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# ESCAPE AND ABSORPTION PROBABILITIES FOR OBLIQUELY REFLECTED BROWNIAN MOTION IN A QUADRANT

PHILIP A. ERNST AND SANDRO FRANCESCHI

ABSTRACT. We consider an obliquely reflected Brownian motion  $Z$  with positive drift in a quadrant stopped at time  $T$ , where  $T := \inf\{t > 0 : Z(t) = (0, 0)\}$  is the first hitting time of the origin. Such a process can be defined even in the non-standard case where the reflection matrix is not completely- $\mathcal{S}$ . We show that in this case the process has two possible behaviors: either it tends to infinity or it hits the corner (origin) in a finite time. Given an arbitrary starting point  $(u, v)$  in the quadrant, we consider the escape (resp. absorption) probabilities  $\mathbb{P}_{(u,v)}[T = \infty]$  (resp.  $\mathbb{P}_{(u,v)}[T < \infty]$ ). We establish the partial differential equations and the oblique Neumann boundary conditions which characterize the escape probability and provide a functional equation satisfied by the Laplace transform of the escape probability. We then give asymptotics for the absorption probability in the simpler case where the starting point in the quadrant is  $(u, 0)$ . We exhibit a remarkable geometric condition on the parameters which characterizes the case where the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the skew symmetry condition for the stationary distribution. We then obtain an explicit integral expression for the Laplace transform of the escape probability. We conclude by presenting exact asymptotics for the escape probability at the origin.

## 1. INTRODUCTION

1.1. **Model and goal.** Let  $Z(t) = (Z_1(t), Z_2(t))$  be a reflected Brownian motion (RBM) in the quadrant, starting from the point  $(u, v)$ , with positive drift  $\mu = (\mu_1, \mu_2)$ ; that is,  $\mu_1 > 0, \mu_2 > 0$ . The covariance matrix is  $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$  and the reflection matrix is  $\begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix}$ . We further assume that

$$r_1 > 0, r_2 > 0 \text{ and } 1 \leq r_1 r_2. \quad (1)$$

See Figure 1 for a representation of the parameters. We define this reflected process up to the first hitting time  $T$  of the corner, defined as

$$T := \inf\{t > 0 : Z(t) = 0\}.$$

For  $t \leq T$ , this process may be written as

$$\begin{cases} Z_1(t) := u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t), \\ Z_2(t) := v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t), \end{cases} \quad (2)$$

where  $l_1(t)$  (resp.  $l_2(t)$ ) is a continuous non-decreasing process which increases only when  $Z_1(t) = 0$  (resp.  $Z_2(t) = 0$ ). Under condition (1), when the process hits the corner, it is immediately absorbed. Further details on the existence and uniqueness of this process will be given in Section 1.2.

The objective of the present paper is to study the escape probability to infinity for a process starting from  $(u, v)$ . We denote this probability as

$$\mathbb{P}_{(u,v)}[T = \infty].$$

The corresponding absorption probability at the origin is  $\mathbb{P}_{(u,v)}[T < \infty] = 1 - \mathbb{P}_{(u,v)}[T = \infty]$ .

Since its introduction in the eighties by Harrison, Reiman, Varadhan and Williams [17, 18, 30, 29, 31] reflected Brownian motion in the quarter plane has received significant attention from

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*Key words and phrases.* Escape probability; Absorption probability; Obliquely reflected Brownian motion in a quadrant; Functional equation; Carleman boundary value problem; Laplace transform; Neumann's condition; Asymptotics.

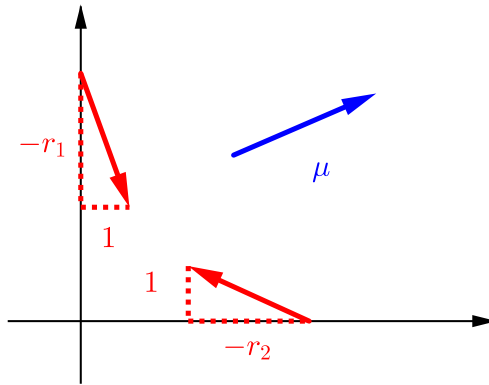


FIGURE 1. Reflection vectors and drift.

probabilists. Recurrence and transience of obliquely reflecting Brownian motion were examined in [21, 30], and the process has also been considered in planar domains [14, 19] as well as in general dimensions in orthants [18, 28, 32]. The stationary distribution of obliquely reflecting Brownian motion has been studied in [4, 5, 13] and its Green's functions have been studied in [11]. Obliquely reflecting Brownian motion has played an important role in applications concerning heavy traffic approximations for open queueing networks ([15, 26]). It has also been utilized in queueing models as diffusion approximations for tandem queues ([22, 23, 25]).

Previous works ([2, 7, 10, 13]) have adapted an analytic method initially developed for random walks by Fayolle and Iasnogorodski [8] and Malyshev [24] for studying obliquely reflected Brownian motion. This method is based on the boundary value problem theory documented by the books of Fayolle et al. [9] and Cohen and Boxma [3]. The present article is in part inspired by this analytic approach.

**1.2. Definition of the process given in (2).** Brownian motion in a quadrant with oblique reflection is usually defined as a process which behaves as a standard Brownian motion in the quadrant. The process reflects instantaneously on the edges with constant direction and the amount of time spent at the origin has Lebesgue measure zero (Varadhan and Williams [29]). Such a process is defined as a solution of a submartingale problem [29]. An interesting case arises when the process is a semimartingale reflecting Brownian motion (SRBM). Reiman and Williams [27] showed that a necessary condition for the process to be a SRBM is for the reflection matrix to be completely- $\mathcal{S}^1$ . Taylor and Williams [28] showed that this condition was also sufficient for the existence of an SRBM, which is unique in law.

Due to condition (1), the reflection matrix of the process in (2) is not completely- $\mathcal{S}$ . The process indeed is not a standard SRBM as it may be trapped at the origin. Nonetheless, it is possible to define this absorbed process up until the stopping time  $T$ . The existence and uniqueness as a solution of a submartingale problem is given for the absorbed process is given in [29, §2.1, Thm 2.1]. Further, in Taylor and Williams [28, §4.2 and §4.3], the existence and uniqueness of an SRBM absorbed at the origin are proven without assuming that the reflection matrix is completely- $\mathcal{S}$ .

**1.3. From the quadrant to the wedge.** Franceschi and Raschel [13, Appendix] recently showed that studying reflected Brownian motion in a quadrant is equivalent to studying reflected Brownian motion in a wedge with angle  $\beta$ , with identity covariance matrix, with two reflection angles  $\delta$  and  $\epsilon$ , and with drift angle  $\theta$  (see Figure 2). The angles  $\delta, \epsilon, \beta$  and  $\theta$  (when the drift is nonzero) are in  $(0, \pi)$  and are defined by

$$\tan \delta = \frac{\sin \beta}{-r_2 + \cos \beta}, \quad \tan \epsilon = \frac{\sin \beta}{-r_1 + \cos \beta}, \quad \tan \theta = \frac{\sin \beta}{\mu_1/\mu_2 + \cos \beta}, \quad \cos \beta = -\rho. \quad (3)$$

<sup>1</sup> A square matrix  $R$  is said to be completely- $\mathcal{S}$  if for each principal sub-matrix  $\tilde{R}$  there exists  $\tilde{x} \geq 0$  such that  $\tilde{R}\tilde{x} > 0$ .

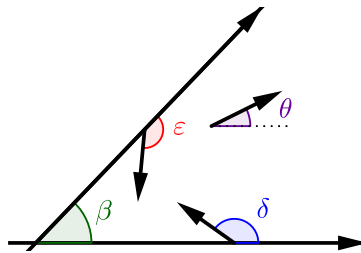


FIGURE 2. Reflected Brownian motion with zero drift in a wedge of angle  $\beta$ , reflection angles  $\delta$  and  $\epsilon$ , and drift angle  $\theta$ .

Condition (1) is equivalent to  $\delta + \epsilon - \beta \geq \pi$ . Finally, we denote  $\alpha$ , now a standard quantity in the SRBM literature, to be

$$\alpha := \frac{\delta + \epsilon - \pi}{\beta}. \quad (4)$$

**1.4. The case of zero drift.** The case of zero drift  $\mu = 0$  was treated by Varadhan and Williams [29]. In this case the absorption probability does not depend on the starting point. We have from [29, Thm. 2.2]

$$\mathbb{P}[T < \infty] = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leq 0. \end{cases}$$

If  $\alpha \leq 0$ , the corner is not reached. If  $0 < \alpha < 2$ , the corner is reached but the amount of time spent by the process in the corner is Lebesgue measure zero. If  $\alpha \geq 2$ , the process reaches the corner and remains there. Condition (1) is equivalent to  $\alpha > 1$ . Under condition (1), the case of positive drift poses a quandary, as  $0 < \mathbb{P}_{(u,v)}[T < \infty] < 1$ .

**1.5. Escape probability and stationary distribution of the dual process.** Harrison [15] and Foddy [10] showed that the hitting time on one of the axes is intrinsically connected to the stationary distribution of a certain dual process. As the present article was nearing completion, it came to our attention that Harrison [16] has extended the results in his earlier work ([15]) by introducing a dual RBM in the quadrant with drift  $-\mu$  and reflection matrix

$$\begin{pmatrix} r_2 & -1 \\ -1 & r_1 \end{pmatrix}$$

when  $1 < r_1 r_2$ . This is depicted in Figure 3 below. This dual process has an explicit connection with the study of the escape probability. In particular, Harrison [16, Corollary 2] states that

$$\mathbb{P}_{(u,v)}[T = \infty] = \pi(\mathcal{S}(u, v))$$

where  $\pi$  is the stationary distribution of the dual process and  $\mathcal{S}(u, v) := \{(u - r_2 z_1 + z_2, v + z_1 - r_1 z_2) \in \mathbb{R}_+^2 : (z_1, z_2) \in \mathbb{R}_+^2\}$  is a trapezoid as pictured in Figure 3.

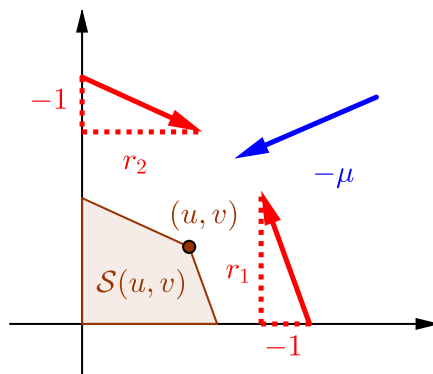


FIGURE 3. Dual process parameters and trapezoid  $\mathcal{S}(u, v)$  in brown.

The remainder of this paper is organized as follow. In Section 2 we explore some general properties of the process of interest given in (2). This section's key result is Theorem 9, which states that the process has only two possible behaviors: either  $T < \infty$ , which means that the process is absorbed at the origin in finite time, or  $T = \infty$ , in which case the process escapes to infinity, namely  $Z(t) \rightarrow \infty$  when  $t \rightarrow \infty$ . In Section 3 we present Proposition 10, which gives a partial differential equation characterizing the escape probability. Later in this section, we give Proposition 11, which provides a functional equation satisfied by the Laplace transform of the escape probability. In Section 4, we study the kernel of this functional equation and obtain asymptotics results for the absorption probability in the simpler case where the starting point is  $(u, 0)$  (Proposition 16). In Section 5 we find a geometric condition which characterize the cases where the absorption probability has a product form and is exponential (Theorem 18). Such a result recall the famous skew symmetry condition studied a lot for invariant measures. In Section 6 we establish a boundary value problem (BVP) satisfied by the Laplace transform (Proposition 19) and conclude with Theorem 24, which gives an explicit integral formula for the Laplace transform of the escape probability. In Section 7 we obtain exact asymptotics for the escape probability at the origin.

**In memory of Larry Shepp** We dedicate this article in memory of our colleague, mentor, and friend, Professor Larry Shepp. Professor Shepp indelibly contributed to many areas of applied probability, and one of the areas that interested him most concerned RBM in a quadrant as well in a strip ([14, 19]).

## 2. GENERAL PROPERTIES OF THE PROCESS IN (2)

In this section we investigate a few key properties of the process given in (2). We prove three key results. The first is that if the starting point tends to infinity, then the probability that the process does not hit the origin tends towards 1 (Theorem 3). The second is that when the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards 1 (Theorem 5). The third key result is that the process has only two possible behaviors: either  $T < \infty$ , which means that the process is absorbed at the origin in finite time, or  $T = \infty$ , in which case the process escapes to infinity, namely  $Z(t) \rightarrow \infty$  when  $t \rightarrow \infty$  (Theorem 9).

**2.1. Theorems 3 and 5.** Our first key results of the section (Theorems 3 and 5) concern the probability of the process hitting the origin. Before stating these results, we begin by introducing Lemma 1 and Proposition 2.

