured by comprehension applied to the formula $x = \#[x : x \neq x]$. On the other side, the existence of the totality of numeral numbers is proved by proving, by HP and impredictive comprehension, the successor axioms, which would be false, if these numbers did not exist.

⁵²In fact, neologicists usually take their definition of natural numbers to be analytic, though not logical. Still, we made clear from the very beginning why we do not endorse this distinction here—see § 1 above. Let us notice, however, that the argument just advanced is emblematic of the reason we advanced to justify our attitude. Since, if this distinction were admitted, this argument should allow to conclude also that our definition of real numbers is no more analytic in the same sense in which neologicists take their definition of natural numbers to be so.

⁵³We leave here apart the question of whether the neologicist definition of natural numbers or our definition of the $E_{\mathcal{O}}$ -abstracta are actually logical or analytical. In Panza (2016), pp. 420-423, the point is made that the former definition might be deemed so in a quite peculiar sense, quite different than those that are current in the discussion on logicism and neologism, and because of a completely different argument than the neologicist's. There is no need to come back on this point, here. It is only important to observe that it does in no way apply to our definition of domains of magnitudes and real numbers.

meantime whether these strategies can make this definition non-arithmetical.

7.1.1 About the Inflationary Strategy

A proof following the inflationary strategy may be deemed non-logical just because of its inflationary nature. The reason is obvious: insofar as no proof by countable induction is possible here, such a proof cannot but grant that the abstraction principle introducing the continuously many objects it concerns (FP, in our case) *eo ipso* entails the existence of these objects, which appears to be incompatible with its being logical, and, then, part of a logical proof.

One could object that the argument merely points out that the proof is not logical, because it requires means other than countable induction to prove the existence of continuously many objects, which is unfair, at best. After all, real numbers must be continuously many, so that accepting this argument would amount to principledly excluding the possibility of a logical definition of real numbers ensuring their existence. The objection is not convincing. It is entirely possible that no such definition be logical. Still, if a definition of these numbers is offered with the aim of being so, it should, at least, avoid requiring an existence proof for continuously many objects other than the reals. This would leave room for arguing that proving the existence of continuously many objects is not part of its job, but should be left for further meta-theoretical considerations. The point is, then, that the inflationary strategy is not suitable for entering a logical definition of real numbers because it requires an existence proof of continuously many objects other than the reals: a proof which cannot but appeal to independent resources from those involved in the definition itself.

Another essential feature of the inflationary strategy is also relevant for the present discussion: its delivering an arithmetical copy of the additive group of the reals as a condition for making their definition appropriate. This makes clear both its arithmetical nature, and its essential mathematical circularity. Let us consider the two allegations in turn.

To reply to the arithmeticity allegation, it is not enough to argue that proving the existence of a nonempty domain of magnitudes arithmetically does not make a definition of real numbers as ratios of magnitudes arithmetical. The fact that an existence proof of such a domain is an indispensable part of the definition immediately entails, indeed, that this definition can be deemed non-arithmetical only in the presence of an existence proof of a non-arithmetical nonempty domain of magnitudes. Since, if one could only prove the existence of arithmetical such domains, defining real numbers as ratios on them would make them arithmetical items, after all⁵⁴.

⁵⁴The point might be softened by observing that our Fregean definition of domains of magnitudes differs from other possibly arithmetical ones for not appealing to any specific property of the objects on which the relevant binary relations are defined. To better see this, we can compare this definition to one mimicking Dedekind's definition by cuts in terms of binary relations (we thank Andrew Moshier for his suggestion). Let $(\mathbf{O}, <_{\mathbf{O}})$ be a totally ordered set without endpoints, whose elements count as objects. By adopting a third-order logical system with third-order predicative comprehension, the following explicit definition can be provided, where the index ' \mathbf{O} ' restricts the quantifiers to binary \mathbf{O} -relations (i.e. binary relations among the elements of \mathbf{O}) and to these very elements, respectively:

$$\forall_{\mathbf{O}} R \left\{ \mathcal{C}_{\mathbf{O}} R \Leftrightarrow \left[\begin{array}{l} \forall_{\mathbf{O}} x, y [xRy \Rightarrow x <_{\mathbf{O}} y] \wedge \\ \forall_{\mathbf{O}} x, y, z, w [(xRy \wedge z <_{\mathbf{O}} x \wedge y <_{\mathbf{O}} w) \Rightarrow zRw] \wedge \\ \forall_{\mathbf{O}} x, y [xRy \Rightarrow \exists_{\mathbf{O}} z, w [zRw \wedge x <_{\mathbf{O}} z \wedge w <_{\mathbf{O}} y]] \wedge \\ \forall_{\mathbf{O}} x, y [x <_{\mathbf{O}} y \Rightarrow (\exists_{\mathbf{O}} z [xRz] \vee \exists_{\mathbf{O}} w [wRy])] \wedge \\ \forall_{\mathbf{O}} x, y, z, w [(xRy \wedge zRw) \Rightarrow xRw] \end{array} \right] \right\}.$$

To reply to the circularity allegation, one should argue that the copy of the additive group of the reals is just a copy, since, though structurally coincident with real numbers, its elements intrinsically differ from them. We can imagine Frege advancing this argument. But we can hardly follow him in this without making any working mathematician sarcastically smile.

Let us recap. The inflationary strategy suffers from two problems: in absence of a further existence proof of non-arithmetical domains of magnitudes, it makes real numbers themselves arithmetical objects, after all; it requires a preliminary structural definition of the real numbers, in order to make the planned definition of these same numbers suitable⁵⁵.

7.1.2 About the Non-Inflationary Strategy

At least four reasons might be advanced to argue that, in the light of the existence proof in § 6.2 above, our Fregean definition of real numbers is hardly both logical and non-arithmetical: *i*) that proof is based on the symmetric group $\Sigma_{\mathbb{N}}$ on the natural numbers; *ii*) it allows to conclude that a nonempty domain of magnitudes exists only if it is admitted that the symmetric group on an infinite countable set exists if this set exists; *iii*) it essentially appeals to the additive group $(\mathbb{R}, +)$ of the real numbers themselves; *iv*) it appeals to Zorn's lemma, and is, then, not constructive.

The first reason cannot be dismissed by merely observing that, in our reconstruction of the proof, $\Sigma_{\mathbb{N}}$ has been replaced by the symmetric group on the set of E_{\emptyset} -abstracta. Since, once $\Sigma_{\mathbb{N}}$ is

The third conjunct of the right-hand side entails that no binary \mathbf{O} -relation has the property $\mathcal{C}_{\mathbf{O}}$ if $(\mathbf{O}, <_{\mathbf{O}})$ is not dense. Let us suppose that it be so. Call a relation ' $\mathcal{C}_{\mathbf{O}}$ -relation' if it is a binary relation having $\mathcal{C}_{\mathbf{O}}$. We can say that any $\mathcal{C}_{\mathbf{O}}$ -relation defines a cut on $(\mathbf{O}, <_{\mathbf{O}})$. The collection of the $\mathcal{C}_{\mathbf{O}}$ -relations does not form a domain of magnitudes, in the sense established above, since the $\mathcal{C}_{\mathbf{O}}$ -relations are not permutations. Still, we might weaken Frege's requirement on domains of magnitudes, and take such domains as constituted by totally ordered, dense and Dedekind-complete groups of first-level binary relations, independently of their being permutations. The $\mathcal{C}_{\mathbf{O}}$ -relations might, then, form a domain of magnitudes if a commutative (and associative) addition admitting a neutral element be defined on them. To this purpose, let us suppose that an addition $+_{\mathbf{O}}$ be defined on $(\mathbf{O}, <_{\mathbf{O}})$, so as to make $(\mathbf{O}, <_{\mathbf{O}}, +_{\mathbf{O}})$ a totally ordered, Abelian and dense additive group. We can easily define an addition $+_{\mathcal{C}_{\mathbf{O}}}$ on the $\mathcal{C}_{\mathbf{O}}$ -relations, by stating that

$$\forall_{\mathcal{C}_{\mathbf{O}}} R, S \forall_{\mathbf{O}} x, y [x (R +_{\mathcal{C}_{\mathbf{O}}} S) y \Leftrightarrow \forall_{\mathbf{O}} z, w, v, u [(zRv \wedge wSu) \Rightarrow (x <_{\mathbf{O}} z + w \wedge v + u <_{\mathbf{O}} y)]],$$

where the index ' $\mathcal{C}_{\mathbf{O}}$ ' to the first universal quantifier restricts it to these relations. The $\mathcal{C}_{\mathbf{O}}$ -relation $\mathbf{Z}_{\mathbf{O}}$ defined by

