Escape and absorption probabilities for obliquely reflected Brownian motion in a quadrant

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ESCAPE AND ABSORPTION PROBABILITIES FOR OBLIQUELY REFLECTED BROWNIAN MOTION IN A QUADRANT

PHILIP A. ERNST, SANDRO FRANCESCHI, AND DONGZHOU HUANG

Abstract. We consider an obliquely reflected Brownian motion Z with positive drift in a quadrant stopped at time T, where $T := \inf\{t > 0 : Z(t) = (0,0)\}$ is the first hitting time of the origin. Such a process can be defined even in the non-standard case where the reflection matrix is not completely-$\mathcal{S}$. We show that in this case the process has two possible behaviors: either it tends to infinity or it hits the corner (origin) in a finite time. Given an arbitrary starting point (u,v) in the quadrant, we consider the escape (resp. absorption) probabilities $\mathbb{P}_{(u,v)}[T = \infty]$ (resp. $\mathbb{P}_{(u,v)}[T < \infty]$). We establish the partial differential equations and the oblique Neumann boundary conditions which characterize the escape probability and provide a functional equation satisfied by the Laplace transform of the escape probability. We then give asymptotics for the absorption probability in the simpler case where the starting point in the quadrant is $(u,0)$. We exhibit a remarkable geometric condition on the parameters which characterizes the case where the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the skew symmetry condition for the stationary distribution. We then obtain an explicit integral expression for the Laplace transform of the escape probability. We conclude by presenting exact asymptotics for the escape probability at the origin.

1. Introduction

1.1. **Model and goal.** Let $Z(t) = (Z_1(t), Z_2(t))$ be a reflected Brownian motion (RBM) in the quadrant, starting from the point (u,v), with positive drift $\mu = (\mu_1, \mu_2)$; that is, $\mu_1 > 0, \mu_2 > 0$. The covariance matrix is $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $|\rho| < 1$ and the reflection matrix is $\begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix}$. We further assume that

$$r_1 > 0,\ r_2 > 0 \text{ and } 1 \leqslant r_1 r_2. \tag{1}$$

See Figure 1 for a representation of the parameters. We define this reflected process up to the first hitting time T of the corner, defined as

$$T := \inf\{t > 0 : Z(t) = 0\}.$$

For $t \leqslant T$, this process may be written as

$$\begin{cases} Z_1(t) := u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t), \\ Z_2(t) := v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t), \end{cases} \tag{2}$$

where $l_1(t)$ (resp. $l_2(t)$) is a local time on the vertical (resp. horizontal) axis and is a continuous non-decreasing process starting from 0 which increases only when $Z_1(t) = 0$ (resp. $Z_2(t) = 0$). Under condition (1), when $t > T$, that is after that the process Z hits the corner, the process is no longer defined by (2) for reasons of convexity. In lieu, for $t > T$, we define $Z(t) = (0,0)$ and say that the process is absorbed when $T < \infty$. Further details on the existence and uniqueness of this process will be given in Section 1.2.

The objective of the present paper is to study the probability of escape to infinity for a process starting from (u,v). We denote this probability as

$$\mathbb{P}_{(u,v)}[T = \infty].$$

The corresponding absorption probability at the origin is $\mathbb{P}_{(u,v)}[T < \infty] = 1 - \mathbb{P}_{(u,v)}[T = \infty]$.

Key words and phrases. Escape probability; Absorption probability; Obliquely reflected Brownian motion in a quadrant; Functional equation; Carleman boundary value problem; Laplace transform; Neumann's condition; Asymptotics.

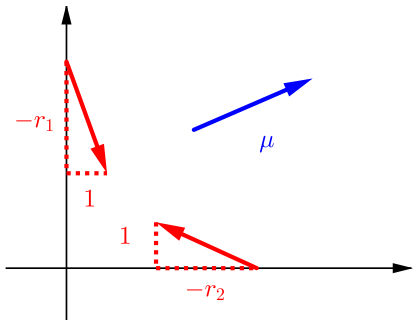


FIGURE 1. Reflection vectors and drift.

Since its introduction in the eighties by Harrison, Reiman, Varadhan and Williams [25, 26, 44, 42, 45], reflected Brownian motion in the quarter plane has received significant attention from probabilists. Recurrence and transience of obliquely reflected Brownian motion were examined in [29, 44], and the process has also been considered in planar domains [22, 27] as well as in general dimensions in orthants [26, 41, 46]. The stationary distribution of obliquely reflected Brownian motion has been studied in [9, 10, 21, 31, 12] and its Green's functions have been studied in [18]. The roughness of its paths has been studied in [32]. Obliquely reflected Brownian motion has played an important role in applications concerning heavy traffic approximations for open queueing networks ([23, 39]). It has also been utilized in queueing models as diffusion approximations for tandem queues ([33, 34, 37]).

There are several possible interpretations in insurance risk of models involving reflected Lévy processes in a quadrant ([1, 4, 30]). For example, suppose there are two funds, each of whose free surplus is modelled by a Cramér-Lundberg process, and which have the following agreement: a deficit in one fund is instantly covered by the other fund, with ruin occuring when neither company can cover the deficit of the other. In the case of our problem, the absorption probability could be interpreted to be the probability of ruin; the escape probability may be interpreted as the probability of survival and infinite capital expansion. The aforementioned process also arises in the study of queueing models as diffusion approximations for some Lévy tandem queues ([7, 17, 43]).

Previous works ([3, 13, 16, 21, 17]) have adapted an analytic method initially developed for random walks by Fayolle and Iasnogorodski [14] and Malyshev [36] for studying obliquely reflected Brownian motion. Above all we mention [17] which focus on a non-standard regime where the reflected process escapes to infinity along one of the axes. The techniques and the results employed to solve this problem are very similar to this article. This method is based on the boundary value problem theory documented by the books of Fayolle et al. [15] and Cohen and Boxma [8]. The present article is in part inspired by this analytic approach.

1.2. **Definition of the process given in (2).** Brownian motion in a quadrant with oblique reflection is usually defined as a process which behaves as a standard Brownian motion in the interior of the quadrant, reflects instantaneously on the edges with constant direction and the amount of time spent at the origin has Lebesgue measure zero (Varadhan and Williams [42]). Such a process is defined as a solution of a submartingale problem [42]. An interesting case arises when the process is a semimartingale reflecting Brownian motion (SRBM). Reiman and Williams [40] showed that a necessary condition for the process to be an SRBM is for the reflection matrix to be completly-$\mathcal{S}$[1]. Taylor and Williams [41] showed that this condition was also sufficient for the existence of an SRBM, which is unique in law.

Due to condition (1), the reflection matrix of the process in (2) is not completely-$\mathcal{S}$. The process indeed is not a standard SRBM as it may be trapped at the origin. Nonetheless, it is

[1] A square matrix R is said to be completly-$\mathcal{S}$ if for each principal sub-matrix $\widetilde{R}$ there exists $\widetilde{x} \geqslant 0$ such that $\widetilde{R}\widetilde{x} > 0$.

possible to define this absorbed process up to the stopping time T. The existence and uniqueness as a solution of a submartingale problem for the absorbed process is given in [42, §2.1, Thm 2.1]. Further, in Taylor and Williams [41, §4.2 and §4.3], the existence and uniqueness of an SRBM absorbed at the origin are proven without assuming that the reflection matrix is completely-$\mathcal{S}$.

1.3. **From the quadrant to the wedge.** Franceschi and Raschel [21, Appendix] recently showed that studying reflected Brownian motion in a quadrant is equivalent to studying reflected Brownian motion in a wedge with angle β, with identity covariance matrix, with two reflection angles δ and ϵ, and with drift angle θ (see Figure 2). The angles δ, ϵ, β and θ (when the drift is nonzero) are in $(0, \pi)$ and are defined by

$$\tan\delta = \frac{\sin\beta}{-r_2 + \cos\beta}, \quad \tan\epsilon = \frac{\sin\beta}{-r_1 + \cos\beta}, \quad \tan\theta = \frac{\sin\beta}{\mu_1/\mu_2 + \cos\beta}, \quad \cos\beta = -\rho. \tag{3}$$

The angles are equal to $\pi/2$ when the denominators of the tangents are equal to 0.

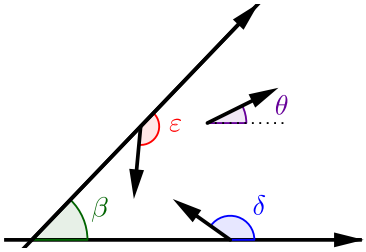


FIGURE 2. Reflected Brownian motion in a wedge of angle β, reflection angles δ and ϵ, and drift angle θ.

Finally, we denote α, now a standard quantity in the SRBM literature, to be

$$\alpha := \frac{\delta + \epsilon - \pi}{\beta}. \tag{4}$$

Condition (1) is equivalent to $\delta + \epsilon - \beta \geqslant \pi$ (or equivalently $\alpha \geqslant 1$) and $\delta > \beta$, $\epsilon > \beta$.

1.4. **The case of zero drift.** The case of zero drift $\mu = 0$ was treated by Varadhan and Williams [42]. In this case the absorption probability does not depend on the starting point. We have from [42, Thm 2.2]

$$\mathbb{P}[T < \infty] = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leqslant 0. \end{cases}$$

If $\alpha \leqslant 0$, the corner is not reached. If $0 < \alpha < 2$, the corner is reached but the amount of time spent by the process in the corner is Lebesgue measure zero. If $\alpha \geqslant 2$, the process reaches the corner and remains there. The previous properties are valid in the case of zero drift. Under condition (1), the case of positive drift poses a new challenge, as $0 < \mathbb{P}_{(u,v)}[T < \infty] < 1$. Remark that condition (1) is equivalent to $\alpha \geqslant 1$.

1.5. **Escape probability and stationary distribution of the dual process.** Harrison [23] and Foddy [16] showed that the hitting time on one of the axes is intrinsically connected to the stationary distribution of a certain dual process. As the present article was nearing completion, it came to our attention that Harrison [24] has extended the results in his earlier work ([23]) by introducing a dual RBM in the quadrant with drift $-\mu$ and reflection matrix

$$\begin{pmatrix} r_2 & -1 \\ -1 & r_1 \end{pmatrix}$$

when $1 < r_1 r_2$. This configuration of parameters is depicted in Figure 3 below. This dual process has an explicit connection with the study of the escape probability. In particular, Harrison [24, Cor. 2] states that

$$\mathbb{P}_{(u,v)}[T = \infty] = \pi(\mathcal{S}(u, v)),$$

where π is the stationary distribution of the dual process and $\mathcal{S}(u, v) := \{(u - r_2 z_1 + z_2, v + z_1 - r_1 z_2) \in \mathbb{R}^2_+ : (z_1, z_2) \in \mathbb{R}^2_+\}$ is a trapezoid as pictured in Figure 3.

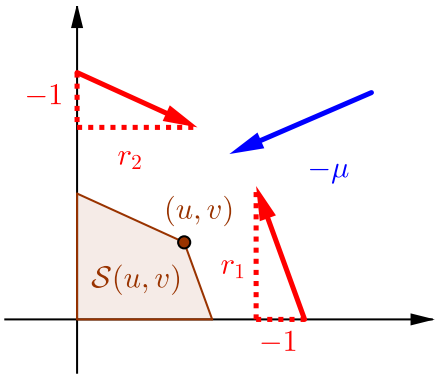


FIGURE 3. Dual process parameters and trapezoid $\mathcal{S}(u,v)$ in brown.

1.6. **Plan.** The remainder of this paper is organized as follows. In Section 2 we explore some general properties of the process of interest given in (2). This section's key result is Theorem 10, which states that the process has only two possible behaviors: either $T < \infty$, which means that the process is absorbed at the origin in finite time, or $T = \infty$, in which case the process escapes to infinity, namely $Z(t) \to \infty$ when $t \to \infty$. In Section 3 we present Proposition 11, which gives a partial differential equation characterizing the escape probability. Later in this section, we give Proposition 12, which provides a functional equation satisfied by the Laplace transform of the escape probability. In Section 4, we study the kernel of this functional equation and obtain asymptotic results for the absorption probability in the simpler case where the starting point is $(u, 0)$ (Proposition 17). In Section 5 we find a geometric condition which characterizes the cases where the absorption probability has a product form and is exponential (Theorem 20). Such a result recalls the famous skew symmetry condition studied a lot for invariant measures ([25, 28]). In Section 6 we establish a boundary value problem (BVP) satisfied by the Laplace transform of the escape probability (Proposition 22) and conclude with Theorem 30, which gives an explicit integral formula of this transform. In Section 7 we obtain exact asymptotics for the escape probability at the origin.

In memory of Larry Shepp. We dedicate this article in memory of our colleague, mentor, and friend, Professor Larry Shepp. Professor Shepp indelibly contributed to many areas of applied probability, and one of the areas that interested him most concerned RBM in a quadrant as well as in a strip ([22, 27]).

2. General properties of process Z

In this section we investigate a few key properties of the process given in (2). We prove three key results. The first one is that if the starting point tends to infinity, then the probability that the process does not hit the origin tends towards 1 (Theorem 4). The second one is that when the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards 1 (Theorem 6). The third key result is that the process has only two possible behaviors: either $T < \infty$, which means that the process is absorbed at the origin in finite time, or $T = \infty$, in which case the process escapes to infinity, namely $Z(t) \to \infty$ when $t \to \infty$ (Theorem 10).

2.1. **Limits of the hitting probability.** Our first key results of the section (Theorems 4 and 6) concern the probability of the process hitting the origin. We wish to show that $\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T = \infty] = 1$. We shall prove this with the aid of Lemma 1 and Proposition 3.

For ease of notation, let us define $\tau_1^\xi := \inf\{t : Z_1(t\wedge T) \le \xi\}$ and $\tau_2^\xi := \inf\{t : Z_2(t\wedge T) \le \xi\}$. Further, let $X_1(t) := u + W_1(t) + \mu_1 t$ and let $X_2(t) := v + W_2(t) + \mu_2 t$.

Suppose $Z(t)$ is a one-dimensional reflected Brownian motion. The analysis of $Z(t)$ is converted to that of one-dimensional Brownian motion with a drift by the Skorokhod map. However,

in the case of obliquely reflected Brownian motion in a quadrant, this method does not generally work due to the presence of $l_1(t)$ and $l_2(t)$. However, on the event $\{\tau_1^\xi = \infty\}$, note that $l_1(t) = 0$, the previously reflected Brownian motion becomes an obliquely reflected Brownian motion in a half-plane. This allows the one-dimensional techniques to be applied in our case. This motivates us to consider the event $\{\tau_1^\xi\}$ below.

Lemma 1. *For $u > \xi > 0$, we have*

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] = \mathbb{P}_{(u,v)}\left[X_1(t\wedge T) - r_2 \sup_{0\le s\le t\wedge T}(-X_2(s))^+ > \xi \text{ for every } t \ge 0\right], \tag{5}$$

where x^+ equals x if $x > 0$ and is 0 otherwise. Hence,

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] \ge \mathbb{P}_{(u,v)}\left[X_1(t) - r_2 \sup_{0\le s\le t}(-X_2(s))^+ > \xi \text{ for every } t \ge 0\right]. \tag{6}$$

A symmetrical result holds for $v > \xi > 0$ and $\mathbb{P}_{(u,v)}[\tau_2^\xi = \infty]$.