For ease of notation, let us define  $\tau_1^\xi := \inf\{t : Z_1(t \wedge T) \leq \xi\}$  and  $\tau_2^\xi := \inf\{t : Z_2(t \wedge T) \leq \xi\}$ . Further, let  $X_1(t) := u + W_1(t) + \mu_1 t$  and let  $X_2(t) := v + W_2(t) + \mu_2 t$ .

**Lemma 1.** *For  $u > \xi > 0$ , we have*

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] = \mathbb{P}_{(u,v)} \left[ X_1(t \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \right], \quad (5)$$

where  $x^+$  equals  $x$  if  $x > 0$  and is 0 otherwise. Hence,

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] \geq \mathbb{P}_{(u,v)} \left[ X_1(t) - r_2 \sup_{0 \leq s \leq t} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \right]. \quad (6)$$

A symmetrical result holds for  $v > \xi > 0$  and  $\mathbb{P}_{(u,v)}[\tau_2^\xi = \infty]$ .

*Proof.* On the event  $\{\tau_1^\xi = \infty\}$ , for every  $t \geq 0$ , we have  $l_1(t) = 0, \mathbb{P}_{(u,v)}$ -a.s.. Then

$$\begin{aligned} Z_1(t \wedge T) &= X_1(t \wedge T) - r_2 l_2(t \wedge T), \\ Z_2(t \wedge T) &= X_2(t \wedge T) + l_2(t \wedge T). \end{aligned}$$

Note that  $l_2(t \wedge T)$  increases only when  $Z_2(t \wedge T) = 0$ . By uniqueness of the Skorokhod map,

$$l_2(t \wedge T) = \sup_{0 \leq s \leq t} (-X_2(s \wedge T))^+ = \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+.$$

Thus

$$Z_1(t \wedge T) = X_1(t \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+.$$

We may then write

$$\begin{aligned} & \{\tau_1^\xi = \infty\} = \{Z_1(t \wedge T) > \xi \text{ for every } t \geq 0\} \\ &= \{Z_1(t \wedge T) > \xi \text{ for every } t \geq 0 \text{ and } l_1(T) = 0\} \\ &= \left\{ X_1(t \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \text{ and } l_1(T) = 0 \right\}, \end{aligned} \quad (7)$$

$\mathbb{P}_{(u,v)}$ -a.s. We now wish to show that

$$\mathbb{P}_{(u,v)} \left[ X_1(t \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \text{ and } l_1(T) > 0 \right] = 0. \quad (8)$$

Note that there is a set  $N$  such that  $\mathbb{P}_{(u,v)}(N) = 1$  and for every  $\omega \in N$ , we have

$$Z_1(t \wedge T) = X_1(t \wedge T) + l_1(t \wedge T) - r_2 l_2(t \wedge T) \geq 0, \quad (9)$$

$$Z_2(t \wedge T) = X_2(t \wedge T) - r_1 l_1(t \wedge T) + l_2(t \wedge T) \geq 0, \quad (10)$$

$$l_1(t \wedge T) \text{ increases only when } Z_1(t \wedge T) = 0, \quad (11)$$

$$l_2(t \wedge T) \text{ increases only when } Z_2(t \wedge T) = 0. \quad (12)$$

Let us  $\omega \in N$ . We claim that the following statements

- a)  $X_1(t \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+ > \xi$  for every  $t \geq 0$ ;
- b)  $l_1(T) > 0$ ,

cannot hold simultaneously. The proof is by contradiction. For sake of contradiction, assume that statements a) and b) hold simultaneously. By (10), (12), and the uniqueness of Skorokhod map, we have

$$\begin{aligned} l_2(t \wedge T) &= \sup_{0 \leq s \leq t} (r_1 l_1(s \wedge T) - X_2(s \wedge T))^+ \\ &\leq \sup_{0 \leq s \leq t} (r_1 l_1(t \wedge T))^+ + \sup_{0 \leq s \leq t} (-X_2(s \wedge T))^+ \\ &= r_1 l_1(t \wedge T) + \sup_{0 \leq s \leq t \wedge T} (-X_2(s))^+. \end{aligned}$$

Let  $\eta := \inf\{t : l_1(t \wedge T) \geq \xi/(2r_1r_2)\}$ . Then for every  $t \geq 0$ ,

$$\begin{aligned} & Z_1(t \wedge \eta \wedge T) = X_1(t \wedge \eta \wedge T) - r_2 l_2(t \wedge \eta \wedge T) \\ & \geq X_1(t \wedge \eta \wedge T) - r_2 \sup_{0 \leq s \leq t \wedge \eta \wedge T} (-X_2(s))^+ - r_1 r_2 l_1(t \wedge \eta \wedge T) \\ & > \xi - \xi/2 = \xi/2, \end{aligned}$$

where in the last inequality we have invoked statement a). Since  $l_1(t \wedge T)$  increases only when  $Z_1(t \wedge T) = 0$ , we have

$$l_1(t \wedge \eta \wedge T) = 0 \text{ for every } t \geq 0,$$

which contradicts statement b) and the definition of  $\eta$ . By contradiction, (8) holds. Combining (7) and (8), (5) follows. Note that (6) follows directly from (5).  $\square$

We now turn to Proposition 2 below.

**Proposition 2.** *Let  $B(t)$  be a one dimensional Brownian motion started from the origin under  $\mathbb{P}$ . For  $\mu > 0$  and  $x > 0$ , we have*

$$\mathbb{P}(B(t) + \mu t > -x \text{ for every } t \geq 0) = 1 - e^{-2x\mu}.$$

*Proof.* Let  $H_{-x} := \inf\{t \geq 0 : B(t) + \mu t = -x\}$ , then  $H_{-x}$  is the hitting time of Brownian motion with drift. By the definition of  $H_{-x}$ , we have

$$\mathbb{P}(B(t) + \mu t > -x \text{ for every } t \geq 0) = \mathbb{P}(H_{-x} = \infty) = 1 - e^{-2x\mu}.$$

□

With Lemma 1 and Proposition 2 in hand, we state Theorem 3 below.

**Theorem 3.** *When the starting point tends to infinity, the probability that process does not hit the origin tends to one. Namely,*

$$\lim_{(u,v) \rightarrow \infty} \mathbb{P}_{(u,v)}[T = \infty] = 1.$$

Equivalently,

$$\lim_{(u,v) \rightarrow \infty} \mathbb{P}_{(u,v)}[T < \infty] = 0.$$

*Proof.* Fix  $\xi > 0$ . For  $\|(u, v)\|$  is sufficiently large, we have  $u > 2\xi$  or  $v > 2\xi$ . If  $u > 2\xi$ , by Lemma 1, we have

$$\begin{aligned} & \mathbb{P}_{(u,v)}[T = \infty \text{ for some } n \in \mathbb{N}_+] \geq \mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] \\ & \geq \mathbb{P}_{(u,v)} \left[ X_1(t) - r_2 \sup_{0 \leq s \leq t} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \right] \\ & \geq \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \geq 0 \text{ and } X_2(t) > -u/(2r_2) \text{ for every } t \geq 0] \\ & \geq \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \geq 0] \\ & \quad + \mathbb{P}_{(u,v)}[X_2(t) > -u/(2r_2) \text{ for every } t \geq 0] - 1 \\ & = \mathbb{P}_{(u,v)}[W_1(t) + \mu_1 t > -(u - 2\xi)/2 \text{ for every } t \geq 0] \\ & \quad + \mathbb{P}_{(u,v)}[W_2(t) + \mu_2 t > -u/(2r_2) - v \text{ for every } t \geq 0] - 1 \\ & = 1 - e^{-(u-2\xi)\mu_1} + 1 - e^{-(u/r_2+2v)\mu_2} - 1 \\ & = 1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2}, \end{aligned}$$

where the second to last equality invokes Proposition 2. Similarly, if  $v > 2\xi$ , we have

$$\mathbb{P}_{(u,v)}[T = \infty \text{ for some } n \in \mathbb{N}_+] \geq 1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1}.$$

Hence,

$$\begin{aligned} & \mathbb{P}_{(u,v)}[T = \infty \text{ for some } n \in \mathbb{N}_+] \\ & \geq \max\{1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2}, 1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1}\}. \end{aligned}$$

Letting  $(u, v)$  tend to  $\infty$ , the desired result follows. □

We now turn to Proposition 4 below, which shall be needed to prove Theorem 5.

**Proposition 4.** *We have the following subset relationship*

$$\{u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \subset \{T < \infty\}.$$

*Proof.* We prove this claim by contradiction. For the sake of contradiction, let us fix  $\omega \in \{u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \cap \{T = \infty\}$ . Assuming  $T = \infty$ , the process can be written as

$$\begin{cases} Z_1(t) = u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t), \\ Z_2(t) = v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t). \end{cases}$$

Solving the linear system for  $l_1$  and  $l_2$ , we obtain

$$\begin{cases} (r_1 r_2 - 1)l_1(t) = (u + W_1(t) + \mu_1 t - Z_1(t)) + r_2(v + W_2(t) + \mu_2 t - Z_2(t)), \\ (r_1 r_2 - 1)l_2(t) = r_1(u + W_1(t) + \mu_1 t - Z_1(t)) + (v + W_2(t) + \mu_2 t - Z_2(t)). \end{cases}$$

For all  $t \in \mathbb{R}_+$  such that

$$u + W_1(t) + \mu_1 t < 0,$$

and

$$v + W_2(t) + \mu_2 t < 0,$$

we have  $(r_1 r_2 - 1)l_1(t) < 0$  and  $(r_1 r_2 - 1)l_2(t) < 0$ , which is not possible since  $l_1(t)$  and  $l_2(t) \geq 0$  and as we assumed  $(r_1 r_2 - 1) \geq 0$ . A contradiction has been reached.  $\square$

Theorem 5 below considers the behavior of the process when the starting point tends to the origin.

**Theorem 5.** *When the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards one. That is,*

$$\lim_{(u,v) \rightarrow (0,0)} \mathbb{P}_{(u,v)} [T < \infty] = 1,$$

or equivalently,

$$\lim_{(u,v) \rightarrow (0,0)} \mathbb{P}_{(u,v)} [T = \infty] = 0.$$

*Proof.* By Proposition 4, we have that

$$\mathbb{P}_{(u,v)} [T < \infty] \geq \mathbb{P}[\exists t \in \mathbb{R}_+ : u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0].$$

By the properties of planar Brownian motion, we have

$$\mathbb{P}[\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0] = 1.$$

Let  $(u_n, v_n) \in \mathbb{R}_+^2$  be a sequence of points such that  $(u_n, v_n) \rightarrow 0$ . Note that

$$\begin{aligned} & \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0\} \\ & \supset \{\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0\}. \end{aligned}$$

Applying Fatou's Lemma yields

$$\liminf_{n \rightarrow \infty} \mathbb{P}[\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0] \geq 1.$$

We may therefore conclude that

$$\mathbb{P}_{(u_n, v_n)} [T < \infty] \xrightarrow{n \rightarrow \infty} 1,$$

and the desired result follows.  $\square$

**2.2. Proving Theorem 9.** We now turn to Theorem 9, which that the process has only two possible behaviors: either  $T < \infty$ , which means that the process is absorbed at the origin in finite time, or  $T = \infty$ , in which case the process escapes to infinity, namely  $Z(t) \rightarrow \infty$  when  $t \rightarrow \infty$ . The result first requires the proofs of Proposition 6, Lemma 7, and Lemma 8, which we give below.

**Proposition 6.** *Suppose  $B(t)$  is a one dimensional Brownian motion starting from the origin under the measure  $\mathbb{P}$ . Let  $a, b$  be two positive numbers. Then*

$$\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0) > 0.$$

*Proof.* Let  $\lambda = \ln 2 / (2b) + 1$ . Note that  $1 - 2e^{-2\lambda b} > 0$ . Then

$$\begin{aligned} & \mathbb{P}(-\lambda - bt < B(t) < \lambda + bt \text{ for every } t \geq 0) \\ & \geq \mathbb{P}(B(t) > -\lambda - bt \text{ for every } t \geq 0) + \mathbb{P}(B(t) < \lambda + bt \text{ for every } t \geq 0) - 1 \\ & = 2(1 - e^{-2\lambda b}) - 1 = 1 - 2e^{-2\lambda b} > 0. \end{aligned}$$

Let  $H_a := \inf\{t : |B(t)| = a\}$ . By standard exit time properties of Brownian motion,  $\mathbb{P}(H_a > \lambda/b + 1) > 0$ . Then

$$\begin{aligned} & \mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0) \\ & = \mathbb{P}(H_a > \lambda/b + 1) \mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0 \mid H_a > \lambda/b + 1). \end{aligned}$$



By the strong Markov property of Brownian motion,

$$\begin{aligned}
& \mathbb{P}(-a - bt < B(t) < a + bt, \forall t \mid H_a > \lambda/b + 1) \\
&= \mathbb{P}(-a - b(t + H_a) < B(t + H_a) < a + b(t + H_a), \forall t \mid H_a > \lambda/b + 1) \\
&= \mathbb{P}(-a - b(t + H_a) - B(H_a) < B(t + H_a) - B(H_a) < a + b(t + H_a) - B(H_a), \forall t \mid H_a > \lambda/b + 1) \\
&\geq \mathbb{P}(-\lambda - bt < B(t + H_a) - B(H_a) < \lambda + bt, \forall t \mid H_a > \lambda/b + 1) \\
&= \mathbb{P}(-\lambda - bt < B(t) < \lambda + bt, \forall t) \\
&> 0,
\end{aligned}$$

from which the desired result follows.  $\square$

We now turn to Lemma 7.