$$\forall_{\mathbf{O}} x, y [x \mathbf{Z}_{\mathbf{O}} y \Leftrightarrow x <_{\mathbf{O}} 0_{\mathbf{O}} <_{\mathbf{O}} y]$$

is the neutral element of $+_{\mathcal{C}_{\mathbf{O}}}$, and another $\mathcal{C}_{\mathbf{O}}$ -relation \mathbf{R} , is deemed positive if and only if

$$\exists_{\mathbf{O}} x, y [x \mathbf{R} y \wedge 0_{\mathbf{O}} <_{\mathbf{O}} x <_{\mathbf{O}} y],$$

and negative otherwise. One could, then, define an order relation $\sqsubset_{\mathcal{C}_{\mathbf{O}}}$ on the $\mathcal{C}_{\mathbf{O}}$ -relations by stating that

$$\forall_{\mathcal{C}_{\mathbf{O}}} R, S \left[R \sqsubset_{\mathcal{C}_{\mathbf{O}}} S \Leftrightarrow \exists_{\mathcal{C}_{\mathbf{O}}^+} T [R +_{\mathcal{C}_{\mathbf{O}}} T = S] \right],$$

where the index ' $\mathcal{C}_{\mathbf{O}}^+$ ' restricts the existential quantifier to positive $\mathcal{C}_{\mathbf{O}}$ -relations. These would form a totally ordered, dense and Dedekind-complete group under $+_{\mathcal{C}_{\mathbf{O}}}$ and $\sqsubset_{\mathcal{C}_{\mathbf{O}}}$, and, then, a domain of magnitudes, in the previous weakened sense. One might, then, define real numbers as ratios on such a group. Still, so defined, real numbers would be, structurally speaking, nothing more than ratios on cuts-relations on the additive group of the rational numbers, and this would make them intrinsically arithmetical items, in a much stronger sense than the real numbers defined in FMR or FMR', under the condition that the only nonempty domains of magnitudes whose existence can be proved were arithmetical.

⁵⁵Both problems also arise if the inflationary strategy is implemented by E_{\emptyset} -abstracta. As for the first, this is obvious. As for the second, notice that these *abstracta* could enter the existence proof only because of their features that make them structurally coincide with the natural numbers.

replaced in Karras and Solitar’s proof by any other symmetric group over any infinite countable set, the problem becomes that of justifying the existence of this set, and even its cardinality, by no appeal to \mathbb{N} itself. That $\Sigma_{\mathbb{N}}$ is isomorphic with the symmetric group over any infinite countable set is, indeed, simply because any such set can be put into a bijection with \mathbb{N} , so that its elements can be taken either to count as natural numbers or, at least, to be encoded by them.

The second reason is similar to one discussed above as for the inflationary strategy: on what logical ground can we argue that the existence of a countable infinite set entails the existence of the symmetric group over this set—or, more in general, of an uncountable set somehow generated by it by considering at once some totality of properties, relations or functions defined on the elements of this set? The fact that Frege himself suggests making a similar admission, in order to prove the existence of a nonempty domain of magnitudes, in no way makes it logically licensed. Rather, it seems to show that the very proof Frege suggested would have actually been not logical.

The third and fourth reasons are by far more delicate, and somehow interconnected. Since, if a constructive proof of Karras and Solitar’s theorem were available, one could hope to rely on it in order to constructively define a totally ordered, dense and Dedekind-complete subgroup of $\Sigma_{\mathbb{N}}$, without recurring to $(\mathbb{R}, +)$.

