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# A damage criterion based on energy balance for isotropic cohesive zone model

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1 The objective of this paper is to present an energy damage criterion for cohesive zone models (CZM) within the  
2 framework of the non-linear thermodynamics of irreversible processes (TIP). An isotropic elastic damageable material  
3 is considered for isothermal transformations. Damage is then the only irreversible effect accompanying the deformation  
4 process and this mechanism is supposed to be fully dissipative. Once a separation law and a damage state variable  
5 have been chosen, the paper shows that the damage evolution law can be automatically derived from the energy  
6 balance. From this observation, a CZM is derived for a given choice of traction-separation law and damage state  
7 variable and the quality of its numerical predictions is analyzed using an experimental benchmark bending test  
8 extracted from literature. Damage, elastic and dissipated energy fields around the crack path are shown during this  
9 rupture test. Finally, a numerical simulation of a Brazilian test is proposed where no pre-crack is present in the  
10 specimen. Then, as before, the evolution of the dissipated energy fields are plotted during the loading until the total  
11 failure of the specimen.

12 Keywords cohesive zone, damage, fracture, thermodynamics of irreversible processes, energy balance, Finite element analy-  
13 sis, Brazilian test

14

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## 15 1 Introduction

16 In many engineering applications, the fracture behavior of the structure is crucial, which is why damage  
17 mechanisms have been studied over the last decades, from a theoretical, numerical and experimental point  
18 of view, using different frameworks [Amor et al. 2009](#); [Lemaitre 1996](#). Since the pioneering work of L. M.  
19 Kachanov [Kachanov 1986](#), continuum damage mechanics has become a scientific discipline focusing on the  
20 effects of various microdefects on the macroscopic behavior of materials and structures. Volume damage  
21 descriptions often use a damage variable linked to loss of stiffness [Chaboche et al. 2001](#); [Kondo et al. 2007](#).  
22 For a given elastic material, the stiffness tensor  $\mathbf{E}_d$  at a certain level of isotropic damage, denoted by  $d$ ,  
23 is very often related to the stiffness tensor  $\mathbf{E}_0$  of the undamaged material as follows:  $\mathbf{E}_d = (1 - d)\mathbf{E}_0$ .  
24 Thermodynamically speaking the elastic free energy of damageable material,  $\psi$ , is also defined in the  
25 same way  $\psi = (1 - d)\psi_0$ , where  $\psi_0$  is the elastic free energy of the non damaged material. The conjugate  
26 variable associated with the damage state variable,  $d$ , is by definition  $Y_d = \frac{\partial \psi}{\partial d}$ , leading to a thermodynamic  
27 force  $X_d$  which is given by  $X_d = -Y_d = \psi_0$ . Classically, for material behavior using threshold criterion, the  
28 thermodynamic force  $X_d$  is then used to define the damage rate  $\dot{d}$ , through a yield function  $F$  and its  
29 associated flow rule [Lemaitre 1996](#).

30 An alternative numerical strategy to predict the evolution of damage and crack propagation is to  
31 introduce a 3D cohesive zone model (CZM) between two elastic layers. The CZM model is considered as a  
32 zero-thickness medium where the traction-separation law can be derived from a surface free energy.  
33 This numerical approach has been widely used in many areas of computational mechanics [Allix et al.  
34 1995](#); [Suo et al. 1990](#); [Needleman 1990](#); [Daridon et al. 2011](#). To model elastic damageable materials, the  
35 main assumption is that all the damage that occurs in the bulk is gathered in a cohesive zone. In a finite  
36 element approach the cohesive zone is then located between two elements where elastic behavior remains  
37 linear [Blal et al. 2011](#). Since the pioneering works carried out by Dugdale and Barenblatt [Dugdale 1960](#),  
38 many cohesive-zone models were proposed in the literature [Alfano et al. 2006](#); [Corigliano et al. 2001](#);  
39 [Chen et al. 2009](#). Cohesive-zone models taking fatigue into account were also developed to simulate  
40 crack propagation under cyclic loading conditions [Roth et al. 2014](#); [Nguyen et al. 2001](#) while others were  
41 developed to combine two irreversible phenomena such as damage and plasticity [Kolluri et al. 2014](#). From  
42 an experimental standpoint, the identification of CZM requires the coupling of experimental data and  
43 numerical studies to accurately define the traction separation law [Huon, V. et al. 2010](#); [Chrysochoos et al.  
44 2014](#); [Richefeu et al. 2012](#).

45  
46 A classic criticism of the cohesive zone methods is their relative dependency on mesh size. For example,  
47 intrinsic CZM approach for crack propagation has a deficiency of introducing artificial compliance to  
48 the model and crack path dependency because the cohesive elements are inserted between every 2D  
49 or 3D elements [Zhou et al. 2004](#). To remedy this mesh dependency, associated with the vanishing of  
50 stiffness, a new class of so-called "non-local methods" has appeared in the domains of damage and fracture  
51 mechanics. Two main regularization techniques exist to avoid pathological localization, namely the  
52 integral [Pijaudier-Cabot et al. 1987](#) or the gradient [Lorentz et al. 1999](#); [Amor et al. 2009](#) damage approaches.  
53 Both consist in introducing non-local terms in the formulae of the cohesive model associated with a  
54 characteristic length. For example, in the Thick Level Set method, which is an integral damage approach,  
55 the undamaged zone is separated from the totally damaged zone by a level set [Moës et al. 2011](#). The  
56 damage variable is then an explicit function of the level set. The damage growth in solids is based on the  
57 movement of a layer of finite thickness  $l_c$  within which the damage varies continuously. Then, the damage  
58 rate is directly linked to the set propagation. The non-local aspect of this method is essentially due to the  
59 fact that the configurational force driving the damage front is an average value over the thickness of the  
60 level set in the wake of the front [Lé et al. 2018](#); [Bernard et al. 2012](#). As in bulk damage mechanics, this  
61 approach allows the cracks' initiation and propagation within the same framework.

62  
63 Another way to regularize the damage progress is to add a gradient-dependent term and to derive  
64 the problem of damage evolution from a variational approach based on an energy formulation [Marigo  
65 et al. 2016](#). This approach has also been used to couple the models for gradient-damage and those for  
66 plasticity [Alessi et al. 2015](#) or to develop a cohesive zone model suitable for fatigue fracture [Cazes et al.](#)

67 2015; Abdelmoula et al. 2009; Pham et al. 2012. The macroscopic behavior can be seen as brittle fracture  
68 with a Griffith-like criterion associated with cohesive fracture of the Barenblatt or Dugdale types Daridon  
69 et al. 1997. The variational formulation of fracture mechanics framework has also been used to develop the  
70 Eigen-erosion scheme Pandolfi et al. 2013. In this finite element approximation scheme, the crack tracking  
71 problem is done by successively eroding elements when the attendant elastic energy releaseable exceeds  
72 the critical fracture energy.

73

74 The objective of the following sections is to construct an energy damage criterion for an isotropic elastic  
75 damageable material within the TIP framework. The damage law is based on the premise that the damage  
76 progress is linked to a prescribed evolution in the maximum elastic energy that can be stored within the  
77 material for a given damage state. The damage mechanisms are the only microstructural irreversible effects  
78 accompanying the deformation processes and these mechanisms are fully dissipative (no energy storage is  
79 induced by the material degradation). Naturally, damage dissipation may induce self heating leading to  
80 non-isothermal deformation processes that are consequently irreversible due to heat diffusion. However  
81 for sake of simplicity, only isothermal transformations are considered and the chosen state variables are the  
82 displacement jump  $\mathbf{u}$  and a scalar damage variable, denoted by  $u_d$ . The damage variable  $u_d$  can be related  
83 to the effective displacement as used in Daridon et al. 2011; Park et al. 2009; Bosch et al. 2006; Blal et al. 2011.

84

85 The layout of the paper is as follows. The energy criterion of the damageable elastic cohesive  
86 zone model is presented in Section 2 through a 1D scenario within the TIP framework. In Section 3, a  
87 vectorial extension of the cohesive zone law is proposed for an isotropic damage evolution. In Section  
88 4, the capability of the model is investigated using an experimental benchmark test (i.e. a single-edge  
89 notch-bending specimen for fracture toughness testing) Moës et al. 2011; Wojtacki et al. 2015; Galvez et al.  
90 1996. Mechanical and energy responses are shown and discussed. In particular, several damage, elastic  
91 and dissipated energy fields around the fracture paths are plotted during the crack propagation. The  
92 computation of the dissipated energy fields is of special interest inasmuch as they can be compared with  
93 the ones derived from quantitative IR techniques. Indeed, it is possible to use infrared data to derive heat  
94 sources fields. Corresponding image processing techniques can be found in Chrysochoos 2012; Benaarbia  
95 et al. 2017. In this perspective, a numerical simulation of a Brazilian disc test is finally proposed. This  
96 kind of test is well adapted to infrared imaging inasmuch as the flat surface of the specimen remains  
97 perpendicular to the optical axis of the camera until the crack occurs.

## 98 2 1D scenario

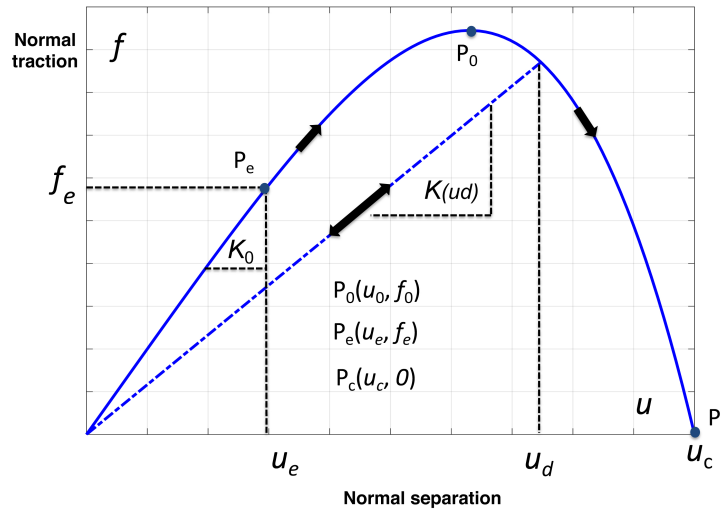
99 The objective of the following section is to briefly review the mechanical concepts classically introduced  
100 with CZM in the case of a 1D monotonic traction and to embed them into the TIP framework to derive,  
101 through an energy criterion, a damage evolution law.



