Large time asymptotics of the wave fronts length II: surfaces with integrable Hamiltonians

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In the paper [Vi-20], the author proves that the length $|S_t|$ of the wave front S_t at time t of a wave propagating in an Euclidean disk $\mathbb{D}$ of radius 1, starting from a source A, admits a linear asymptotics as $t \to +\infty$: $|S_t| \sim (2\arcsin a)t$ with $a = d(0, A)$. In the paper [Co-Vi-20], we gave a more direct proof and some improvements of that result.

Here, we will explain that this result is quite general for surfaces with an *integrable Hamiltonian*. We discuss only the 2D case for simplicity. The main idea is to use *action-angle coordinates* (section 2) in order to get a nice integral expression for $|S_t|$ (section 4). Integrable systems have in general singularities, therefore we need to make some genericity assumptions (section 2) and to study what happens to the action-angle coordinates (section 3) near these generic singularities. We need then to evaluate some oscillatory integrals (section 6) using an ergodic lemma (Appendix B).

For the geodesic flow on closed manifolds of negative curvature, Margulis [Ma-69] proved that the asymptotics of the length is exponential. The generic behaviour is not known. Here we study the integrable case which is highly non generic.

Before starting, let us give a rough version of the main theorem 5.1:

Let (X, g) be a 2D-Riemannian manifold. Let $H : T^\star X \to \mathbb{R}$ be an integrable Hamiltonian near a given energy E. Assume that the

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energy shell $\Sigma := H^{-1}(E)$ is compact and that dH does not vanish on Σ. We assume also that H satisfies some "generic properties". The wave front starting from a point A at energy E is the projection onto X of $\phi_t(\Sigma^A)$ where ϕ_t is the Hamiltonian flow of H and $\Sigma^A := \{(A,\xi)|H(A,\xi) = E\}$ is assumed to be smooth. If the point A in X is "generic" (see section 4), the g-length of the wave front starting from A, at energy E, admits a linear asymptotics $|S_t| \sim \lambda(A)t$ as $t \to +\infty$, where $\lambda(A)$ expresses in terms of the action-angle coordinates.

1 Wave fronts

Let us consider a smooth 2D Riemannian manifold (X, g) without boundary and fix a real number E. Let $H : T^\star X \to \mathbb{R}$ be a smooth Hamiltonian. Assume that $H^{-1}([E-a, E+a])$ is compact for some positive a and that dH does not vanish on $\Sigma := H^{-1}(E)$. Let us fix some point $A \in X$ and put $\Sigma^A := \{(A,\xi) \in \Sigma\}$. We denote by ω the generic point of Σ^A. Assume that $d(H_{|T_A^\star X})$ does not vanish on Σ^A. This implies that Σ^A is a 1D-compact submanifold of Σ. We denote by π_X the canonical projection of Σ onto X and by $\phi_t : \Sigma \to \Sigma,\ t \in \mathbb{R}$, the flow of $\vec{H}$, the Hamiltonian vector field derived from H. For any positive t, we define the *wave front* S_t at time t as the set of points of X of the form $\pi_X(\phi_t(\Sigma^A))$. The wave front S_t has a smooth parametrization by Σ^A. This allows to define its length $|S_t|$ using the Riemannian metric g, assumed to be continuous and possibly degenerate:

$$|S_t| = \int_{\Sigma^A} \gamma^{\frac{1}{2}} \left(\phi_t(\omega); \frac{d}{d\omega}\phi_t(\omega) \right) |d\omega|$$

where $\gamma = \pi_X^\star(g)$. Note that S_t admits in general some singular points as a subset of X, namely cusps and transversal self-intersections. We are interested in the asymptotic behaviour of $|S_t|$ as $t \to +\infty$.

Examples:

1. *Geodesic flows:* $H := \frac{1}{2}g^\star$ is the Hamiltonian of the geodesic flow of a closed Riemannian manifold (X, g). Let us fix $E = 2$. Then Σ is the unit cotangent bundle and, on Σ, ϕ_t is the geodesic flow with speed 1. In this case, S_t is the image by the exponential map at the point A of the circle of radius t in the tangent plane $T_A X$.

2. *Schrödinger Hamiltonians:* Let (X, g) be a Riemannian manifold without boundary, $V : X \to \mathbb{R}$ a smooth function and E a real number. We take $H := \frac{1}{2}g^\star + V$. Our assumptions are satisfied if $V^{-1}(]-\infty, E+a])$ is compact for some $a > 0$, dV does not vanish on $V^{-1}(E)$ and $V(A) < E$.