**Lemma 7.** *For  $\alpha$  a positive number,*

$$\inf_{u \geq \alpha} P_{(u,0)}[\tau_1^0 = \infty] > 0, \quad (13)$$

$$\inf_{v \geq \alpha} P_{(0,v)}[\tau_2^0 = \infty] > 0. \quad (14)$$

*Proof.* We need only prove (13), since the proof of (14) is completely symmetric. Let us consider  $\xi < \alpha$ . By Lemma 1,

$$\begin{aligned}
& \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] \geq \mathbb{P}_{(u,0)}[\tau_1^\xi = \infty] \\
&\geq \mathbb{P}_{(u,0)} \left[ X_1(t) - r_2 \sup_{0 \leq s \leq t} (-X_2(s))^+ > \xi \text{ for every } t \geq 0 \right] \\
&= \mathbb{P}_{(u,0)} \left[ u + W_1(t) + \mu_1 t - r_2 \sup_{0 \leq s \leq t} (-W_2(s) - \mu_2 t)^+ > \xi \text{ for every } t \geq 0 \right] \\
&\geq \mathbb{P}_{(u,0)} [W_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0 \\
&\quad \text{and } W_2(t) + \mu_2 t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0]. \quad (15)
\end{aligned}$$

Let  $B_1(t)$  and  $B_2(t)$  be two independent Brownian motions starting from 0 under  $\mathbb{P}_{(u,0)}$ . Then, under  $\mathbb{P}_{(u,0)}$ , the process  $(W_1(t), W_2(t))$  has the same law as  $(B_1(t), \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t))$ . We now show that (13) holds in three separate cases:  $\rho = 0$ ,  $0 < \rho < 1$  and  $-1 < \rho < 0$ .

Case I:  $\rho = 0$ . If  $\rho = 0$ , then  $W_1(t)$  and  $W_2(t)$  are two independent Brownian motions. Then

$$\begin{aligned}
(15) &= \mathbb{P}_{(u,0)} [W_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0] \\
&\quad \times \mathbb{P}_{(u,0)} [W_2(t) + \mu_2 t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0] \\
&= \left(1 - e^{-(u-\xi)\mu_1}\right) \cdot \left(1 - e^{-(u-\xi)\mu_2/r_2}\right),
\end{aligned}$$

where the last equality invokes Proposition 2. Taking infimums yields

$$\inf_{u \geq \alpha} P_{(u,0)}[\tau_1^0 = \infty] \geq \left(1 - e^{-(\alpha-\xi)\mu_1}\right) \cdot \left(1 - e^{-(\alpha-\xi)\mu_2/r_2}\right) > 0.$$

Case II:  $0 < \rho < 1$ . If  $0 < \rho < 1$ , then

$$\begin{aligned}
(15) &= \mathbb{P}_{(u,0)} [B_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0 \\
&\quad \text{and } \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t) + \mu_2 t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0] \\
&\geq \mathbb{P}_{(u,0)} [B_1(t) + (\mu_1 \wedge \mu_2)t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0 \\
&\quad \text{and } \sqrt{1 - \rho^2} B_2(t) + (1 - \rho)\mu_2 t > -(1 - \rho)(u - \xi)/(2r_2) \text{ for every } t \geq 0].
\end{aligned}$$

Using the same argument in the case for  $\rho = 0$ , (13) follows.

Case III:  $-1 < \rho < 0$ . If  $-1 < \rho < 0$ , then for  $u \geq \alpha$

$$(15) = \mathbb{P}_{(u,0)} [B_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0]$$

$$\begin{aligned}
& \text{and } \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t) + \mu_2 t > -(u - \xi)/(2r_2) \text{ for every } t \geq 0 \\
\geq & \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u - \xi)/2 \text{ for every } t \geq 0, \\
& \rho B_1(t) - \rho(\mu_1 \wedge \mu_2)t > -|\rho|(u - \xi)/(2r_2) \text{ for every } t \geq 0 \\
& \text{and } \sqrt{1 - \rho^2} B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2))t > -(1 - |\rho|)(u - \xi)/(2r_2) \text{ for every } t \geq 0] \\
\geq & \mathbb{P}_{(u,0)}[-(u - \xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u - \xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t \\
& \text{and } \sqrt{1 - \rho^2} B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2))t > -(1 - |\rho|)(u - \xi)/(2r_2), \forall t] \\
= & \mathbb{P}_{(u,0)}[-(u - \xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u - \xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\
& \times \mathbb{P}_{(u,0)}[\sqrt{1 - \rho^2} B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2))t > -(1 - |\rho|)(u - \xi)/(2r_2), \forall t] \\
\geq & \mathbb{P}_{(u,0)}[-(\alpha - \xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (\alpha - \xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\
& \times \mathbb{P}_{(u,0)}[\sqrt{1 - \rho^2} B_2(t) + (\mu_2 + \rho(\mu_1 \wedge \mu_2))t > -(1 - |\rho|)(\alpha - \xi)/(2r_2), \forall t].
\end{aligned}$$

Taking infimums and invoking Proposition 6, (13) follows. This concludes the proof.  $\square$

Let us denote  $T_r := \inf\{t \geq 0 : \|Z(t \wedge T)\| \leq r\}$ .

**Lemma 8.** *On the event  $\{T_{1/n} = \infty\}$ , we have  $\mathbb{P}_{(u,v)}$ -a.s.*

$$\lim_{t \rightarrow \infty} Z(t) = \infty.$$

That is,

$$\mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0. \quad (16)$$

*Proof.* We will first show (16) holds when  $v = 0$ . Then (16) will follow immediately in the case that  $u = 0$ . We conclude by showing that (16) holds when  $u \neq 0$  and  $v \neq 0$ .

Case I:  $v = 0$ . When  $v = 0$ , let

$$K := \sup_{u \geq 0} \mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right].$$

For  $u \leq 1/n$ ,

$$\mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0.$$

Then

$$K = \sup_{u \geq 1/n} \mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right]. \quad (17)$$

We now define a stopping time

$$\eta_1^0 := \begin{cases} \inf\{t \geq \tau_1^0 : Z_2(t) = 0\}, & \tau_1^0 < \infty, \\ \infty, & \tau_1^0 = \infty. \end{cases}$$

By Lemma 7,

$$\inf_{u \geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 = \infty] \geq \inf_{u \geq 1/n} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] > 0,$$

and hence,

$$\sup_{u \geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty] < 1. \quad (18)$$

Note that

$$\begin{aligned}
& \mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\
= & \mathbb{P}_{(u,0)} \left[ \tau_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\
& + \mathbb{P}_{(u,0)} \left[ \tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\
& + \mathbb{P}_{(u,0)} \left[ \eta_1^0 < \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right].
\end{aligned} \quad (19)$$

On the event  $\{\tau_1^0 = \infty\}$ , for all  $t \geq 0$ ,  $T = \infty$  and  $l_1(t) = 0$ . Then

$$Z_2(t) = X_2(t) + l_2(t) \geq X_2(t) = W_2(t) + \mu_2 t \rightarrow \infty,$$

$\mathbb{P}_{(u,0)}$ -a.s., by the law of the iterated logarithm for Brownian motion. Hence, the first term on the right-hand side of (19) is 0. We now consider the second term on the right-hand side of (19). On the event  $\{\tau_1^0 < \infty\}$ , let us define  $\tilde{\eta}_1^0 := \inf\{t \geq 0 : Z_2(t + \tau_1^0) = 0\}$  and  $\tilde{T}_{1/n} := \inf\{t \geq 0 : \|Z(t + \tau_1^0)\| \leq 1/n\}$ . By the strong Markov property, we have

$$\begin{aligned} & \mathbb{P}_{(u,0)} \left[ \tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\ &= \mathbb{P}_{(u,0)} \left[ \tau_1^0 < \infty, \inf_{0 \leq s \leq \tau_1^0} \|Z(s)\| > \frac{1}{n}, \tilde{\eta}_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t + \tau_1^0) < \infty, \tilde{T}_{\frac{1}{n}} = \infty \right] \\ &= \mathbb{E}_{(u,0)} \left[ \mathbb{1}_{\{\tau_1^0 < \infty, \inf_{0 \leq s \leq \tau_1^0} \|Z(s)\| > 1/n\}} \mathbb{P}_{Z(\tau_1^0)} \left[ \eta_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \right] \\ &= 0. \end{aligned}$$

By the same argument used to show that the first term on the right-hand side of (19) is 0, for  $v > 0$ ,

$$\mathbb{P}_{(0,v)} \left[ \eta_1^0 = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0.$$

This proves that the second term on the right-hand side of (19) is also 0. We now consider the third term on the right-hand side of (19). On the event  $\{\eta_1^0 < \infty\}$ , let  $\hat{T}_{1/n} := \inf\{t \geq 0 : Z(t + \eta_1^0) \leq 1/n\}$ . By the strong Markov property,

$$\begin{aligned} & \mathbb{P}_{(u,0)} \left[ \eta_1^0 < \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\ &= \mathbb{P}_{(u,0)} \left[ \eta_1^0 < \infty, \inf_{0 \leq s \leq \eta_1^0} \|Z(s)\| > \frac{1}{n}, \liminf_{t \rightarrow \infty} Z(t + \eta_1^0) < \infty, \hat{T}_{\frac{1}{n}} = \infty \right] \\ &= \mathbb{E}_{(u,0)} \left[ \mathbb{1}_{\{\eta_1^0 < \infty, \inf_{0 \leq s \leq \eta_1^0} \|Z(s)\| > 1/n\}} \mathbb{P}_{Z(\eta_1^0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \right] \\ &\leq K \cdot \mathbb{E}_{(u,0)} \left[ \mathbb{1}_{\{\eta_1^0 < \infty, \inf_{0 \leq s \leq \eta_1^0} \|Z(s)\| > 1/n\}} \right] \\ &\leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty]. \end{aligned}$$

Combining (19) and the above estimates yields

$$\mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].$$

Taking supremums and invoking (17), we obtain

$$K = \sup_{u \geq 1/n} \mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \leq K \cdot \sup_{u \geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].$$

Together with (18), we have  $K = 0$ . Hence, for every  $u \geq 0$ ,

$$\mathbb{P}_{(u,0)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0. \quad (20)$$

Similarly, for every  $v \geq 0$ ,

$$\mathbb{P}_{(0,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0. \quad (21)$$

Case II:  $u \neq 0$  and  $v \neq 0$ . For the case when  $u \neq 0$  and  $v \neq 0$ , let  $\tau := \inf\{t \geq 0 : Z_1(t) = 0 \text{ or } Z_2(t) = 0\}$ . Then

$$\begin{aligned} & \mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\ &= \mathbb{P}_{(u,v)} \left[ \tau = \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \end{aligned}$$

$$+ \mathbb{P}_{(u,v)} \left[ \tau < \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right]. \quad (22)$$

On the event  $\{\tau = \infty\}$ ,  $T = \infty$  and, for every  $t \geq 0$ ,  $l_1(t) = l_2(t) = 0$ . Then, as  $t \rightarrow \infty$ ,

$$Z_1(t) = u + W_1(t) + \mu_1 t \rightarrow \infty,$$

$\mathbb{P}_{(u,v)}$ -a.s. Hence the first term on the right-hand side of (22) is 0. We now consider the second term on the right-hand side of (22). By the strong Markov property,

$$\begin{aligned} & \mathbb{P}_{(u,v)} \left[ \tau < \infty, \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \\ & \leq \mathbb{E}_{(u,v)} \left[ \mathbb{1}_{\{\tau < \infty\}} \mathbb{P}_{Z(\tau)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] \right] \\ & = 0, \end{aligned}$$

where (20) and (21) have been invoked in the last equality. Hence the second term on the right-hand side of (22) is also 0. Thus for  $u \neq 0$  and  $v \neq 0$ ,

$$\mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty \right] = 0.$$

The proof is now complete.  $\square$

With the above results in hand, we are now ready to state Theorem 9.