Proof. On the event $\{\tau_1^\xi = \infty\}$, for every $t \ge 0$, we have $l_1(t) = 0$, $\mathbb{P}_{(u,v)}$-a.s. Then

$$\begin{aligned} Z_1(t\wedge T) &= X_1(t\wedge T) - r_2\, l_2(t\wedge T),\\ Z_2(t\wedge T) &= X_2(t\wedge T) + l_2(t\wedge T). \end{aligned}$$

Note that $l_2(t\wedge T)$ increases only when $Z_2(t\wedge T) = 0$. By uniqueness of the Skorokhod map (see e.g. [38] and references therein)

$$l_2(t\wedge T) = \sup_{0\le s\le t}(-X_2(s\wedge T))^+ = \sup_{0\le s\le t\wedge T}(-X_2(s))^+.$$

Thus

$$Z_1(t\wedge T) = X_1(t\wedge T) - r_2 \sup_{0\le s\le t\wedge T}(-X_2(s))^+.$$

We may then write

$$\begin{aligned} &\{\tau_1^\xi = \infty\} = \{Z_1(t\wedge T) > \xi \text{ for every } t \ge 0\}\\ =\ &\{Z_1(t\wedge T) > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) = 0\}\\ =\ &\left\{X_1(t\wedge T) - r_2 \sup_{0\le s\le t\wedge T}(-X_2(s))^+ > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) = 0\right\}, \end{aligned} \tag{7}$$

$\mathbb{P}_{(u,v)}$-a.s. We now wish to show that

$$\mathbb{P}_{(u,v)}\left[X_1(t\wedge T) - r_2 \sup_{0\le s\le t\wedge T}(-X_2(s))^+ > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) > 0\right] = 0. \tag{8}$$

Note that there is a set N such that $\mathbb{P}_{(u,v)}(N) = 1$ and for every $\omega \in N$, we have

$$Z_1(t\wedge T) = X_1(t\wedge T) + l_1(t\wedge T) - r_2\, l_2(t\wedge T) \ge 0, \tag{9}$$

$$Z_2(t\wedge T) = X_2(t\wedge T) - r_1\, l_1(t\wedge T) + l_2(t\wedge T) \ge 0, \tag{10}$$

$$l_1(t\wedge T) \text{ increases only when } Z_1(t\wedge T) = 0, \tag{11}$$

$$l_2(t\wedge T) \text{ increases only when } Z_2(t\wedge T) = 0. \tag{12}$$

Let $\omega \in N$. We claim that the following statements

a) $X_1(t\wedge T) - r_2 \sup_{0\le s\le t\wedge T}(-X_2(s))^+ > \xi$ for every $t \ge 0$;
b) $l_1(T) > 0$,

cannot hold simultaneously. The proof is by contradiction. For sake of contradiction, assume that statements a) and b) hold simultaneously. By (10), (12), and the uniqueness of Skorokhod map, we have

$$l_2(t\wedge T) = \sup_{0\le s\le t}(r_1\, l_1(s\wedge T) - X_2(s\wedge T))^+$$

$$\begin{aligned} &\leq \sup_{0\leq s\leq t}(r_1\, l_1(s\wedge T))^+ + \sup_{0\leq s\leq t}(-X_2(s\wedge T))^+ \\ &= r_1\, l_1(t\wedge T) + \sup_{0\leq s\leq t\wedge T}(-X_2(s))^+. \end{aligned}$$

Let $\eta := \inf\{t : l_1(t\wedge T) \geq \xi/(2r_1r_2)\}$. Then for every $t \geq 0$,

$$\begin{aligned} & Z_1(t\wedge\eta\wedge T) = X_1(t\wedge\eta\wedge T) + l_1(t\wedge\eta\wedge T) - r_2 l_2(t\wedge\eta\wedge T) \\ \geq\ & X_1(t\wedge\eta\wedge T) - r_2\, l_2(t\wedge\eta\wedge T) \\ \geq\ & X_1(t\wedge\eta\wedge T) - r_2 \sup_{0\leq s\leq t\wedge\eta\wedge T}(-X_2(s))^+ - r_1r_2\, l_1(t\wedge\eta\wedge T) \\ >\ & \xi - \xi/2 = \xi/2, \end{aligned}$$

where in the last inequality we have invoked statement a). Since $l_1(t\wedge T)$ increases only when $Z_1(t\wedge T) = 0$, we have

$$l_1(t\wedge\eta\wedge T) = 0 \text{ for every } t \geq 0,$$

which contradicts statement b) and the definition of η. By contradiction, (8) holds. Combining (7) and (8), (5) follows. Note that (6) follows directly from (5). □

Remark 2. *To estimate the probability of the event*

$$\{X_1(t) - r_2 \sup_{0\leq s\leq t}(-X_2(s))^+ > \xi \text{ for every } t \geq 0\},$$

we note that the above event contains the intersection of the event $\{X_1(t) > \xi + c \text{ for every } t\}$ *and the event* $\{\sup_{0\leq s\leq t}(-X_2(s))^+ < c/r_2 \text{ for every } t\}$ *for every positive* c*, both of which correspond to the first hitting problems of one-dimensional Brownian motion with a drift. We will use the idea repeatedly in the proofs of Theorem 4 and Lemma 8.*

We now turn to Proposition 3 below, which is a reformulation of the formula 1.2.4(1) on p. 252 of [5].

Proposition 3. *Let* $B(t)$ *be a one dimensional Brownian motion started from the origin under* $\mathbb{P}$*. For* $\mu > 0$ *and* $x > 0$*, we have*

$$\mathbb{P}(B(t) + \mu t > -x \text{ for every } t \geq 0) = 1 - e^{-2x\mu}.$$

With Lemma 1 and Proposition 3 in hand, we state Theorem 4 below.

Theorem 4. *When the starting point tends to infinity, the probability that the process does not hit the origin tends to one. Namely,*

$$\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T = \infty] = 1.$$

Equivalently,

$$\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T < \infty] = 0.$$

Proof. Fix $\xi > 0$. For $\|(u,v)\|$ sufficiently large, we have $u > 2\xi$ or $v > 2\xi$. If $u > 2\xi$, by Lemma 1, we have

$$\begin{aligned} & \mathbb{P}_{(u,v)}[T = \infty\ ] \geq \mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] \\ \geq\ & \mathbb{P}_{(u,v)}\left[X_1(t) - r_2 \sup_{0\leq s\leq t}(-X_2(s))^+ > \xi \text{ for every } t \geq 0\right] \\ \geq\ & \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \geq 0 \text{ and } X_2(t) > -u/(2r_2) \text{ for every } t \geq 0] \\ \geq\ & \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \geq 0] \\ & + \mathbb{P}_{(u,v)}[X_2(t) > -u/(2r_2) \text{ for every } t \geq 0] - 1 \\ =\ & \mathbb{P}_{(u,v)}[W_1(t) + \mu_1 t > -(u-2\xi)/2 \text{ for every } t \geq 0] \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}_{(u,v)}[W_2(t)+\mu_2 t > -u/(2r_2) - v \text{ for every } t \geq 0] - 1\\
=\ & 1 - e^{-(u-2\xi)\mu_1} + 1 - e^{-(u/r_2+2v)\mu_2} - 1\\
=\ & 1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2},
\end{aligned}
$$

where the second to last equality invokes Proposition 3. Similarly, if $v > 2\xi$, we have

$$\mathbb{P}_{(u,v)}[T = \infty\ ] \geq 1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1}.$$

Hence,

$$
\begin{aligned}
&\mathbb{P}_{(u,v)}[T=\infty]\\
\geq\ & \max\{(1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2})1_{\{u>2\xi\}}, (1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1})1_{\{v>2\xi\}}\}.
\end{aligned}
$$

Letting (u, v) tend to ∞, the desired result follows. □

We now turn to Proposition 5 below, which shall be needed to prove Theorem 6.

Proposition 5. *We have the following subset relationship*

$$\{u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \subset \{T < \infty\}.$$

Proof. We prove this claim by contradiction. For the sake of contradiction, let us fix $\omega \in \{u+W_1(t)+\mu_1 t < 0 \text{ and } v+W_2(t)+\mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \cap \{T = \infty\}$. Assuming $T = \infty$, the process can be written as

$$
\begin{cases}
Z_1(t) = u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t),\\
Z_2(t) = v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t).
\end{cases}
$$

Solving the linear system for l_1 and l_2, we obtain

$$
\begin{cases}
(r_1 r_2 - 1) l_1(t) = (u + W_1(t) + \mu_1 t - Z_1(t)) + r_2(v + W_2(t) + \mu_2 t - Z_2(t)),\\
(r_1 r_2 - 1) l_2(t) = r_1(u + W_1(t) + \mu_1 t - Z_1(t)) + (v + W_2(t) + \mu_2 t - Z_2(t)).
\end{cases}
$$

For all $t \in \mathbb{R}_+$ such that

$$u + W_1(t) + \mu_1 t < 0,$$

and

$$v + W_2(t) + \mu_2 t < 0,$$

we have $(r_1 r_2 - 1) l_1(t) < 0$ and $(r_1 r_2 - 1) l_2(t) < 0$, which is not possible since $l_1(t)$ and $l_2(t) \geqslant 0$ and as we assumed $(r_1 r_2 - 1) \geqslant 0$. A contradiction has been reached. □

Theorem 6 below considers the behavior of the process when the starting point tends to the origin.

Theorem 6. *When the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards one. That is,*

$$\lim_{(u,v)\to(0,0)} \mathbb{P}_{(u,v)}\left[T < \infty\right] = 1,$$

or equivalently,

$$\lim_{(u,v)\to(0,0)} \mathbb{P}_{(u,v)}\left[T = \infty\right] = 0.$$

Proof. By Proposition 5, we have that

$$\mathbb{P}_{(u,v)}\left[T < \infty\right] \geqslant \mathbb{P}\left[\exists t \in \mathbb{R}_+ : u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0\right].$$

By the properties of planar Brownian motion, we have

$$\mathbb{P}\left[\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0\right] = 1.$$

Let $(u_n, v_n) \in \mathbb{R}_+^2$ be a sequence of points such that $(u_n, v_n) \to (0, 0)$. Note that

$$
\begin{aligned}
&\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0\}\\
\supset\ & \{\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0\}.
\end{aligned}
$$

Applying Fatou's Lemma yields

$$\liminf_{n\to\infty} \mathbb{P}\left[\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0\right] \geqslant 1.$$

We may therefore conclude that

$$\mathbb{P}_{(u_n,v_n)}\left[T < \infty\right] \underset{n\to\infty}{\longrightarrow} 1,$$

and the desired result follows. □

2.2. **Complementarity of absorption and escape.** We now turn to Theorem 10, which states that the process has only two possible behaviors: either $T < \infty$, or $T = \infty$, in which case $Z(t) \to \infty$ when $t \to \infty$. The result first requires the proofs of three auxiliary statements which we give below.

Proposition 7. *Suppose $B(t)$ is a one dimensional Brownian motion starting from the origin under the measure $\mathbb{P}$. Let a, b be two positive numbers. Then*

$$\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0) > 0.$$

Proof. Let $\lambda = \ln 2/(2b) + 1$. Note that $1 - 2\,e^{-2\lambda b} > 0$. Then

$$\begin{aligned}
&\mathbb{P}(-\lambda - bt < B(t) < \lambda + bt \text{ for every } t \geq 0)\\
\geq\ &\mathbb{P}(B(t) > -\lambda - bt \text{ for every } t \geq 0) + \mathbb{P}(B(t) < \lambda + bt \text{ for every } t \geq 0) - 1\\
=\ &2(1 - e^{-2\lambda b}) - 1 = 1 - 2\,e^{-2\lambda b} > 0.
\end{aligned}$$

Let $H_a := \inf\{t : |B(t)| = a\}$. By standard exit time properties of Brownian motion, $\mathbb{P}(H_a > \lambda/b + 1) > 0$. Then

$$\begin{aligned}
&\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0)\\
=\ &\mathbb{P}(H_a > \lambda/b + 1)\,\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0 \mid H_a > \lambda/b + 1).
\end{aligned}$$

By the strong Markov property of Brownian motion,

$$\begin{aligned}
&\mathbb{P}(-a - bt < B(t) < a + bt, \forall t \mid H_a > \lambda/b + 1)\\
=\ &\mathbb{P}(-a - b(t + H_a) < B(t + H_a) < a + b(t + H_a), \forall t \mid H_a > \lambda/b + 1)\\
=\ &\mathbb{P}(-a - b(t + H_a) - B(H_a) < B(t + H_a) - B(H_a) < a + b(t + H_a) - B(H_a), \forall t \mid H_a > \lambda/b + 1)\\
\geq\ &\mathbb{P}(-\lambda - bt < B(t + H_a) - B(H_a) < \lambda + bt, \forall t \mid H_a > \lambda/b + 1)\\
=\ &\mathbb{P}(-\lambda - bt < B(t) < \lambda + bt, \forall t\,)\\
>\ &0,
\end{aligned}$$

from which the desired result follows. □

We now turn to Lemma 8.

Lemma 8. *For α a positive number,*

$$\inf_{u\geq\alpha} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] > 0, \tag{13}$$

$$\inf_{v\geq\alpha} \mathbb{P}_{(0,v)}[\tau_2^0 = \infty] > 0. \tag{14}$$

Proof. We need only prove (13), since the proof of (14) is completely symmetric. Let us consider $\xi < \alpha$. By Lemma 1,

$$\begin{aligned}
&\mathbb{P}_{(u,0)}[\tau_1^0 = \infty] \geq \mathbb{P}_{(u,0)}[\tau_1^\xi = \infty]\\
\geq\ &\mathbb{P}_{(u,0)}\left[X_1(t) - r_2 \sup_{0\leq s\leq t} (-X_2(s))^+ > \xi \text{ for every } t \geq 0\right]\\
=\ &\mathbb{P}_{(u,0)}\left[u + W_1(t) + \mu_1 t - r_2 \sup_{0\leq s\leq t} (-W_2(s) - \mu_2 t)^+ > \xi \text{ for every } t \geq 0\right]\\
\geq\ &\mathbb{P}_{(u,0)}[W_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0\\
&\qquad \text{and } W_2(t) + \mu_2 t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0].
\end{aligned} \tag{15}$$

Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions starting from 0 under $\mathbb{P}_{(u,0)}$. Then, under $\mathbb{P}_{(u,0)}$, the process $(W_1(t), W_2(t))$ has the same law as $(B_1(t), \rho B_1(t) + \sqrt{1-\rho^2}B_2(t))$. We now show that (13) holds in three separate cases: $\rho = 0$, $0 < \rho < 1$ and $-1 < \rho < 0$.

Case I: $\rho = 0$. If $\rho = 0$, then $W_1(t)$ and $W_2(t)$ are two independent Brownian motions. Then

$$\begin{aligned}(15) &= \mathbb{P}_{(u,0)}[W_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0] \\ &\quad \times \mathbb{P}_{(u,0)}[W_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\ &= \left(1 - e^{-(u-\xi)\mu_1}\right) \cdot \left(1 - e^{-(u-\xi)\mu_2/r_2}\right),\end{aligned}$$

where the last equality invokes Proposition 3. Taking infimums yields

$$\inf_{u \geq \alpha} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] \geq \left(1 - e^{-(\alpha-\xi)\mu_1}\right) \cdot \left(1 - e^{-(\alpha-\xi)\mu_2/r_2}\right) > 0.$$

Case II: $0 < \rho < 1$. If $0 < \rho < 1$, then

$$\begin{aligned}(15) &= \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0 \\ &\qquad \text{and } \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\ &\geq \mathbb{P}_{(u,0)}[B_1(t) + (\mu_1 \wedge \mu_2)t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0 \\ &\qquad \text{and } \sqrt{1-\rho^2}B_2(t) + (1-\rho)\mu_2 t > -(1-\rho)(u-\xi)/(2r_2) \text{ for every } t \geq 0].\end{aligned}$$

Using the same argument in the case for $\rho = 0$, (13) follows.

Case III: $-1 < \rho < 0$. If $-1 < \rho < 0$, then for $u \geq \alpha$

$$\begin{aligned}(15) &= \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0 \\ &\qquad \text{and } \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\ &\geq \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0, \\ &\qquad \rho B_1(t) - \rho(\mu_1 \wedge \mu_2)t > -|\rho|(u-\xi)/(2r_2) \text{ for every } t \geq 0 \\ &\qquad \text{and } \sqrt{1-\rho^2}B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\ &\geq \mathbb{P}_{(u,0)}[-(u-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t \\ &\qquad \text{and } \sqrt{1-\rho^2}B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2), \forall t] \\ &= \mathbb{P}_{(u,0)}[-(u-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\ &\qquad \times \mathbb{P}_{(u,0)}[\sqrt{1-\rho^2}B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2), \forall t] \\ &\geq \mathbb{P}_{(u,0)}[-(\alpha-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (\alpha-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\ &\qquad \times \mathbb{P}_{(u,0)}[\sqrt{1-\rho^2}B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(\alpha-\xi)/(2r_2), \forall t].\end{aligned}$$

Taking infimums and invoking Proposition 7, (13) follows. This concludes the proof. □

Let us denote $T_r := \inf\{t \geq 0 : \|Z(t \wedge T)\| \leq r\}$.