2 Integrable Hamiltonian flows

For this section, one can look at the chapter 4 of [Vu-06] and the section 1 of [Co-Vu-03]. We will assume that the Hamiltonian H is integrable near the energy E. "Integrability" means that there exists a positive number a and a smooth map $M = (I, J) : H^{-1}(]E - a, E + a[) \to \mathbb{R}^2$, called the *moment map*, so that

- The Poisson bracket $\{I, J\}$ vanishes identically.
- The critical points of M are of measure 0, i.e. the differentials dI and dJ are almost everywhere independent.
- There exists a smooth function $\Phi : M(H^{-1}(]E - a, E + a[)) \to \mathbb{R}$ so that $H = \Phi(I, J)$.

Note that dI and dJ cannot vanish at the same point of Σ because dH does not vanish there.

The main examples with the geodesic flows are the surfaces of revolution, the tri-axial ellipsoïds ([Ja-39]) and the Liouville metrics on 2D tori (Liouville metrics are of the form $ds^2 = (f(u) + g(v))(du^2 + dv^2)$, see [B-S-K-97] Chap. 7). Usually, integrable systems have singularities. We will make the following "generic" assumption which is already used in [Co-Vu-03]: we assume that the moment map M satisfies the

(A1) Morse-Bott condition: at any point of Σ where dI and dJ are linearly dependent, i.e. where $\lambda dI + \mu dJ = 0$ for some pair $(\lambda, \mu) \neq (0, 0)$, the function $\lambda I + \mu J$, restricted to Σ, admits a critical manifold of dimension 1 with a transversally non degenerate Hessian.

This implies that the singular set $Z_0 \subset \Sigma$, i.e. the set of critical points of M located in Σ, is a finite union of periodic orbits of $\vec{H}$. These periodic orbits are either hyperbolic or elliptic according to the signature of the transversal Hessian. We denote by $Z \subset \Sigma$ the part of the preimage by M of the critical values of M which is the union of Z_0 and all the stable and unstable manifolds of the hyperbolic periodic orbits. The open set $\Sigma \setminus Z_0$ admits a smooth Lagrangian foliation given by the level sets of M.

The open set $\Sigma \setminus Z$ is foliated by 2D-tori on which the Hamiltonian flow of H is quasi-periodic. The set of these tori is a smooth 1D-manifold. We denote it by $\mathcal{L}$ and by σ the generic point of $\mathcal{L}$. The manifold $\mathcal{L}$ is a 1D-torus in the case where there are no singularities, i.e. if Z is empty, and a finite union of real lines $D_j,\ j = 1, \cdots N$ if there are some singularities. If $\sigma \in D_j$ tends to one of the infinity of D_j, the corresponding torus $\mathbb{T}_\sigma$ converges to a compact connected set $\mathbb{T}_{j,\infty}$ of Σ which is either an elliptic periodic orbit of H or the union of a finite set of hyperbolic periodic orbits of $\vec{H}$ and some cylinders which are connected components of their stable manifolds. In the last case, $\mathbb{T}_{j,\infty}$ is homeomorphic to a 2D torus or to a Klein bottle.

Let us denote by U_j the open connected component of $\Sigma \setminus Z$ which is the union of the tori associated to the line D_j. The projection of U_j onto D_j is a smooth fibration by 2D-tori which is trivial, because it is a fibration on the real line. There exist global coordinates $(\theta, \sigma) \in \mathbb{T}^2 \times \mathcal{L}$ on $\Sigma \setminus Z$ so that the torus $\mathbb{T}_\sigma$ is mapped onto $\mathbb{T}^2 \times \{\sigma\}$ and the Hamiltonian flow is mapped on a vector field $V(\sigma) = A(\sigma)\partial_{\theta_1} + B(\sigma)\partial_{\theta_2}$ on $\mathbb{T}^2$ with some smooth functions A and B. Note that A and B have no common zeroes because the Hamiltonian flow does not vanish on Σ.