## 102 2.1 Mechanical aspects

103 In the literature, the mechanical response of the cohesive zone is described by the correspondence between  
 104 the “normal traction” force  $f$  supported by the interface and its normal opening displacement often called  
 105 “separation” during a monotonic opening. Depending on the chosen form of the traction-separation  
 106 diagram, the relationships are called bilinear, polynomial or exponential cohesive laws. In Figure 1 a  
 107 polynomial form has been chosen to illustrate the most commonly characteristics of these curves. We find  
 108 the cohesive strength  $f_0$  corresponding to the maximum of the traction–separation curve or its associated  
 109 opening displacement  $u_0$ , the maximum value of separation  $u_c$  corresponding to the crack opening. An  
 110 energy parameter is also often mentioned [Ortiz et al. 1999](#): this is the fracture energy  $A_c = \int_0^{u_c} f(u)du$   
 111 (work of separation), which is the area below the traction-separation curve.

112 This traction-separation curve is considered as a threshold over which the damage develops irreversibly.  
 113 This threshold is an intrinsic characteristic of the cohesive zone behavior. When unloading is considered,  
 114 it is supposed to be purely elastic, assuming that the damage progress stops as soon as the loading point is  
 115 below the threshold curve. For convenience, the elastic unloading paths are often directed towards the  
 116 origin of the traction–separation diagram (see Figure 1). This implies that the elasticity remains linear and  
 117 that there is no residual opening at the end of the unloading.



**Figure 1:** Traction-separation diagram. Monotonic envelope (continuous line), elastic unload or reload (dashed line). An arbitrary polynomial cohesive law has been chosen.

118 The progress of the damage can be depicted by a continuous decrease of the secant stiffness  $K = \frac{f}{u}$   
 119 towards zero until rupture at  $u_c$ . A classical scalar definition of the damage variable can then be given by:

$$D_k = \frac{K_0 - K}{K_0}, \quad (1)$$

120 where  $K_0$  is the initial stiffness of the cohesive zone. The variable  $D_k$  progressively increases from 0 to  
 121 1 when the opening displacement increases from 0 to  $u_c$  (or from  $u_e$  to  $u_c$  when a pure elastic domain,

122  $[0, u_e]$ , is introduced in the traction-separation curve (see Figure 1)).

123 A second possibility is to consider a normalized deformation energy definition of the damage Ortiz  
124 et al. 1999 :

$$D_A = \frac{A}{A_c}, \text{ where } A = \int_0^u f(v)dv . \quad (2)$$

125 Here again, this last definition slightly changes when an elastic domain limited by the point  $(u_e, f_e)$  is  
126 introduced. In such a case, Eq.(2) requires a renormalization:

$$D_A^* = \frac{A^*}{A_c^*}, \text{ where } A^* = \int_{u_e}^u f(v)dv \text{ and } A_c^* = \int_{u_e}^{u_c} f(v)dv . \quad (3)$$

127 Then by construction  $D_A$  and  $D_A^*$  belong to  $[0, 1]$ . In fact, there are many ways to define damage. The  
128 damage process being assumed irreversible, the damage variable rate is often chosen to be non-negative  
129 whatever the loading history, to depict its monotonic evolution. Damage develops when the mechanical  
130 state  $(u, f)$  corresponds to a point of the cohesive threshold curve. In what follows we have chosen a  
131 kinematic definition of the damage variable. Like previously done by numerous authors (e.g. Serpieri et al.  
132 2015), we have chosen the maximum value of the separation  $u_d$  ever reached by the cohesive zone until  
133 instant  $t$ . This damage variable is then defined at instant  $t$  by:

$$u_d = \max \{u(\tau), \forall \tau \leq t\} . \quad (4)$$

134 This variable monotonically increases during the damage progress from 0 to  $u_c$  whatever the loading  
135 path (see Fig.1)) .

## 136 2.2 Energy aspects

137 Usually during a load cycle, the deformation energy  $w_{def}$ , which corresponds to the area surrounded by  
138 the loading curve Eq.(7) is transformed into dissipated energy, denoted by  $w_d$ , and stored energy, denoted  
139 by  $w_s$ , due to the irreversible microstructural transformations accompanying the deformation process.  
140 Part of  $w_{def}$  can also involve strong thermomechanical coupling energy (heat)  $w_{thm}$  Chrysochoos 2012.  
141 An illustrative example of the coupling effects on the mechanical response can be given by the famous  
142 thermoelastic damping presented by Zener in Zener 1938. The general form of the energy balance over a  
143 loading cycle can then be written as:

$$w_{def} = w_d + w_s + w_{thm} . \quad (5)$$

144 For any other loading the elastic energy,  $w_e$ , has to be added so that :

$$w_{def} = w_e + w_d + w_s + w_{thm} , \quad (6)$$

145  $w_e$  vanishing, by construction, over a loading cycle. In the present situation, we only consider  
 146 isothermal transformations with no thermomechanical coupling. Moreover, we assume that damage is a  
 147 pure dissipative mechanism and that, consequently, no energy storage or release of stored energy, due to  
 148 microstructural changes, occurs during the loading. These assumptions imply  $w_s = 0$  and  $w_{thm} = 0$ . For  
 149 any kind of separation-controlled loading  $\{u(\tau), \forall \tau \leq t\}$ , the deformation energy at instant  $t$  is here  
 150 defined by:

$$w_{def}(t) = \int_0^t f(\tau) \dot{u}(\tau) d\tau . \quad (7)$$

151 For monotonic loadings, the mechanical state follows the traction-separation curve. The deformation  
 152 energy then represents the mechanical energy required to reach the damage state  $u_d = u(t)$ . This cost in  
 153 deformation energy can be defined by:

$$w_{def}^d(u_d) = \int_0^{u_d} f(v) dv . \quad (8)$$

154 Another important mechanical energy term is the elastic energy,  $w_e(u, u_d)$ , in the cohesive zone at a  
 155 given state of damage  $u_d$ . It is defined by:

$$w_e(u, u_d) = \frac{1}{2} K(u_d) u^2 . \quad (9)$$

156 Note that this energy is mechanically recoverable during the unloading. This is the reason why it did  
 157 not appear in the general form of the energy balance proposed in Eq.(5) for a complete loading cycle.

158 As previously done for the deformation energy during monotonic loading, we can define the elastic  
 159 energy  $w_e^d(u_d)$  by:

$$w_e^d(u_d) = \frac{1}{2} K(u_d) u_d^2 = w_e(u_d, u_d) , \quad (10)$$

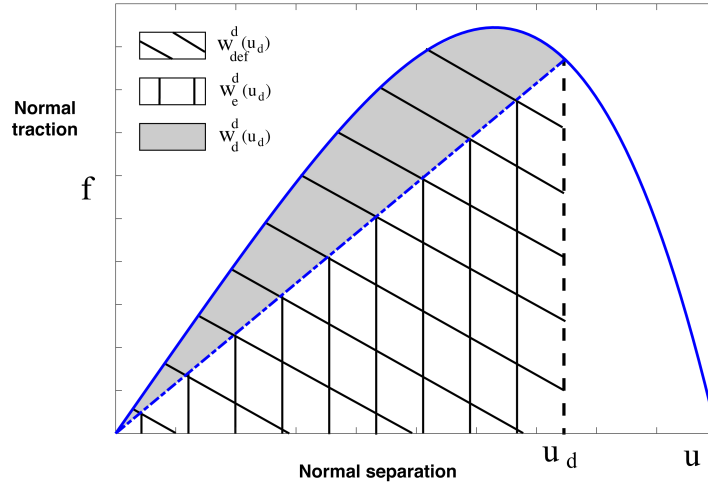
160 which represents the maximum elastic energy mechanically recoverable for a given damage state,  
 161 defined by  $u_d$ .

162 As previously supposed (no thermomechanical coupling energy, no energy storage) the difference

163 between  $w_{def}^d(u_d)$  and  $w_e^d(u_d)$  is attributed to the energy dissipation accompanying the irreversibility of  
 164 damage mechanisms. The dissipated energy,  $w_d^d(u_d)$ , is then defined by :

$$w_d^d(u_d) = w_{def}^d(u_d) - w_e^d(u_d), \quad (11)$$

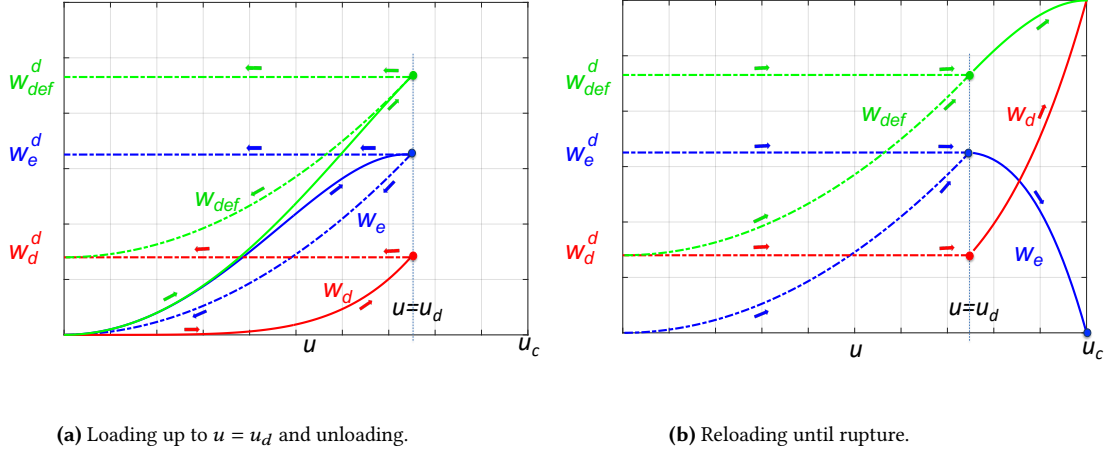
165  $w_d^d(u_d)$ ,  $w_e^d(u_d)$  and  $w_{def}^d(u_d)$  are illustrated in Figure 2.