Lemma 9. *For fixed n, on the event $\{T_{1/n} = \infty\}$, we have $\mathbb{P}_{(u,v)}$-a.s.*

$$\lim_{t \to \infty} Z(t) = \infty.$$

That is,

$$\mathbb{P}_{(u,v)}\left[\liminf_{t \to \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{16}$$

Proof. We will first show (16) holds when $v = 0$. Then (16) will follow immediately in the case $u = 0$. We conclude by showing that (16) holds when $u \neq 0$ and $v \neq 0$.

Case I: $v = 0$. When $v = 0$, let

$$K := \sup_{u \geq 0} \mathbb{P}_{(u,0)}\left[\liminf_{t \to \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right].$$

For $u \leq 1/n$,

$$\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.$$

Then

$$K = \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \tag{17}$$

We now define a stopping time

$$\eta_1^0 := \begin{cases} \inf\{t \geq \tau_1^0 : Z_2(t) = 0\}, & \tau_1^0 < \infty, \\ \infty, & \tau_1^0 = \infty. \end{cases}$$

By Lemma 8,

$$\inf_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 = \infty] \geq \inf_{u\geq 1/n} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] > 0,$$

and hence,

$$\sup_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty] < 1. \tag{18}$$

Note that

$$\begin{aligned} & \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ = \; & \mathbb{P}_{(u,0)}\left[\tau_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ & + \mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ & + \mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \end{aligned} \tag{19}$$

On the event $\{\tau_1^0 = \infty\}$, for all $t \geq 0$, $T = \infty$ and $l_1(t) = 0$. Then

$$Z_2(t) = X_2(t) + l_2(t) \geq X_2(t) = W_2(t) + \mu_2 t \to \infty,$$

$\mathbb{P}_{(u,0)}$-a.s., by the law of the iterated logarithm for Brownian motion. Hence, the first term on the right-hand side of (19) is 0. We now consider the second term on the right-hand side of (19). On the event $\{\tau_1^0 < \infty\}$, let us define $\tilde{\eta}_1^0 := \inf\{t \geq 0 : Z_2(t+\tau_1^0) = 0\}$ and $\tilde{T}_{1/n} := \inf\{t \geq 0 : \|Z(t+\tau_1^0)\| \leq 1/n\}$. By the strong Markov property, we have

$$\begin{aligned} & \mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ = \; & \mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \inf_{0\leq s\leq \tau_1^0} \|Z(s)\| > \frac{1}{n}, \tilde{\eta}_1^0 = \infty, \liminf_{t\to\infty} Z(t+\tau_1^0) < \infty, \tilde{T}_{\frac{1}{n}} = \infty\right] \\ = \; & \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\tau_1^0<\infty, \inf_{0\leq s\leq \tau_1^0}\|Z(s)\|>1/n\}} \mathbb{P}_{Z(\tau_1^0)}\left[\eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \\ = \; & 0. \end{aligned}$$

By the same argument used to show that the first term on the right-hand side of (19) is 0, for $v > 0$,

$$\mathbb{P}_{(0,v)}\left[\eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.$$

This proves that the second term on the right-hand side of (19) is also 0. We now consider the third term on the right-hand side of (19). On the event $\{\eta_1^0 < \infty\}$, let $\hat{T}_{1/n} := \inf\{t \geq 0 : Z(t+\eta_1^0) \leq 1/n\}$. By the strong Markov property,

$$\begin{aligned} & \mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ = \; & \mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \inf_{0\leq s\leq \eta_1^0} \|Z(s)\| > \frac{1}{n}, \liminf_{t\to\infty} Z(t+\eta_1^0) < \infty, \hat{T}_{\frac{1}{n}} = \infty\right] \\ = \; & \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\eta_1^0<\infty, \inf_{0\leq s\leq \eta_1^0}\|Z(s)\|>1/n\}} \mathbb{P}_{Z(\eta_1^0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \end{aligned}$$

$$\begin{aligned} &\leq K \cdot \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\eta_1^0<\infty, \inf_{0\leq s\leq \eta_1^0}\|Z(s)\|>1/n\}}\right] \\ &\leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty]. \end{aligned}$$

Combining (19) and the above estimates yields

$$\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].$$

Taking supremums and invoking (17), we obtain

$$K = \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \leq K \cdot \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].$$

Together with (18), we have $K = 0$. Hence, for every $u \geq 0$,

$$\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{20}$$

Similarly, for every $v \geq 0$,

$$\mathbb{P}_{(0,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{21}$$

Case II: $u \neq 0$ and $v \neq 0$. For the case when $u \neq 0$ and $v \neq 0$, let $\tau := \inf\{t \geq 0 : Z_1(t) = 0 \text{ or } Z_2(t) = 0\}$. Then

$$\begin{aligned} &\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ = \ &\mathbb{P}_{(u,v)}\left[\tau = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ &+ \mathbb{P}_{(u,v)}\left[\tau < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \end{aligned} \tag{22}$$

On the event $\{\tau = \infty\}$, $T = \infty$ and, for every $t \geq 0$, $l_1(t) = l_2(t) = 0$. Then, as $t \to \infty$,

$$Z_1(t) = u + W_1(t) + \mu_1 t \to \infty,$$

$\mathbb{P}_{(u,v)}$-a.s. Hence the first term on the right-hand side of (22) is 0. We now consider the second term on the right-hand side of (22). By the strong Markov property,

$$\begin{aligned} &\mathbb{P}_{(u,v)}\left[\tau < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ \leq \ &\mathbb{E}_{(u,v)}\left[\mathbb{1}_{\{\tau<\infty\}} \mathbb{P}_{Z(\tau)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \\ = \ &0, \end{aligned}$$

where (20) and (21) have been invoked in the last equality. Hence the second term on the right-hand side of (22) is also 0. Thus for $u \neq 0$ and $v \neq 0$,

$$\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.$$

The proof is now complete. □

With the above results in hand, we are now ready to state Theorem 10.

Theorem 10. *On the event $\{T = \infty\}$, $\mathbb{P}_{(u,v)}$-a.s. the process $Z(t)$ tends to infinity when $t \to \infty$, namely*

$$\mathbb{P}_{(u,v)}\left[\lim_{t\to\infty} Z(t) = \infty \,\middle|\, T = \infty\right] = 1.$$

Equivalently,

$$\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T = \infty\right] = 0.$$

Proof. We deduce from Lemma 9 that for every $n \in \mathbb{N}_+^*$

$$\begin{aligned} & \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\, T = \infty\right] \\ = \;& \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\, T_{\frac{1}{n}} < \infty,\, T = \infty\right]. \end{aligned}$$

Applying the strong Markov property yields

$$\begin{aligned} & \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\, T_{\frac{1}{n}} < \infty,\, T = \infty\right] \\ = \;& \mathbb{E}_{(u,v)}\left[\mathbb{1}_{\{T_{1/n}<\infty\}}\, \mathbb{P}_{Z(T_{1/n})}\left[\liminf_{t\to\infty} Z(t) < \infty,\, T = \infty\right]\right] \\ \leq \;& \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\, T = \infty\right] \\ \leq \;& \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)}\left[T = \infty\right]. \end{aligned}$$

Applying Theorem 6 and letting $n \to \infty$, the desired result follows. □

3. Partial differential equation and functional equation

We now turn to the study of the escape probability $\mathbb{P}_{(u,v)}[T = \infty]$. We begin with Proposition 11, which provides partial differential equations characterizing the escape probability. We proceed with Proposition 12, which gives a functional equation satisfied by the Laplace transform of the escape probability. Note that there is no particular difficulty in defining the process starting from the edge (except the origin).

Let us define the infinitesimal generator of the process inside the quarter plane as

$$\mathcal{G}f(u,v) := \lim_{t\to 0} \frac{1}{t}(\mathbb{E}_{(u,v)}[f(Z(t\wedge T)] - f(u,v)),$$

where f must be a bounded function in the quadrant to ensure that the above limit exists and is uniform. For f twice differentiable, the infinitesimal generator inside the quadrant is

$$\mathcal{G}f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} + 2\rho\frac{\partial^2 f}{\partial z_1 \partial z_2}\right) + \mu_1\frac{\partial f}{\partial z_1} + \mu_2\frac{\partial f}{\partial z_2}.$$

This leads us to Proposition 11.

Proposition 11 (Partial differential equation)**.** *The absorption probability*

$$f(u,v) = \mathbb{P}_{(u,v)}[T < \infty],$$

is the only function which is both (i) bounded and continuous in the quarter plane and on its boundary and (ii) continuously differentiable in the quarter plane and on its boundary (except perhaps at the corner), and which satisfies the partial differential equation

$$\mathcal{G}f(u,v) = 0, \quad \forall (u,v) \in \mathbb{R}_+^2,$$

with oblique Neumann boundary conditions

$$\begin{cases} \partial_{r_1} f(0,v) := \frac{\partial f}{\partial u}(0,v) - r_1\frac{\partial f}{\partial v}(0,v) = 0 & \forall v > 0, \\ \partial_{r_2} f(u,0) := -r_2\frac{\partial f}{\partial u}(u,0) + \frac{\partial f}{\partial v}(u,0) = 0 & \forall u > 0, \end{cases} \tag{23}$$

and with limit values

$$\begin{cases} f(0,0) = 1, \\ \lim_{(u,v)\to\infty} f(u,v) = 0. \end{cases}$$

The same result holds for the escape probability

$$g(u,v) = 1 - f(u,v) = \mathbb{P}_{(u,v)}[T = \infty]$$

but with the following limit values

$$\begin{cases} g(0,0) = 0, \\ \lim_{\|(u,v)\|\to\infty} f(u,v) = 1. \end{cases}$$

Remark that a similar partial differential equation with different limit values could be obtained for the domination probability considered in [17].

Proof. This proof is inspired by Foddy [16, p. 86-89]. We assume that f satisfies the hypotheses of the Proposition. Applying Dynkin's formula, we obtain

$$\begin{aligned}\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] &= f(u,v) + \mathbb{E}_{(u,v)}\int_0^{t\wedge T} \mathcal{G}f(Z(s))\,\mathrm{d}s + \sum_{i=1}^{2}\mathbb{E}_{(u,v)}\int_0^{t\wedge T}\partial_{r_i}f(Z(s))\,\mathrm{d}l_i(s)\\ &= f(u,v).\end{aligned}$$

But,

$$\begin{aligned}\mathbb{E}_{(u,v)}[f(Z(t\wedge T)] &= \mathbb{E}_{(u,v)}[f(Z(t\wedge T))\mathbb{1}_{T\leqslant t}] + \mathbb{E}_{(u,v)}[f(Z(t\wedge T))\mathbb{1}_{T>t}]\\ &= f(0,0)\mathbb{P}_{(u,v)}[T\leqslant t] + \mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}]\\ &\underset{t\to\infty}{\longrightarrow} \mathbb{P}_{(u,v)}[T<\infty] + \lim_{t\to\infty}\mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}]\\ &= \mathbb{P}_{(u,v)}[T<\infty].\end{aligned}$$

Note that $\lim_{|z|\to\infty} f(z) = 0$ and that for $T > t$, $Z(t) \underset{t\to\infty}{\to} \infty$ a.s. By dominated convergence and by Theorem 10,

$$\lim_{t\to\infty}\mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}] = \mathbb{E}_{(u,v)}[\lim_{t\to\infty} f(Z(t))\mathbb{1}_{T=\infty}] = 0.$$

We may thus conclude that

$$f(u,v) = \mathbb{P}_{(u,v)}[T<\infty].$$

Conversely, denote $f(u,v) := \mathbb{P}_{(u,v)}[T<\infty]$. The function f is bounded. By the Markov property, we have

$$\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] = f(u,v).$$

Since

$$\mathcal{G}f(u,v) = \lim_{t\to 0}\frac{1}{t}(\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] - f(u,v)) = 0,$$

we may conclude that $\mathcal{G}f = 0$ on the quarter plane. The continuity and differentiability properties of f are immediate from [2, Thm 2.2 and Cor 2.4]. One can also refer to [35] which establishes these properties in a greater generality. The Neumann boundary condition is satisfied by applying [2, Cor 3.3]. The desired limit values at 0 and at infinity are obtained by invoking Theorem 4 and Theorem 6. The result for $g = 1 - f$ is straightforward, and this completes the proof. □

In preparation for Proposition 12, let us define the Laplace transform of the escape probability starting from (u,v) as

$$\psi(x,y) := \int_0^\infty\int_0^\infty e^{-xu-yv}\mathbb{P}_{(u,v)}[T=\infty]\,\mathrm{d}u\mathrm{d}v.$$

Further, let

$$\psi_1(x) := \int_0^\infty e^{-xu}\mathbb{P}_{(u,0)}[T=\infty]\,\mathrm{d}u \quad\text{and}\quad \psi_2(y) := \int_0^\infty e^{-yv}\mathbb{P}_{(0,v)}[T=\infty]\,\mathrm{d}v. \tag{24}$$

We also define the kernel

$$K(x,y) := \frac{1}{2}(x^2+y^2+2\rho xy) + \mu_1 x + \mu_2 y, \tag{25}$$

and let

$$k_1(x,y) := \frac{1}{2}(r_2x+y) + \rho x + \mu_2, \quad k_2(x,y) := \frac{1}{2}(x+r_1y) + \rho y + \mu_1. \tag{26}$$

We now give a functional equation satisfied by the Laplace transform of the escape probability.

Proposition 12 (Functional equation)**.** *For $(x, y) \in \mathbb{C}^2$ such that $\Re x > 0$ and $\Re y > 0$ we have*

$$K(x, y)\psi(x, y) = k_1(x, y)\psi_1(x) + k_2(x, y)\psi_2(y). \tag{27}$$

This functional equation recalls the one obtained in [17, (32)] to compute an escape probability along one of the axes for another range of parameters.