**Theorem 9.** *On the event  $\{T = \infty\}$ ,  $\mathbb{P}_{(u,v)}$ -a.s. the process  $Z(t)$  tends to infinity when  $t \rightarrow \infty$ , namely*

$$\mathbb{P}_{(u,v)} \left[ \lim_{t \rightarrow \infty} Z(t) = \infty \mid T = \infty \right] = 1.$$

Equivalently,

$$\mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T = \infty \right] = 0.$$

*Proof.* We deduce from Lemma 8 that for every  $n \in \mathbb{N}_+$

$$\begin{aligned} & \mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T = \infty \right] \\ & = \mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} < \infty, T = \infty \right]. \end{aligned}$$

Applying the strong Markov property yields

$$\begin{aligned} & \mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T_{\frac{1}{n}} < \infty, T = \infty \right] \\ & = \mathbb{E}_{(u,v)} \left[ \mathbb{1}_{\{T_{1/n} < \infty\}} \mathbb{P}_{Z(T_{1/n})} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T = \infty \right] \right] \\ & \leq \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)} \left[ \liminf_{t \rightarrow \infty} Z(t) < \infty, T = \infty \right] \\ & \leq \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)} [T = \infty]. \end{aligned}$$

Applying Theorem 5 and letting  $n \rightarrow \infty$ , the desired result follows.  $\square$

### 3. PARTIAL DIFFERENTIAL EQUATION AND FUNCTIONAL EQUATION

We now turn to the study of the escape probability  $\mathbb{P}_{(u,v)}[T = \infty]$ . We begin with Proposition 10, which provides partial differential equations characterizing the escape probability. We proceed with Proposition 11, which gives a functional equation satisfied by the Laplace transform of the escape probability.

Let us define the infinitesimal generator of the process inside the quarter plane as

$$\mathcal{G}f(u, v) := \lim_{t \rightarrow 0} \frac{1}{t} (\mathbb{E}_{(u,v)}[f(Z(t \wedge T))] - f(u, v)),$$

where  $f$  must be a bounded function in the quadrant to ensure that the above limit exists and is uniform. For  $f$  twice differentiable, the infinitesimal generator inside the quadrant is

$$\mathcal{G}f = \frac{1}{2} \left( \frac{\partial^2 f}{\partial^2 z_1} + \frac{\partial^2 f}{\partial^2 z_2} + 2\rho \frac{\partial^2 f}{\partial z_1 \partial z_2} \right) + \mu_1 \frac{\partial f}{\partial z_1} + \mu_2 \frac{\partial f}{\partial z_2}.$$

This leads us to Proposition 10.

**Proposition 10** (Partial differential equation). *The absorption probability*

$$f(u, v) = \mathbb{P}_{(u,v)}[T < \infty],$$

is the only function which is both (i) bounded and continuous in the quarter plane and on its boundary and (ii) continuously differentiable in the quarter plane and on its boundary (except perhaps at the corner), and which satisfies the partial differential equation

$$\mathcal{G}f(u, v) = 0, \quad \forall (u, v) \in \mathbb{R}_+^2,$$

with oblique Neumann boundary conditions

$$\begin{cases} \partial_{r_1} f(0, v) := \frac{\partial f}{\partial u}(0, v) - r_1 \frac{\partial f}{\partial v}(0, v) = 0 & \forall v > 0, \\ \partial_{r_2} f(u, 0) := -r_2 \frac{\partial f}{\partial u}(u, 0) + \frac{\partial f}{\partial v}(u, 0) = 0 & \forall u > 0, \end{cases} \quad (23)$$

and with limit values

$$\begin{cases} f(0, 0) = 1, \\ \lim_{(u,v) \rightarrow \infty} f(u, v) = 0. \end{cases}$$

The same result holds for the escape probability

$$g(u, v) = 1 - f(u, v) = \mathbb{P}_{(u,v)}[T = \infty]$$

but with the following limit values

$$\begin{cases} g(0, 0) = 0, \\ \lim_{|z| \rightarrow \infty} f(z) = 1. \end{cases}$$

*Proof.* This proof is inspired by Foddy [10, p. 86-89]. We assume that  $f$  satisfies the hypotheses of the Proposition. Applying Dynkin's formula, we obtain

$$\begin{aligned} \mathbb{E}_{(u,v)}[f(Z(t \wedge T))] &= f(u, v) + \mathbb{E}_{(u,v)} \int_0^{t \wedge T} \mathcal{G}f(Z(s)) ds + \sum_{i=1}^2 \mathbb{E}_{(u,v)} \int_0^{t \wedge T} \partial_{r_i} f(Z(s)) dl_i(s) \\ &= f(u, v). \end{aligned}$$

But,

$$\begin{aligned} \mathbb{E}_{(u,v)}[f(Z(t \wedge T))] &= \mathbb{E}_{(u,v)}[f(Z(t \wedge T)) \mathbf{1}_{T \leq t}] + \mathbb{E}_{(u,v)}[f(Z(t \wedge T)) \mathbf{1}_{T > t}] \\ &= f(0, 0) \mathbb{P}_{(u,v)}[T \leq t] + \mathbb{E}_{(u,v)}[f(Z(t)) \mathbf{1}_{T > t}] \\ &\xrightarrow{t \rightarrow \infty} \mathbb{P}_{(u,v)}[T < \infty] + \lim_{t \rightarrow \infty} \mathbb{E}_{(u,v)}[f(Z(t)) \mathbf{1}_{T > t}] \\ &= \mathbb{P}_{(u,v)}[T < \infty]. \end{aligned}$$

Note that  $\lim_{|z| \rightarrow \infty} f(z) = 0$  and that for  $T > t$ ,  $Z(t) \xrightarrow{t \rightarrow \infty} \infty$  a.s. By dominated convergence and by Theorem 9,

$$\lim_{t \rightarrow \infty} \mathbb{E}_{(u,v)}[f(Z(t)) \mathbf{1}_{T > t}] = \mathbb{E}_{(u,v)}[\lim_{t \rightarrow \infty} f(Z(t)) \mathbf{1}_{T = \infty}] = 0.$$

We may thus conclude that

$$f(u, v) = \mathbb{P}_{(u,v)}[T < \infty].$$

Conversely, denote  $f(u, v) := \mathbb{P}_{(u,v)}[T < \infty]$ . The function  $f$  is bounded. By the Markov property, we have

$$\mathbb{E}_{(u,v)}[f(Z(t \wedge T))] = f(u, v).$$

Since

$$\mathcal{G}f(u, v) = \lim_{t \rightarrow 0} \frac{1}{t} (\mathbb{E}_{(u,v)}[f(Z(t \wedge T))] - f(u, v)) = 0,$$

we may conclude that  $\mathcal{G}f = 0$  on the quarter plane. The continuity and differentiability properties of  $f$  are immediate from [1, Thm 2.2 and Cor 2.4]. The Neumann boundary condition is satisfied by applying [1, Cor 3.3]. The desired limit values at 0 and at infinity are obtained by invoking Theorem 3 and Theorem 5. The result for  $g = 1 - f$  is straightforward, and this completes the proof.  $\square$

In preparation for Proposition 11, let us define the Laplace transform of the escape probability starting from  $(u, v)$  as

$$\psi(x, y) := \int_0^\infty \int_0^\infty e^{-xu-yv} \mathbb{P}_{(u,v)}[T = \infty] \, dudv.$$

Further, let

$$\psi_1(x) := \int_0^\infty e^{-xu} \mathbb{P}_{(u,0)}[T = \infty] \, du \quad \text{and} \quad \psi_2(y) := \int_0^\infty e^{-yv} \mathbb{P}_{(0,v)}[T = \infty] \, dv. \quad (24)$$

We also define the kernel

$$K(x, y) := \frac{1}{2}(x^2 + y^2 + 2\rho xy) + \mu_1 x + \mu_2 y. \quad (25)$$

and let

$$k_1(x, y) := \frac{1}{2}(r_2 x + y) + \rho x + \mu_2, \quad k_2(x, y) := \frac{1}{2}(x + r_1 y) + \rho y + \mu_1. \quad (26)$$

We now give a functional equation satisfied by the Laplace transform of the escape probability.

**Proposition 11** (Functional equation). *For  $(x, y) \in \mathbb{C}^2$  such that  $\Re x > 0$  and  $\Re y > 0$  we have*

$$K(x, y)\psi(x, y) = k_1(x, y)\psi_1(x) + k_2(x, y)\psi_2(y). \quad (27)$$

*Proof.* Recall the partial differential equation in Proposition 10 with the oblique Neumann boundary condition and limit values satisfied by  $g(u, v) := \mathbb{P}_{(u,v)}[T = \infty]$ . Employing integration by parts yields

$$\begin{aligned} 0 &= \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} \mathcal{G}g(z_1, z_2) \, dz_1 dz_2 \\ 0 &= \int_0^\infty \frac{1}{2} e^{-yz_2} \left( -\frac{\partial g}{\partial z_1}(0, z_2) + x \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1, z_2) \, dz_1 \right) \, dz_2 \\ &\quad + \int_0^\infty \frac{1}{2} e^{-xz_1} \left( -\frac{\partial g}{\partial z_2}(z_1, 0) + y \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_2}(z_1, z_2) \, dz_2 \right) \, dz_1 \\ &\quad + \int_0^\infty \rho e^{-xz_1} \left( -\frac{\partial g}{\partial z_1}(z_1, 0) + y \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_1}(z_1, z_2) \, dz_2 \right) \, dz_1 \\ &\quad + \int_0^\infty \mu_1 e^{-yz_2} \left( -g(0, z_2) + x \int_0^\infty e^{-xz_1} g(z_1, z_2) \, dz_1 \right) \, dz_2 \\ &\quad + \int_0^\infty \mu_2 e^{-xz_1} \left( -g(z_1, 0) + y \int_0^\infty e^{-yz_2} g(z_1, z_2) \, dz_2 \right) \, dz_1 \\ 0 &= -\frac{1}{2} r_1 \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_2}(0, z_2) \, dz_2 + \frac{x}{2} \int_0^\infty e^{-yz_2} \left( -g(0, z_2) + x \int_0^\infty e^{-xz_1} g(z_1, z_2) \, dz_1 \right) \, dz_2 \\ &\quad - \frac{1}{2} r_2 \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1, 0) \, dz_1 + \frac{y}{2} \int_0^\infty e^{-xz_1} \left( -g(z_1, 0) + y \int_0^\infty e^{-yz_2} g(z_1, z_2) \, dz_2 \right) \, dz_1 \\ &\quad - \rho \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1, 0) \, dz_1 + \rho y \int_0^\infty e^{-yz_2} \left( -g(0, z_2) + x \int_0^\infty e^{-xz_1} g(z_1, z_2) \, dz_1 \right) \, dz_2 \\ &\quad - \mu_1 \int_0^\infty e^{-yz_2} g(0, z_2) \, dz_2 + \mu_1 x \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} g(z_1, z_2) \, dz_1 dz_2 \\ &\quad - \mu_2 \int_0^\infty e^{-xz_1} g(z_1, 0) \, dz_1 + \mu_2 y \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} g(z_1, z_2) \, dz_1 dz_2 \end{aligned}$$

$$\begin{aligned}
0 &= \left( \frac{1}{2}(x^2 + y^2 + 2\rho xy) + \mu_1 x + \mu_2 y \right) \int_0^\infty \int_0^\infty e^{-xz_1 - yz_2} g(z_1, z_2) dz_1 dz_2 \\
&\quad - \left( \frac{1}{2}(r_2 x + y) + \rho x + \mu_2 \right) \int_0^\infty e^{-xz_1} g(z_1, 0) dz_1 \\
&\quad - \left( \frac{1}{2}(x + r_1 y) + \rho y + \mu_1 \right) \int_0^\infty e^{-yz_2} g(0, z_2) dz_2 \\
0 &= K(x, y)\psi(x, y) - k_1(x, y)\psi_1(x) - k_2(x, y)\psi_2(y).
\end{aligned}$$

This concludes the proof.  $\square$

#### 4. KERNEL AND ASYMPTOTICS

We begin by studying some properties of the kernel  $K$  defined in (25). Note that this kernel is similar to that in [13] except that in the present paper the drift is positive. We define the functions  $X$  and  $Y$  satisfying

$$K(X(y), y) = 0 \quad \text{and} \quad K(x, Y(x)) = 0.$$

The branches are given by

$$\begin{cases} X^\pm(y) = -(\rho y + \mu_1) \pm \sqrt{y^2(\rho^2 - 1) + 2y(\mu_1\rho - \mu_2) + \mu_1^2}, \\ Y^\pm(x) = -(\rho x + \mu_2) \pm \sqrt{x^2(\rho^2 - 1) + 2x(\mu_2\rho - \mu_1) + \mu_2^2}. \end{cases} \quad (28)$$

and the branch points of  $X$  and  $Y$  are given, respectively, by

$$\begin{cases} y^\pm = \frac{\mu_1\rho - \mu_2 \pm \sqrt{(\mu_1\rho - \mu_2)^2 + \mu_1^2(1 - \rho^2)}}{(1 - \rho^2)}, \\ x^\pm = \frac{\mu_2\rho - \mu_1 \pm \sqrt{(\mu_2\rho - \mu_1)^2 + \mu_2^2(1 - \rho^2)}}{(1 - \rho^2)}. \end{cases} \quad (29)$$

The functions  $X^\pm$  and  $Y^\pm$  are analytic, respectively, on the cut planes  $\mathbb{C} \setminus ((-\infty, y^-] \cup [y^+, \infty))$  and  $\mathbb{C} \setminus ((-\infty, x^-] \cup [x^+, \infty))$ . Figure 4 below depicts the functions  $Y^\pm$  on  $[x^-, x^+]$ .