In what follows, we fix some component U_j. Let us describe the *action-angle coordinates* in some neighbourhood of U_j in $T^\star X$: there exists a symplectic diffeomorphism χ_j of some neighbourhood V_j of U_j onto an open set $\mathbb{T}^2 \times \Omega_j$, with $\Omega_j \subset \mathbb{R}^2$, contained in $T^\star\mathbb{T}^2 \setminus 0$ with canonical coordinates (θ, p), so that $H \circ \chi_j^{-1}(\theta, p) = K_j(p)$ with K_j a smooth function from Ω_j into $\mathbb{R}$. In these coordinates, the vector field V_j is given by $V_j = (\partial K_j/\partial p_1)\partial_{\theta_1} + (\partial K_j/\partial p_2)\partial_{\theta_2}$. We note $\tilde{\nabla} K$ this non vanishing vector field. The vector field $\tilde{\nabla} K_j$ does not vanish and hence the curve $C_j := \{p \in \mathbb{R}^2 | K_j(p) = 1\}$ is a smooth submanifold of Ω_j. The line D_j identifies smoothly to the curve C_j. The manifold $\mathcal{L}$ can be identified to the disjoint union of the curves C_j. The coordinates p are called the actions: they are given by action integrals $p_j := \int_{\gamma_j} \alpha$, where $d\alpha$ is the symplectic form, and the loops $\gamma_j,\ j = 1, 2$ form a basis of $H_1(\mathbb{T}_\sigma, \mathbb{Z})$ varying continuously in V_j. Note that if α' is another primitive of the symplectic form, the difference $\alpha - \alpha'$ is closed, hence the action integrals differ by some constants. There are many choices for the coordinates θ: if $\Lambda \subset U_j$ is a Lagrangian manifold transversal to the foliation by the tori, one can choose θ vanishing on Λ.

We will need one more "generic" assumption on the Hamiltonian flow:

(A2) For any $j = 1, \cdots, N$, there exists, at any point p of C_j, two integers $k \geq 1$ and $l \geq 1$, so that the derivatives of order k and l of the vector field ∇K_j along C_j are linearly independent.

Note that this condition is independent of the parametrization of C_j. For example, $k = 1, l = 2$ means that the curvature of the curve $\{\nabla K_j(p) | p \in C_j\}$ does not vanish while $k = 2, l = 3$ means a generic cusp for that curve.

The assumption **(A2)** implies that, for any $\nu \in \mathbb{R}^2 \setminus 0$, the map from C_j into $\mathbb{R}$ defined by $p \to \langle \nu | \nabla K_j \rangle$ has only critical points of finite order.

3 The behaviour of ∇K near Z

In this section, we forget about the index j: K denotes the expression of H in some of the action-angle coordinates. We are interested at the behaviour of ∇K near Z.

3.1 The Elliptic case

Lemma 3.1 *Let γ be an elliptic periodic orbit of H (included in Z_0), then K is a smooth function of p_1 and p_2 up to γ.*

Proof.– There exists a symplectic chart of a neighborhood of the elliptic periodic orbit of H so that $H = \Phi(\xi, y^2 + \eta^2)$ in $(T^\star\mathbb{T})_{x,\xi} \times (T^\star\mathbb{R})_{y,\eta}$ (see [Vu-00]). The invariant tori are the level surfaces of the moment function $M(x, \xi; y, \eta) = (\xi, y^2 + \eta^2)$. Let us choose $\gamma_1 = \{s \to (s, \xi; y, \eta) | s \in \mathbb{R}/\mathbb{Z}\}$ and $\gamma_2 = \{s \to (x, \xi, \sqrt{y^2 + \eta^2} \cos 2\pi s, \sqrt{y^2 + \eta^2} \sin 2\pi s) | s \in \mathbb{R}/\mathbb{Z}\}$. If $\alpha = \xi dx + \eta dy$, we get the action integrals $p_1 = \xi$ and $p_2 = \pi(y^2 + \eta^2)$. Hence $K(p_1, p_2) \equiv \Phi(p_1, p_2/\pi)$. □

We have the following

Corollary 3.1 *The manifold $\mathcal{L}$ admits an extension as a manifold with boundary at the elliptic periodic orbits of H and the 1-form $d\nabla K$ is smooth and hence integrable on $\mathcal{L}$ near that boundary.*

3.2 The Hyperbolic case

In this section, we will use Section 1 of [Co-Vu-03].

3.2.1 Functions of type (L)

Let us start with a

Definition 3.1 *A function $f : [0, c[\to \mathbb{R}$, with $c > 0$, is called of type* **(L)** *if there exists two smooth functions $\phi, \psi : [0, c[\to \mathbb{R}$ so that*

$$\forall x \in [0, c[, \;\; f(x) = \phi(x) \log x + \psi(x)$$

with $\phi(0) = 0, \;\; \phi'(0) \neq 0$.

This definition is invariant by any smooth change of variable from $[0, c[$ into $[0, c'[$. Hence it extends to $1D$-manifolds with boundaries. Such a function is invertible in a small enough subinterval of $[0, c[$ and the inverse f^{-1} is C^1 up to the boundary $\psi(0) = \lim_{x \to 0^+} f(x)$.

Now let us describe the application that we have in mind.