Proof. Recall the partial differential equation in Proposition 11 with the oblique Neumann boundary condition and limit values satisfied by $g(u, v) := \mathbb{P}_{(u,v)}[T = \infty]$. Employing integration by parts yields

$$\begin{aligned}
0 &= \int_0^\infty \int_0^\infty e^{-xz_1-yz_2}\mathcal{G}g(z_1, z_2)\,\mathrm{d}z_1\mathrm{d}z_2\\
0 &= \int_0^\infty \frac{1}{2}e^{-yz_2}\left(-\frac{\partial g}{\partial z_1}(0, z_2) + x\int_0^\infty e^{-xz_1}\frac{\partial g}{\partial z_1}(z_1, z_2)\,\mathrm{d}z_1\right)\mathrm{d}z_2\\
&+ \int_0^\infty \frac{1}{2}e^{-xz_1}\left(-\frac{\partial g}{\partial z_2}(z_1, 0) + y\int_0^\infty e^{-yz_2}\frac{\partial g}{\partial z_2}(z_1, z_2)\,\mathrm{d}z_2\right)\mathrm{d}z_1\\
&+ \int_0^\infty \rho e^{-xz_1}\left(-\frac{\partial g}{\partial z_1}(z_1, 0) + y\int_0^\infty e^{-yz_2}\frac{\partial g}{\partial z_1}(z_1, z_2)\,\mathrm{d}z_2\right)\mathrm{d}z_1\\
&+ \int_0^\infty \mu_1 e^{-yz_2}\left(-g(0, z_2) + x\int_0^\infty e^{-xz_1}g(z_1, z_2)\,\mathrm{d}z_1\right)\mathrm{d}z_2\\
&+ \int_0^\infty \mu_2 e^{-xz_1}\left(-g(z_1, 0) + y\int_0^\infty e^{-yz_2}g(z_1, z_2)\,\mathrm{d}z_2\right)\mathrm{d}z_1\\
0 &= -\frac{1}{2}r_1\int_0^\infty e^{-yz_2}\frac{\partial g}{\partial z_2}(0, z_2)\,\mathrm{d}z_2 + \frac{x}{2}\int_0^\infty e^{-yz_2}\left(-g(0, z_2) + x\int_0^\infty e^{-xz_1}g(z_1, z_2)\,\mathrm{d}z_1\right)\mathrm{d}z_2\\
&- \frac{1}{2}r_2\int_0^\infty e^{-xz_1}\frac{\partial g}{\partial z_1}(z_1, 0)\,\mathrm{d}z_1 + \frac{y}{2}\int_0^\infty e^{-xz_1}\left(-g(z_1, 0) + y\int_0^\infty e^{-yz_2}g(z_1, z_2)\,\mathrm{d}z_2\right)\mathrm{d}z_1\\
&- \rho\int_0^\infty e^{-xz_1}\frac{\partial g}{\partial z_1}(z_1, 0)\,\mathrm{d}z_1 + \rho y\int_0^\infty e^{-yz_2}\left(-g(0, z_2) + x\int_0^\infty e^{-xz_1}g(z_1, z_2)\,\mathrm{d}z_1\right)\mathrm{d}z_2\\
&- \mu_1\int_0^\infty e^{-yz_2}g(0, z_2)\,\mathrm{d}z_2 + \mu_1 x\int_0^\infty\int_0^\infty e^{-xz_1-yz_2}g(z_1, z_2)\,\mathrm{d}z_1\mathrm{d}z_2\\
&- \mu_2\int_0^\infty e^{-xz_1}g(z_1, 0)\,\mathrm{d}z_1 + \mu_2 y\int_0^\infty\int_0^\infty e^{-xz_1-yz_2}g(z_1, z_2)\,\mathrm{d}z_1\mathrm{d}z_2\\
0 &= \left(\frac{1}{2}(x^2 + y^2 + 2\rho xy) + \mu_1 x + \mu_2 y\right)\int_0^\infty\int_0^\infty e^{-xz_1-yz_2}g(z_1, z_2)\,\mathrm{d}z_1\mathrm{d}z_2\\
&- \left(\frac{1}{2}(r_2 x + y) + \rho x + \mu_2\right)\int_0^\infty e^{-xz_1}g(z_1, 0)\,\mathrm{d}z_1\\
&- \left(\frac{1}{2}(x + r_1 y) + \rho y + \mu_1\right)\int_0^\infty e^{-yz_2}g(0, z_2)\,\mathrm{d}z_2\\
0 &= K(x, y)\psi(x, y) - k_1(x, y)\psi_1(x) - k_2(x, y)\psi_2(y).
\end{aligned}$$

This concludes the proof. □

4. Kernel and asymptotics

We begin by studying some properties of the kernel K defined in (25). Note that this kernel is similar to that in [21] except that in the present paper the drift is positive. We define the functions X and Y satisfying

$$K(X(y), y) = 0 \quad \text{and} \quad K(x, Y(x)) = 0.$$

The branches are given by

$$\begin{cases} X^{\pm}(y) := -(\rho y + \mu_1) \pm \sqrt{y^2(\rho^2-1) + 2y(\mu_1\rho - \mu_2) + \mu_1^2}, \\ Y^{\pm}(x) := -(\rho x + \mu_2) \pm \sqrt{x^2(\rho^2-1) + 2x(\mu_2\rho - \mu_1) + \mu_2^2}, \end{cases} \tag{28}$$

and the branch points of X and Y (which are roots of the polynomials in the square roots of (28)) are given, respectively, by

$$\begin{cases} y^{\pm} := \dfrac{\mu_1\rho - \mu_2 \pm \sqrt{(\mu_1\rho-\mu_2)^2 + \mu_1^2(1-\rho^2)}}{(1-\rho^2)}, \\ x^{\pm} := \dfrac{\mu_2\rho - \mu_1 \pm \sqrt{(\mu_2\rho-\mu_1)^2 + \mu_2^2(1-\rho^2)}}{(1-\rho^2)}. \end{cases} \tag{29}$$

By (3) we obtain that

$$y^+ = \mu_1 \frac{1-\cos(\beta-\theta)}{\sin\beta\sin(\beta-\theta)}. \tag{30}$$

The functions $X^{\pm}$ and $Y^{\pm}$ are analytic, respectively, on the cut planes $\mathbb{C}\setminus((-\infty,y^-]\cup[y^+,\infty))$ and $\mathbb{C}\setminus((-\infty,x^-]\cup[x^+,\infty))$. Figure 4 below depicts the functions $Y^{\pm}$ on $[x^-,x^+]$.

Recall k_1 and k_2 as defined in (26). Consider the intersection points between the ellipse $K=0$ and the lines $k_1=0$ and $k_2=0$. We define

$$x_0 := -2\mu_1 < 0 \quad \text{and} \quad y_0 := -2\mu_2 < 0, \tag{31}$$

$$x_1 := -\frac{2(r_2\mu_2+\mu_1)}{1+r_2^2+2\rho r_2} < 0 \quad \text{and} \quad y_2 := -\frac{2(r_1\mu_1+\mu_2)}{1+r_1^2+2\rho r_1} < 0. \tag{32}$$

These points are represented on Figure 4 and satisfy the following:

- $K(x_0,0) = k_2(x_0,0) = 0$, $K(0,y_0) = k_1(0,y_0) = 0$.
- $\exists y_1 \in \mathbb{R}$ such that $K(x_1,y_1) = k_2(x_1,y_1) = 0$.
- $\exists x_2 \in \mathbb{R}$ such that $K(x_2,y_2) = k_1(x_2,y_2) = 0$.

Let us define the curve $\mathcal{H}$, which is the boundary of the BVP established in Section 6.1

$$\mathcal{H} = X^{\pm}([y^+,\infty)) = \{x\in\mathbb{C}\colon K(x,y)=0 \text{ and } y\in[y^+,\infty)\}. \tag{33}$$

Lemma 13 (Hyperbola $\mathcal{H}$)**.** *The curve $\mathcal{H}$ is a branch of the hyperbola of equation*

$$(\rho^2-1)x^2 + \rho^2y^2 - 2(\mu_1-\rho\mu_2)x = \mu_1(\mu_1-2\rho\mu_2). \tag{34}$$

The curve $\mathcal{H}$ is symmetrical with respect to the horizontal axis and is the right branch of the hyperbola if $\rho<0$. Further, it is the left branch if $\rho>0$ and it is a straight line if $\rho=0$.

Proof. A similar kernel has already been studied; we refer the reader to [21, Lemma 4] and [3, Lemma 9], where the equation of such a hyperbola is derived. □

Let $\mathcal{H}^+$ denote the part of the hyperbola $\mathcal{H}$ with positive imaginary part. We also define the domain $\mathcal{G}$ bounded by $\mathcal{H}$ and containing x^+. This is depicted in Figure 5 below.

4.1. **Meromorphic continuation.** This section focuses on establishing the boundary value problem. We begin by meromorphically continuing the Laplace transform $\psi_1(x)$ (which converges for $x>0$).

Lemma 14 (Meromorphic continuation)**.** *By the formula*

$$\psi_1(x) = \frac{-k_2(x,Y^+(x))\psi_2(Y^+(x))}{k_1(x,Y^+(x))}, \tag{35}$$

the Laplace transform $\psi_1(x)$ can be meromorphically continued to the set

$$S := \{x\in\mathbb{C} : \Re x > 0 \text{ or } \Re Y^+(x) > 0\} \cup \{0\}, \tag{36}$$

where the domain $\mathcal{G}$ and its boundary $\mathcal{H}$ are included in the set defined in (36). Then ψ_1 is meromorphic on $\mathcal{G}$ and is continuous on $\overline{\mathcal{G}}$.

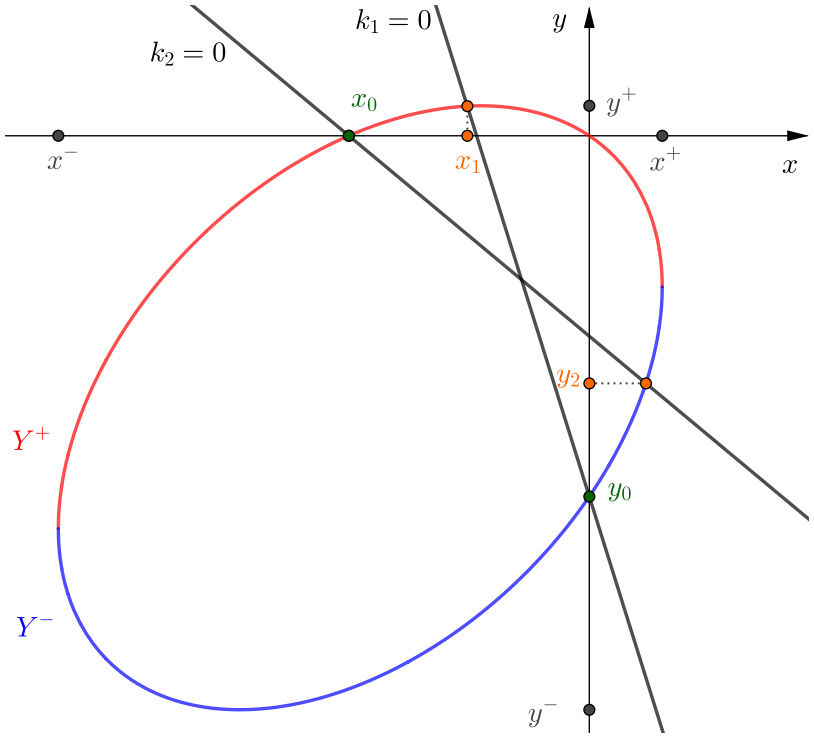
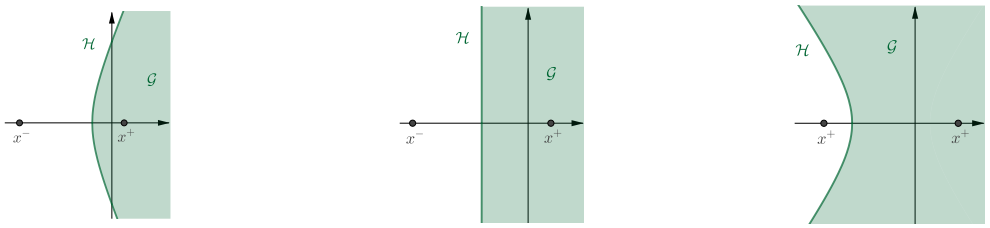


FIGURE 4. The ellipse $K = 0$, the function Y^- in blue, the function Y^+ in red, the two lines $k_1 = 0$ and $k_2 = 0$, the branch points $x^\pm$ and $y^\pm$, the points x_0 and y_0 in green, the points x_1 and y_2 in orange. This figure is drawn for the following parameters: $\mu_1 = 2$, $\mu_2 = 3$, $\rho = -0.4$, $r_1 = 2$, $r_2 = 4$.



(A) $\rho < 0$ (B) $\rho = 0$ (C) $\rho > 0$

FIGURE 5. Hyperbola $\mathcal{H}$ and domain $\mathcal{G}$.

Proof. The Laplace transforms $\psi_1(x)$ and $\psi_2(y)$ are analytic, respectively, on $\{x \in \mathbb{C}\colon \Re x > 0\}$ and $\{y \in \mathbb{C}\colon \Re y > 0\}$. The functional equation (27) implies that for (x, y) in the set $\widetilde{S} := \{(x, y) \in \mathbb{C}^2 : \Re x > 0,\ \Re y > 0 \text{ and } \psi(x, y) = 0\}$, we have

$$0 = k_1(x, y)\psi_1(x) + k_2(x, y)\psi_2(y). \tag{37}$$

The open connected set

$$S_1 := \{x \in \mathbb{C}\colon \Re Y^+(x) > 0\},$$

intersects the open set $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$. For $x \in S_1 \cap S_2$, $(x, Y^+(x)) \in \widetilde{S}$; equation (37) implies that the continuation formula in (35) is satisfied for all $x \in S_1 \cap S_2$. Figure 6 represents

these sets. With $\psi_1(x)$ defined as in (35), we invoke the principle of analytic continuation and meromorphically extend ψ_1 to $S = S_1 \cup S_2$. Note that the inclusion of $\mathcal{G}$ in the set S defined in (36) is similar to that in [21, Lemma 5]. This inclusion is depicted below in Figure 6. □

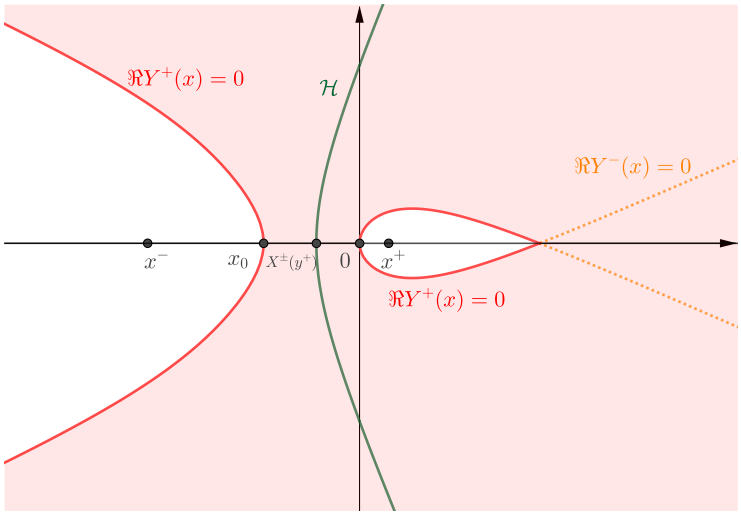


FIGURE 6. The complex plane of x. The red curve of equation $\Re Y^+(x) = 0$ bounds the red domain $S_1 := \{x \in \mathbb{C} : \Re Y^+(x) > 0\}$. The orange dotted curve corresponds to the equation $\Re Y^-(x) = 0$. The domain $\mathcal{G}$ is bounded on the left by the green hyperbola $\mathcal{H}$, contains x^+ (see Figure 5), and is included in $S = S_1 \cup S_2$, where $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$. This figure is drawn for the parameters $\mu_1 = 2$, $\mu_2 = 3$, $\rho = -0.4$.

4.2. **Poles and geometric conditions.**

Lemma 15 (Poles)**.** *On the set S defined in (36), the Laplace transform ψ_1 has either one or two poles, as follows:*

- *(One pole:) If $k_1(x^-, Y^\pm(x^-)) \geqslant 0$, the point 0 is the unique pole of ψ_1 in S and this pole is simple.*
- *(Two poles:) If $k_1(x^-, Y^\pm(x^-)) < 0$, the points 0 and x_1 (defined in (32)) are the only possible poles of ψ_1 in S and these poles are simple; $x_1 \in S$ if and only if $x_1 > x_0$.*

In addition, $\lim_{x\to 0} x\psi_1(x) = 1$. Further, the point x_1 is a pole of ψ_1 and belongs to the domain $\mathcal{G}$ if and only if $k_1(X^\pm(y^+), y^+) < 0$.

Proof. The final value theorem for the Laplace transform, together with Theorem 4, imply that

$$\lim_{x\to 0} x\psi_1(x) = \lim_{u\to\infty} \mathbb{P}_{(u,0)}[T = \infty] = 1.$$

We may thus conclude that 0 is a simple pole. On the set $\{x \in \mathbb{C} : \Re x > 0\}$, ψ_1 is defined as a Laplace transform which converges (and thus has no poles). Therefore, with the exception of 0, the only possible poles in S are the zeros of $k_1(x, Y^+(x))$, which are the zeros of the denominator of the continuation formula in (35). Straightforward calculations show that equation $k_1(x, Y^+(x)) = 0$ has either no roots or one (simple) root, and that this depends on the sign of $k_1(x^-, Y^\pm(x^-))$. When the root exists, it is x_1 (see (32)). The condition for the existence of this root is depicted in Figure 7 below. It now only remains to remark that when x_1 is a pole, x_1 is in $\mathcal{G}$ if and only if $x_1 > X^\pm(y^+)$. The latter holds if and only if $k_1(X^\pm(y^+), y^+) < 0$ (see Figure 8). □

Before turning to Lemma 16, recall that the angles δ, β and θ were defined above in (3) and that k_1 was defined in (26).