Recall  $k_1$  and  $k_2$  as defined in (26). Consider the intersection points between the ellipse  $K = 0$  and the lines  $k_1 = 0$  and  $k_2 = 0$ . We define

$$x_0 := -2\mu_1 < 0 \quad \text{and} \quad y_0 := -2\mu_2 < 0, \quad (30)$$

$$x_1 := -\frac{2(r_2\mu_2 + \mu_1)}{1 + r_2^2 + 2\rho r_2} < 0 \quad \text{and} \quad y_2 := -\frac{2(r_1\mu_1 + \mu_2)}{1 + r_1^2 + 2\rho r_1} < 0. \quad (31)$$

These points are represented on Figure 4 and satisfy the following:

- $K(x_0, 0) = k_2(x_0, 0) = 0$ ,  $K(0, y_0) = k_1(0, y_0) = 0$ .
- $\exists y_1 \in \mathbb{R}$  such that  $K(x_1, y_1) = k_2(x_1, y_1) = 0$
- $\exists x_2 \in \mathbb{R}$  such that  $K(x_2, y_2) = k_1(x_2, y_2) = 0$ .

Let us define the curve  $\mathcal{H}$ , which is the boundary of the BVP established in Section 6.1

$$\mathcal{H} = X^\pm([y^+, \infty)) = \{x \in \mathbb{C} : K(x, y) = 0 \text{ and } y \in [y^+, \infty)\}.$$

**Lemma 12** (Hyperbola  $\mathcal{H}$ ). *The curve  $\mathcal{H}$  is a branch of the hyperbola of equation*

$$(\rho^2 - 1)x^2 + \rho^2 y^2 - 2(\mu_1 - \rho\mu_2)x = \mu_1(\mu_1 - 2\rho\mu_2). \quad (32)$$

*The curve  $\mathcal{H}$  is symmetrical with respect to the horizontal axis and is the right branch of the hyperbola if  $\rho < 0$ . Further, it is the left branch if  $\rho > 0$  and it is a straight line if  $\rho = 0$ .*

*Proof.* A similar kernel has already been studied; we refer the reader to [13, Lemma 4] and [2, Lemma 9], where the equation of such a hyperbola is derived.  $\square$

Let  $\mathcal{H}^-$  denote the part of the hyperbola  $\mathcal{H}$  with imaginary part negative. We also define the domain  $\mathcal{G}$  bounded by  $\mathcal{H}$  and containing  $x^+$ . This is depicted in Figure 5 below.

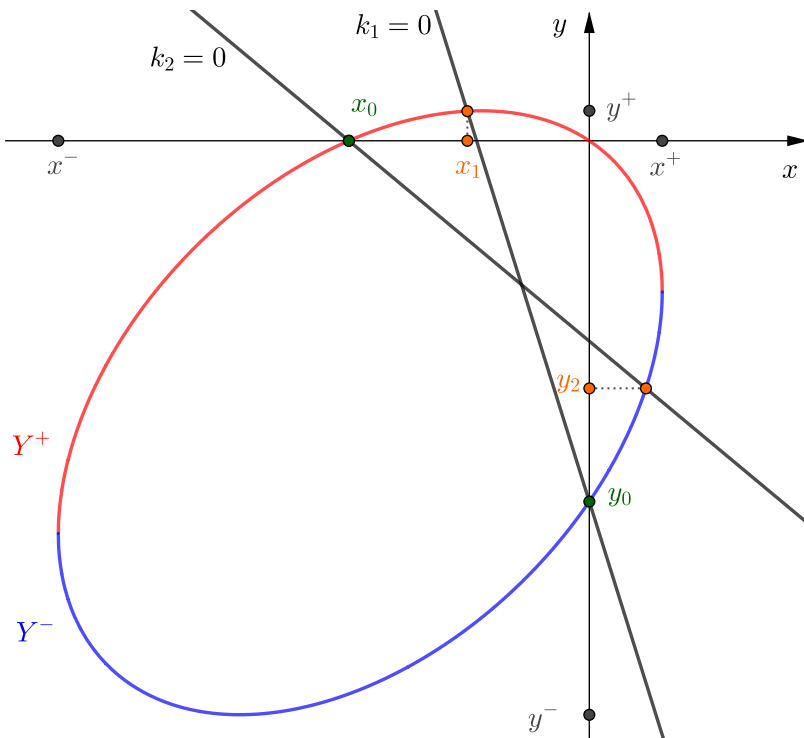


FIGURE 4. The ellipse  $K = 0$ , the function  $Y^-$  in blue, the function  $Y^+$  in red, the two lines  $k_1 = 0$  and  $k_2 = 0$ , the branch points  $x^\pm$  and  $y^\pm$ , the points  $x_0$  and  $y_0$  in green, the points  $x_1$  and  $y_2$  in orange. This figure is drawn for the following parameters:  $\mu_1 = 2$ ,  $\mu_2 = 3$ ,  $\rho = -0.4$ ,  $r_1 = 2$ ,  $r_2 = 4$ .

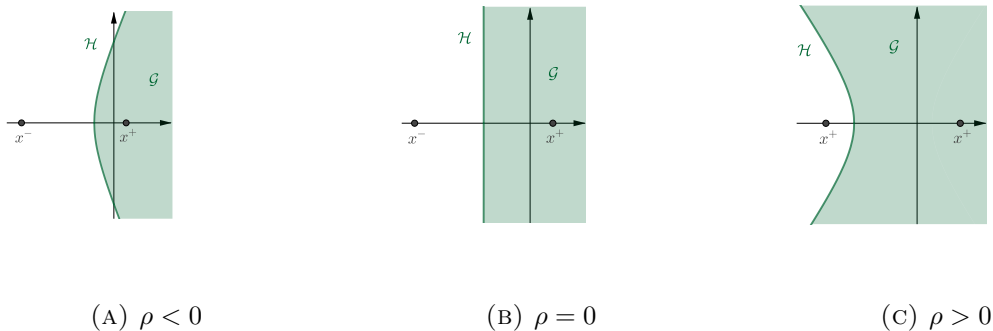


FIGURE 5. Hyperbola  $\mathcal{H}$  and domain  $\mathcal{G}$ .

**4.1. Meromorphic continuation.** This section focuses on establishing the boundary value problem. We begin by meromorphically continuing the Laplace transform  $\psi_1(x)$  (which converges for  $x > 0$ ).

**Lemma 13** (Meromorphic continuation). *By the formula*

$$\psi_1(x) = \frac{-k_2(x, Y^+(x))\psi_2(Y^+(x))}{k_1(x, Y^+(x))}, \quad (33)$$

the Laplace transform  $\psi_1(x)$  can be meromorphically continued to the set

$$S := \{x \in \mathbb{C} : \Re x > 0 \text{ or } \Re Y^+(x) > 0\} \cup \{0\}, \quad (34)$$



where the domain  $\mathcal{G}$  and its boundary  $\mathcal{H}$  are included in the set defined in (34). Then  $\psi_1$  is meromorphic on  $\mathcal{G}$  and is continuous on  $\bar{\mathcal{G}}$ .

*Proof.* The Laplace transforms  $\psi_1(x)$  and  $\psi_2(y)$  are analytic, respectively, on  $\{x \in \mathbb{C} : \Re x > 0\}$  and  $\{y \in \mathbb{C} : \Re y > 0\}$ . The functional equation (27) implies that for  $(x, y)$  in the set  $\tilde{S} := \{(x, y) \in \mathbb{C}^2 : \Re x > 0, \Re y > 0 \text{ and } \psi(x, y) = 0\}$ , we have

$$0 = k_1(x, y)\psi_1(x) + k_2(x, y)\psi_2(y). \quad (35)$$

The open connected set

$$S_1 := \{x \in \mathbb{C} : \Re Y^+(x) > 0\},$$

intersects the open set  $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$ . For  $x \in S_1 \cap S_2$ ,  $(x, Y^+(x)) \in \tilde{S}$ ; equation (35) implies that the continuation formula in (33) is satisfied for all  $x \in S_1 \cap S_2$ . Figure 6 represent these sets. With  $\psi_1(x)$  defined as in (33), we invoke the principle of analytic continuation and meromorphically extend  $\psi_1$  to  $S = S_1 \cup S_2$ . Note that the inclusion of  $\mathcal{G}$  in the set  $S$  defined in (34) is similar to that in [13, Lemma 5]. This inclusion is depicted below in Figure 6.  $\square$

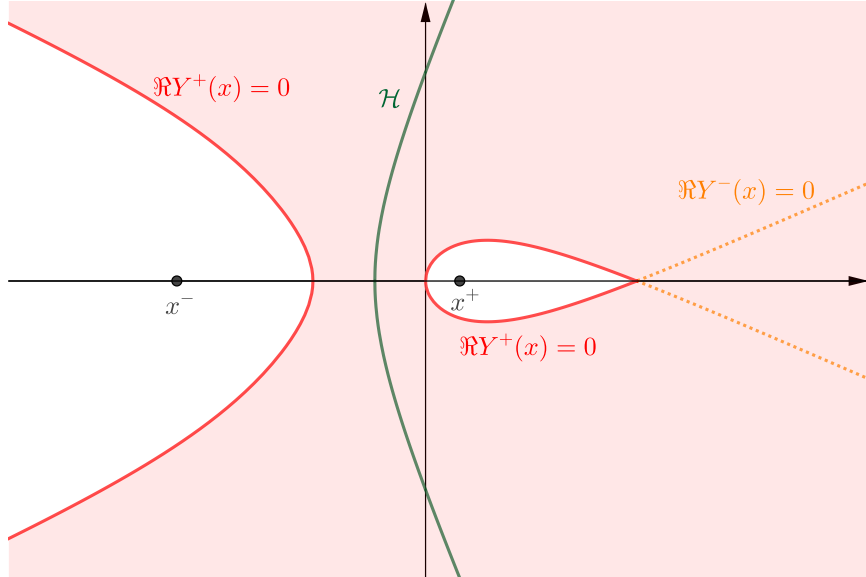


FIGURE 6. The complex plane of  $x$ . The red curve of equation  $\Re Y^+(x) = 0$  bounds the red domain  $S_1 := \{x \in \mathbb{C} : \Re Y^+(x) > 0\}$ . The orange dotted curve corresponds to the equation  $\Re Y^-(x) = 0$ . The domain  $\mathcal{G}$  is bounded on the left by the green hyperbola  $\mathcal{H}$ , contains  $x^+$  (see Figure 5), and is included in  $S = S_1 \cup S_2$ , where  $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$ . This figure is drawn for the parameters  $\mu_1 = 2$ ,  $\mu_2 = 3$ ,  $\rho = -0.4$ .

#### 4.2. Poles and geometric conditions.

**Lemma 14** (Poles). *On the set  $S$  defined in (34), the Laplace transform  $\psi_1$  has either one or two poles, as follows:*

- (One pole): If  $k_1(x^-, Y^\pm(x^-)) \geq 0$ , the point 0 is the unique pole of  $\psi_1$  in  $S$  and this pole is simple.
- (Two poles): If  $k_1(x^-, Y^\pm(x^-)) < 0$ , the points 0 and  $x_1$  (defined in (31)) are the only poles of  $\psi_1$  in  $S$  and these poles are simple.

In addition,  $\lim_{x \rightarrow 0} x\psi_1(x) = 1$ . Further, the point  $x_1$  is a pole of  $\psi_1$  and belongs to the domain  $\mathcal{G}$  if and only if  $k_1(X^\pm(y^+), y^+) < 0$ .