To dismiss the third reason, and the circularity allegation that goes with it, one might replace Karras and Solitar’s theorem with a more general result not involving $(\mathbb{R}, +)$. A natural candidate is a result by de Bruijn (1964, p. 594), according to which, for any infinite cardinal κ , every Abelian group of order 2^κ can be “embedded into” the symmetric group of a set of cardinality κ ⁵⁶. Still, the basic idea of de Bruijn’s proof is not so different from Karras and Solitar’s and makes this proof also depend on the axiom of Choice, though avoiding appealing to vector spaces. The fact that the theorem does not specifically involve $(\mathbb{R}, +)$ is, moreover, far from being an advantage in our perspective. Since it makes this theorem unable to provide a ground for the required existence proof. For the purpose of this latter proof is establishing that $\Sigma_{\mathbb{N}}$ (or, more generally, the group of permutations on a countable set) actually includes a subgroup complying with the relevant structure, while this theorem merely ensures that, if there is such a group, then it can be embedded into $\Sigma_{\mathbb{N}}$, and can be regarded as a subgroup of it. This makes, of course, de Bruijn’s theorem immediately entail that $(\mathbb{R}, +)$ can be embedded into $\Sigma_{\mathbb{N}}$. This cannot but make the circularity even more evident, since it is only the existence of \mathbb{R} that can ensure that a subgroup of $\Sigma_{\mathbb{N}}$ complying with the relevant structure exists. In order to solve the issue, one should prove that $\Sigma_{\mathbb{N}}$ includes a totally ordered, dense and Dedekind-complete subgroup, without assuming the existence of this group. To the best of our knowledge, this has not yet been done.

This does not mean, of course, that this result, or any other entailing it, has not actually been proved or, even less so, that this cannot be done. The fourth reason suggests, however, that the relevant question is not whether this has been or might be done, but, rather, whether this can be done constructively, i.e. without appealing to a form of the Axiom of Choice, which might hardly be taken as a logical principle. When put in a clear mathematical form, the question is whether it is provable in ZF alone (or in some other appropriate setting that neither presupposes nor entails the Axiom of Choice) that $\Sigma_{\mathbb{N}}$ contains a totally ordered, dense and Dedekind-complete subgroup, and whether, moreover, this can be done without assuming the existence of $(\mathbb{R}, +)$. To the best of our knowledge, this question also has not been answered yet.

⁵⁶This theorem was firstly published one year later than Karras and Solitar’s (de Bruijn 1957, pp. 560-61 and 566), but it was then erroneously proved. The error lied with a lemma proved by erroneously supposing that a certain arbitrary group could be non-Abelian. The proof was later corrected and made independent of this lemma—and in fact simplified.

To begin enquiring about it, one might wonder whether Karrass and Solitar’s proof can be freed from Zorn’s lemma or any equivalent assumption. Such an assumption enters the proof to ensure that any vector space has a basis—i.e. that such a basis exists though it cannot be constructively displayed. This makes it relevant to observe that Blass (1984) proves that the assumption that any vector space has a basis is (ZF-)equivalent to the Axiom of Choice. This is still not enough to ensure that the appeal to a form of the Axiom of Choice cannot be avoided in Karrass and Solitar’s proof, and that this proof is, then, both non-constructive and intrinsically dependent on such an axiom. Since what is required for this proof to work is not, properly, that any vector space has a basis, but rather that this is so for the two relevant such spaces, i.e. $\prod_{i \in \mathbf{I}} (\mathbb{Q}, +)$ and $(\mathbb{R}, +)$. The issue becomes, then, whether one can prove that these very vector spaces have a basis without appealing to a form of the Axiom of Choice or to any other non-constructive means. To the best of our knowledge, anew, this is still unknown.

Still, even if this could not be done, it would not follow that Karrass and Solitar’s theorem cannot be proved without appealing to a form of this axiom. The only occurrence of Zorn’s lemma in the previous proof is, indeed, in its very last step, which is the only one involving vector spaces. It is, then, natural to wonder whether the theorem could be proved by avoiding this step (and, then, presumably any reference to vector spaces), by replacing it with another step not depending on the Axiom of Choice or some other non-constructive assumptions.

We could imagine two scenarios. In the first, the question is whether ZF alone is capable of proving Karrass and Solitar’s theorem: to this extent, either it is, or the theorem is undecidable there. In the second scenario, the question is whether ZF augmented with some axioms incompatible with the Axiom of Choice, such as the Axiom of Determinateness,⁵⁷ is capable of proving this theorem. To the best of our knowledge, these issues have also not been settled yet⁵⁸.

The conclusion to be drawn from all these remarks cannot be but prudent. Still, it can certainly no more suggest that Karrass and Solitar’s theorem provide a basis to argue that our Fregean definition of real numbers is logical. Since, circularity issues aside, arguing that this theorem can enter a non-inflationary existence proof for a nonempty domain of magnitudes, suitable for making our definition logical, would be quite premature, at best. And it would be even more so to argue that a more general theorem, asserting that $\Sigma_{\mathbb{N}}$ includes an appropriate group identified without appealing to \mathbb{R} , can enter such a non-inflationary existence proof.