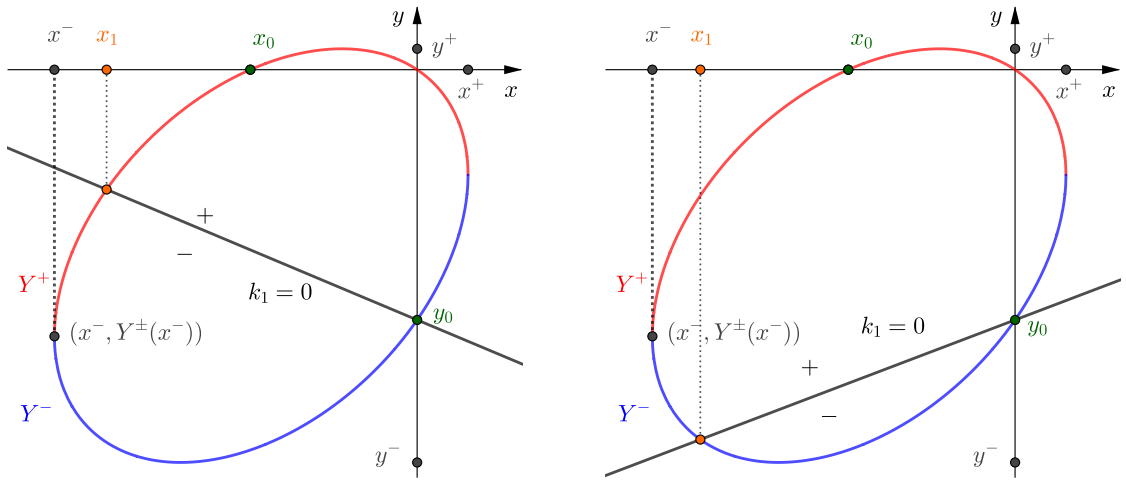


FIGURE 7. On the left, we see that $k_1(x^-, Y^\pm(x_-)) < 0$ and x_1 is a simple pole of ψ_1. On the right, we see that $k_1(x^-, Y^\pm(x^-)) > 0$ and ψ_1 has no pole in S.

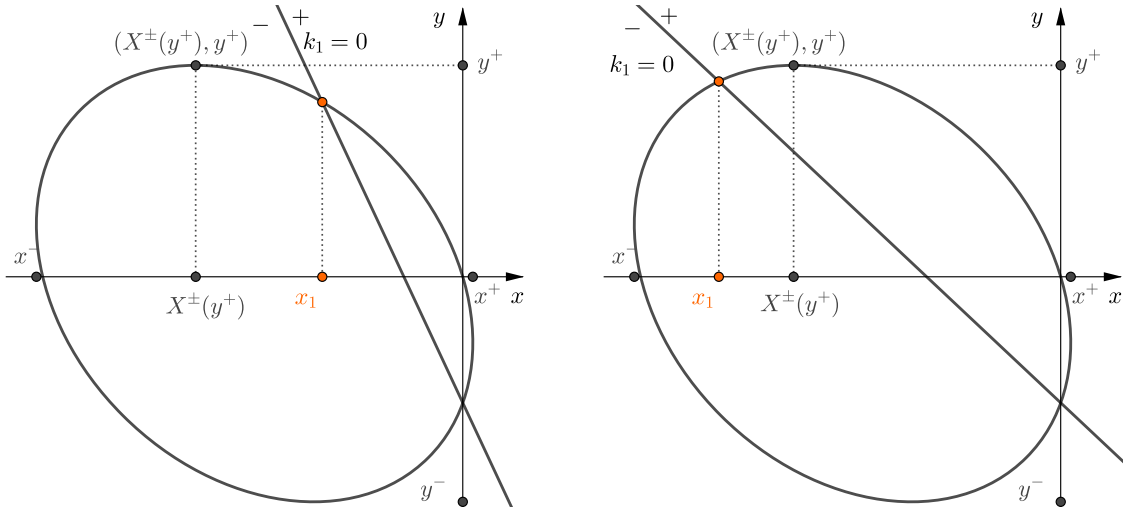


FIGURE 8. On the left, we see that $k_1(X^\pm(y^+), y^+) < 0$ and x_1 is in $\mathcal{G}$. On the right, we see that $k_1(X^\pm(y^+), y^+) > 0$ and x_1 is not in $\mathcal{G}$.

Lemma 16 (Geometric conditions)**.** *The condition $k_1(x^-, Y^\pm(x^-)) > 0$ (resp. $= 0$ and < 0) is equivalent to*

$$2\delta - \theta < \pi,$$

(resp. $= \pi$ and $> \pi$). The condition $k_1(X^\pm(y^+), y^+) > 0$ (resp. $= 0$ and < 0) is equivalent to

$$2\delta - \theta + \beta < 2\pi,$$

(resp. $= 2\pi$ and $> 2\pi$).

Proof. By condition (1) and by the fact that the drift is positive, we have $0 < \theta < \beta < \delta < \pi$. By (3) and (29),

$$x^-/\mu_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}} - \sqrt{\left(\frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}}\right)^2 + 1}\right) = \frac{-\cot(\theta) - \sqrt{\cot^2(\theta) + 1}}{\sin(\beta)}. \quad (38)$$

We begin by proving the first equivalence for $\delta \geqslant \pi/2$. In this case we have

$$k_1(x^-, Y^\pm(x^-)) > 0 \Leftrightarrow \frac{1}{2}(r_2 x^- + Y^\pm(x^-)) + \rho x^- + \mu_2 > 0$$

$$
\begin{aligned}
&\Leftrightarrow r_2 + \rho < -\mu_2/x^- \text{ since } Y^{\pm}(x^-) = -\rho x^- - \mu_2 \text{ by (28) and (29)}\\
&\Leftrightarrow r_2 - \cos(\beta) < \sin(\beta)\left(\cot(\theta) + \sqrt{\cot^2(\theta)+1}\right)^{-1} \text{ by (38)}\\
&\Leftrightarrow -\cot(\delta)\left(\cot(\theta) + \sqrt{\cot^2(\theta)+1}\right) < 1\\
&\Leftrightarrow 0 < -\cot(\delta)\sqrt{\cot^2(\theta)+1} < 1 + \cot(\delta)\cot(\theta) \text{ since we assumed } \delta \geqslant \pi/2\\
&\Leftrightarrow \cot^2(\delta)(\cot^2(\theta)+1) < (1+\cot(\delta)\cot(\theta))^2\\
&\Leftrightarrow 2\cot(\delta)\cot(\theta) - \cot^2(\delta) + 1 > 0\\
&\Leftrightarrow 2\sin(\delta)\cos(\delta)\cos(\theta) - (\cos^2(\delta) - \sin^2(\delta))\sin(\theta) > 0\\
&\Leftrightarrow \sin(2\delta)\cos(\theta) - \cos(2\delta)\sin(\theta) > 0\\
&\Leftrightarrow \sin(2\delta - \theta) > 0\\
&\Leftrightarrow 2\delta - \theta < \pi \text{ since } 0 < 2\delta - \theta < 2\pi.
\end{aligned}
$$

It is straightforward to see that if $\delta < \pi/2$, then $2\delta - \theta < \pi$. Further, $\delta < \pi/2$ is equivalent to $r_2 + \rho < 0$ by (3), which implies that $r_2 + \rho < -\mu_2/x^-$. Therefore, $k_1(X^{\pm}(y^+), y^+) < 0$. This proves the first equivalence. The second equivalence is proved in exactly the same way, so the details are omitted. This concludes the proof. □

4.3. **Absorption asymptotics along the axes.** In this section, we establish asymptotic results for the absorption probability (and escape probability) in a simpler case where the starting point is $(u, 0)$.

Proposition 17 (Absorption asymptotics)**.** *Let us assume that $x^- \in S$. For some constant C, the asymptotic behavior of $\mathbb{P}_{(u,0)}[T < \infty]$ as $u \to \infty$ is given by*

$$
\mathbb{P}_{(u,0)}[T < \infty] \sim C \begin{cases} e^{ux_1} & \text{if } 2\delta - \theta > \pi,\\ u^{-\frac{3}{2}} e^{ux^-} & \text{if } 2\delta - \theta < \pi,\\ u^{-\frac{1}{2}} e^{ux^-} & \text{if } 2\delta - \theta = \pi. \end{cases}
$$

Proof. The largest singularity of the Laplace transform of $\mathbb{P}_{(u,0)}[T < \infty]$ determines its asymptotics. We proceed by invoking a classical transfer theorem, see [11, Theorem 37.1]. This theorem says that if a is the largest singularity of order k of the Laplace transform (that is, the Laplace transform behaves as $(s-a)^{-k}$ up to additive and multiplicative constants in the neighborhood of a), then when $u \to \infty$, the probability $\mathbb{P}_{(u,0)}[T < \infty]$ is equivalent (up to a constant) to $u^{k-1}e^{au}$. The Laplace transform of $\mathbb{P}_{(u,0)}[T < \infty]$ is $1/x - \psi_1(x)$. By Lemma 15, the point 0 is not a singularity and the point x_1 is a simple pole. When that pole exists, the asymptotics are given by Ce^{ux_1} for some constant C. When there is no pole, that is, when $k_1(x^-, Y^{\pm}(x^-)) \geqslant 0$, the largest singularity is given by the branch point x^-. The definition of Y^+ and (35) together imply that for some constants C_i we have

$$
\psi_1(x) \underset{x \to x^-}{=} \begin{cases} C_1 + C_2\sqrt{x - x^-} + \mathrm{O}(x - x^-) & \text{if } k_1(x^-, Y^{\pm}(x^-)) > 0,\\ \dfrac{C_3}{\sqrt{x - x^-}} + \mathrm{O}(1) & \text{if } k_1(x^-, Y^{\pm}(x^-)) = 0. \end{cases}
$$

The proof is then completed by applying Lemma 16 and invoking the classical transfer theorem. □

Remark 18 (Asymptotics along the vertical axis)**.** *Studying the singularities of ϕ_1 we obtained in Proposition 17 the asymptotics of the absorption probability (and then of the escape probability which is equal to $1 - \mathbb{P}_{(u,0)}[T < \infty]$) along the horizontal axis. A similar study for ψ_2 would lead*

to the following asymptotics along the vertical axis. As $v \to \infty$

$$\mathbb{P}_{(0,v)}[T < \infty] \sim C \begin{cases} e^{vy_2} & \text{if } 2\epsilon + \theta - \beta > \pi, \\ v^{-\frac{3}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta < \pi, \\ v^{-\frac{1}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta = \pi. \end{cases}$$

Remark 19 (Bivariate asymptotics)**.** *The bivariate asymptotics of the absorption probability could be derived using the saddle point method and studying the singularities, see [19] and [13]. Such a study is very technical and requires to distinguish a lot of different cases. We would obtain some functions* a*,* b*,* c *depending on the parameters, such that for* $(u, v) = (r\cos(t), r\sin(t))$ *in polar coordinates,*

$$\mathbb{P}_{(u,v)}[T < \infty] \underset{r\to\infty}{\sim} a(t) r^{b(t)} e^{-c(t)r}.$$

Typically b *would take the values* 0 *or* $-1/2$*.*

5. Product form and exponential absorption probability

In this section, we consider a remarkable geometric condition on the parameters characterizing the case where the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the famous skew symmetry condition studied by Harrison, Reiman and Williams [25, 28], which characterizes the cases where the stationary distribution has a product form and is exponential. The *dual skew symmetry* condition gives a criterion for the solution to the partial differential equation of Proposition 11 (dual to that satisfied by the invariant measure) to be of product form. The following Theorem states a simple geometric criterion on the parameters for the absorption probability to be of product form; the absorption probability is then exponential.

Theorem 20 (Dual skew symmetry)**.** *Let* $f(u, v) = \mathbb{P}_{(u,v)}[T < \infty]$ *be the absorption probability. The following statements are equivalent:*

(1) The absorption probability has a product form, i.e. there exist f_1 *and* f_2 *such that*

$$f(u, v) = f_1(u) f_2(v);$$

(2) The absorption probability is exponential, i.e. there exist x *and* y *in* $\mathbb{R}$ *such that*

$$f(u, v) = e^{ux+vy};$$

(3) The reflection vectors are in opposite directions, i.e.

$$r_1 r_2 = 1;$$

(4) The reflection angles in the wedge satisfy $\alpha = 1$*, i.e.*

$$\delta + \epsilon - \beta = \pi.$$

In this case we have

$$f(u, v) = e^{ux_1+vy_2}$$

where x_1 *and* y_2 *are given in* (32)*.*

Proof. (1) ⇒ (2): Neumann boundary conditions (23) imply that $f_1'(0) f_2(y) - r_1 f_1(0) f_2'(y) = 0$ and $-r_2 f_1'(u) f_2(0) + f_1(u) f_2'(0) = 0$. Solving these differential equations imply that f_1 and f_2 (and thus f) are exponential.
(2) ⇒ (1): This implication is straightforward.
(2) ⇒ (3): Neumann boundary conditions (23) imply that for all $v > 0$, $xe^{vy} - r_1 y e^{vy} = 0$ and that for all $u > 0$, $-r_2 x e^{ux} + y e^{ux} = 0$. It follows that $r_1 = x/y$, $r_2 = y/x$, and thus $r_1 r_2 = 1$.
(3) ⇒ (2): Let us define $f(u, v) = e^{ux_1+vy_2}$. We need to show that f satisfies the partial differential equation of Proposition 11. This will imply that f is the absorption probability. The fact that $r_1 = 1/r_2$, combined with (32), gives $r_1 = x_1/y_2$. This implies that f satisfies the Neumann boundary conditions in (23). The limit values are satisfied because $f(0, 0) = 1$ and

$\lim_{(u,v)\to\infty} f(u,v) = 0$ for $x_1 < 0$ and $y_2 < 0$. It now only remains to show that $\mathcal{G}f = 0$. We now only need verify that $K(x_1, y_2) = 0$, see Figure 9. By definition of y_2 (see (32)), we have

$$\begin{aligned} K(x_1,y_2) &= y_2\left(\frac{y_2}{2}\left(\left(\frac{x_1}{y_2}\right)^2 + 1 + 2\rho\frac{x_1}{y_2}\right) + \mu_1\frac{x_1}{y_2} + \mu_2\right) \\ &= y_2\left(\frac{y_2}{2}\left(r_1^2 + 1 + 2\rho r_1\right) + \mu_1 r_1 + \mu_2\right) = 0. \end{aligned}$$

(3) $\Leftrightarrow$ (4): The following equivalences hold:

$$\begin{aligned} r_1 r_2 = 1 &\Leftrightarrow (\sin(\beta)/\tan(\delta) - \cos(\beta))\,(\sin(\beta)/\tan(\epsilon) - \cos(\beta)) = 1 \quad \text{by (3)} \\ &\Leftrightarrow \frac{\sin(\beta)}{\tan(\epsilon} = \frac{\tan(\delta)}{\sin(\beta) - \cos(\beta)\tan(\delta)} + \cos(\beta) = \frac{\tan(\delta)(1-\cos^2(\beta) + \cos(\beta)\sin(\beta)}{\sin(\beta) - \cos(\beta)\tan(\delta)} \\ &\Leftrightarrow \tan(\epsilon) = \frac{\tan(\beta) - \tan(\delta)}{1 + \tan(\delta)\tan(\beta)} \\ &\Leftrightarrow \tan(\epsilon) = \tan(\beta - \delta) \\ &\Leftrightarrow \epsilon = \beta - \delta + \pi. \end{aligned}$$

□

Remark 21 (Standard and dual skew symmetry)**.** *The standard skew symmetry condition for the matrix* $\begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix}$ *is* $2\rho = -r_1 - r_2$ *or equivalently* $\epsilon + \delta = \pi$*. The standard skew symmetry condition for the dual matrix* $\begin{pmatrix} r_2 & -1 \\ -1 & r_1 \end{pmatrix}$ *defined in Section 1.5 is* $2\rho = -1/r_1 - 1/r_2$ *or equivalently* $\epsilon + \delta - 2\beta = \pi$*. Note that the dual skew symmetry condition obtained in Theorem 20 is different from these two conditions. Further properties of the dual skew symmetry condition will be explored in future work.*

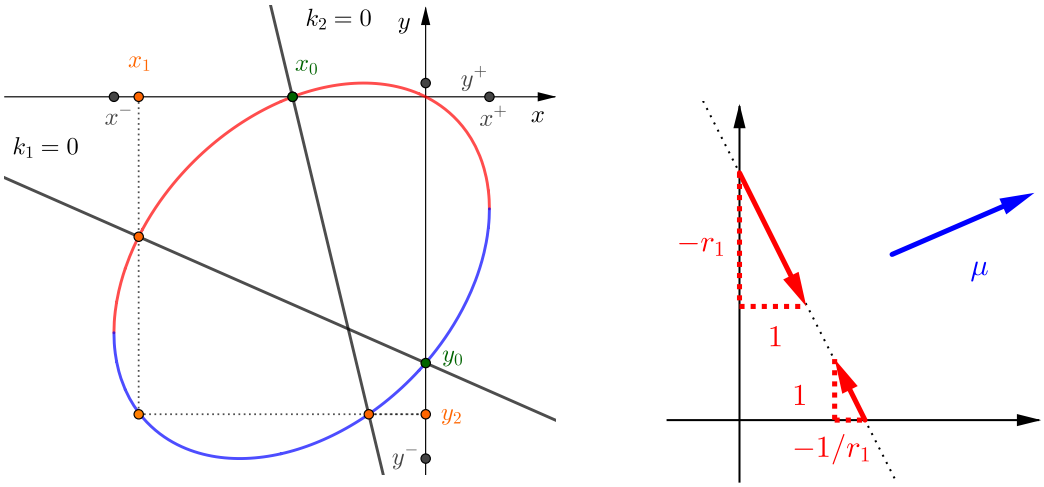


FIGURE 9. Dual skew symmetry: on the left, we see that $K(x_2, y_2) = 0$; on the right, we see that condition $r_1 r_2 = 1$ implies that the reflection vectors are in opposite directions.