Lemma 3.2 *Let α be a smooth 1-form so that $d\alpha$ is a volume form in some neighbourhood $[-d, d]^2$ of the origin in the (y, η) plane. Let $m_j(s) = (y_j(s), \eta_j s))$, $j = 1, 2$, be two smooth curves with $m_1(0) = (e, 0)$, $m_2(0) = (0, f)$ and $e, f \in]0, d[$, $\eta_1'(0) > 0$, $y_2'(0) > 0$ which are arcs transverse to each of the coordinates axes. Then consider the integral $I(t) = \int_{\Gamma_t} \alpha$ where Γ_t is, for t small enough, the part of the hyperbola $y\eta = t$ $(t > 0)$ between the curves m_1 and m_2 oriented in any of the two possible directions. Then $I(t)$ is a function of type* **(L)**.

This lemma follows from the Stokes formula: the isochoric Morse lemma (see [Co-Ve-79]) allows to reduce to the case where $d\alpha = dy \wedge d\eta$ and to the change of variable $t \to F(t)$.

3.2.2 The lines D_j as 1D-manifolds with boundary

Let us put a structure of a 1D-manifold with boundary on the line D_j in the "hyperbolic case". Let us recall that we denote be $\mathbb{T}_\infty$ the limit of the tori $\mathbb{T}_\sigma$ as σ tends to one of the infinities. We showed that $\mathbb{T}_\infty$ is the union of a finite number of closed hyperbolic orbits and of a finite number of cylinders which are parts of the stable an unstable manifolds of these orbits. Near $\mathbb{T}_\infty \setminus Z_0$, the foliation by the level sets of the moment map is smooth. We can choose any local transverse arc to that foliation. They are all equivalent up to diffeomorphism along any connected component of $\mathbb{T}_\infty \setminus Z_0$ and give

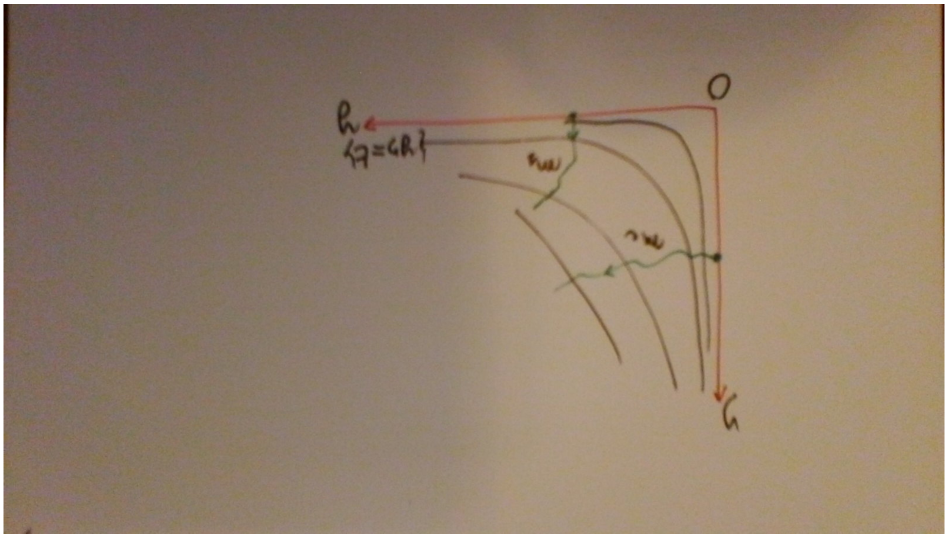

Figure 1: the foliation by hyperbolae and the transverse arcs m_1 and m_2

local parametrization of D_j near that boundary by intervals $[0, c[$. How do we pass from one component to the next by crossing Z_0? We choose a Poincaré section at a point of Z_0 and use the Morse lemma which gives local coordinates (y, η) in that section so that $(\lambda I + \mu J)(y, \eta) = cte + y\eta$. The local parameter is then the evaluation of the function $y\eta$ which allows to pass from the transversal $\eta = 1$ to the transversal $y = 1$. Both are locally parametrized by the restriction of the function $y\eta$. This gives to D_j the structure of a 1D compact manifold with boundary. Note that this holds in a smooth way with respect to E' close to E.

3.2.3 The asymptotic behaviour of the action integrals

There exists, in a neighborhood V_j of $\mathbb{T}_\infty$, invariant by the flows of $\vec{I}$ and $\vec{J}$, an Hamiltonian P, Poisson commuting with I and J, whose orbits are periodic of period 1 (Theorem 1.6 of [Co-Vu-03]). P is constant on Z_0. This gives a smooth action of the group S^1 on V_j. Note that this action is principal on $(V_j \cap \Sigma) \setminus Z_0$, but can get some non trivial isotropy $\mathbb{Z}/2\mathbb{Z}$ on Z_0. Let $\gamma_1(z),\ z \in V_j$, be the S^1-orbits. They are all homotopic. If z lies in some invariant torus, γ_1 is a homotopically non trivial loop in this torus. We denote by p_1 the action integral on $\gamma_1(z)$ which is clearly smooth in V_j. Note that p_1 is a function of P which is a local diffeomorphism.