*Proof.* The final value theorem for the Laplace transform, together with Theorem 3, imply that

$$\lim_{x \rightarrow 0} x \psi_1(x) = \lim_{u \rightarrow \infty} \mathbb{P}_{(u,0)}[T = \infty] = 1.$$

We may thus conclude that 0 is a simple pole. On the set  $\{x \in \mathbb{C} : \Re x > 0\}$ ,  $\psi_1$  is defined as a Laplace transform which converges (and thus has no poles). Therefore, with the exception of 0, the only possible poles in  $S$  are the zeros of  $k_1(x, Y^+(x))$ , which are the zeros of the denominator of the continuation formula in (33). Straightforward calculations show that equation  $k_1(x, Y^+(x)) = 0$  has either no roots or one (simple) root, and that this depends on the sign of  $k_1(x^-, Y^\pm(x^-))$ . When the root exists, it is  $x_1$  (see (31)). The condition for the existence of this root is depicted in Figure 7 below. It now only remains to remark that when  $x_1$  is a pole,  $x_1$  is in  $\mathcal{G}$  if and only if  $x_1 > X^\pm(y^+)$ . The latter holds if and only if  $k_1(X^\pm(y^+), y^+) < 0$  (see Figure 8).  $\square$

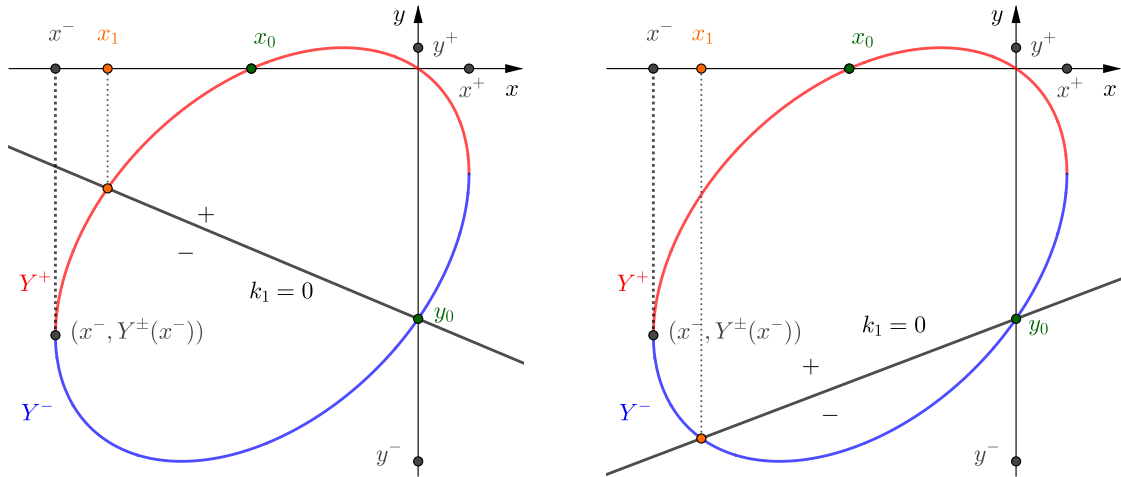


FIGURE 7. On the left, we see that  $k_1(x^-, Y^\pm(x^-)) < 0$  and  $x_1$  is a simple pole of  $\psi_1$ . On the right, we see that  $k_1(x^-, Y^\pm(x^-)) > 0$  and  $\psi_1$  has no pole in  $S$ .

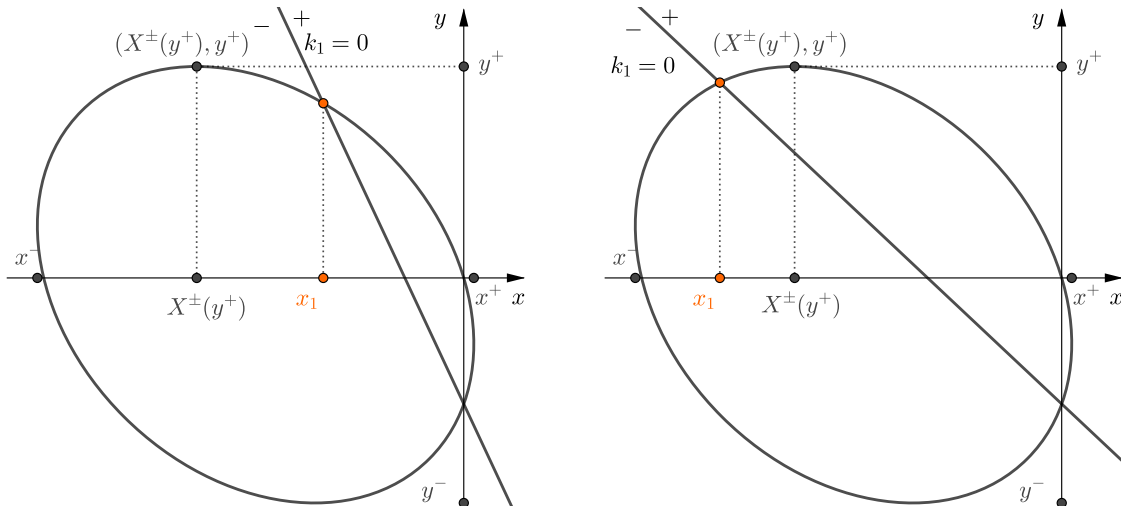


FIGURE 8. On the left, we see that  $k_1(X^\pm(y^+), y^+) < 0$  and  $x_1$  is in  $\mathcal{G}$ . On the right, we see that  $k_1(X^\pm(y^+), y^+) > 0$  and  $x_1$  is not in  $\mathcal{G}$ .

Before turning to Lemma 15, recall that the angles  $\delta$ ,  $\beta$  and  $\theta$  were defined above in (3) and that  $k_1$  was defined in (26).

**Lemma 15** (Geometric conditions). *The condition  $k_1(x^-, Y^\pm(x^-)) > 0$  (resp.  $= 0$  and  $< 0$ ) is equivalent to*

$$2\delta - \theta < \pi.$$

(resp.  $= \pi$  and  $> \pi$ ). *The condition  $k_1(X^\pm(y^+), y^+) > 0$  (resp.  $= 0$  and  $< 0$ ) is equivalent to*

$$2\delta - \theta + \beta < 2\pi.$$

(resp.  $= 2\pi$  and  $> 2\pi$ ).

*Proof.* By condition (1) and by the fact that the drift is positive, we have  $0 < \theta < \beta < \delta < \pi$ . By (3) and (29),

$$x^-/\mu_2 = \frac{1}{\sqrt{1-\rho^2}} \left( \frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}} - \sqrt{\left( \frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}} \right)^2 + 1} \right) = \frac{-\cot(\theta) - \sqrt{\cot^2(\theta) + 1}}{\sin(\beta)}. \quad (36)$$

We begin by proving the first equivalence for  $\delta \geq \pi/2$ . In this case we have

$$\begin{aligned} k_1(x^-, Y^\pm(x^-)) > 0 &\Leftrightarrow \frac{1}{2}(r_2 x^- + Y^\pm(x^-)) + \rho x^- + \mu_2 > 0 \\ &\Leftrightarrow r_2 + \rho < -\mu_2/x^- \text{ since } Y^\pm(x^-) = -\rho x^- - \mu_2 \text{ by (28) and (29)} \\ &\Leftrightarrow r_2 - \cos(\beta) < \sin(\beta) \left( \cot(\theta) + \sqrt{\cot^2(\theta) + 1} \right)^{-1} \text{ by (36)} \\ &\Leftrightarrow -\cot(\delta) \left( \cot(\theta) + \sqrt{\cot^2(\theta) + 1} \right) < 1 \\ &\Leftrightarrow 0 < -\cot(\delta) \sqrt{\cot^2(\theta) + 1} < 1 + \cot(\delta) \cot(\theta) \text{ since we assumed } \delta \geq \pi/2 \\ &\Leftrightarrow \cot^2(\delta)(\cot^2(\theta) + 1) < (1 + \cot(\delta) \cot(\theta))^2 \\ &\Leftrightarrow 2 \cot(\delta) \cot(\theta) - \cot^2(\delta) + 1 > 0 \\ &\Leftrightarrow 2 \sin(\delta) \cos(\delta) \cos(\theta) - (\cos^2(\delta) - \sin^2(\delta)) \sin(\theta) > 0 \\ &\Leftrightarrow \sin(2\delta) \cos(\theta) - \cos(2\delta) \sin(\theta) > 0 \\ &\Leftrightarrow \sin(2\delta - \theta) > 0 \\ &\Leftrightarrow 2\delta - \theta < \pi \text{ since } 0 < 2\delta - \theta < 2\pi. \end{aligned}$$

It is straightforward to see that if  $\delta < \pi/2$ , then  $2\delta - \theta < \pi$ . Further,  $\delta < \pi/2$  is equivalent to  $r_2 + \rho < 0$  by (3), which implies that  $r_2 + \rho < -\mu_2/x^-$ . Therefore,  $k_1(X^\pm(y^+), y^+) < 0$ . This proves the first equivalence. The second equivalence is proved in exactly the same way, so the details are omitted. This concludes the proof.  $\square$

**4.3. Absorption Asymptotics along the axes.** In this section, we establish asymptotics results for the absorption probability (and escape probability) in a simpler case where the starting point is  $(u, 0)$ .

**Proposition 16** (Absorption asymptotics). *Let us assume that  $x^- \in S$ . For some constant  $C$ , the asymptotic behavior of  $\mathbb{P}_{(u,0)}[T < \infty]$  as  $u \rightarrow \infty$  is given by*

$$\mathbb{P}_{(u,0)}[T < \infty] \sim C \begin{cases} e^{ux_1} & \text{if } 2\delta - \theta > \pi, \\ u^{-\frac{3}{2}} e^{ux^-} & \text{if } 2\delta - \theta < \pi, \\ u^{-\frac{1}{2}} e^{ux^-} & \text{if } 2\delta - \theta = \pi. \end{cases}$$

*Proof.* The largest singularity of the Laplace transform of  $\mathbb{P}_{(u,0)}[T < \infty]$  determines its asymptotics. We proceed by invoking a classical transfer theorem, see [6, Theorem 37.1]. This theorem says that if  $a$  is the largest singularity of order  $k$  of the Laplace transform (that is, the Laplace transform behaves as  $(s-a)^{-k}$  up to additive and multiplicative constants in the neighborhood of  $a$ ), then when  $u \rightarrow \infty$ , the probability  $\mathbb{P}_{(u,0)}[T < \infty]$  is equivalent (up to a constant) to  $u^{k-1} e^{au}$ . The Laplace transform of  $\mathbb{P}_{(u,0)}[T < \infty]$  is  $1/x - \psi_1(x)$ . By Lemma 14, the point 0 is

not a singularity and the point  $x_1$  is a simple pole. When that pole exists, the asymptotics are given by  $Ce^{ux_1}$  for some constant  $C$ . When there is no pole, that is, when  $k_1(x^-, Y^\pm(x^-)) \geq 0$ , the largest singularity is given by the branch point  $x^-$ . The definition of  $Y^+$  and (33) together imply that for some constants  $C_i$  we have

$$\psi_1(x) \underset{x \rightarrow x^-}{=} \begin{cases} C_1 + C_2\sqrt{x - x^-} + O(x - x^-) & \text{if } k_1(x^-, Y^\pm(x^-)) > 0, \\ \frac{C_3}{\sqrt{x - x^-}} + O(1) & \text{if } k_1(x^-, Y^\pm(x^-)) = 0. \end{cases}$$

The proof is then completed by applying Lemma 15 and invoking the classical transfer theorem.  $\square$

**Remark 17** (Asymptotics along the vertical axis). *Studying the singularities of  $\phi_1$  we obtained in Proposition 16 the asymptotics of the absorption probability (and then of the escape probability which is equal to  $1 - \mathbb{P}_{(u,0)}[T < \infty]$ ) along the horizontal axis. A similar study for  $\psi_2$  would lead to the following asymptotics along the vertical axis. As  $v \rightarrow \infty$*

$$\mathbb{P}_{(0,v)}[T < \infty] \sim C \begin{cases} e^{vy_2} & \text{if } 2\epsilon + \theta - \beta > \pi, \\ v^{-\frac{3}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta < \pi, \\ v^{-\frac{1}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta = \pi. \end{cases}$$

## 5. PRODUCT FORM AND EXPONENTIAL ABSORPTION PROBABILITY

In this section, we consider a remarkable geometric condition on the parameters characterizing the case where the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the famous skew symmetry condition studied by Harrison, Reiman and Williams [17, 20], which characterizes the cases where the stationary distribution has a product form and is exponential. The *dual skew symmetry* condition gives a criterion for the solution to the partial differential equation of Proposition 10 (dual to that satisfied by the invariant measure) to be of product form. The following Theorem state a simple geometric criterion on the parameters for the absorption probability to be of product form; the absorption probability is then exponential.