6. Integral expression of the Laplace transform ψ_1

In this section, we establish a boundary value problem (BVP) satisfied by the Laplace transform (Proposition 22). The section's key result is Theorem 30, which gives an explicit integral formula for the Laplace transform of the escape probability.

6.1. **Carleman boundary value problem.** We state a Carleman BVP satisfied by the Laplace transform ψ_1.

Proposition 22 (Carleman BVP)**.** *The Laplace transform ψ_1 satisfies the following boundary value problem:*

(i) $\psi_1(x)$ *is meromorphic on* $\mathcal{G}$ *and continuous on* $\overline{\mathcal{G}}$.
(ii) $\psi_1(x)$ *admits one or two poles in* $\mathcal{G}$. 0 *is always a simple pole and* x_1 *is a simple pole if and only if* $2\delta - \theta + \beta > 2\pi$.
(iii) $\lim_{x\to\infty} x\psi_1(x) = 0$.
(iv) ψ_1 *satisfies the boundary condition*

$$\psi_1(\overline{x}) = G(x)\psi_1(x), \quad \forall\, x \in \mathcal{H},$$

where

$$G(x) := \frac{k_1}{k_2}(x, Y^+(x))\frac{k_2}{k_1}(\overline{x}, Y^+(x)). \tag{39}$$

Proof. Statement (i) immediately follows from Lemma 14. Statement (ii) immediately follows from Lemmas 15 and 16. Statement (iii) follows from the initial value theorem for the Laplace transform, which implies that $\lim_{x\to\infty} x\psi_1(x) = \mathbb{P}_{(0,0)}[T = \infty] = 0$. To prove statement (iv), we recall the functional equation (27). For $x \in \mathcal{H}$, we evaluate this equation for $(x, Y^+(x))$ and $(\overline{x}, Y^+(\overline{x}))$. By the definition of Y^+, we have $K(x, Y^+(x)) = K(\overline{x}, Y^+(\overline{x})) = 0$. By the definition of the hyperbola $\mathcal{H}$ in (34), we have $Y^+(\overline{x}) = Y^+(x)$. This enables us to obtain the following system of equations

$$\begin{cases} 0 = k_1(x, Y^+(x))\psi_1(x) + k_2(x, Y^+(x))\psi_2(Y^+(x)), \\ 0 = k_1(\overline{x}, Y^+(x))\psi_1(\overline{x}) + k_2(\overline{x}, Y^+(x))\psi_2(Y^+(x)). \end{cases}$$

Solving this system of equations and eliminating $\psi_2(Y^+(x))$, we obtain the boundary condition in statement (iv). □

6.2. **Gluing function.** To solve the BVP, we need a conformal gluing function which glues together the upper and lower parts of the hyperbola. This conformal gluing function was introduced in [20, 21]. For $a \geqslant 0$ and for $x \in \mathbb{C} \setminus (-\infty, -1]$, the generalized Chebyshev polynomial is defined by

$$T_a(x) := \cos(a \arccos(x)) = \frac{1}{2}\left((x + \sqrt{x^2-1})^a + (x - \sqrt{x^2-1})^a\right).$$

We define the angle

$$\beta := \arccos(-\rho).$$

We also define the functions

$$w(x) := T_{\frac{\pi}{\beta}}\left(\frac{2x - (x^+ + x^-)}{x^+ - x^-}\right), \tag{40}$$

and

$$W(x) := \frac{w(x) - w(X^{\pm}(y^+))}{w(x) - w(0)}.$$

We now recall a useful lemma from [21] for the conformal gluing function W.

Lemma 23 (Lemma 9, [21])**.** *The function W satisfies the following properties*

(i) W *is holomorphic in* $\mathcal{G} \setminus \{0\}$, *continuous in* $\overline{\mathcal{G}} \setminus \{0\}$ *and bounded at infinity.*
(ii) W *is bijective from* $\mathcal{G} \setminus \{0\}$ *to* $\mathbb{C} \setminus [0, 1]$.
(iii) W *satisfies the gluing property on the hyperbola*

$$W(x) = W(\overline{x}), \quad \forall\, x \in \mathcal{H}.$$

6.3. **Index.** We begin with some necessary notation. Let the angle Δ be the variation of the argument of $G(x)$ when x lies on $\mathcal{H}^+$:

$$\Delta := [\arg G(x)]_{\mathcal{H}^+} = \left[\arg \frac{k_1}{k_2}(x, Y^+(x))\right]_{\mathcal{H}}.$$

Further, let d be the argument of G at the real point of the hyperbola $\mathcal{H}$:

$$d := \arg G(X^+(y^+)) \in (-\pi, \pi].$$

We define the index κ as

$$\kappa := \left\lfloor \frac{d+\Delta}{2\pi} \right\rfloor.$$

The index shall prove useful to solving the boundary value problem given in Proposition 22.

Lemma 24. *We have*

$$d = \begin{cases} 0 & \text{if } k_1(x^-, Y^\pm(x^-)) \neq 0 \text{ i.e. } 2\delta - \theta + \beta \neq 2\pi, \\ \pi & \text{if } k_1(x^-, Y^\pm(x^-)) = 0 \text{ i.e. } 2\delta - \theta + \beta = 2\pi, \end{cases}$$

and

$$\tan\frac{d+\Delta}{2} = \frac{(1-(r_1+2\rho)(r_2+2\rho))\sqrt{1-\rho^2}}{r_1+r_2+3\rho-r_1r_2\rho-2(r_1+r_2)\rho^2-4\rho^3} = \tan(\epsilon+\delta+\beta).$$

Note also that $\epsilon+\delta+\beta \geqslant 2\pi$ *is equivalent to* $1-(r_1+2\rho)(r_2+2\rho) \leqslant 0$.

Proof. The proof is in each step similar to the proof of [21, Lemma 13]. Firstly, note that the value of d is obtained by the fact that $G(X^+(y^+)) = 1$ if $k_1(x^-, Y^\pm(x^-)) \neq 0$ and that $G(X^+(y^+)) = -1$ if $k_1(x^-, Y^\pm(x^-)) = 0$. We recall that by definition we have $\Delta = \lim_{\substack{x\to\infty \\ x\in\mathcal{H}^+}} \arg G(x) - d$ and that by (39) we have

$$G(x) = \frac{(\frac{1}{2}(r_2+Y^+(x)/x)+\rho+\mu_2/x)(\frac{1}{2}(1+r_1Y^+(x)/\overline{x})+\rho Y^+(x)/\overline{x}+\mu_1/\overline{x})}{(\frac{1}{2}(r_2+Y^+(x)/\overline{x})+\rho+\mu_2/\overline{x})(\frac{1}{2}(1+r_1Y^+(x)/x)+\rho Y^+(x)/x+\mu_1/x)}.$$

By (28), we may compute the limit

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{H}^+}} \frac{Y^+(x)}{x} = -\rho + i\sqrt{1-\rho^2},$$

from which we obtain

$$\begin{aligned} e^{i(\Delta+d)} &= \lim_{\substack{x\to\infty \\ x\in\mathcal{H}^+}} G(x) \\ &= \frac{(r_2+\rho+i\sqrt{1-\rho^2})(1-r_1\rho-2\rho^2-i(r_1+2\rho)\sqrt{1-\rho^2})}{(r_2+\rho-i\sqrt{1-\rho^2})(1-r_1\rho-2\rho^2+i(r_1+2\rho)\sqrt{1-\rho^2})} \\ &= \frac{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)+i(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)-i(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}. \end{aligned}$$

We then see that

$$\tan\frac{d+\Delta}{2} = \frac{(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)} = \tan(\epsilon+\delta+\beta),$$

where the last equality follows from (3) and straightforward calculations. The proof concludes by recalling the two following facts:

(1) For $\alpha = \frac{\epsilon+\delta-\pi}{\beta} \geqslant 1$ and for ϵ, δ and $\beta \in (0,\pi)$, we have that $-\pi < 2\beta-\pi \leqslant \epsilon+\delta+\beta-2\pi < \pi$.

(2) By (3), $\sin(\epsilon+\delta+\beta)$ has the same sign as that $(r_1+2\rho)(r_2+2\rho)-1$, where $(r_1+2\rho)(r_2+2\rho)-1 = \sin(\epsilon+\delta+\beta)\frac{\sin(\beta)}{\sin(\epsilon)\sin(\delta)}$.

□

We now prepare to state Lemma 25 below. For $1-(r_1+2\rho)(r_2+2\rho)\neq 0$, let us define

$$\widetilde{y} := 2\frac{\mu_2-\mu_1(r_2+2\rho)}{(r_1+2\rho)(r_2+2\rho)-1} = 2\mu_1\frac{\sin(\beta+\delta-\theta)\sin(\epsilon)}{\sin(\beta-\theta)\sin(\epsilon+\delta+\beta)}, \tag{41}$$

where the last equality holds by (3).

Lemma 25. *If $\widetilde{y}-y^+\leqslant 0$ or if $1-(r_1+2\rho)(r_2+2\rho)=0$ then*

$$(G(x)=1 \text{ and } x\in\mathcal{H}) \Leftrightarrow x=X^{\pm}(y^+),$$

and thus $d+\Delta\in(-2\pi,2\pi)$. If $\widetilde{y}-y^+>0$ then

$$(G(x)=1 \text{ and } x\in\mathcal{H}) \Leftrightarrow (x=X^{\pm}(y^+) \text{ or } x=X^{\pm}(\widetilde{y})),$$

and thus $d+\Delta\in(-4\pi,4\pi)$.

Proof. Assume that $x\in\mathcal{H}$, where $x=a+ib$ for $a,b\in\mathbb{R}$ and $y=Y^{\pm}(x)$. Then by (39), $G(x)=1$ is equivalent to $\Im(k_1(a+ib,y)k_2(a-ib,y))=0$. Straightforward calculations yield

$$\Im(k_1(a+ib,y)k_2(a-ib,y)) = \frac{b}{4}\left[\frac{y}{2}((r_1+2\rho)(r_2+2\rho)-1)-2\mu_2+2\mu_1(r_2+2\rho)\right],$$

from which we may obtain that $G(x)=1$ is equivalent to $b=0$ or to $y=\widetilde{y}$. We conclude the proof by noting that

(1) $b=0$ and $x\in\mathcal{H}$ together imply that $x=X^{\pm}(y^+)$, the latter being the only real point of the hyperbola.
(2) By the definition of (33), $x\in\mathcal{H}$ and $y=\widetilde{y}$ imply that $\widetilde{y}\in[y^+,\infty)$.

□

We continue with Lemma 26 below.

Lemma 26. *Assume that $2\delta-\theta+\beta>2\pi$. Then $\widetilde{y}>y^+$ is equivalent to $\epsilon+\delta+\beta<2\pi$.*

Proof. We first note that $2\delta-\theta+\beta>2\pi$ implies that $\pi<\delta-\theta+\beta<2\pi$, and thus that $\sin(\delta-\theta+\beta)<0$. Recall that we have previously seen that the conditions in (1) are equivalent to $\alpha\geqslant 1$ and $\delta>\beta$, $\epsilon>\beta$, and thus that $\pi<\epsilon+\delta+\beta<3\pi$. We employ the following steps to conclude the proof:

(1) Assume that $\widetilde{y}>y^+$. Then for $y^+>0$, we have that $\widetilde{y}>0$. Then by (41) we have that $\sin(\epsilon+\delta+\beta)<0$ and thus $\epsilon+\delta+\beta<2\pi$.
(2) We now assume that $\epsilon+\delta+\beta<2\pi$. Hence $\sin(\epsilon+\delta+\beta)<0$. By hypothesis we have $\beta<\epsilon<2\pi-\beta-\delta$. Using (41), we may easily see that $\epsilon\mapsto\widetilde{y}$ is increasing for $\beta<\epsilon<2\pi-\beta-\delta$. Replacing ϵ by β in (41), we may deduce that

$$\widetilde{y}>y_\delta := 2\mu_1\frac{\sin(\beta+\delta-\theta)\sin(\beta)}{\sin(\beta-\theta)\sin(2\beta+\delta)}.$$

By hypothesis, we have that $\pi+\frac{\theta-\beta}{2}<\delta<2\pi-2\beta$. Note that $\delta\mapsto y_\delta$ is increasing in this interval. We then see that

$$\widetilde{y}>y_{\inf} := 2\mu_1\frac{\sin(\beta+\pi+\frac{\theta-\beta}{2}-\theta)\sin(\beta)}{\sin(\beta-\theta)\sin(2\beta+\pi+\frac{\theta-\beta}{2})}.$$

Employing (30) and performing straightforward calculations, we obtain

$$\widetilde{y}-y^+>y_{\inf}-y^+ = \mu_1\frac{-2\sin(\frac{\beta-\theta}{2})\sin^2(\frac{\beta+\theta}{2})}{\sin(\beta-\theta)\sin(\epsilon+\delta+\beta)\sin(\beta)}>0.$$

□

Before stating the main lemma of this section, we introduce the following indicator variable χ, which is associated with the results of Lemma 15 and Lemma 16.

$$\chi := \begin{cases} -1 & \text{if } 2\delta-\theta+\beta>2\pi \Leftrightarrow x_1 \text{ is a pole of } \psi_1 \text{ in } \mathcal{G}, \\ 0 & \text{if } 2\delta-\theta+\beta\leqslant 2\pi \Leftrightarrow \psi_1 \text{ has no pole but 0 in } \mathcal{G}. \end{cases} \tag{42}$$

Lemma 27 (Index)**.** *The index κ satisfies*

$$\kappa := \begin{cases} \chi & \text{if } \epsilon+\delta+\beta \geqslant 2\pi, \\ \chi - 1 & \text{if } \epsilon+\delta+\beta < 2\pi. \end{cases}$$

The value of the index appears below in Table 1.