**Figure 2:** Energy illustration of the traction-separation diagram. Monotonic envelope (continuous blue line), elastic unload (dashed blue line).

166 Based on the mechanical response chosen in Figure 1, the evolutions of the three different energies  
 167 associated with a loading-unloading tensile testing are shown in Figure 3. The deformation, elastic, and  
 168 dissipated energies are plotted in green, blue, and red respectively. The deformation energy  $w_{def}$  is  
 169 naturally the sum of the dissipated  $w_d$  and elastic  $w_e$  energies Eq.(6) since the damage is supposed to  
 170 be the only microstructural transformation which is fully dissipative during loading, see Eq.(11) (no  
 171 energy storage is induced by the microstructural transformations). Figure 3(a) illustrates that during the  
 172 elastic unloading  $w_e^d$  remains constant (no evolution of damage) while  $w_e$  returns to zero. In parallel,  
 173 the deformation energy  $w_{def}$  also decreases and tends towards the energy previously dissipated during  
 174 the first loading cycle,  $w_d^d$ . In Figure 3(b) the elastic reloading while  $u \leq u_d$  is shown (dashed lines) and  
 175 extended by a monotonic loading until rupture for  $u_d = u_c$  (full lines).

176 Under these restrictive assumptions, the area under the traction-separation curve, Figure 1 (equivalent  
 177 to a monotonic traction rupture) is completely dissipated when the cohesive zone vanishes. In the next  
 178 sub-section once the thermodynamic working framework has been specified, this important property is  
 179 discussed. Then, another point to underline is that if the traction-separation curve is classically considered  
 180 as the constituent element of the behavior of the cohesion zone, it is thus the same for the evolutions of  
 181  $w_{def}^d(u_d)$  and  $w_e^d(u_d)$ . Therefore, instead of using the tension-separation curve to describe the damage  
 182 progress, associated with the loss of stiffness, it is also possible to use the evolution of the allowable  
 183 maximum elastic energy  $w_e^d$  to define the threshold function associated with the damage rate.



**Figure 3:** Energy balance evolution during a load-unload-reload process - Continuous lines are associated with the monotonic envelope, dashed lines correspond to the elastic unload and reload.

### 184 2.3 Thermodynamics aspects

185 In this sub-section, the above results and comments are integrated into the TIP framework.

#### 186 2.3.1 Cohesive zone potential and state laws

In the case of isothermal transformations, the chosen state variables are  $(u, u_d)$ . A first gambling of the thermomechanical approach is to assume the existence of a potential  $\psi(u, u_d)$  capable of gathering all the state laws. Here we identify this potential to the elastic energy  $w_e$  defined in Eq.(10):

$$187 \quad \psi(u, u_d) = \frac{1}{2}K(u_d)u^2. \quad (12)$$

187 The state laws are by construction the partial derivatives of the potential with respect to the state  
188 variables. We then define the conjugate variable  $f^r$ , associated with  $u$  which represents the reversible part  
189 of the traction force, and  $A_d$  the conjugate variable associated with  $u_d$  respectively :

$$\left\{ \begin{array}{l} f^r = \frac{\partial \psi}{\partial u} = K(u_d)u \\ A_d = \frac{\partial \psi}{\partial u_d} = \frac{1}{2}K'(u_d)u^2 \end{array} \right., \quad (13)$$

190 where  $K'(u_d) = \frac{dK(u_d)}{du_d}$ .

#### 191 2.3.2 Clausius-Duhem inequality

The irreversibility of the mechanisms accompanying the opening of the cohesive zone is depicted by the Clausius-Duhem inequality which enables the definition of the intrinsic dissipation  $w_d^o$ . In the present framework, it can be written as :

$$w_d^o = w_{def}^o - \dot{\psi} = f\dot{u} - \frac{\partial \psi}{\partial u}\dot{u} - \frac{\partial \psi}{\partial u_d}\dot{u}_d = f^{ir}\dot{u} + X_d\dot{u}_d \geq 0. \quad (14)$$

192 The terms  $w_d^o$  and  $w_{def}^o$  determine the dissipated and deformation energy rates, respectively. The symbol  
 193  $(-)^o$  is introduced to underline that  $w_d$  and  $w_{def}$  are not a priori state functions and are then path  
 194 dependent. Eq.(14) also introduce the irreversible part of the traction force,  $f^{ir} = f - f^r$ , and the  
 195 thermodynamic force  $X_d$  associated with  $\dot{u}_d$ . Note that during an irreversible transformation  $\dot{u}_d > 0$  we  
 196 get  $X_d = -A_d$ . If damage is the only irreversible process, no dissipation has to be associated with  $\dot{u}$ . In  
 197 such a case the irreversible traction force vanishes  $f^{ir} = 0$ . The traction force  $f$  can then be directly  
 198 defined via the state law:

$$f = f^r = K(u_d)u . \quad (15)$$

199 Moreover, the intrinsic dissipation becomes with Eq.(13) and Eq.(14) :

$$w_d^o = X_d \dot{u}_d = -\frac{1}{2} K'(u_d) u^2 \dot{u}_d \geq 0 . \quad (16)$$

200 The fact that  $\dot{u}_d \geq 0$  implies  $K'(u_d) \leq 0$  what is physically consistent. The irreversible nature of  
 201 damage leads to a degradation of the secant stiffness.

### 202 2.3.3 Threshold function and damage evolution law

203 In the TIP framework the thermodynamic forces are supposed to be function of the state variable rates. In  
 204 the case of the linear TIP proposed by Onsager [Onsager 1931](#), the correspondence between thermodynamic  
 205 forces and state variable fluxes is linear. The Onsager matrix is supposed to be symmetric positive definite  
 206 in order to verify the Clausius-Duhem inequality (positive dissipation) whatever the thermodynamic  
 207 process. Extension to non-linear theory exists as for example the formalism of Generalized Standard  
 208 Materials [Halphen et al. 1975](#). Based on the hypothesis of normal dissipation, the thermodynamic forces  
 209 derive from a convex dissipation potential or equivalently, state variables rates derived from a dual  
 210 dissipation potential, function of the thermodynamic forces. This dissipation potential can also involve the  
 211 state variables of the model as parameters [Lemaitre 1996](#). A common approach is then:

- 212 - to define a threshold function depending on the thermodynamic forces (and possibly state variables)
- 213 - to write that irreversibility occurs and develops if the thermodynamic state is on the threshold and  
 214 remains on it during a time increment.

215 Note that once the state laws (derived from the thermodynamic potential) and complementary laws  
 216 (derived from the dissipation potential) have been written, it is then possible to deduce the evolution of the  
 217 energy balance associated with the transformation.

In what follows in as much as our approach is directly based on the energy balance form imposing by  
 construction non-negative dissipation, the existence of the threshold function will not be associated with  
 the normal dissipation hypothesis. Indeed, the current elastic domain is characterized by  $w_e^d(u_d)$  the

maximum elastic energy available for a given damage state which also corresponds to the energy required to further damage the material. Then the damage energy criterion based on the energy balance is defined by :

$$w_e(u, u_d) \leq w_e^d(u_d) . \quad (17)$$

218 The evolution law for  $u_d$  is then derived from the fact that for the damage to occur the maximum  
219 elastic energy allowable in the material has to be and remain on the threshold during the loading step, i.e.

$$\begin{cases} w_e(u, u_d) = w_e^d(u_d) & \text{(a)} \\ \dot{w}_e(u, u_d) = \dot{w}_e^d(u_d) & \text{(b)} \end{cases} . \quad (18)$$

220 The first equality gives naturally  $u_d = u$ . The second equality leads to a proposal of evolution equation  
221 for the damage :

$$\dot{u}_d = \begin{cases} \dot{u} & \text{if } u = u_d \text{ and } \dot{u} \geq 0 \\ 0 & \text{if } u < u_d \text{ or } \dot{u} \leq 0 \end{cases} , \quad (19)$$

222 what is consistent if we remind the definition of the damage state variable *Eq.(4)* and the fact that the  
223 damage increases irreversibly,  $\dot{u}_d \geq 0$ .

224 To be fully compatible with non-linear TIP framework, the final step is to propose a threshold function  
225 that takes the thermodynamic force  $X_d$  into account. As previously stated, we consider a derivative form  
226 of the energy balance to get this threshold function *Eq.(18)b*. By using *Eq.(13)* and *Eq.(16)*, we get:

$$\frac{d w_e}{d t} = -X_d \dot{u}_d + K(u_d) u \dot{u} . \quad (20)$$

227 On the threshold, *Eq.(20)* becomes :

$$\frac{d w_e^d}{d u_d} \dot{u}_d = (-X_d + K(u_d) u_d) \dot{u}_d . \quad (21)$$

228 Then a threshold function  $F$  involving the thermodynamic force  $X_d$  and the state variables can be  
229 taken under the form :

$$F(X_d; u, u_d) = K(u_d) u - X_d - \frac{d w_e^d}{d u_d} \leq 0 . \quad (22)$$

230 To be consistent with the incremental form of the energy balance, the equality  $F(X_d; u, u_d) = 0$ , gives  
 231 once again  $u_d = u$  while the consistency condition  $dF = 0$  leads to  $du = du_d$ , or equivalently to Eq.(19).