**Theorem 18** (Dual skew symmetry). *Let  $f(u, v) = \mathbb{P}_{(u,v)}[T < \infty]$  be the absorption probability. The following statement are equivalent:*

(1) *The absorption probability has a product form, i.e. it exists  $f_1$  and  $f_2$  such that*

$$f(u, v) = f_1(u)f_2(v);$$

(2) *The absorption probability is exponential, i.e. there exists  $x$  and  $y$  in  $\mathbb{R}$  such that*

$$f(u, v) = e^{ux+vy};$$

(3) *The reflection vectors are in opposite direction, i.e.*

$$r_1 r_2 = 1;$$

(4) *The reflection angles in the wedge satisfy  $\alpha = 1$ , i.e.*

$$\delta + \epsilon - \beta = \pi.$$

*In this case we have*

$$f(u, v) = e^{ux_1+vy_2}$$

*where  $x_1$  and  $y_2$  are given in (31).*

*Proof.* (1)  $\Rightarrow$  (2): Neumann boundary conditions (23) imply that  $f_1'(0)f_2(y) - r_1 f_1(0)f_2'(y) = 0$  and  $-r_2 f_1'(u)f_2(0) + f_1(u)f_2'(0) = 0$ . Solving these differential equation imply that  $f_1$  and  $f_2$  (and thus  $f$ ) are exponential.

(2)  $\Rightarrow$  (1): This implication is straightforward.

(2)  $\Rightarrow$  (3): Neumann boundary conditions (23) imply that for all  $v > 0$ ,  $xe^{vy} - r_1 ye^{vy} = 0$  and that for all  $u > 0$ ,  $-r_2 xe^{ux} + ye^{ux} = 0$ . It follows that  $r_1 = x/y$ ,  $r_2 = y/x$ , and thus  $r_1 r_2 = 1$ .

(3)  $\Rightarrow$  (2): Let us define  $f(u, v) = e^{ux_1+vy_2}$ . We need to show that  $f$  satisfies the partial

differential equation of Proposition 10. This will imply that  $f$  is the absorption probability. The fact that  $r_1 = 1/r_2$ , combined with (31), gives  $r_1 = x_1/y_2$ . This implies that  $f$  satisfies the Neumann boundary conditions in (23). The limit values are satisfied because  $f(0,0) = 1$  and  $\lim_{(u,v) \rightarrow \infty} f(u,v) = 0$  for  $x_1 < 0$  and  $y_2 < 0$ . It now only remains to show that  $\mathcal{G}f = 0$ . We now only need verify that  $K(x_1, y_2) = 0$ , see Figure 9. By definition of  $y_2$  (see (31)), we have

$$\begin{aligned} K(x_1, y_2) &= y_2 \left( \frac{y_2}{2} \left( \left( \frac{x_1}{y_2} \right)^2 + 1 + 2\rho \frac{x_1}{y_2} \right) + \mu_1 \frac{x_1}{y_2} + \mu_2 \right) \\ &= y_2 \left( \frac{y_2}{2} (r_1^2 + 1 + 2\rho r_1) + \mu_1 r_1 + \mu_2 \right) = 0. \end{aligned}$$

(3)  $\Leftrightarrow$  (4): The following equivalences hold:

$$\begin{aligned} r_1 r_2 = 1 &\Leftrightarrow (\sin(\beta)/\tan(\delta) - \cos(\beta)) (\sin(\beta)/\tan(\epsilon) - \cos(\beta)) = 1 \quad \text{by (3)} \\ &\Leftrightarrow \frac{\sin(\beta)}{\tan(\epsilon)} = \frac{\tan(\delta)}{\sin(\beta) - \cos(\beta)\tan(\delta)} + \cos(\beta) = \frac{\tan(\delta)(1 - \cos^2(\beta)) + \cos(\beta)\sin(\beta)}{\sin(\beta) - \cos(\beta)\tan(\delta)} \\ &\Leftrightarrow \tan(\epsilon) = \frac{\tan(\beta) - \tan(\delta)}{1 + \tan(\delta)\tan(\beta)} \\ &\Leftrightarrow \tan(\epsilon) = \tan(\beta - \delta) \\ &\Leftrightarrow \epsilon = \beta - \delta + \pi. \end{aligned}$$

□

The skew symmetry condition is  $2\rho = -r_1 - r_2$  and  $\epsilon + \delta = \pi$ . Note that the condition  $r_1 r_2 = 1$  is independent of the covariance  $\rho$ ; however, the condition  $\delta + \epsilon - \beta = \pi$  involves the wedge angle  $\beta$ . Further properties of the *dual skew symmetry* condition will be explored in future work.

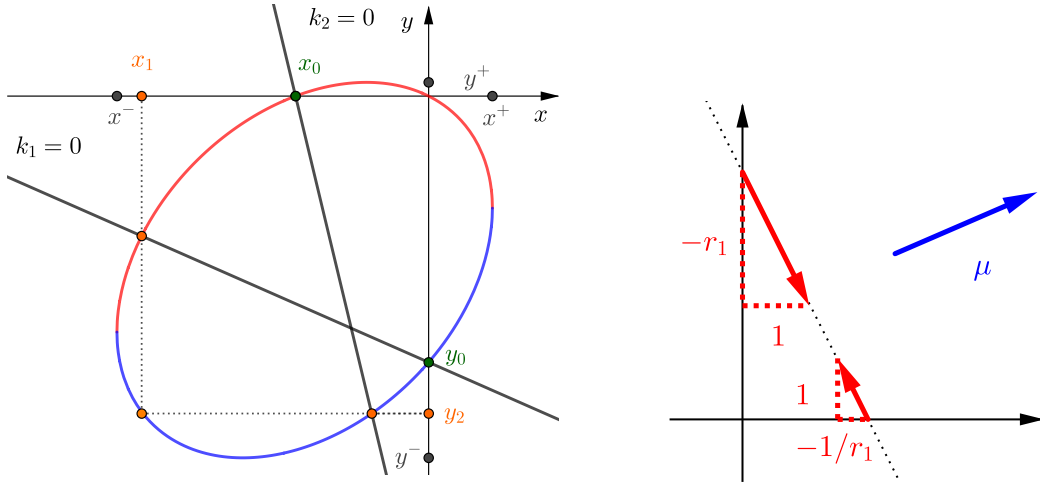


FIGURE 9. Dual skew symmetry: on the left, we see that  $K(x_2, y_2) = 0$ ; on the right, we see that condition  $r_1 r_2 = 1$  imply that the reflection vectors are in opposite direction.

## 6. INTEGRAL EXPRESSION OF THE LAPLACE TRANSFORM $\psi_1$

In this section, we establish a boundary value problem (BVP) satisfied by the Laplace transform (Proposition 19). The section's key result is Theorem 24, which gives an explicit integral formula for the Laplace transform of the escape probability.

**6.1. Carleman boundary value problem.** We state a Carleman BVP satisfied by the Laplace transform  $\psi_1$ .

**Proposition 19** (Carleman BVP). *The Laplace transform  $\psi_1$  satisfies the following boundary value problem:*

- (i)  $\psi_1(x)$  is meromorphic on  $\mathcal{G}$  and continuous on  $\overline{\mathcal{G}}$ .
- (ii)  $\psi_1(x)$  admits one or two poles in  $\mathcal{G}$ . 0 is always a simple pole and  $x_1$  is a simple pole if and only if  $2\delta - \theta + \beta > 2\pi$ .
- (iii)  $\lim_{x \rightarrow \infty} x\psi_1(x) = 0$ .
- (iv)  $\psi_1$  satisfies the boundary condition

$$\psi_1(\bar{x}) = G(x)\psi_1(x), \quad \forall x \in \mathcal{H},$$

where

$$G(x) := \frac{k_1}{k_2}(x, Y^+(x)) \frac{k_2}{k_1}(\bar{x}, Y^+(x)). \quad (37)$$

*Proof.* Statement (i) immediately follows from Lemma 13. Statement (ii) immediately follows from Lemmas 14 and 15. Statement (iii) follows from the initial value theorem for the Laplace transform, which implies that  $\lim_{x \rightarrow \infty} x\psi_1(x) = \mathbb{P}_{(0,0)}[T = \infty] = 0$ . To prove statement (iv), we recall the functional equation (27). For  $x \in \mathcal{H}$ , we evaluate this equation for  $(x, Y^+(x))$  and  $(\bar{x}, Y^+(\bar{x}))$ . By the definition of  $Y^+$ , we have  $K(x, Y^+(x)) = K(\bar{x}, Y^+(\bar{x})) = 0$ . By the definition of the hyperbola  $\mathcal{H}$  in (32), we have  $Y^+(\bar{x}) = Y^+(x)$ . This enables us to obtain the following system of equations

$$\begin{cases} 0 = k_1(x, Y^+(x))\psi_1(x) + k_2(x, Y^+(x))\psi_2(Y^+(x)), \\ 0 = k_1(\bar{x}, Y^+(x))\psi_1(\bar{x}) + k_2(\bar{x}, Y^+(x))\psi_2(Y^+(x)). \end{cases}$$

Solving this system of equations and eliminating  $\psi_2(Y^+(x))$ , we obtain the boundary condition in statement (iv).  $\square$

**6.2. Gluing function.** To solve the BVP, we need a conformal gluing function which glues together the upper and lower parts of the hyperbola. This conformal gluing function was introduced in [12, 13]. For  $a \geq 0$  and for  $x \in \mathbb{C} \setminus (-\infty, -1]$ , the generalized Chebyshev polynomial is defined by

$$T_a(x) := \cos(a \arccos(x)) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right).$$

We define the angle

$$\beta := \arccos(-\rho).$$

We also define the functions

$$w(x) := T_{\frac{\pi}{\beta}} \left( \frac{2x - (x^+ + x^-)}{x^+ - x^-} \right), \quad (38)$$

and

$$W(x) := \frac{w(x) - w(X^\pm(y^+))}{w(x) - w(0)}.$$

We now recall a useful Lemma from [13] for the conformal gluing function  $W$ .

**Lemma 20** (Lemma 9, [13]). *The function  $W$  satisfies the following properties*

- (i)  $W$  is holomorphic in  $\mathcal{G} \setminus \{0\}$ , continuous in  $\overline{\mathcal{G}} \setminus \{0\}$  and bounded at infinity.
- (ii)  $W$  is bijective from  $\mathcal{G} \setminus \{0\}$  to  $\mathbb{C} \setminus [0, 1]$ .
- (iii)  $W$  satisfies the gluing property on the hyperbola

$$W(x) = W(\bar{x}), \quad \forall x \in \mathcal{H}.$$

**6.3. Index.** We introduce the  $\Delta$  which is the variation of the argument of  $G(x)$  when  $x$  lies on  $\mathcal{H}^-$  and  $d$  which is the argument of  $G$  at the real point of the hyperbola  $\mathcal{H}$ . Let

$$\Delta := [\arg G(x)]_{\mathcal{H}^-} = \left[ \arg \frac{k_1}{k_2}(x, Y^+(x)) \right]_{\mathcal{H}} \quad \text{and} \quad d := \arg G(X^+(y^+)).$$

We now define the index  $\kappa$  such as

$$\kappa := \left\lfloor \frac{d + \Delta}{2\pi} \right\rfloor.$$

The index will prove to be useful in solving the boundary value problem given in Proposition 19.

**Lemma 21.** *We have*

$$d = \begin{cases} 0 & \text{if } k_1(x^-, Y^\pm(x^-)) \neq 0 \text{ i.e. } 2\delta - \theta + \beta \neq 2\pi, \\ \pi & \text{if } k_1(x^-, Y^\pm(x^-)) = 0 \text{ i.e. } 2\delta - \theta + \beta = 2\pi, \end{cases}$$

and the angle  $d + \Delta \in (-4\pi, 2\pi)$  and satisfy

$$\tan \frac{d + \Delta}{2} = \tan(\epsilon + \delta + \beta).$$

*Proof.* The proof is in each step similar to the proof of [13, Lemma 13].  $\square$

Before to state the main Lemma of this section, combining Lemma 14 and Lemma 15 we define

$$\chi := \begin{cases} -1 & \text{if } 2\delta - \theta + \beta > 2\pi \Leftrightarrow x_1 \text{ is a pole of } \psi_1 \text{ in } \mathcal{G}, \\ 0 & \text{if } 2\delta - \theta + \beta \leq 2\pi \Leftrightarrow \psi_1 \text{ has no pole but } 0 \text{ in } \mathcal{G}. \end{cases} \quad (39)$$

**Lemma 22 (Index).** *The index  $\kappa$  satisfies*

$$\kappa := \begin{cases} \chi & \text{if } \epsilon + \delta + \beta \geq 2\pi, \\ \chi - 1 & \text{if } \epsilon + \delta + \beta < 2\pi. \end{cases}$$

*Proof.* The proof follows immediately by combining Lemma 14, Lemma 15, and [13, Lemma 14] (which studies a similar quantity).  $\square$

**Lemma 23.** *We have*

$$\left( -\frac{d + \Delta}{2\pi} + \chi - 1 \right) \frac{\pi}{\beta} = -\alpha - 1$$