	$\epsilon+\delta+\beta \geqslant 2\pi$	$\epsilon+\delta+\beta < 2\pi$
$2\delta-\theta+\beta > 2\pi$	$\kappa = -1$	$\kappa = -2$
$2\delta-\theta+\beta \leqslant 2\pi$	$\kappa = 0$	$\kappa = -1$

TABLE 1. Value of the index κ.

Remark 28 (Index and argument principle)**.** *Notice that the index can take the values 0, -1 and -2 while in [21, Lemma 14] the index takes only the values 0 and -1. The difference comes from the fact that ψ_1 can have two distinct poles while in [21] the Laplace transform has at most one simple pole. The index is deeply connected to number of zeros and poles of ψ_1. In the case of a closed curve, the argument principle implies that the index is equal to the number of zeros minus the number of poles counted with multiplicity of the function of the BVP. See [17, Lemma 6.9] which presents a case where the boundary of the BVP is a circle. In our case, the boundary is an (unbounded) hyperbola and ψ_1 is not meromorphic at infinity, therefore we cannot apply directly the argument principle and the index κ is not always equal to the opposite of the number of poles χ.*

Proof. The proof proceeds with two separate cases.

Case I: $\widetilde{y} - y^+ \leqslant 0$. In this case, by Lemma 25, we have that $d+\Delta \in (-2\pi, 2\pi)$ and that $G(x) \neq 1$ for all $x \in \mathcal{H}$ such that $x \neq X^{\pm}(y^+)$. Then $\kappa = 0$ or -1 depending on the sign of $d+\Delta$. This sign is given by the sign of $\arg G(x)$ when $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$. Note that $x = a+ib \in \mathcal{H}^+$ and $y = Y^+(x)$. We then compute

$$k_1(a+ib,y)k_2(\overline{a+ib},y) = k_1(a,y)k_2(a,y) + \frac{b^2}{4}(r_2+2\rho) - i\frac{b}{4}(1-(r_1+2\rho)(r_2+2\rho))(y-\widetilde{y}).$$

Figure 10 represents the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$. It is useful to remark that $\arg \frac{k_1}{k_2}(x, Y^+(x)) = \arg k_1(x, Y^+(x))/k_2(\overline{x}, Y^+(x))$. We may thus deduce that

$$\begin{aligned} \operatorname{sgn}\arg G(x) &= \operatorname{sgn}\arg(k_1(a+ib,y)k_2(a-ib,y)) \\ &= \operatorname{sgn}\frac{-b(1-(r_1+2\rho)(r_2+2\rho))(y-\widetilde{y})}{k_1(a,y)k_2(a,y) + \frac{b^2}{4}(r_2+2\rho)}. \end{aligned}$$

For $x \in \mathcal{H}^+$, we have $k_2(X^{\pm}(y^+), y^+) > 0$, $b > 0$. When $x \to X^{\pm}(y^+)$, we have that $b \to 0$ and $a \to X^{\pm}(y^+)$. Thus for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$,

$$\begin{aligned} \operatorname{sgn}\arg G(x) &= -\operatorname{sgn}(k_1(X^{\pm}(y^+), y^+)(1-(r_1+2\rho)(r_2+2\rho))(y-\widetilde{y})) \\ &= -\operatorname{sgn}(2\delta-\theta+\beta-2\pi)\operatorname{sgn}(\epsilon+\delta+\beta-2\pi), \end{aligned}$$

where the last equality comes from Lemmas 16 and 24, as well as from the fact that in this case $y - \widetilde{y} > 0$ for $y > y^+$. This allows us to conclude the following

- If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta > 2\pi$, then for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is negative. We may thus deduce that $\kappa = -1$, see Figure 10a.
- If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is positive. We may thus deduce that $\kappa = 0$, see Figure 10b.

- If $\epsilon+\delta+\beta<2\pi$ and $2\delta-\theta+\beta\leqslant 2\pi$, then for $x\in\mathcal{H}^+$ and $x\to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is positive. We may thus deduce that $\kappa=-1$, see Figure 10c.

We pause to note that by Lemma 26 it is not possible to have $\epsilon+\delta+\beta<2\pi$ and $2\delta-\theta+\beta>2\pi$. This is because we have assumed $\widetilde{y}\leqslant y^+$.

Case II: $\widetilde{y}-y^+>0$. In this case by, Lemma 25 we have that $d+\Delta\in(-4\pi,4\pi)$ and $(G(x)=1$ and $x\in\mathcal{H})\Leftrightarrow(x=X^{\pm}(y^+)$ or $x=X^{\pm}(\widetilde{y}))$. Then $\kappa\in\{-2,-1,0,1\}$. To obtain the value of the index we study the curve $\mathcal{C}:=\{\frac{k_1}{k_2}(x,Y^+(x)):x\in\mathcal{H}\}$. By straightforward calculations we see that $\widetilde{A}:=\frac{k_1}{k_2}(X^{\pm}(\widetilde{y}),\widetilde{y})$ is positive. The study of the sign of the real and the imaginary parts of $\frac{k_1}{k_2}(x,Y^+(x))$ for $x\in\mathcal{H}^+$ and $x\to X^{\pm}(y^+)$ gives the value of κ. Following the same logic as that of Case I above, we see that the real part of $\frac{k_1}{k_2}(x,Y^+(x))$ for $x\in\mathcal{H}^+$ and $x\to X^{\pm}(y^+)$ has the same sign as $-(2\delta-\theta+\beta-2\pi)$. Further, the imaginary part has the same sign that $-(\epsilon+\delta+\beta-2\pi)$. We may then conclude as follows:

- If $\epsilon+\delta+\beta<2\pi$ and $2\delta-\theta+\beta>2\pi$, $\kappa=-2$, see Figure 11a.
- If $\epsilon+\delta+\beta\geqslant 2\pi$ and $2\delta-\theta+\beta\leqslant 2\pi$, $\kappa=0$, see Figure 11b.
- If $\epsilon+\delta+\beta<2\pi$ and $2\delta-\theta+\beta\leqslant 2\pi$, $\kappa=-1$, see Figure 11c.

Note that by Lemma 26 it is not possible to have $\epsilon+\delta+\beta\geqslant 2\pi$ and $2\delta-\theta+\beta>2\pi$. This is because we have assumed that $\widetilde{y}>y^+$.

□

We now state a technical lemma which shall be invoked in Section 7.

Lemma 29. *The following equality holds*

$$\left(-\frac{d+\Delta}{2\pi}+\chi-1\right)\frac{\pi}{\beta}=-\alpha-1.$$

Proof. First, we recall by Lemma 24 that

$$\tan\frac{d+\Delta}{2}=\tan(\epsilon+\delta+\beta).$$

For $\alpha=\frac{\epsilon+\delta-\pi}{\beta}\geqslant 1$ and ϵ,δ and $\beta\in(0,\pi)$,

$$2\beta-\pi\leqslant\epsilon+\delta+\beta-2\pi<\pi.$$

Further, recall that by definition, $\kappa=\lfloor\frac{d+\Delta}{2\pi}\rfloor$.

We now consider two cases for the value of $\epsilon+\delta+\beta-2\pi$. The first case considers $\epsilon+\delta+\beta-2\pi\geqslant 0$. In this case,

$$\frac{d+\Delta}{2}=\begin{cases}\epsilon+\delta+\beta-2\pi & \text{if } \frac{d+\Delta}{2\pi}\geqslant 0 \text{ i.e. } \kappa=0,\\ \epsilon+\delta+\beta-3\pi & \text{if } \frac{d+\Delta}{2\pi}<0 \text{ i.e. } \kappa=-1.\end{cases}$$

By Lemma 27, we have $\kappa=\chi$. We may thus deduce that

$$\frac{d+\Delta}{2}=\epsilon+\delta+\beta+(\chi-2)\pi.$$

The second case considers $\epsilon+\delta+\beta-2\pi<0$. In this case,

$$\frac{d+\Delta}{2}=\begin{cases}\epsilon+\delta+\beta-2\pi & \text{if } -\pi\leqslant\frac{d+\Delta}{2\pi}<0 \text{ i.e. } \kappa=-1,\\ \epsilon+\delta+\beta-3\pi & \text{if } -2\pi\leqslant\frac{d+\Delta}{2\pi}<-\pi \text{ i.e. } \kappa=-2.\end{cases}$$

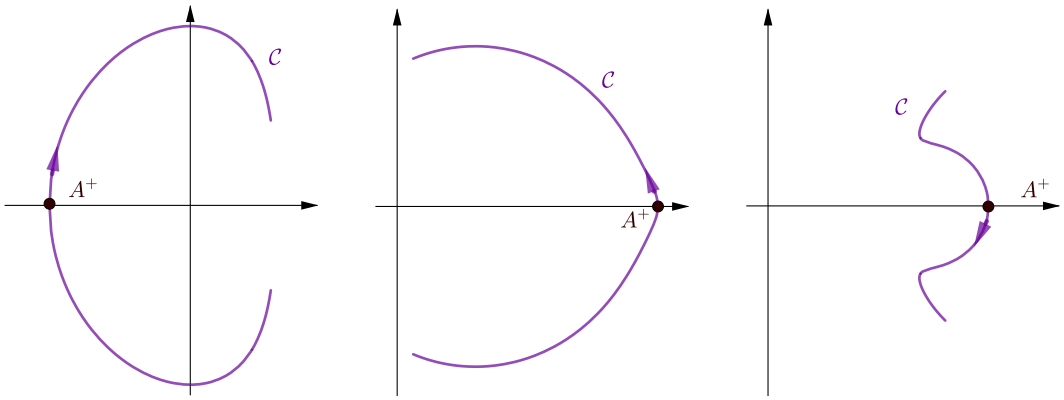
By Lemma 27, we have $\kappa=\chi-1$. We may thus deduce that

$$\frac{d+\Delta}{2}=\epsilon+\delta+\beta+(\chi-2)\pi.$$

Thus, in both cases we have

$$\left(-\frac{d+\Delta}{2\pi}+\chi-1\right)\frac{\pi}{\beta}=(-\epsilon-\delta-\beta-(\chi-2)\pi+\chi\pi-\pi)\frac{1}{\beta}=-\alpha-1.$$

This concludes the proof. □

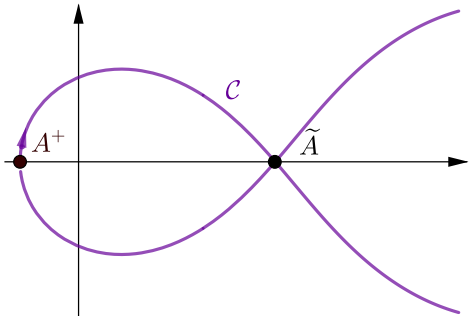


(A) If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta > 2\pi$, then $\kappa=-1$.

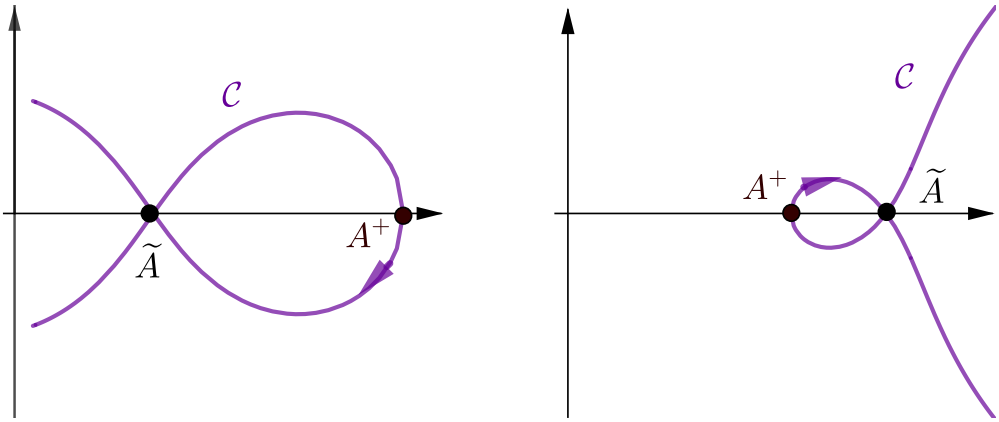
(B) If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa=0$.

(C) If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa=-1$.

FIGURE 10. When $\widetilde{y}-y^+ \leqslant 0$: a plot of the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$ and the point $A^+ := \frac{k_1}{k_2}(X^+(y^+), y^+)$.



(A) If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta > 2\pi$, then $\kappa=-2$.



(B) If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa=0$.

(C) If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa=-1$.

FIGURE 11. When $\widetilde{y}-y^+ > 0$: a plot of the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$, the point $A^+ := \frac{k_1}{k_2}(X^+(y^+), y^+)$, and the point $\widetilde{A} := \frac{k_1}{k_2}(X^+(\widetilde{y}), \widetilde{y})$.

6.4. **Solution of the BVP.** The following theorem gives an explicit integral formula for the Laplace transform of the escape probability ψ_1.

Theorem 30 (Explicit expression for ψ_1)**.** *The Laplace transform ψ_1 is given for $x \in \mathcal{G}$ by*

$$\psi_1(x) = \frac{w'(0)}{w(x)-w(0)} \left(\frac{w(0)-w(x_1)}{w(x)-w(x_1)}\right)^{-\chi} \exp\left(\frac{1}{2i\pi}\int_{\mathcal{H}^+} \log G(t) \left[\frac{w'(t)}{w(t)-w(x)} - \frac{w'(t)}{w(t)-w(0)}\right] \mathrm{d}t\right), \tag{43}$$

where x_1 is defined in (32), G is defined (39), w is defined (40), χ is defined in (42) and $\mathcal{H}$ is defined in (34).

Remark 31. *Let us now give some remarks about Theorem 30.*

- *The poles 0 and x_1 found in Lemma 15 can be easily visualized in the formula of Theorem 30. The indicator variable χ defined in (42) indicates clearly on the formula if the pole x_1 is in $\mathcal{G}$ or not.*
- *A symmetrical result holds for ψ_2. Using the functional equation (27) we obtain an explicit formula for ψ. By inverting this Laplace transform we obtain the escape probability which is the main motivation of our work. But such an inversion is not easy neither very explicit except in some special cases.*
- *But it is still possible to deduce some concrete results from the integral formula obtained in Theorem 30. In Section 7 we derive thanks to this explicit expression a very explicit and simple expression for the asymptotics of the escape probability at the origin.*
- *It can also be used to show some differential properties of the Laplace transform. More precisely, similarly to [6, Thm 2.3, §9.1] we can show that ψ_1 is differentially algebraic if $\beta \in \pi\mathbb{Q}$. Such results on the algebraic nature of a generating function are very classical in analytic combinatorics to obtain concrete results. When ψ_1 is differentially algebraic, it satisfies a differential equation from which it is possible to deduce a polynomial recurrence relation for the moments of the escape/absorption probability. See [6, §6.3] which gives an explicit example for the SRBM stationary distribution in the recurrent case.*
- *The methods and techniques employed to prove this theorem are inspired by the one used to study random walks in the quarter plane [15].*

Proof. Let

$$\widetilde{\psi}_1(y) := \frac{(y-W(x_1))^{-\chi}}{(y-1)^{1+\kappa-\chi}} \psi_1 \circ W^{-1}(y).$$

Proposition 22, Lemma 23 and Lemma 27 together imply that

- $\widetilde{\psi}_1$ is analytic on $\mathbb{C}\setminus[0,1]$.
- $\widetilde{\psi}_1(y) \sim_\infty cy^{-\kappa}$ for some constant c.
- $\widetilde{\psi}_1(1) = 0$.
- For $y \in [0,1]$, $\widetilde{\psi}_1$ satisfies the boundary condition

  $$\widetilde{\psi}_1^+(y) = \widetilde{G}(y)\widetilde{\psi}_1^-(y),$$

  where $\widetilde{\psi}_1^+(y)$ is the left limit and $\widetilde{\psi}_1^-(y)$ is the right limit of $\widetilde{\psi}_1$ on $[0,1]$, $(W^{-1})^-$ is the right limit of W^{-1} on $[0,1]$, and $\widetilde{G}(y) = G\circ(W^{-1})^-(y)$.