7.2 Might the Existence Proof be Avoided?

The previous considerations suggest that there is no way to prove the existence of nonempty domains of magnitudes without wiping out both the logicality and non-arithmeticity of our Fregean definition of real numbers. In light of this conclusion, one might suggest changing the rules of the game. Even if there is no way to prove, by (higher-order) logic, suitable existentially innocent abstraction principles and appropriate algebraic and/or set-theoretical constructive arguments, that nonempty domains of magnitudes exist, still we know they do. For we can show or prove it,

⁵⁷First introduced by Mycielski & Steinhaus (1962), this axiom asserts that “certain infinite, deterministic 2-person games with complete information [...] are determinate, i.e., that one of the players has a winning strategy”. See also Herrlich (2006, p. 151), which also provides a proof of the incompatibility between the Axiom of Choice and the Axiom of Determinateness.

⁵⁸There still might be clues on the second issue: while the Axiom of Choice entails that \mathbb{R} , as a vector space over $(\mathbb{Q}, +, \cdot)$, has bases, the Axiom of Determinateness entails that it has not (Herrlich 2006, Theorem 4.44 and Corollary 7.20)—and one might even guess that it be the same for an infinite product of copies of $(\mathbb{Q}, +)$.

for example, by empirically-tied geometric or mechanical considerations; or, as above, by trusting non-constructive set-theoretical principles; or by assuming that natural numbers exist and granting the previous abstraction principles an existential import. Hence, one could argue, defining the real numbers *à la* Frege within FMR or FMR', even with no existential proof, allows one to show that ratios on any externally given domain of magnitudes, whether intrinsically arithmetical or not, are real numbers. Since, as we have already seen, taking a real number to be a ratio on distinct domains of magnitudes is nothing but describing the same object in different ways—or giving different names to it.

The problem with this move is that applying our definition to whatever externally given domain of magnitudes would certainly warrant that the ratios on it are real numbers, but not that real numbers are intrinsically such ratios, let alone that they are non-arithmetical items. If we reasoned this way, we would do nothing essentially different from appealing to a representation theorem to draw the conclusion that real numbers measure the magnitudes in the relevant domains, in the spirit of the measurement theory⁵⁹. In both cases, all we do is recognize that some externally given systems (arithmetical or not) comply with some fixed structural conditions. The fact that these structural conditions are fixed by our definition in FMR or FMR', or by recurring to algebraic axioms as those of a totally-ordered complete Abelian field (as usually supposed in measurement theory), or, again, by alternative definitions (as Cantor's and Dedekind's, by Cauchy's sequence and cuts on rationals, respectively, or even as the one grounded on FP) makes no essential difference on this matter.

In the eyes of a Frege partisan, there would be a crucial difference only if the existence proof were deemed an essential, though supplementary, part of the definition itself, as we did above. Since this would make the numbers so defined intrinsically ratios on domains of magnitudes, and their application to measurement “built into” their nature and/or their very definition, as required by the application constraint (Wright, 2000, p. 325). In this respect, the previous remarks on the arithmeticity and logicity of our definition in FMR or FMR' should be intended to suggest that compliance with this constraint is incompatible not only with offering a logical definition of real numbers, as already argued in Panza & Sereni (2019), but also with defining them non-arithmetically, despite Frege's adherence to the same constraint as the main source of his quest for a non-arithmetical definition of these numbers.

7.3 About Euclid's Principle

Up to now, we have only considered the existence proof of nonempty domains of magnitudes. Still, the indispensability and the nature of this proof are not the only reasons suggesting that our Fregean definition of real numbers is neither logical nor non-arithmetical. Since, once domains of magnitudes have been defined and somehow proved to exist, the question remains open of defining real numbers as ratios on them. In our setting, this is done by means of EP*⁶⁰. The questions

⁵⁹About the tension between considering applications of real numbers in agreement with the measurement theory and taking them to be ratios on domains of magnitudes, see the Hale-Batitsky discussion in Hale (2000, 2002) and Batitsky (2002). On this matter see also Panza & Sereni (2019, pp. 122-123 and 126-130).