232 To be precise, the full calculation of  $dF = 0$  at  $u = u_d$  leads to:

$$(K(u_d) + 2K'(u_d)u_d)(du - du_d) = 0, \quad (23)$$

233 then  $du = du_d$ , except possibly when  $u_d = -\frac{K(u_d)}{2K'(u_d)}$ .

#### 234 2.3.4 Some comments about the damage evolution equations

235 To depict the evolution of damage, in addition to the traction-separation curve data, the literature often  
 236 proposes a specific evolution equation in the form of  $\dot{D} = \dot{D}(f, D, \dot{u})$  whatever the definition of the damage  
 237 variable  $D$  [Roe et al. 2003](#); [Bouvard et al. 2009](#); [Kuna et al. 2015](#).

238 In the foregoing, because of the hypotheses explicitly made on the energy balance (i.e. damage is  
 239 the only dissipative mechanism and it is totally dissipative), the damage evolution law is fixed by the  
 240 definition of the damage variable itself and by the explicit form of the energy balance. Note that the  
 241 damage evolution law Eq.(19) deduced from the energy criterion Eq.(18) is perfectly compatible with the  
 242 definition of the damage variable itself given in Eq.(4). We can also note that this evolution law is an  
 243 extremely simple form of the general equation proposed by [Roe et al. 2003](#), but here this law is totally  
 244 imposed by the shape of the traction-separation curve or equivalently by the threshold  $w_e^d(u_d)$  of Eq.(10)

To set ideas, let's consider the following simple case: let  $f(u)$  be a 1D traction-separation law. We suppose that the elastic energy is, as often, written as:  $\psi(u, D) = (1 - D)K_0 \frac{u^2}{2}$ , where  $D$  is the isotropic damage variable,  $u$  the displacement jump, and  $K_0$  the elastic stiffness of the virgin cohesive zone. We consider a monotonic loading. The deformation energy rate is given by definition:

$$(w_{def})^0 = f(u) \frac{du}{dt}$$

where  $f(u)$  follows the traction-separation curve. The elastic energy rate can be split in two parts:

$$(w_e)^0 = (1 - D)K_0 u \frac{du}{dt} - K_0 \frac{u^2}{2} \frac{dD}{dt}$$

If we assume now that the damage is the only irreversible mechanism, then the traction force is the conjugate variable of the displacement jump, where  $f(u) = \frac{\partial \psi(u, D)}{\partial u} = (1 - D)K_0 u$  and  $(w_e)^0 = (w_{def})^0 - K_0 \frac{u^2}{2} \frac{dD}{dt}$

If now the damage is supposed to be exclusively dissipative (no internal stored energy), then the dissipation is given by:

$$(w_d)^0 = K_0 \frac{u^2}{2} \frac{dD}{dt} = (w_{def})^0 - (w_e)^0$$



Following the traction-separation curve, the damage evolution has to verify:

$$\frac{dD}{dt} = 2 \frac{[(w_{def})^0 - (w_e)^0]}{K_0 \frac{u^2}{2}}$$

Noting that for each current point  $(u, f(u))$  of the traction separation curve, we have  $w_e = \frac{1}{2}f(u)u$ , the time derivation, following the curve, reads :

$$(w_e)^0 = \frac{1}{2}f(u) \frac{du}{dt} + \frac{1}{2} \frac{df(u)}{dt} u$$

Then,

$$\frac{dD}{dt} = \frac{f(u) \frac{du}{dt} - \frac{df(u)}{dt} u}{K_0 \frac{u^2}{2}}$$

245 The right-hand member of this equation is fully determined by the traction-separation curve. Any form of  
 246 damage evolution law, incompatible with this previous equation, would lead to an energy balance form  
 247 incompatible with the initial energy assumptions (i.e. form of the free energy, damage unique and exclusive  
 248 dissipative mechanism). The consequences could be the appearance of energy storage mechanisms, i.e.  
 249  $\dot{w}_d^d < \dot{w}_{def}^d - \dot{w}_e^d$ , or internal energy transformation into dissipated energy (release of stored energy), i.e.  
 250  $\dot{w}_d^d > \dot{w}_{def}^d - \dot{w}_e^d$ . Taking into account this stored energy variations should lead to the introduction of new  
 251 internal state variables and/or to a change of the deformation energy rate definition [Fremond 2002](#).

## 252 3 3D cohesive zone model

253 In this paragraph, we propose an extension to a 3D vectorial version of the CZM where the isotropic  
 254 damage is controlled by the evolution of the maximum recoverable elastic energy,  $w_d^e(u_d)$ . Isotropic  
 255 damage means here that a scalar state variable is solely used to describe the damage evolution. This  
 256 generalization has been made by following the same approach as the one previously proposed, namely  
 257 define a damage variable and a energy balance where the damage is the only dissipative phenomenon.

### 258 3.1 Mechanical variables

259 Regarding the mechanical description of the cohesive zone, the traction force and the separation become  
 260 now vectors. Let us introduce a frame of reference where directions 1 and 2 correspond to the tangent  
 261 plane of the cohesive zone while direction 3, is the normal direction. The traction vector,  $\mathbf{f}$ , whose  
 262 components are  $(f_{t_1}, f_{t_2}, f_n)$  and the separation vector,  $\mathbf{u}$ , which has 3 components denoted by  $(u_{t_1}, u_{t_2}, u_n)$   
 263 are introduced. As is conventionally admitted in CZM, the normal move jump denoted by  $u_n$  is positive  
 264 or null. This unilateral condition is taken into account by a Signori type relationship in the numerical  
 265 simulations, using the open source software [LMGc90 Dubois et al. 2006](#), performed at the end of this article.

### 266 3.2 Cohesive zone potential and state equations

A set of state variables has first to be chosen. Here we selected the components  $(u_{t_1}, u_{t_2}, u_n)$  of the separation vector and a scalar damage variable denoted by  $u_d$ . Then, to generalize the form of the cohesive zone potential proposed in Eq.(12), the following form, inspired by Bouvard et al. 2009, is adopted :

$$\begin{aligned}
 \psi(\mathbf{u}, u_d) &= w_e(\mathbf{u}, u_d) \\
 &= \frac{1}{2} \left( K_n(u_d)u_n^2 + K_t(u_d)u_{t_1}^2 + K_t(u_d)u_{t_2}^2 \right) , \\
 &= \frac{1}{2} K_n(u_d)(u_n^2 + \alpha u_{t_1}^2 + \alpha u_{t_2}^2) \\
 &= \frac{1}{2} K_n(u_d)u_{eq}^2
 \end{aligned} \tag{24}$$

where:

$$u_{eq} = (u_n^2 + \alpha u_{t_1}^2 + \alpha u_{t_2}^2)^{\frac{1}{2}} . \tag{25}$$

267 The parameter  $\alpha$  is the ratio between  $K_t(u_d)$ , the tangential and  $K_n(u_d)$ , the normal secant stiffnesses  
 268 at a given  $u_d$ . In the case of isotropic damage  $\alpha$  is a constant.

269 In Eq.(26), a 3D formulation of the scalar depicting the isotropic damage is given. By construction,  $u_d$   
 270 takes the 3D aspect of the separation vector  $\mathbf{u}$  into account and then  $\dot{u}_d$  is non-negative and de facto  
 271 respects the irreversibility of the damage progress.

$$u_d = \max \{ u_{eq}(\tau), \forall \tau \leq t \} . \tag{26}$$

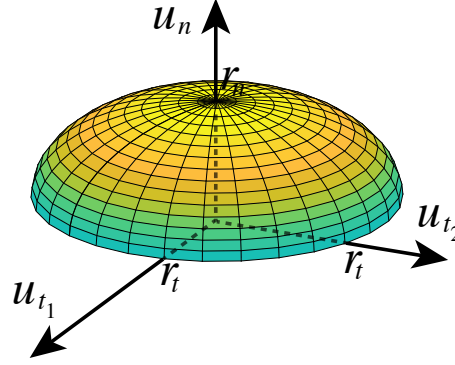
272 By definition, the state laws are the partial derivatives of the cohesive zone potential Eq.(24). They  
 273 introduce the components of the reversible traction vector  $\mathbf{f}^r$  and the conjugate variable  $A_d$  associated  
 274 with  $(u_{t_1}, u_{t_2}, u_n)$  and  $u_d$  respectively :

$$\left\{ \begin{array}{l}
 f_n^r = \frac{\partial \psi}{\partial u_n} = K_n(u_d)u_n \\
 f_{t_1}^r = \frac{\partial \psi}{\partial u_{t_1}} = \alpha K_n(u_d)u_{t_1} = K_t(u_d)u_{t_1} \\
 f_{t_2}^r = \frac{\partial \psi}{\partial u_{t_2}} = \alpha K_n(u_d)u_{t_2} = K_t(u_d)u_{t_2} \\
 A_d = \frac{\partial \psi}{\partial u_d} = \frac{1}{2} K'(u_d)u_{eq}^2
 \end{array} \right. . \tag{27}$$

275 Because only damage induces irreversibility, no dissipation has to be associated with the component of  
 276 the separation vector. The reversible part  $\mathbf{f}^r$  of the separation can therefore be identified with  $\mathbf{f}$ , then  
 277  $\mathbf{f} = \mathbf{f}^r$  .

### 278 3.2.1 Energy definition of the damage threshold

279 To extend the damage energy criterion to a 3D isotropic damageable CZM, it is possible to choose the  
 280 damage variable,  $u_d$ , whose evolution is directly related to that of the elastic energy  $w_e^d$ . In the 3D case, this  
 281 maximum elastic energy, for a given damage state  $u_d$ , describes in the displacement space a half spheroid of  
 282 radii  $r_n(u_d) = \left(\frac{2w_e^d(u_d)}{k_n(u_d)}\right)^{\frac{1}{2}}$  and  $r_t(u_d) = \left(\frac{2w_e^d(u_d)}{\alpha k_n(u_d)}\right)^{\frac{1}{2}} = \frac{r_n(u_d)}{\sqrt{\alpha}}$  as shown in Figure 4. As the normal jump  
 283 denoted  $u_n$  is by definition positive or null only half of the spheroid is reachable for any separation states.