We now define

$$\widetilde{S}(y) := (y-1)^{-\kappa} \exp\left(\frac{1}{2i\pi}\int_0^1 \frac{\log \widetilde{G}(u)}{u-y}\right).$$

Following the classical boundary theory results in [15, (5.2.24) and Theorem 5.2.3], the above function is analytic and does not cancel on $\mathbb{C}\setminus[0,1]$ and is such that $\widetilde{S}(y) \sim_\infty c' y^{-\kappa}$ for some constant c'. Furthermore, for $y \in [0,1]$, it satisfies the boundary condition

$$\widetilde{S}^+(y) = \widetilde{G}(y)\widetilde{S}^-(y),$$

where $\widetilde{S}^+(y)$ is the left limit and $\widetilde{S}^-(y)$ is the right limit of $\widetilde{S}$ on $[0,1]$. By the properties of $\widetilde{\psi}_1$ and $\widetilde{S}$ detailed above, the function $\widetilde{\psi}_1/\widetilde{S}$ is analytic on $\mathbb{C}$ and bounded at infinity. Therefore there must exist a constant C such that

$$\widetilde{\psi}_1(y) = C\widetilde{S}(y).$$

Invoking the definition of $\widetilde{\psi}_1$, we have that

$$\frac{(W(x)-W(x_1))^{-\chi}}{(W(x)-1)^{1+\kappa-\chi}}\psi_1(x) = C(W(x)-1)^{-\kappa}\exp\left(\frac{1}{2i\pi}\int_0^1 \frac{\log\widetilde{G}(u)}{u-W(x)}\right). \tag{44}$$

Noting that

$$W(x)-1 = \frac{w(0)-w(X^{\pm}(y^+))}{w(x)-w(0)} \quad \text{and} \quad W(x)-W(x_1) = \frac{w(x)-w(x_1)}{w(x)-w(0)}\frac{w(X^{\pm}(y^+))-w(0)}{w(x_1)-w(0)},$$

and making a change of variable $u = w(t)$ in the integral in (44), we obtain for some constant C'

$$\psi_1(x) = C'\left(\frac{1}{w(x)-w(0)}\right)\left(\frac{1}{w(x)-w(x_1)}\right)^{-\chi}\exp\left(\frac{1}{2i\pi}\int_{\mathcal{H}^+}\log G(t)\frac{w'(t)}{w(t)-w(x)}\mathrm{d}t\right).$$

The final value theorem for the Laplace transform gives

$$\lim_{x\to 0} x\psi_1(x) = \lim_{u\to\infty}\mathbb{P}_{(u,0)}[T=\infty] = 1.$$

This enables us to compute the constant

$$C' = w'(0)\left(w(0)-w(x_1)\right)^{-\chi}\exp\left(\frac{-1}{2i\pi}\int_{\mathcal{H}^+}\log G(t)\frac{w'(t)}{w(t)-w(0)}\mathrm{d}t\right),$$

which gives us (43), completing the proof. □

7. Asymptotics of the escape probability at the origin

In this section we use the explicit expression in Theorem 30 to obtain the asymptotics of the escape probability at the origin. We begin with computing the asymptotics of ψ_1 at infinity.

Lemma 32 (Asymptotics of ψ_1)**.** *Let α be defined as in (4). For ease of notation, allow C to be a constant which may change from one line to the next. For some positive constant C,*

$$\psi_1(x) \underset{x\to\infty}{\sim} Cx^{-\alpha-1}.$$

A symmetrical result holds for ψ_2. That is, for some positive constant C,

$$\psi_2(y) \underset{y\to\infty}{\sim} Cy^{-\alpha-1}.$$

Proof. This proof follows the same steps as those of [21, Prop 19]. The key is to invoke [15, (5.2.20)], which states that

$$\exp\left(\frac{1}{2i\pi}\int_0^1\frac{\log\widetilde{G}(u)}{u-y}\right) \underset{y\to 1}{\sim} C(y-1)^{\frac{d+\Delta}{2\pi}}.$$

Recall that $w(x) \underset{x\to\infty}{\sim} Cx^{\frac{\pi}{\beta}}$ and that $W(x)-1 \underset{x\to\infty}{\sim} Cx^{-\frac{\pi}{\beta}}$. The explicit expressions of ψ_1 obtained in (43) and in (44) imply that

$$\psi_1(x) \underset{x\to\infty}{\sim} Cx^{(-\frac{d+\Delta}{2\pi}+\chi-1)\frac{\pi}{\beta}}.$$

The proof concludes by invoking Lemma 29, which states that $\left(-\frac{d+\Delta}{2\pi}+\chi-1\right)\frac{\pi}{\beta} = -\alpha-1$. □

Lemma 33 (Asymptotics of ψ)**.** *Let α defined as in (4). For $t\in[0,\frac{\pi}{2}]$ and some positive constant C_t,*

$$\psi(r\cos t, r\sin t) \underset{r\to\infty}{\sim} C_t r^{-\alpha-2}.$$

Proof. The result is immediate from the functional equation (27) and Lemma 32. □

Proposition 34 (Asymptotics at the origin)**.** *For positive constants c_0 and c_1 we have the following asymptotics*

$$\mathbb{P}_{(u,0)}[T=\infty] \underset{u\to 0}{\sim} c_0 u^\alpha \quad \text{and} \quad \mathbb{P}_{(0,v)}[T=\infty] \underset{v\to 0}{\sim} c_1 v^\alpha.$$

Proof. The result follows by combining the asymptotic results of ψ_1 and ψ_2 at infinity that we computed in Lemma 32 with the reciprocal of the result in [11, Thm 33.3][2]. We begin by denoting $g(u) := \mathbb{P}_{(u,0)}[T=\infty]$. Then, by definition, $\psi_1(x) = \int_0^\infty e^{-xu} g(u)\mathrm{d}u$. As $\psi_1(x)$ has no singularities greater than 0, for every $A > 0$, the inverse Laplace transform gives

$$g(u) = \frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty} e^{ux}\psi_1(x)\mathrm{d}x.$$

By Lemma 32, we have $\psi_1(x) = \frac{C+\eta(x)}{x^{\alpha+1}}$, where η is a function such that $\lim_\infty \eta = 0$. Recalling that the Laplace transform of $u^\alpha/\Gamma(\alpha+1)$ is $x^{-\alpha-1}$ and performing a change of variables $s = ux$, we obtain

$$\begin{aligned} g(u) &= \frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty} e^{ux}\frac{C+\eta(x)}{x^{\alpha+1}}\mathrm{d}x \\ &= u^\alpha\left(\frac{C}{\Gamma(\alpha+1)} + \frac{1}{2i\pi}\int_{Au-i\infty}^{Au+i\infty} e^s \frac{\eta(s/u)}{s^{\alpha+1}}\mathrm{d}s\right). \end{aligned}$$

It remains to show that the last integral tends to 0 when $u \to 0$. To do so, consider $\epsilon > 0$ arbitrarily small. Then there exists $B > 0$ sufficiently large such that $\eta(x) < \epsilon$ for all $|x| > B$. For all u such that $u < 1/B$, let us consider $A := 1/u$. This gives

$$\left|\frac{1}{2i\pi}\int_{Au-i\infty}^{Au+i\infty} e^s \frac{\eta(s/u)}{s^{\alpha+1}}\mathrm{d}s\right| < \frac{\epsilon}{2i\pi}\int_{1-i\infty}^{1+i\infty}\frac{1}{s^{\alpha+1}}\mathrm{d}s,$$

where the last integral converges for $\alpha \geqslant 1$. The proof concludes by letting ϵ tend towards 0. □

Theorem 35 (Asymptotics at the origin)**.** *For $t \in (0, \frac{\pi}{2})$ and some positive constant c_t we have*

$$\mathbb{P}_{(r\cos t, r\sin t)}[T=\infty] \underset{r\to 0}{\sim} c_t r^\alpha.$$

Proof. This proof follows directly from the asymptotics of the double Laplace transform ψ computed in Lemma 33. Recall the result used in the proof of Proposition 34 linking the asymptotics of a function at 0 to the asymptotics of its Laplace transform at infinity. The only necessary modification is to apply this result with a polar coordinate transformation. The desired asymptotics then follows with nearly identical calculations. □

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[2]Doetsch [11, Thm 33.3] establishes that if for some constant a a function is equivalent to u^a at 0, then at infinity, its Laplace transform is equivalent (up to a multiplicative constant) to x^{-a-1}.

References

[1] Albrecher, H., Azcue, P., and Muler, N. (2017). Optimal dividend strategies for two collaborating insurance companies. *Advances in Applied Probability*, 49(2):515–548.

[2] Andres, S. (2009). Pathwise differentiability for SDEs in a convex polyhedron with oblique reflection. *Annales de l'IHP Probabilités et Statistiques*, 45(1):104–116.

[3] Baccelli, F. and Fayolle, G. (1987). Analysis of models reducible to a class of diffusion processes in the positive quarter plane. *SIAM Journal on Applied Mathematics*, 47(6):1367–1385.

[4] Badila, E. S., Boxma, O. J., Resing, J. A. C., and Winands, E. M. M. (2014). Queues and risk models with simultaneous arrivals. *Advances in Applied Probability*, 46(3):812–831.

[5] Borodin, A. N. and Salminen, P. (2012). *Handbook of Brownian Motion-Facts and Formulae*. Birkhäuser.

[6] Bousquet-Mélou, M., Price, A. E., Franceschi, S., Hardouin, C., and Raschel, K. (2021). The stationary distribution of reflected brownian motion in a wedge: differential properties. arXiv:2101.01562.

[7] Boxma, O. and Ivanovs, J. (2013). Two coupled lévy queues with independent input. *Stoch. Syst.*, 3(2):574–590.

[8] Cohen, J. W. and Boxma, O. J. (1983). *Boundary Value Problems in Queueing System Analysis*, volume 79. North-Holland Publishing Co., Amsterdam.

[9] Dai, J. G. and Miyazawa, M. (2011). Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution. *Stochastic Systems*, 1(1):146–208.

[10] Dieker, A. B. and Moriarty, J. (2009). Reflected Brownian motion in a wedge: sum-of-exponential stationary densities. *Electronic Communications in Probability*, 14:1–16.

[11] Doetsch, G. (1974). *Introduction to the Theory and Application of the Laplace Transformation*. Springer Berlin Heidelberg.

[12] Dupuis, P. and Ramanan, K. (2002). A time-reversed representation for the tail probabilities of stationary reflected brownian motion. *Stochastic Processes and their Applications*, 98(2):253 – 287.

[13] Ernst, P. A. and Franceschi, S. (2020). Asymptotic behavior of the occupancy density for obliquely reflected Brownian motion in a half-plane and Martin boundary. *arXiv preprint arXiv:2004.06968*.

[14] Fayolle, G. and Iasnogorodski, R. (1979). Two coupled processors: The reduction to a Riemann–Hilbert problem. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 47(3):325–351.

[15] Fayolle, G., Iasnogorodski, R., and Malyshev, V. (2017). *Random Walks in the Quarter-plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics*. Springer Publishing Company, Incorporated, 2nd edition.

[16] Foddy, M. E. (1984). *Analysis of Brownian motion with drift, confined to a quadrant by oblique reflection (diffusions, Riemann-Hilbert problem)*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Stanford University.

[17] Fomichov, V., Franceschi, S., and Ivanovs, J. (2020). Probability of total domination for transient reflecting processes in a quadrant. *arXiv preprint arXiv:2006.11826*.

[18] Franceschi, S. (2020). Green's functions with oblique neumann boundary conditions in the quadrant. *Journal of Theoretical Probability*.

[19] Franceschi, S. and Kourkova, I. (2017). Asymptotic expansion of stationary distribution for reflected brownian motion in the quarter plane via analytic approach. *Stochastic Systems*, 7(1):32–94.

[20] Franceschi, S. and Raschel, K. (2017). Tutte's invariant approach for Brownian motion reflected in the quadrant. *ESAIM: Probability and Statistics*, 21:220–234.

[21] Franceschi, S. and Raschel, K. (2019). Integral expression for the stationary distribution of reflected Brownian motion in a wedge. *Bernoulli*, 25(4B):3673–3713.

[22] Harrison, J., Landau, H., and Shepp, L. A. (1985). The stationary distribution of reflected Brownian motion in a planar region. *The Annals of Probability*, 13(3):744–757.

[23] Harrison, J. M. (1978). The diffusion approximation for tandem queues in heavy traffic. *Advances in Applied Probability*, 10(4):886–905.
[24] Harrison, J. M. (2020). Reflected Brownian motion in the quarter plane: An equivalence based on time reversal. https://gsb-faculty.stanford.edu/michael-harrison/files/2021/01/singlespaced-harrison_reflected.pdf.
[25] Harrison, J. M. and Reiman, M. I. (1981a). On the distribution of multidimensional reflected Brownian motion. *SIAM Journal on Applied Mathematics*, 41(2):345–361.
[26] Harrison, J. M. and Reiman, M. I. (1981b). Reflected Brownian motion on an orthant. *The Annals of Probability*, 9(2):302–308.
[27] Harrison, J. M. and Shepp, L. A. (1984). A tandem storage system and its diffusion limit. *Stochastic Processes and their Applications*, 16(3):257–274.
[28] Harrison, J. M. and Williams, R. J. (1987). Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.*, 15(1):115–137.
[29] Hobson, D. G. and Rogers, L. C. G. (1993). Recurrence and transience of reflecting Brownian motion in the quadrant. *Mathematical Proceedings of the Cambridge Philosophical Society*, 113(2):387–399.
[30] Ivanovs, J. and Boxma, O. (2015). A bivariate risk model with mutual deficit coverage. *Insurance: Mathematics and Economics*, 64:126 – 134.
[31] Kang, W. and Ramanan, K. (2014). Characterization of stationary distributions of reflected diffusions. *Ann. Appl. Probab.*, 24(4):1329–1374.
[32] Lakner, P., Reed, J., and Zwart, B. (2019). On the roughness of the paths of RBM in a wedge. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(3):1566 – 1598.
[33] Lieshout, P. and Mandjes, M. (2007). Tandem Brownian queues. *Mathematical Methods of Operations Research*, 66(2):275–298.
[34] Lieshout, P. and Mandjes, M. (2008). Asymptotic analysis of Lévy-driven tandem queues. *Queueing Systems*, 60(3-4):203.
[35] Lipshutz, D. and Ramanan, K. (2019). Pathwise differentiability of reflected diffusions in convex polyhedral domains. *Ann. Inst. H. Poincaré Probab. Statist.*, 55(3):1439–1476.
[36] Malyshev, V. A. (1972). An analytic method in the theory of two-dimensional positive random walks. *Sibirski Matematiceski Zurnal*, 13:1314–1329.
[37] Miyazawa, M. and Rolski, T. (2009). Tail asymptotics for a Lévy-driven tandem queue with an intermediate input. *Queueing Systems*, 63(1-4):323.
[38] Pilipenko, A. (2014). *An introduction to stochastic differential equations with reflection*, volume 1. Universitätsverlag Potsdam.
[39] Reiman, M. I. (1984). Open queueing networks in heavy traffic. *Mathematics of Operations Research*, 9(3):441–458.
[40] Reiman, M. I. and Williams, R. J. (1988). A boundary property of semimartingale reflecting Brownian motions. *Probability Theory and Related Fields*, 77(1):87–97.
[41] Taylor, L. M. and Williams, R. J. (1993). Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probability Theory and Related Fields*, 96(3):283–317.
[42] Varadhan, S. R. and Williams, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Communications on Pure and Applied Mathematics*, 38(4):405–443.
[43] Whitt, W. (2002). *Stochastic-Process Limits*. Springer New York.
[44] Williams, R. (1985a). Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *The Annals of Probability*, 13(3):758–778.
[45] Williams, R. (1985b). Reflected Brownian motion in a wedge: semimartingale property. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 69(2):161–176.
[46] Williams, R. J. (1995). Semimartingale reflecting Brownian motions in the orthant. In *Stochastic Networks*, volume 71, pages 125–137. Springer, New York.

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