⁶⁰Possibly with the help of $(CA_{\Sigma_1}^2)$, if an explicit definition like (5.2) is required. We do not want to enter here a discussion about the logicity of (the different sorts of) comprehension. We merely remark that the high impredicativity of $(CA_{\Sigma_1}^2)$ might make many doubt not only its logicity, but also its licitness. Who doubts both has no other choice but rejecting definition (5.2) and rest content with (5.2'). Who doubts only the former can either admit definition (5.2), but take it as non-logical, or rejecting, again, this definition in favor of (5.2').

are, then, obvious: is this principle logical? Is it actually independent of natural numbers?

Let us start from the latter. Undoubtedly, EP depends on natural numbers, as it explicitly quantifies over them. Still, this principle also depends on the use of other linguistic means foreign to $L_2PCA_{\Delta_0}^2$, by involving a piece of informal language allowing for predicate variables like ‘ xR ’. The two features are connected, since the quantification over natural numbers just operates on the individual variables occurring in these predicate variables. Replacing EP with EP* allows avoiding both the quantification over natural numbers and the appeal to informal language at once. Surely, EP* involves no second-order predicate constant supposedly designating the property of being a natural number. Still, this does not ensure, yet, its independence from natural numbers, since it is far from clear that the trick used to avoid the quantification on these numbers is actually independent of them. What might make one suspect it is not that the right-hand side of (5.1) is just an appropriate third-order rephrasing of the right-hand side of HP. Hence, if it were admitted that, when applied to finite concepts, HP is intrinsically inherent to natural numbers—not only because the objects not complying with it are not natural numbers, but also because its assumption *ipso facto* brings these latter about—, one should infer that, in spite of appearance, also EP* depends on these numbers. As a matter of fact, this is a strong assumption, but one that can be made in a Fregean vein, and which might bring, then, *ipso facto*—that is, independently of any consideration on the existence proof of nonempty domain of magnitudes—, to the conclusion that our definition of real numbers, whether in FMR or FMR’, is essentially arithmetical.

Someone admitting this assumption might still argue against this conclusion by observing that (5.1) essentially differs from HP for being a (metalinguistic) abbreviation stipulation, rather than an axiom providing an implicit definition of a functional constant. This is enough, one might continue, to make EP* appeal to no variable ranging on objects that might count as the natural numbers. This is unquestionably so. However, any instance of ‘ $_{(x, x')} \mathcal{E}(R, T, R', T')$ ’ asserts that a certain first-level binary relation is the same multiple of another such relation over a certain positive class as a third such relation of a fourth one, over the same or another positive class. This is, in turn, the same as asserting that the iterations of the composition operation on such a relation within the former class are into a bijection with the iterations of composition on such a relation within the latter class. If this is not the same as making natural numbers enter into play, it is, at least, the same as making the equinumerosity relation so. Hence, if EP* is not dependent on natural numbers, it seems to be, at least, dependent on counting. There is no easy way to settle whether this is enough to make EP* an arithmetical principle. Here, we just observe that this makes our Fregean definition of real numbers, whether in FMR or FMR’, dependent on an essential ingredient of any Fregean definition of natural numbers. Even if this, as such, does not make our definition arithmetical, it is plausibly enough for making it much more related to natural numbers than Frege might have desired his definition be.

Let us come, now, to the first question: can EP* be deemed logical? A simple way to tackle the question might be that of choosing between two quite natural options: either any abstraction principle is logical if it is stated through a logical language, or it is so only because of the peculiar nature of its right-hand side. In the former case, EP* is logical if $L_2PCA_{\Delta_0}^2$ is so. In the latter case, EP* cannot be logical on the same grounds on which HP or a consistent version of BLV might be so. If this simple alternative is rejected, if only for argument’s sake, what criterion might be provided to distinguish logical abstraction principles stated in a logical language from non-logical ones? Consistency is surely not enough. But, then, what? We cannot dwell on this issue here. We simply contend that the burden of the proof seems to be on anyone arguing that

EP* is logical, despite its being essentially akin to Euclid’s definition of proportionality, which has been considered for centuries as the cornerstone of the most fundamental mathematical theory on which classical geometry was crucially grounded.

8 Concluding Remarks

Though some of them are certainly far from knock down ones, we think we advanced enough arguments in favor of the claim that our rendering of Frege’s envisaged definition of real numbers is neither logical nor non-arithmetical. As our rendering is arguably the closest possible to it, this conclusion questions the possibility that Frege’s own definition could be achieved logically and non-arithmetically. It remains to establish whether this was actually Frege’s intent.