*Proof.* Recall that by Lemma 21 we have

$$\tan \frac{d + \Delta}{2} = \tan(\epsilon + \delta + \beta)$$

and as  $\alpha = \frac{\epsilon + \delta - \pi}{\beta} \geq 1$  and that  $\epsilon, \delta$  and  $\beta \in (0, \pi)$  we have

$$2\beta - \pi \leq \epsilon + \delta + \beta - 2\pi < \pi.$$

Recall also that by definition  $\kappa = \lfloor \frac{d + \Delta}{2\pi} \rfloor$ . First case,  $\epsilon + \delta + \beta - 2\pi > 0$  then

$$\frac{d + \Delta}{2} = \begin{cases} \epsilon + \delta + \beta - 2\pi & \text{if } \frac{d + \Delta}{2\pi} \geq 0 \text{ i.e. } \kappa = 0, \\ \epsilon + \delta + \beta - 3\pi & \text{if } \frac{d + \Delta}{2\pi} < 0 \text{ i.e. } \kappa = -1. \end{cases}$$

Furthermore by Lemma 22 we have  $\kappa = \chi$ . We deduce that

$$\frac{d + \Delta}{2} = \epsilon + \delta + \beta + (\chi - 2)\pi.$$

Second case  $\epsilon + \delta + \beta - 2\pi < 0$  then

$$\frac{d + \Delta}{2} = \begin{cases} \epsilon + \delta + \beta - 2\pi & \text{if } -\pi \leq \frac{d + \Delta}{2\pi} < 0 \text{ i.e. } \kappa = -1, \\ \epsilon + \delta + \beta - 3\pi & \text{if } -2\pi \leq \frac{d + \Delta}{2\pi} < -\pi \text{ i.e. } \kappa = -2. \end{cases}$$

Furthermore by Lemma 22 we have  $\kappa = \chi - 1$ . We deduce again that we have

$$\frac{d + \Delta}{2} = \epsilon + \delta + \beta + (\chi - 2)\pi.$$

Then, in both cases we have

$$\begin{aligned} \left(-\frac{d + \Delta}{2\pi} + \chi - 1\right) \frac{\pi}{\beta} &= (-\epsilon - \delta - \beta - (\chi - 2)\pi + \chi\pi - \pi) \frac{1}{\beta} \\ &= -\alpha - 1. \end{aligned}$$

□

#### 6.4. Solution of the BVP.

**Theorem 24** (Explicit expression for  $\psi_1$ ). *The Laplace transform  $\psi_1$  is given for  $x \in \mathcal{G}$  by*

$$\psi_1(x) = \frac{w'(0)}{w(x) - w(0)} \left( \frac{w(0) - w(x_1)}{w(x) - w(x_1)} \right)^{-\chi} \exp \left( \frac{1}{2i\pi} \int_{\mathcal{H}^-} \log G(t) \left[ \frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right] dt \right), \quad (40)$$

where  $x_1$  is defined in (31),  $G$  is defined (37),  $w$  is defined (38),  $\chi$  is defined in (39) and  $\mathcal{H}$  is defined in (32).

*Proof.* Let

$$\tilde{\psi}_1(y) := \frac{(y - W(x_1))^{-\chi}}{(y - 1)^{1+\kappa-\chi}} \psi_1 \circ W^{-1}(y).$$

Proposition 19 and Lemma 20 together imply that

- $\tilde{\psi}_1$  is analytic on  $\mathbb{C} \setminus [0, 1]$ .
- $\tilde{\psi}_1(y) \sim_{\infty} cy^{-\kappa}$  for some constant  $c$ .
- $\tilde{\psi}_1(1) = 0$ .
- For  $y \in [0, 1]$ ,  $\tilde{\psi}_1$  satisfies the boundary condition

$$\tilde{\psi}_1^+(y) = \tilde{G}(y)\tilde{\psi}_1^-(y),$$

where  $\tilde{\psi}_1^+(y)$  is the left limit and  $\tilde{\psi}_1^-(y)$  is the right limit of  $\tilde{\psi}_1$  on  $[0, 1]$ ,  $(W^{-1})^-$  is the right limit of  $W^{-1}$  on  $[0, 1]$ , and  $\tilde{G}(y) = G \circ (W^{-1})^-(y)$ .

We now define

$$\tilde{S}(y) := (y - 1)^{-\kappa} \exp \left( \frac{1}{2i\pi} \int_0^1 \frac{\log \tilde{G}(u)}{u - y} \right).$$

Following the classical boundary theory results in [9, (5.2.24) and Theorem 5.2.3], the above function is analytic and does not cancel on  $\mathbb{C} \setminus [0, 1]$  and is such that  $\tilde{S}(y) \sim_{\infty} c'y^{-\kappa}$  for some constant  $c'$ . Furthermore, for  $y \in [0, 1]$ , it satisfies the boundary condition

$$\tilde{S}^+(y) = \tilde{G}(y)\tilde{S}^-(y),$$

where  $\tilde{S}^+(y)$  is the left limit and  $\tilde{S}^-(y)$  is the right limit of  $\tilde{S}$  on  $[0, 1]$ . By the properties of  $\tilde{\psi}_1$  and  $\tilde{S}$  detailed above, the function  $\tilde{\psi}_1/\tilde{S}$  is analytic on  $\mathbb{C}$  and bounded at infinity. Therefore there must exist a constant  $C$  such that

$$\tilde{\psi}_1(y) = C\tilde{S}(y).$$

Invoking the definition of  $\tilde{\psi}_1$ , we have that

$$\frac{(W(x) - W(x_1))^{-\chi}}{(W(x) - 1)^{1+\kappa-\chi}} \psi_1(x) = C(W(x) - 1)^{-\kappa} \exp \left( \frac{1}{2i\pi} \int_0^1 \frac{\log \tilde{G}(u)}{u - W(x)} \right). \quad (41)$$

Noting that

$$W(x) - 1 = \frac{w(0) - w(X^\pm(y^+))}{w(x) - w(0)} \quad \text{and} \quad W(x) - W(x_1) = \frac{w(x) - w(x_1)}{w(x) - w(0)} \frac{w(X^\pm(y^+)) - w(0)}{w(x_1) - w(0)},$$



and making a change of variable  $u = w(t)$  in the integral in (41), we obtain for some constant  $C'$

$$\psi_1(x) = C' \left( \frac{1}{w(x) - w(0)} \right) \left( \frac{1}{w(x) - w(x_1)} \right)^{-\chi} \exp \left( \frac{1}{2i\pi} \int_{\mathcal{H}^-} \log G(t) \frac{w'(t)}{w(t) - w(x)} dt \right).$$

The final value theorem for the Laplace transform gives

$$\lim_{x \rightarrow 0} x\psi_1(x) = \lim_{u \rightarrow \infty} \mathbb{P}_{(u,0)}[T = \infty] = 1.$$

This enables us to compute the constant

$$C' = w'(0) (w(0) - w(x_1))^{-\chi} \exp \left( \frac{-1}{2i\pi} \int_{\mathcal{H}^-} \log G(t) \frac{w'(t)}{w(t) - w(0)} dt \right).$$

which gives us (40), completing the proof.  $\square$

## 7. ASYMPTOTICS OF THE ESCAPE PROBABILITY AT THE ORIGIN

In this section we use the explicit expression in Theorem 24 to obtain the asymptotics of the escape probability at the origin. We begin with computing the asymptotics of  $\psi_1$  at infinity.

**Lemma 25** (Asymptotics of  $\psi_1$ ). *Let  $\alpha$  be defined as in (4). For ease of notation, allow  $C$  to be a constant which may change from one line to the next. For some positive constant  $C$ ,*

$$\psi_1(x) \underset{x \rightarrow \infty}{\sim} Cx^{-\alpha-1}.$$

A symmetrical result holds for  $\psi_2$ . That is, for some positive constant  $C$ ,

$$\psi_2(y) \underset{y \rightarrow \infty}{\sim} Cy^{-\alpha-1}.$$

*Proof.* This proof follows the same steps as those of [13, Prop 19]. The key is to invoke [9, (5.2.20)], which states that

$$\exp \left( \frac{1}{2i\pi} \int_0^1 \frac{\log \tilde{G}(u)}{u - y} \right) \underset{y \rightarrow 1}{\sim} C(y - 1)^{\frac{\Delta}{2\pi}}.$$

Recall that  $w(x) \underset{x \rightarrow \infty}{\sim} Cx^{\frac{\pi}{\beta}}$  and that  $W(x) - 1 \underset{x \rightarrow \infty}{\sim} Cx^{-\frac{\pi}{\beta}}$ . The explicit expressions of  $\psi_1$  obtained in (40) and in (41) imply that

$$\psi_1(x) \underset{x \rightarrow \infty}{\sim} Cx^{(-\frac{\Delta}{2\pi} + \chi - 1)\frac{\pi}{\beta}}.$$

The proof concludes by invoking Lemma 23, which states that  $(-\frac{\Delta}{2\pi} + \chi - 1)\frac{\pi}{\beta} = -\alpha - 1$ .  $\square$

**Lemma 26** (Asymptotics of  $\psi$ ). *Let  $\alpha$  defined as in (4). For  $t \in [0, \frac{\pi}{2}]$  and some positive constant  $C_t$ ,*

$$\psi(r \cos t, r \sin t) \underset{r \rightarrow \infty}{\sim} C_t r^{-\alpha-2}.$$

*Proof.* The result is immediate from the functional equation (27) and Lemma 25.  $\square$

**Proposition 27** (Asymptotics at the origin). *For positive constants  $c_0$  and  $c_1$  we have the following asymptotics*

$$\mathbb{P}_{(u,0)}[T = \infty] \underset{u \rightarrow 0}{\sim} c_0 u^\alpha \quad \text{and} \quad \mathbb{P}_{(0,v)}[T = \infty] \underset{v \rightarrow 0}{\sim} c_1 v^\alpha.$$

*Proof.* The result follows by combining the asymptotic results of  $\psi_1$  and  $\psi_2$  at infinity that we computed in Lemma 25 with the reciprocal of the result in Theorem [6, Thm 33.3]<sup>2</sup>. We begin by denoting  $g(u) := \mathbb{P}_{(u,0)}[T = \infty]$ . Then, by definition,  $\psi_1(x) = \int_0^\infty e^{-xu} g(u) du$ . As  $\psi_1(x)$  has no singularities greater than 0, for every  $A > 0$ , the inverse Laplace transform gives

$$g(u) = \frac{1}{2i\pi} \int_{A-i\infty}^{A+i\infty} e^{ux} \psi_1(x) dx.$$

By Lemma 25, we have  $\psi_1(x) = \frac{C+\eta(x)}{x^{\alpha+1}}$ , where  $\eta$  is a function such that  $\lim_{\infty} \eta = 0$ . Recalling that the Laplace transform of  $u^\alpha/\Gamma(\alpha+1)$  is  $x^{-\alpha-1}$  and performing a change of variables  $s = ux$ , we obtain

$$\begin{aligned} g(u) &= \frac{1}{2i\pi} \int_{A-i\infty}^{A+i\infty} e^{ux} \frac{C+\eta(x)}{x^{\alpha+1}} dx \\ &= u^\alpha \left( \frac{C}{\Gamma(\alpha+1)} + \frac{1}{2i\pi} \int_{Au-i\infty}^{Au+i\infty} e^s \frac{\eta(s/u)}{s^{\alpha+1}} ds \right). \end{aligned}$$

It remains to show that the last integral tends to 0 when  $u \rightarrow 0$ . To do so, consider  $\epsilon > 0$  arbitrarily small. Then there exists  $B > 0$  sufficiently large such that  $\eta(x) < \epsilon$  for all  $|x| > B$ . For all  $u$  such that  $u < 1/B$ , let us consider  $A := 1/u$ . This gives

$$\left| \frac{1}{2i\pi} \int_{Au-i\infty}^{Au+i\infty} e^s \frac{\eta(s/u)}{s^{\alpha+1}} ds \right| < \frac{\epsilon}{2i\pi} \int_{1-i\infty}^{1+i\infty} \frac{1}{s^{\alpha+1}} ds,$$

where the last integral converges for  $\alpha \geq 1$ . The proof concludes by letting  $\epsilon$  tend towards 0.  $\square$

**Theorem 28** (Asymptotics at the origin). *For  $t \in (0, \frac{\pi}{2})$  and some positive constant  $c_t$  we have*

$$\mathbb{P}_{(r \cos t, r \sin t)}[T = \infty] \underset{r \rightarrow 0}{\sim} c_t r^\alpha.$$

*Proof.* This proof follows directly from the asymptotics of the double Laplace transform  $\psi$  computed in Lemma 26. Recall the result used in the proof of Proposition 27 linking the asymptotics of a function at 0 to the asymptotics of its Laplace transform at infinity. The only necessary modification is to apply this result with a polar coordinate transformation. The desired asymptotics then follow with nearly identical calculations.  $\square$

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<sup>2</sup>Doetsch [6, Thm 33.3] establishes that if for some constant  $a$  a function is equivalent to  $u^a$  at 0, then at infinity, its Laplace transform is equivalent (up to a multiplicative constant) to  $x^{-a-1}$ .

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