We need to choose a loop γ_2 on the tori in V_j which, with γ_1, generates a basis of the homology of the invariant tori. Let $R_h := V_j \cap P^{-1}(h)/S^1$ with h close to $P(Z_0)$. The reduced manifolds (see Appendix C) is foliated by the reduction of the integrable foliation restricted to $P^{-1}(h)$. Let us denote by Z_h the singular set of that foliation. As does $V_j \cap \Sigma$, the orbifold R_h consists of a singular part $R_{\mathrm{sing},h}$, the quotient of $Z_h \cap V_j$, which is homeomorphic to a circle, and an open set smoothly foliated by circles which are the reductions of the invariant tori. Together they give a topological foliation of R_h depending smoothly of h. The singular leaf $R_{\mathrm{sing},h}$ is smooth outside the finite set of points which are quotients by the S^1 action of the hyperbolic periodic orbits of H. This foliation is smooth outside these singular points. We take for γ_2 a lift of the projection of $\mathbb{T}_\sigma$ depending continuously of σ.

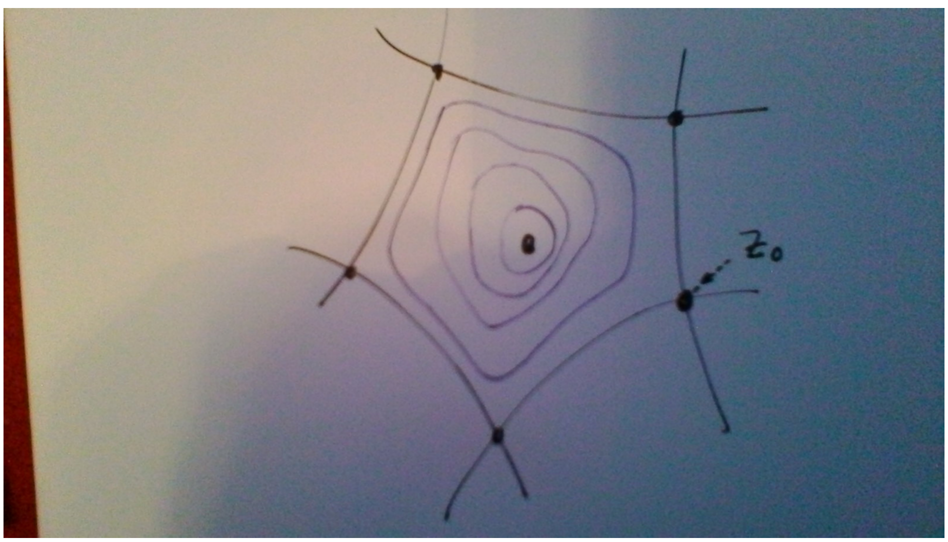

Figure 2: a reduced manifold R_h

We have the following crucial Lemma:

Lemma 3.3 *The action integrals* (p_1, p_2) *on the previously choosen loops* γ_1 *and* γ_2 *satisfy at the boundary*

- *the action* p_1 *is smooth up to the boundary*
- *The action* p_2 *as a function of* σ *is of type* **(L)** *at the boundary and depends smoothly of* h *and hence of* p_1.

Proof.– We saw already the smoothness of p_1. The function $p_2(\sigma, h) - p_2(Z_h)$ is given by the symplectic area in R_h between the reduction of γ_2 and Z_h. Lemma 3.2 implies that p_2 is of type **(L)** depending smoothly of h and hence of p_1.

□

3.2.4 The asymptotic behaviour of ∇K

We have the following important

Corollary 3.2 *The 1-form $d\nabla K$ is integrable at any hyperbolic boundary point of $\mathcal{L}$.*

Proof.–

Near a closed orbit of Z_0, we have the normal form $H = \Phi(\xi, y\eta)$ with $(x, \xi, y, \eta) \in T^\star_{x,\xi}\mathbb{T} \times T^\star_{y,\eta}\mathbb{R}$. We have $p_1 = \xi$ up to a constant. We get $H = K(p_1, p_2) = \Phi(p_1, F(p_1, p_2))$ expressing H in terms of the actions. We get

$$\partial_1 K = \partial_1 \Phi + \partial_2 \Phi \times \partial F/\partial p_1, \quad \partial_2 K = \partial_2 \Phi \times \partial F/\partial p_2$$

which are smooth outside Σ and continuous on Σ. Hence their derivatives are integrable.