That Frege was aiming at a logical definition of real numbers as his main goal for the foundation of real analysis might be questioned for several reasons⁶¹. One of them might be the following.

From our reconstruction, it seems to emerge that arguing for the logicity of a definition of real numbers following Frege’s indications requires arguing that FP, or any akin principle, is both logical and capable of delivering continuously many objects without the assistance of any independent existence proof. But if so, then FP would also be enough for delivering real numbers, if not as logical objects, at least as objects defined in a logical setting. But, then, why did Frege venture himself in a so entangled definition whose logical nature is as suspect as that of real numbers as ratios of magnitudes?

Possibly, far from considering logicity as his ultimate aim, he overall wanted to link real analysis to a general theory of magnitudes. This has been argued for in Panza & Sereni (2019). Or, possibly, he merely wanted to distinguish real from natural numbers, making the former essentially independent of the latter, for their being objects of an essentially different kind. Though the two possibilities are not incompatible with each other, our conclusion might be taken as a piece of evidence that he could not have reached this second aim by following the route envisaged in the *Grundgesetze*. The first aim remains, which is certainly paramount from a purely mathematical perspective. If we admit that this was, after all, his prominent goal, then our rendering of his definition might be taken as an indication of a simple way to accomplish it.

Acknowledgments: We would like to thank Mirna Džamonja, Alain Genestier, Paolo Mancosu, Andrew Moshier, Jamie Tappenden, an anonymous reviewer for *The Review of Symbolic Logic*, and the audiences of the conferences previous drafts of this paper were presented at, for useful comments and criticisms. We are also indebted to Mya MacRae for the linguistic revision of the paper. The work of Marco Panza has been supported by the ANR-DFG project FFIUM.

References

- Adeleke S.A., Dummett, M.A.E, Neumann, P.M. (1987) On a Question of Frege’s about Right-Ordered Groups, *Bull. Londin Maths. Soc.*, 19: 513–521.
- Antonelli, A. & May, R. (2005) Frege’s Other Program, *Notre Dame Journal of Formal Logic*, 46(1): 1–17.

⁶¹Some are advanced in Panza (FC2).

- Batitsky, V. (2002) Some Measurement-Theoretic Concerns about Hale’s ‘Reals by Abstraction’, *Philosophia Mathematica*, 10(3): 286–303.
- Blass, A. (1984) Existence of bases implies the axiom of choice, in J. E. Baumgartner, D. A. Martin and S. Shelah (eds.), *Axiomatic Set Theory*, vol. 31 of *Contemporary Mathematics*, AMS: 31–33.
- Błaszczczyk, P. (2013) Nota o rozprawie Otto Höldera “Die Axiome der Quantität und die Lehre vom Mass”, *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia*, 5: 129–144.
- Boolos, G. (1998a) Iteration Again, in Boolos (1998c): 88–104.
- Boolos, G. (1998b) Saving Frege from Contradiction, in Boolos (1998c): 171–182.
- Boolos, G. (1998c) *Logic, Logic, and Logic*, J. Burgess and R. Jeffrey (eds.), Cambridge, MA: Harvard University Press.
- Bourbaki, N. (Algebra I) *Elements of Mathematics, Algebra I*, Springer, Berlin, Heidelberg, etc., without date.
- de Bruijn, N.G. (1957) Embedding Theorems for Infinite Groups, *Nederl. Akad. Wetensch. Proceedings*, Ser. A, 60 (= *Indagationes Math.*, 19(5)): 560–570.
- de Bruijn, N.G. (1964) Addendum to ‘Embedding Theorems for Infinite Groups’, *Nederl. Akad. Wetensch. Proceedings*, Ser. A, 67 (= *Indagationes Math.*, 26(5)): 594–595.
- Conti, L. (2019) *Paradossi di Russell e programmi astrazionisti. Spiegazioni e soluzioni a confronto*, PhD thesis.
- Dummett, M. (1991) *Frege: Philosophy of Mathematics*, Duckworth, London.
- Euclid (EH) *The Thirteen books of Euclid’s Elements*, translated with Introduction and Commentary by T. L. Heath, Cambridge University Press, Cambridge 1908 (3 vols.), Duckworth, London.
- Frege, G. (1884) *Grundlagen der Arithmetik*, W. Kōbner, Breslau.
- Frege, G. (1893-1903) *Die Grundgesetze der Arithmetik*, Verlag von H. Pohle, Jena, 1893-1903 (2 vols).
- Frege, G. (1953) *The Foundations of Arithmetic* (translated by J. L. Austin), Blackwell, Oxford, 1953.
- Frege, G. (2013) *Basic Laws of Arithmetic*, translated by P. A. Ebert and M. Rossberg, Oxford University Press, Oxford, 2013.
- Gauss, C. F. (1831) *Theoria residuorum biquadraticorum. Commentatio secunda*, *Göttingische gelehrte Anzeigen*, 1831, vol. I, n. 64, April, 23th, 1831, pp. 624-638.
- Gauss, C. F. (Werke) *Werke*, Herausgeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1863-1933, 12 vols.