**Figure 4:** 3D representation of the reachable separation states for a given damage state  $u_d$ . The color variation represents the value of  $u_n$ .

284 As long as the further separation states,  $\mathbf{u}$ , respect the damage energy criterion (i.e.  $w_e(\mathbf{u}, u_d) <$   
 285  $w_e^d(u_d)$ ), the behavior remains elastic. Then for a given opening such that  $u_{eq} = u_d$ , the elastic energy  
 286 reaches the maximal value associated with this damage state (i.e.  $w_e(\mathbf{u}, u_d) = w_e^d(u_d)$ ). Once the surface  
 287 of the spheroid is reached:

- 288 • either the separation increment  $\delta\mathbf{u}$  is directed towards the inside of the spheroid, and an elastic  
 289 unloading at constant damage can be observed,
- 290 • or  $\delta\mathbf{u}$  is directed towards the outside of the spheroid, and then the damage develops defining a new  
 291 elastic limit surface.

292 For isotropic damage, a single evolution equation for  $u_d$  is required. We have already underlined that for  
 293 threshold behavior law, the yield function depends on the thermodynamic forces and possibly on the states  
 294 variables themselves, acting as parameters. In the present case, the thermodynamic force of the model,  
 295 associated with the damage variable rate, is  $X_d$ . A generalized form of the yield criterion proposed in  
 296 Eq.(22) is chosen where the role of  $u$  used in the 1D scenario is played by  $u_{eq}$ . So, the proposed yield  
 297 energy criterion Eq.(17), may be rewritten using the thermodynamic force,  $X_d$  as:

$$F(X_d; u_{eq}, u_d) = K_n(u_d) u_{eq} - X_d - \frac{d w_e^d}{d u_d} \leq 0. \quad (28)$$

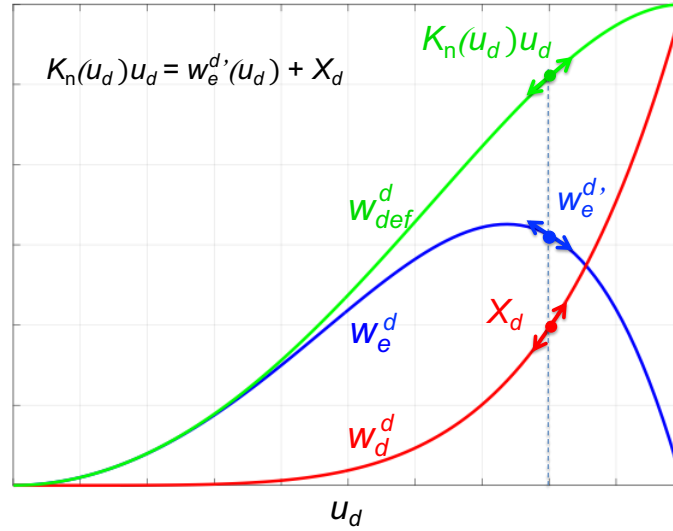
298 Damage develops if the threshold is reached,  $F(X_d; u_{eq}, u_d) = 0$  and if the consistency condition is  
 299 verified,  $\dot{F}(X_d; u_{eq}, u_d) = 0$ . For the same reasons as the ones shown for the 1D model, the evolution law of

300 the parameter  $u_d$  is written as:

$$\dot{u}_d = \dot{u}_{eq} \text{ if } u_d = u_{eq} \text{ and } \dot{u}_{eq} \geq 0, \quad (29)$$

301 results which, as already underlined, are imposed by the very definition of the damage variable.

302 An illustration of the energy criteria is given in Figure 5. Following a monotonic loading (i.e. remaining  
303 on the  $w_e^d(u_d)$  curve),  $K_n(u_d)u_d$  is the slope of the deformation energy  $w_{def}^d$ ,  $X_d$  is the slope of the  
304 dissipated energy  $w_d^d$  and  $(w_e^d)'$  is naturally the slope of the maximal allowable elastic energy  $w_e^d$ .



**Figure 5:** Illustration of the damage energy criterion -  $u_{eq} = u_d$

305 To conclude section 3, we would like to stress once more the fact that the damage evolution law is not  
306 here a matter of choice. It is imposed by the chosen form of the energy balance and by the definition of the  
307 damage state variable.

308 The state equations Eqs(27) and the evolution equation Eq.(29) will be, in what follows, implemented  
309 in a home-made finite element code. The different material parameter of the constitutive equations will be  
310 specified. In order to show the capabilities of such a CZM, two types of simulations are made here after.  
311 The first one is a bending test whose numerical results are compared with experimental one's. The second  
312 one is a Brazilian disc test whose the material is made of heterogeneous elastic grains.

## 313 4 Numerical Implementation

314 To illustrate the potentiality of the proposed model, simulations reproducing a common benchmark  
315 extracted from the literature Galvez et al. 1996 were carried out. It is important to notice that the objective  
316 of this practical comparison is simply to show the operability of the model and not to optimize its  
317 parameters in order to fit the benchmark. The numerical implementation of the previous model is then  
318 done in the code L'Gc90 based on Non-Smooth Contact Dynamics (NSCD) Moreau 1988; Jean 1999; Jean

319 et al. 2001. The NSCD method is dedicated to solving problems related to dynamic systems with unilateral  
 320 constraints. It is therefore particularly suitable for contact friction problems. It proposes a non-smooth  
 321 treatment (no compliance, no penalty) of the conditions of contact Jean 1999, which is explicit in the  
 322 definition of  $u_n$ . The way which adhesion is taken into account in this method makes it possible to  
 323 consider each point of contact as a cohesive zone. Then the mechanical behavior of the cohesive zones  
 324 may vary at any point of the spatial discretization of the problem. This relevant modeling framework was  
 325 then adopted to numerically simulate crack propagation with cohesive zone Champagne et al. 2014 .  
 326

### 327 4.1 Bending test

328 To compare the proposed model with a benchmark found in literature Galvez et al. 1996, the form of the  
 329 maximum storable elastic energy, which we remember is  $w_e^d(u_d)$ , must be specified in order to be able to  
 330 implement it in LMGc90, the open source platform <sup>1</sup> used to carry out the simulations Dubois et al. 2011.  
 331 This benchmark, illustrated Figure 6, traces the evolution of a crack in mixed mode to be followed. In the  
 332 context of this feasibility study, a simple quadratic form of  $w_e^d(u_d)$  is proposed. In what follows, we also  
 333 assumed the existence of a pure elastic domain and thus the existence of a threshold equivalent elastic  
 334 displacement  $u_{eq}^e$ , simply denoted by  $u_e$ . The maximum storable elastic energy as a function of the damage  
 335 parameter  $u_d$  simply reads:

$$w_e^d(u_d) = A(u_d - u_c)^2 + B(u_d - u_c), \text{ if } u_e \leq u_d \leq u_c \quad (30)$$

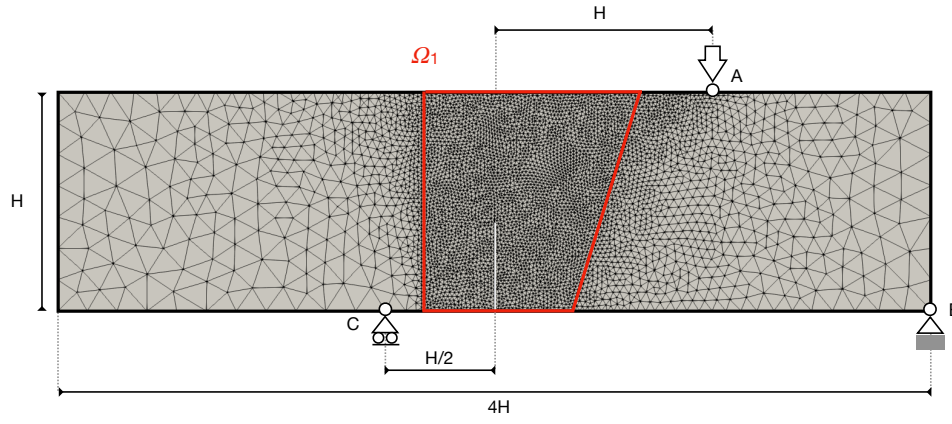
where  $u_c$  is the critical equivalent displacement corresponding to the crack onset. Parameters  $A$  and  $B$   
 are two constants chosen to ensure the  $C_1$  continuity of the maximum storable elastic energy,  $w_e^d(u_d)$ , at  
 the threshold equivalent elastic displacement,  $w_e^d(u_e) = \frac{1}{2} K_n^0 u_e^2$ . They are defined by:

$$\begin{cases} A = -\frac{1}{2} K_n^0 u_e \frac{(2u_c - u_e)}{(u_c - u_e)^2} \\ B = -K_n^0 \frac{u_c u_e}{u_c - u_e} \end{cases}, \quad (31)$$

336 where  $K_n^0$  is the initial normal stiffness of the CZM.