□

4 An integral formula for $|S_t|$

One of the difficulties in extending the result for the disk to this case is the fact that the action-angle coordinates only exist outside Z. Therefore, we need to make some assumptions on the point A.

4.1 Assumptions on the point A

If $\Lambda \subset T^\star X$ is a Lagrangian manifold, the *caustic set* of Λ is the set of critical points of the projection π_X restricted to Λ. We first need a

Lemma 4.1 *Let us take $\omega_0 \in \Sigma^A$ so that $(A, \omega_0) \notin Z_0$ and denote by $\mathbb{F}_0$ the 2D-leaf of the invariant foliation of Σ containing (A, ω_0). If (A, ω_0) does not belong to the caustic set of $\mathbb{F}_0$, then Σ^A and $\mathbb{F}_0$ are transversal at the point (A, ω_0).*

Proof.– Σ^A is a 1D-submanifold of $T^\star_A X$ and hence $\pi_X(\Sigma^A) = \{A\}$. On the other hand, the fact that (A, ω_0) is not in the caustic set means that $(\pi_X)_{|\mathbb{F}_0}$ is a local diffeomorphism onto X near (A, ω_0). The conclusion follows. □

We will assume:

(A3) The intersection of Σ^A with Z is a countable set.

Proposition 4.1 **(A3)** *is satisfied as soon as there is only a finite number of $\omega \in \Sigma^A \cap (Z \setminus Z_0)$ so that (A, ω) is in the caustic set of the Lagrangian leaf in which it lives.*

Proof.– The intersection of Σ^A with Z_0 is a finite set. On the other hand, the points ω so that (A, ω) is not a caustic point of the corresponding leaf are isolated inside Σ^A. Hence there is at most a countable set of such points. □

(A4) The set of critical points of the smooth map $\omega \to \sigma$ from $\Sigma^A \cap (\Sigma \setminus Z)$ into $\mathcal{L}$ is countable.

Proposition 4.2 **(A4)** *is satisfied as soon as there is only a finite number of $\omega \in \Sigma^A \cap (\Sigma \setminus Z)$ so that (A, ω) is a caustic point of the invariant torus containing that point.*

The argument is quite similar to that of the proof of Proposition 4.1

4.2 Exact formulae for $|S_t|$

We will compute the lengths of the wave front using the action-angle coordinates.

We will start with the finite covering of $\Sigma \setminus Z$ by the semi-global action-angle charts. This allows a description of S_t as follows: let $\chi : U \to \mathbb{T}^2 \times C$ be one of these charts and let $\Pi_X : \mathbb{T}^2 \times C \to X$ be the map $\pi_X \circ \chi^{-1}$.

This way, if we call $(\theta(\omega), p(\omega))$ the image of $\omega \in \Sigma^A$ by χ, we can assume that $\theta(\omega)$ vanishes identically, because $T^\star_A X$ is Lagrangian. We get that the corresponding part of the wave front S_t is defined by

$$S_t = \{\Pi_X \left(t\tilde{\nabla} K(p(\omega)), p(\omega) \right) | \omega \in \Sigma^A\}$$

where K is the Hamiltonian H expressed in the action coordinates and $(\theta(\omega), p(\omega)$ are the action-angle coordinates of $\omega \in \Sigma^A$. We get, using the Assumption **(A3)**, the expression

$$|S_t| = t \int_{\Sigma^A} \gamma^{\frac{1}{2}} \left(\left(\tilde{\nabla} K(p(\omega)), p(\omega) \right) ; \frac{d}{d\omega} \tilde{\nabla} K(p(\omega)) \right) d\omega$$

where γ is the pull-back of g by Π_X. We can make a change of variable: instead of ω, one can use $\sigma \in \mathcal{L}$ thanks to assumption **(A4)**. We get the

Proposition 4.3 *The length of the wave front is given by*

$$|S_t| = t \int_{\mathcal{L}} N_A(\sigma)\gamma^{\frac{1}{2}}\left(\left(t\tilde{\nabla}K(\sigma),\sigma\right); d\tilde{\nabla}K(\sigma)\right)$$

where $N_A(\sigma) = \#\{\Sigma^A \cap \mathcal{L}\}$.