- Hale, B. (2000) Reals by Abstraction, *Philosophia Mathematica*, 8: 100–123.
- Hale, B. (2002) Real Numbers, Quantities, and Measurement, *Philosophia Mathematica*, 10(3): 304–323.
- Herrlich, H. (2006) *Axiom of Choice*, Springer, Berlin-Heidelberg.
- Hölder, O. (1901) Die Axiome der Quantität und die Lehre vom Mass. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physischen Classe*, 53: 1–64.
- Karrass, A. & Solitar, D. (1956) Some remarks on the infinite symmetric groups, *Mathematische Zeitschrift*, 66: 64–69.
- Mycielski, J. & Steinhaus, H. (1962) A mathematical axiom contradicting the axiom of choice, *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques*, 10: 1–3.
- Panza, M. (2015) From Lagrange to Frege: Functions and Expressions, in H. Benis-Sinaceur, M. Panza & G. Sandu, *Functions and Generality of Logic. Reflections on Frege's and Dedekind's Logicisms*, Springer, Cham, Heidelberg, New York, Dordrecht, London: 59–95.
- Panza, M. (2016) Abstraction and Epistemic Economy, in S. Costreie (Ed.), *Early Analytic Philosophy. New Perspectives on the Tradition*, Springer: 387–428.
- Panza, M. (2018) Was Frege a Logician for Arithmetic?, in A. Coliva, P. Leonardi, S. Moruzzi (eds), *Eva Picardi on Language, Analysis and History*, Palgrave MacMillan, Springer Nature, Cham, Switzerland: 87–112.
- Panza, M. (FC1) *Anzahlen et Werthverläufe* : quelques remarques sur les §§I.34-40 des *Grundgesetze* de Frege, forthcoming in *L'épistémologie du dedans. Mélanges en l'honneur d'Hourya Benis-Sinaceur*, Garnier, Paris.
- Panza, M. (FC2) Was Frege a Logician?, forthcoming (This is a much extended version of Panza (2018), still unpublished).
- Panza, M. (FC3) *Reals by Abstraction. An enquiry about Epistemic Economy in Mathematics*, forthcoming monograph.
- Panza, M. & Sereni, A. (2019) Frege's Constraint and the Nature of Frege's Foundational Program, *The Review of Symbolic Logic*, 12(1): 97–143.
- Payne, J. (2013) Abstraction Relations Need not be Reflexive, *Thought: A Journal of Philosophy*, 2(2): 137–147.
- Schirn, M. (2013) Frege's approach to the foundations of analysis (1874-1903), *History and Philosophy of Logic*, 34(3): 266–292.
- Scott, D. (1958-59) *A General Theory of Magnitudes*, Unpublished typewritten document drawn from a series of lectures on geometry given at the University of Chicago in 1958-59.

- Sereni, A. (2019) On the Philosophical Significance of Frege's Constraint, *Philosophia Mathematica*, 27(2): 244–275.
- Shapiro, S. & Weir, A. (2000) 'Neo-Logicist' Logic is not Epistemically Innocent, *Philosophia Mathematica*, 8: 160–189.
- Simons, P. (1987) Frege's theory of real numbers, *History and Philosophy of Logic*, 8: 25–44.
- Shapiro, S. & Snyder E. (2020) Frege on the real numbers, In M. Rossberg and P. Ebert (eds), *Essays on Frege's Basic Laws of Arithmetic*, Oxford, Oxford University Press: 343–383.
- Wright, C. (2000) Neo-Fregean Foundations for Real Analysis: Some Reflections on Frege's Constraint, *Notre Dame Journal of Formal Logic*, 41(4): 317–334.