337 The thickness of the sample, denoted by  $H$ , is equal to 0.3 m while its length is equal to 1.2 m. A 0.15 m  
 338 pre-crack is located in the middle on the lower edge. The point B is fixed in both x and y directions  
 339 whereas the point A is only fixed in the y direction. A displacement is imposed on the point A to load  
 340 the structure. The mesh is composed of 3 parts: Two coarse meshes, the left and the right parts of the  
 341 structure composed respectively of 958 and 2 063 T3 elements, where no interface elements have been  
 342 introduced between the different meshes and a finer mesh, assuring the continuity of the structure (domain  
 343  $\Omega_1$  in Figure 6), composed of 6 723 T3 elements where the crack path is supposed to appear and where

<sup>1</sup> <https://git-xen.lmgc.univ-montp2.fr/lmgc90/>



**Figure 6:** Characteristics of the benchmark issued from Galvez et al. 1996 used for simulation

$K_n^0$ (N m <sup>-1</sup> )	$\alpha$	$u_e$ (m)	$u_c$ (m)
$2.48 \cdot 10^9$	0.5	$0.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$

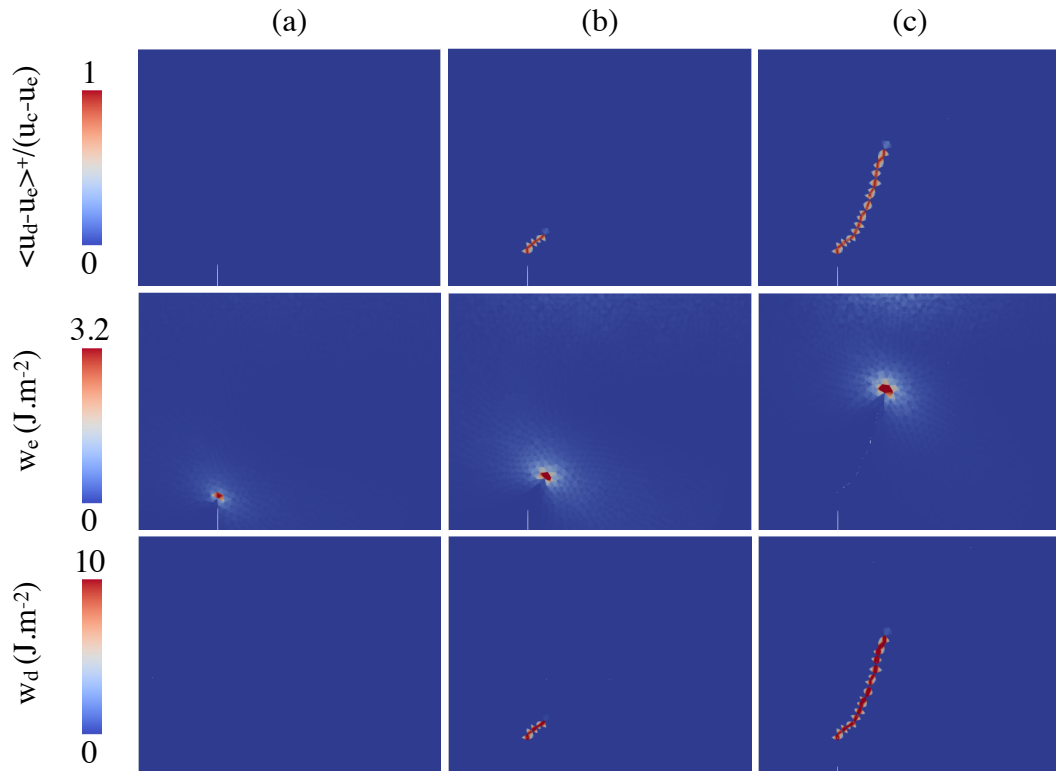
**Table 1:** Parameter values of the CZM

344 interface elements are therefore introduced between each element. The interactions between elements of  
 345  $\Omega_1$  are governed by the proposed cohesive zone model where the initial secant elastic stiffness,  $K_n^0$  and  $K_t^0$ ,  
 346 are chosen to satisfy the criterion proposed in Blal et al. 2011 to limit the reduction of stiffness due to the  
 347 presence of CZM. It is important to underline that the objective of this practical comparison is simply to  
 348 show the operability of the model and not to optimize its parameters in order to fit the benchmark. The  
 349 values of the CZM parameters are summarized in Table 1.

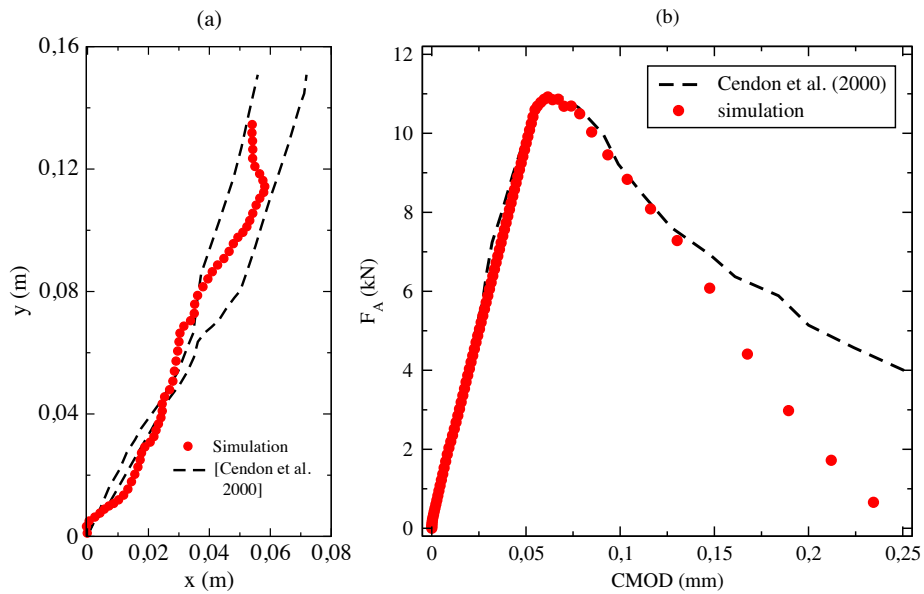
350 Figure 7 shows the evolution, for different simulation times, of different characteristic quantities  
 351 associated with the model : the damage variable, the elastic energy  $w_e$  and the dissipated energy  $w_d$ . In  
 352 order to present a quantity varying from 0 to 1 the damage ratio, as a function the damage variable, is  
 353 introduced and defined by  $\frac{\langle u_d - u_e \rangle^+}{u_c - u_e}$ . To improve the visibility of these different quantities supported by  
 354 the interfaces, they are projected on adjacent elements.

355 Figure 7(a), corresponding to a pre-cracking state, shows a concentration of the elastic energy at the  
 356 outset of the crack tip. However, the damage criterion has not been reached within the cohesive zone so  
 357 that no damage or dissipation has yet occurred (see Eq.(17)). The corresponding map to  $\frac{\langle u_d - u_e \rangle^+}{u_c - u_e}$  and  $w_d$   
 358 are then uniformly equal to 0. As expected, Figure 7 (b) and (c), corresponding to two post-cracking steps,  
 359 highlight the correlation between the evolution of the dissipated energy and the damage ratio. The elastic  
 360 energy is still concentrated ahead of the crack tip, then returns to zero along the crack lips. In contrast, the  
 361 dissipated energy related to the damage evolution can be exhibited all along the crack path. Similarly, the  
 362 damage field allows the cracking path to be tracked.

363 To exhibit the capability of our CZM where only the shape of the cohesive energy associated with a  
 364 simple energy balance is needed (cf. Eq.(30)), different quantities, numerically obtained, are compared with  
 365 experimental measurements present in the literature Cendón et al. 2000. For such comparisons, Figure 8  
 366 presents both the classical crack path monitoring and the load vs. CMOD curve (Crack Mouth Opening  
 367 Displacement).



**Figure 7:** Visualization of the damage ratio (top row), the elastic energy (center) and the dissipated energy (bottom row) during the crack propagation



**Figure 8:** Comparison of numerical macroscopic measurements associated to the crack evolution with experimental results [Cendón et al. 2000](#): a) The crack path and b) the load vs. CMOD curve.

368 In Figure 8(a) and Figure 8(b), the red dot line corresponds to the simulation result while the black  
 369 dashed lines represent the crack envelop obtained experimentally Cendón et al. 2000. In Figure 8a, the  
 370 crack obtained numerically corresponds closely to the experimental envelope. The starting angle is  
 371 strongly related to the discretization around the initiation point, explaining the slight difference at the  
 372 beginning of the initiation. Then, the path is corrected and repositioned in the experimental envelope until  
 373 the end of the simulation.

374 Concerning the force vs. CMOD curves, they fit perfectly in the section corresponding to the linear  
 375 increase. This highlight that the introduction of a 2D interface element, where the values of  $K_n^0$  and  $K_t^0$   
 376 satisfy the criterion proposed in Blal et al. 2011 between each elements of  $\Omega_1$  do not affect the global  
 377 stiffness of the sample. The maximum force obtained is also in good agreement with that obtained in the  
 378 experiment, as well as the beginning of the non-linear decreasing part of the CMOD curve occurring at the  
 379 initiation of cracking. In the last part, the curves diverge. This difference is partly explained by the fact that  
 380 the numerical simulation is two-dimensional while the experiments are three-dimensional. Indeed, not all  
 381 deformation modes are taken into account (especially out-of-plane modes), which explains this different  
 382 behavior at the end of the simulation. Moreover, we have arbitrarily chosen a 2nd degree polynomial to  
 383 characterize the damage of the cohesive zone model, Eq.(30). This choice could be fine-tuned in order to  
 384 better account for experiences by taking a Needleman-type damage, Bosch et al. 2006; Needleman 1990.