5 The main result

Theorem 5.1 *Let (X, g) be a Riemannian manifold of dimension 2 with g continuous, possibly degenerate. Let H be an Hamiltonian integrable at energy E and satisfying the assumptions* **(A1)** *and* **(A2)**. *Let $A \in X$ be a point satisfying the assumptions* **(A3)** *and* **(A4)**. *The length for the metric g of the wave front S_t starting from A has a linear asymptotics $|S_t| \sim \lambda(A)t$ as $t \to \infty$.*

Let us denote be $\mathcal{L}$ the 1D-manifold of all invariant Lagrangian tori L_σ, $\sigma \in \mathcal{L}$, filling $\Sigma \setminus Z$ and consider the continuous density on $\mathcal{L}$ defined by

$$|d\sigma| = \int_{\mathbb{T}_\sigma} \gamma^{\frac{1}{2}}\left((\theta,\sigma); d(\tilde{\nabla}K(\sigma))\right)|d\theta|$$

where γ is the pull-back of g by the projection Π_X. The measure $|d\sigma|$ is independent of A. We have

$$\lambda(A) = \int_{\mathcal{L}} N_A(\sigma)|d\sigma| \tag{1}$$

with $N_A(\sigma) := \#\{\Sigma^A \cap L_\sigma\}$.

Corollary 5.1 *Let H be the Hamiltonian of the geodesic flow of a smooth metric G on a closed manifold X. If D is a smooth domain with boundary in X, the $g-$length of $S_t \cap D$, is given by*

$$|S_t \cap D| \sim t \int_D d\mu_A$$

where $d\mu_A$ is an absolutely continuous density $d\mu_A = F|dx|$ with $F \in L^1(X, |dx|)$. whose integral is $\lambda(A)$.

Proof of the Corollary.– Let ψ be a positive continuous function on X. We can apply the previous theorem with $g' = \psi^2 g$. This way, we see that the asymptotics of the g'-length of S_t is given by replacing the measure $|d\sigma|$ by the measure

$$|d\sigma|' = \int_{\mathbb{T}_\sigma} \psi(\pi_X(\theta,\sigma))\gamma^{\frac{1}{2}}(\Big((\theta,\sigma); d(\tilde{\nabla}K(\sigma))\Big)\,|d\theta|$$

This says that

$$|S_t|_{g'} \sim t\int_X \psi d\mu_A$$

where $d\mu_A$ is the pushforward by π_X of the absolutely continuous (a.c. in short) finite measure $dM_A := N_A(\sigma)\gamma^{\frac{1}{2}}(\Big((\theta,\sigma); d(\tilde{\nabla}K(\sigma))\Big)\,|d\theta|$, supported by Σ.

We need to show that we can apply this when ψ is the characteristic function of a smooth domain. In our situation Σ is the unit cotangent bundle and $\pi_X : \Sigma \to X$ is a submersion. It follows that that $d\mu_A$ is a.c. w.r. to $|dx|$. □

6 Proof of Theorem 5.1

We start from the expression of $|S_t|$ given in Proposition 4.3. Let us show that we can apply Lemma B.1 to the integral giving $|S_t|/t$. In the notations of that lemma, we have $V(\sigma) = \tilde{\nabla}K(\sigma)$. The Assumption **(A2)** implies that the assumption on V of the Lemma is satisfied. The function F is given by $F(\sigma,\theta) = N_A(\sigma)\gamma^{\frac{1}{2}}\left(\theta; d\tilde{\nabla}K(\sigma)/d\sigma\right)$. The integrability assumption follows from the Corollaries 3.1 and 3.2 and the upper bound

$$|N_A(\sigma)\gamma^{\frac{1}{2}}\left(\theta; W\right)| \leq C\|W\|$$

The continuity with respect to θ follows from the continuity of g and the smoothness of the projection of any $\mathbb{T}_\sigma$ onto X. It is shown using Lebesgue's dominated convergence Theorem.

7 Examples

7.1 Surfaces of revolution

Surfaces with a non trivial action of S^1 are tori or spheres. In both case, the metric is given by $g = a(s)^2 d\theta^2 + ds^2$ where $s \in \mathbb{R}/L\mathbb{Z}$ in the first case and $s \in [0, L]$ in the second (in this case $s = 0$ and $s = L$ are the poles).

The assumption **(A1)** is satisfied if and only if a is a Morse function. The assumption **(A2)** is satisfied for a generic a. Assuming **(A1)** and **(A2)**, the assumption **(A3)** is satisfied for any point of the torus and for A not a pole in the case of the sphere. while **(A4)** is always satisfied. If A is a pole, $|S_t|$ is periodic of period $2L$.

7.2 Tri-axial ellipsoïds

The integrability was found by C. Jacobi ([Ja-39], see also [Kl-82] and section 3.2 of [Co-Vu-03]). Assumption **(A1)** and **(A2)** are satisfied. **(A3)** is satisfied for A not an ombilical point while **(A4)** is always satisfied. If A is an ombilical point, $|S_t|$ is periodic.

A Stationary phase

For this section, one can look at [Gu-St-77], chap. 1.