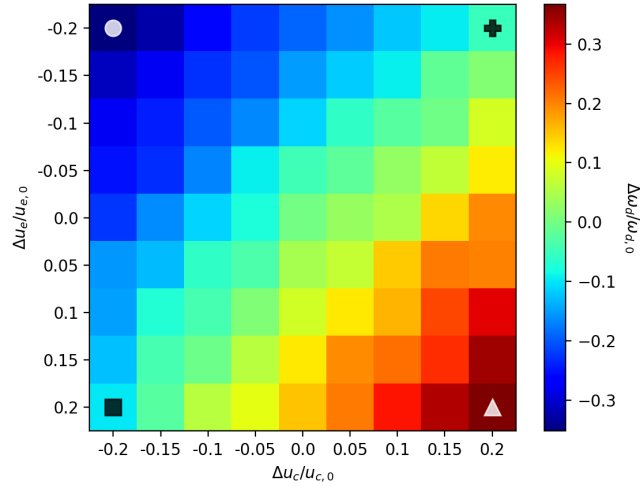
## 385 4.2 Sensitivity study

386 Finally, in order to see the impact of a variation in the parameters  $u_e$  and  $u_c$  on the overall behavior of  
 387 the system and more particularly on the evolution of the force vs. CMOD curves, a sensitivity study is  
 388 proposed. The influence of these parameters on the crack path is not presented because it is not very  
 389 significant. The influence of these parameters on the energy available to be dissipated in the model  
 390 is pointed out in Figure 9. The parametric study is carried out relative to the reference point (0, 0)  
 391 corresponding to the results presented in Figure 8(b) with the parameters define in table 1. With the chosen  
 392 law, a variation of  $u_c$  has almost the same consequence as a variation of  $u_e$  in terms of the energy available  
 393 to be dissipated. Then the map presented in Figure 9 is symmetric in the regard of the circle-triangle  
 394 diagonal. During the different parametric studies, the color code used for the curves will refer to the one  
 395 defined in Figure 9.

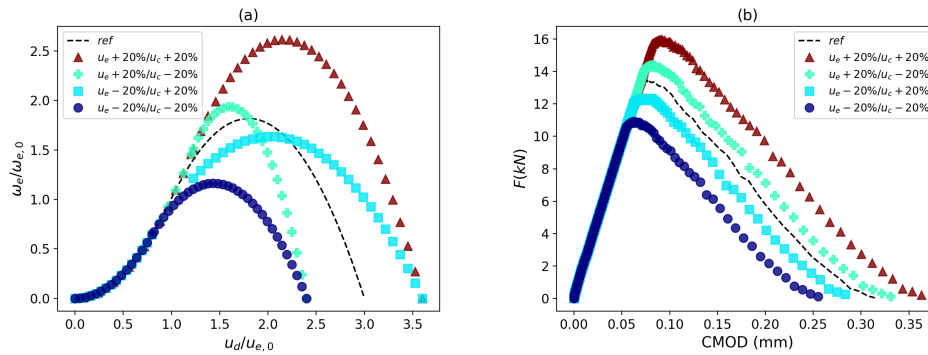
396 Figure 10 presents the normalized plots of  $w_e^d(u_d)$  for different values of  $u_e$  and  $u_c$  (Figure 10(a)) and the  
 397 corresponding force vs. CMOD curves (Figure 10(b)). The normalization parameters are  $u_{e,0} = 0.5 \cdot 10^{-2}$  and  
 398  $w_{e,0} = w_{e,0}^d(u_{e,0})$  using the values of parameters in Tab. 1. Even if the shape of the curves is significantly  
 399 different in Figure 10 (a), the energies available to be dissipated for the case represented by a cross and the  
 400 one represented by a square are of same order of magnitude. The maxima order observed at the scale of  
 401 the CZM models (Figure 10(a)) is conserved at the scale of the structure (Figure 10(b)).

402 In Figures 11 to 13, we observe respectively the influence of  $u_c$  and  $u_e$  on the force vs. CMOD curves.  
 403 These figures show that the influence of the variation of  $u_c$  is less than that of  $u_e$ . Indeed, where we

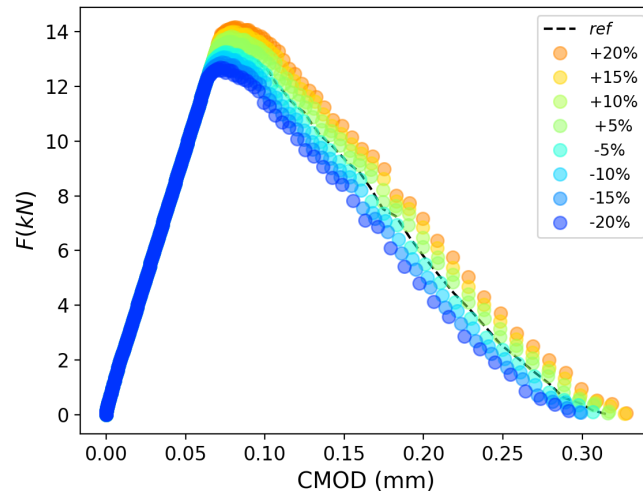




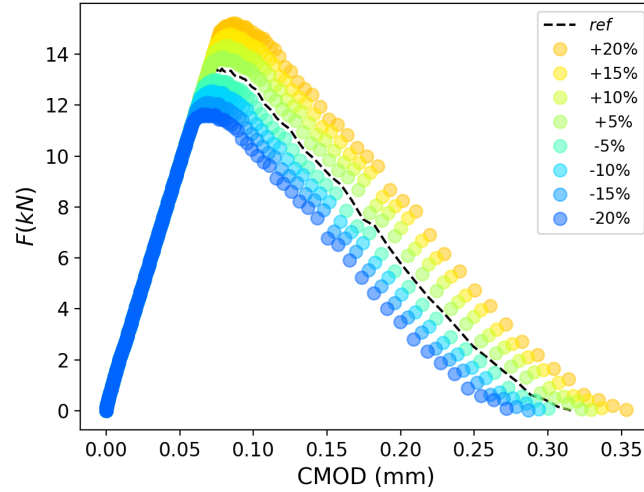
**Figure 9:** Map of the normalized dissipated energy variations as a function of the variations of  $u_e$  and  $u_c$ . The symbols used at the four corners of the map identify the curves shown in Figure 10.



**Figure 10:** (a) Normalized plots of  $w_e^d(u_d)$  for different values of  $u_e$  and  $u_c$ . The normalization parameters are  $u_{e,0} = 0.5 \cdot 10^{-2}$  and  $w_{e,0} = w_{e,0}^d(u_{e,0})$  using the values of parameters in Tab.1. (b) Corresponding force vs. CMOD curves.

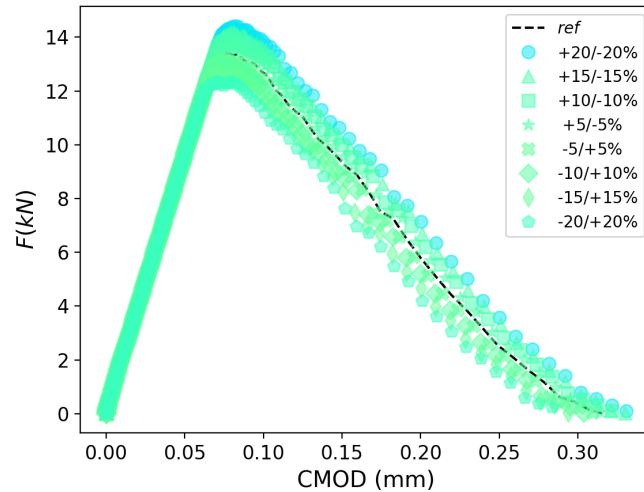


**Figure 11:** Comparison between the reference Load vs. CMOD curve (dash line) and the ones related to the variation of  $u_c$  of  $\pm 20\%$  while  $u_e$  constant.



**Figure 12:** Comparison between the reference Load vs. CMOD curve (dash line) and the ones related to the variation of  $u_e$  of  $\pm 20\%$  while  $u_c$  constant.

404 observe for a variation of  $u_c$  a variation of less than 10% on the critical values of the curve ( $F_{max}$  and  
 405  $CMOD_{max}$ ), while a variation of more than 20% for an equivalent variation of  $u_e$  is observed. Nevertheless,  
 406 in both cases, an increase of the damage energy  $w_d$  induces an increase of the CMOD and loading maxima  
 407 in the Load vs. CMOD curve. In this model where an elastic domain is assumed,  $u_e$  is the threshold where  
 408 the damage begins to occur. This value determines the outset of the non-linear response of the structure.  
 409 This is exhibited in Figure 12 where an increase of  $u_e$  at the local scale induces an increase of the maximal  
 410 force at the macroscopic scale and a delay of the occurrence of the nonlinear response of the curves.



**Figure 13:** Comparison between the reference Load vs. CMOD curve (dash line) and the ones related to the variation of  $u_c$  and  $u_e$  of  $\pm 20\%$  while  $w_d^0$  is constant.

411 Figure 13 demonstrates that the non-linear region of the curve is also governed by the shape of the  
 412 energy curve (cf. Figure 10). Although the dissipative energy in this parametric study is almost constant,  
 413 we observe a variation of about 10% on the characteristic values of the response curve. So by combining  
 414 the effects of  $u_e$ ,  $u_c$  and the shape of the local curve (cf. Figure 10) it is possible to obtain a better optimal

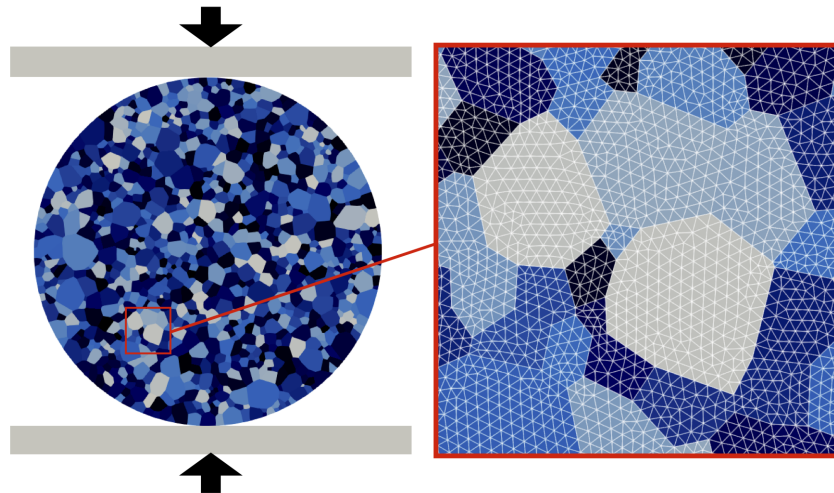
$K_n^0$ (N m <sup>-1</sup> )	$\alpha$	$u_e$ (m)	$u_c$ (m)
$2.48 \cdot 10^{15}$	0.4	$0.8 \cdot 10^{-6}$	$1.0 \cdot 10^{-6}$

**Table 2:** Parameter values of the CZM

415 result to fit experiments. Thus the experimental characterization of this type of local curve depicting the  
 416 micro-structural phenomenon linked to fracture is relevant and is still an ongoing problem.

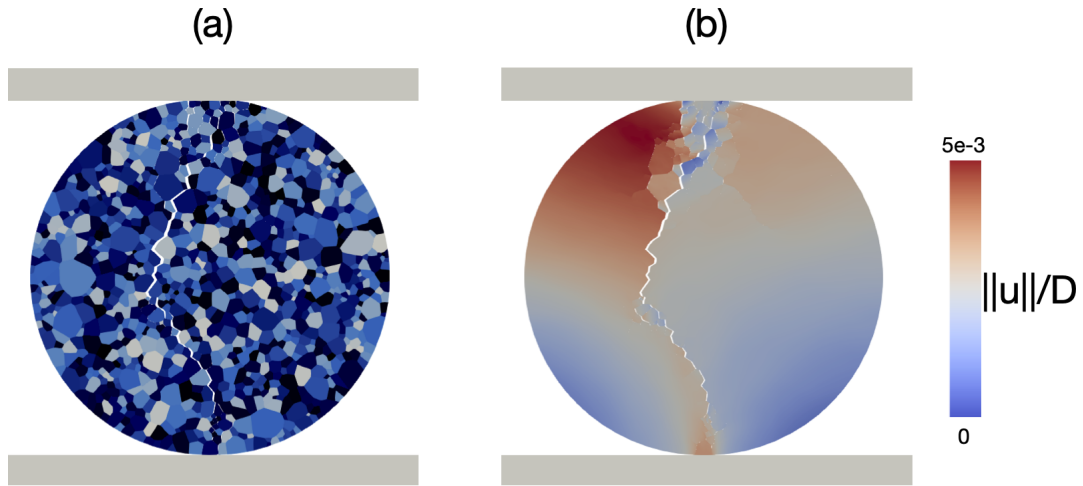
### 417 4.3 Brazilian test

418 As a complement to the previous numerical simulation and as an opening to the continuation of the  
 419 present work, the developed CZM law is used in the simulation of a Brazilian test. This test consists  
 420 of compressing a circular sample located between two rigid plates. Contrary to the previous case, no  
 421 pre-crack is introduced in the numerical model. The microstructure of the sample used is presented on the  
 422 left side of Figure 14. This microstructure has been generated using the open-source Software Neper  
 423 [Quey et al. 2011](#). It is composed of 1 000 elastic grains following a normal size distribution to make the  
 424 microstructure heterogeneous [Ma et al. 2018](#). In these case, the cohesive zones are only introduced at the  
 425 grain boundaries. The mesh size used for meshing is identical for all grains and calibrated so that the  
 426 smallest grains have at least two elements on their smallest side. The total number of elements is 98 378. A  
 427 zoom of the mesh is shown on the right hand side of Figure 14. For the sake of simplicity the diameter of  
 428 the sample is unitary. A vertical velocity is imposed on both walls to compress the sample. The simulation  
 429 is carried out in large deformations in order to manage possible strain localisation and grain rotations. The  
 430 elastic constitutive equations are those of the linear elasticity where stresses and strains are respectively  
 431 represented by the 2<sup>nd</sup>e Piola-Kirchoff stress tensor and the Green-Lagrange strain tensor.

**Figure 14:** Visulation of the meshed microstructure used in the simulation of the Brazilaian disc test

432 For homogeneous material during Brazilian test a tensile state is induced in the center of the disc  
 433 perpendicular to the load direction. Increasing the load leads to an increase in tensile stress until a crack  
 434 appears in the center of the disc. Under the effect of the load, the crack develops until the disc eventually  
 435 separates into at least two parts. For a heterogeneous material the damage occurs near the rigid plates and

436 then develops along the loading axis until the disk breaks [Na et al. 2017](#). Figure 15 shows a visual of the  
 437 sample at the end of the simulation.



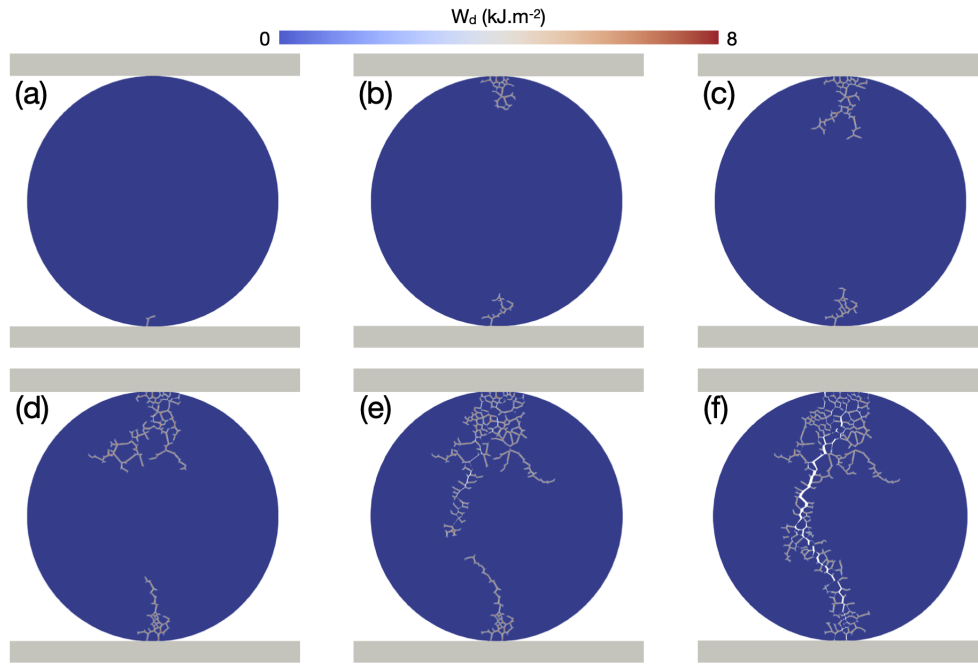
**Figure 15:** Final state of the simulation: Visualization of a) the crack through the microstructure and b) the norm of displacement field normalized by the value,  $D$ , of the diameter within the sample.

438 Figure 15(a) shows the macro crack zigzagging through the microstructure. This extends from the  
 439 contact between the sample and the rigid plates away from the centre of the sample. Figure 15(b) shows  
 440 the norm of the adimensionalized displacement field within the sample. The discontinuities within this  
 441 field allow the observation of multiple cracking paths generated during compression. Numerous disjointed  
 442 fragments can be particularly seen in the volume near the top wall.

443 To complete these observations, the evolution of the dissipated energy fields is presented in Figure 16.  
 444 The image (a) corresponds to the initiation of the crack while the image (f) corresponds to the end of the  
 445 simulation. The other images are captured at intermediate times. Through the figures 16 (a) to (f) the  
 446 damage evolution is exhibited where branching is observed until the coalescence of the macro crack.  
 447 The next step is to experimentally perform the same type of test using an experimental setup coupling  
 448 kinematic and thermal full-field measurements. The kinematic measurements will allow us to locate zones  
 449 of strain localization and even discontinuities of the displacement fields while the thermal measurements  
 450 will be used to determine the zones where the dissipation is localized. The confrontation of this two  
 451 informations should help us to check the relevance of this energy approach of cohesive zones.

## 452 5 Conclusion

453 In this paper we present an energy criterion for cohesive zone models where the damage progress is  
 454 assessed together with the ability of the material to store energy elastically. The damage parameter used  
 455 is  $u_d(t) = \text{Sup} \{u_{eq}(\tau), \tau \leq t\}$  where  $u_{eq}$  is an equivalent displacement compatible with the isotropic  
 456 evolution of the damage progress. The paper shows that if damage is the unique and exclusive dissipative  
 457 mechanisms, the damage evolution law is automatically fixed by the evolution of the maximum storable  
 458 elastic energy  $w_e^d(u_d)$ . We have also underlined that the data of this energy is equivalent in a 1D formalism



**Figure 16:** Evolution of the dissipated energy field from the crack initiation (a) to the end of simulation (f).

459 to the one of a traction-separation law. The interest of this energy approach is its immediate generalization  
 460 to 3D cohesive zone models. In order to check the operational character of this type of approach, the  
 461 isotropic damage model has been implemented in the open source software L<sup>3</sup>Gc90 based on Non-Smooth  
 462 Contact Dynamics (NSCD) and used to perform numerical simulations in the case of bending and Brazilian  
 463 tests. The results obtained for this plain stress modeling are encouraging. Using a simple quadratic function  
 464  $w_e^d(u_d)$  for the interface, we obtained a close correlation between the simulations and the experimental  
 465 observations of the crack path for the bending test, and realistic multicrack propagations in the case of the  
 466 Brazilian test. A parametric study of the macroscopic response of the structure naturally demonstrates the  
 467 importance of the shape of the function  $w_e^d(u_d)$  which characterize the interface behavior between two  
 468 elements. It is indeed this quantity that we will have to identify experimentally. In subsequent theoretical  
 469 developments, first we will consider an extension to a non isotropic degradation of the material elastic  
 470 properties. From an experimental stand point Brazilain test will be performed by using full field techniques  
 471 during monotonic loadings, the goal being to extract from the experimental data valuable information on  
 472 the form of the energy balance and particularly on  $w_e^d(u_d)$ .

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