We want to evaluate the asymptotics as $t \to +\infty$ of integrals of the form

$$I(t) := \int_{\mathbb{R}} e^{itS(x)} a(x) dx$$

where S is a real valued smooth function and $a \in C_o^\infty(\mathbb{R})$. We have the

Proposition A.1 *Let us assume that the critical points of S, i.e. the zeroes of S', are non degenerate, i.e. $S''(x) \neq 0$ if $S'(x) = 0$. Then, if $x_1, \cdots, x_N$ are the critical points of S in the support of a, $I(t)$ admits an asymptotic expansion given by*

$$I(t) = \sum_{j=1}^{N} \frac{\sqrt{2\pi} e^{i\varepsilon_j \pi/4}}{|tS''(x_j)|^{\frac{1}{2}}} e^{itS(x_j)} \left(a(x_j) + O(t) \right) \tag{2}$$

with $\varepsilon_j = \pm 1$ depending on the sign of $S_j''(0)$.

In the case where the critical points are degenerate, we have the following result:

Proposition A.2 *If the zeroes of S' in the support of a are of finite order, we have $I(t) \to 0$ as $t \to \infty$.*

Note also that in equation (2), the remainders "$O(t)$" are uniform if S' (resp. a') is close to S (resp. close to a) in the C^∞ topology and the support of a stays in some fixed bounded intervall.

B An "ergodic" lemma

This section could be of independent interest.

Lemma B.1 *For $s \in J$ where J is an interval of the real line, let $V(s) = A_1(s)\partial_1 + A_2(s)\partial_2$ be a family of constant vector fields on $\mathbb{T}^2$ depending smoothly of s. Assume that, for any $s \in J$, there exists two derivatives $V^{(k)}(s)$ and $V^{(l)}(s)$ which are linearly independent.*

Let F is a function on $J \times \mathbb{T}^2$ with $F \in C^0(\mathbb{T}^2, L^1(J, ds))$ satisfying the following condition: there exists a function $\psi \in L^1(J, ds)$ so that

$$\forall (s,\theta) \in J \times \mathbb{T}, \ |F(s,\theta)| \le \psi(s) \ .$$

Then

$$\lim_{t \to +\infty} \int_J F(s, [tV(s)])|ds| = \int_{J \times \mathbb{T}^2} F|dsd\theta|$$

The assumption on the derivatives of V have the following geometrical meaning: if $V'(s_0) = 0$, we get a cusp point which is of finite order; if $V'(s_0) \neq 0$, the curvature of the curve V vanishes at a finite order. In particular the points where V' and V'' are linearly dependent are isolated.

Proof.– It follows from Lebesgue's dominated convergence theorem, that the map $f : \theta \to F(., \theta)$ is continuous from $\mathbb{T}^2$ into $L^1(J, |ds|)$. Let us choose a finite covering of $\mathbb{T}^2$ by balls of centers θ_j, $1 \le j \le N$, so that the L^1-oscillation of f is each ball is smaller than $\varepsilon/2$ and a smooth finite partition (ψ_j) of unity subordinated to that covering. Let $F_j \in C_o^\infty(J)$ satisfying $\|F(., \theta_j) - F_j\|_{L^1} \le \varepsilon/2$. Such functions do exist (see [Fo-99] Prop. 8.17). If $G(s,\theta) = \sum_j \psi_j(\theta) F_j(s)$, we have

$$\int_J |F(s,\theta) - G(s,\theta)|ds| \le \varepsilon$$

This allows to reduce to prove the result for such a function G. We can again approximate G by a function

$$L(s,\theta) = \sum_{n\in\mathbb{Z}^2,\ \|n\|\leq N} a_n(s)\exp(2\pi i < n|\theta >)$$

uniformly in $L^1(|ds|)$. We have $a_0(s) = \int_{\mathbb{T}^2} L(s,\theta)|d\theta|$. We are left with the integrals

$$\int_J a_n(s) e^{2i\pi t<n|V(s)>}|ds|$$

It follows from the assumption on V that such integrals tend to 0 as $t\to\infty$ for $n\neq 0$.

□

C Symplectic S^1-reduction

Let $P : M \to \mathbb{R}$ be an Hamiltonian on a symplectic manifold (M,ω) so that the vector field $\vec{P}$ is complete and generates an action of $\mathbb{T}$ onto M. Let us assume that this action is almost free: it is free on an open dense subset of M and all the isotropy subgroups are finite. Let us look at an energy shell $S_h := P^{-1}(h)$ for some $h\in\mathbb{R}$. The quotient of S_h by the $\mathbb{T}$-action is an orbifold R_h. Let us denote by π_h the canonical projection of S_h onto R_h. The orbifold R_h admits an unique symplectic structure Ω so that $\pi^\star(\Omega)=\omega$.

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