Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications

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PARTIALLY DISSIPATIVE ONE-DIMENSIONAL HYPERBOLIC SYSTEMS IN THE CRITICAL REGULARITY SETTING, AND APPLICATIONS

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Abstract. Here we develop a method for investigating global strong solutions of partially dissipative hyperbolic systems in the critical regularity setting. Compared to the recent works by Kawashima and Xu, we use *hybrid* Besov spaces with different regularity exponent in low and high frequency. This allows to consider more general data and to track the exact dependency on the dissipation parameter for the solution. Our approach enables us to go beyond the L^2 framework in the treatment of the low frequencies of the solution, which is totally new, to the best of our knowledge.

Focus is on the one-dimensional setting (the multi-dimensional case will be considered in a forthcoming paper) and, for expository purpose, the first part of the paper is devoted to a toy model that may be seen as a simplification of the compressible Euler system with damping. More elaborated systems (including the compressible Euler system with general increasing pressure law) are considered at the end of the paper.

Introduction

The study of the global existence issue for so-called partially dissipative hyperbolic systems of balance laws goes back to the seminal work of Kawashima [20]. Recall that a general n-component systems of balance laws in $\mathbb{R}^d$ reads:

$$\frac{\partial w}{\partial t}+\sum_{j=1}^{d}\frac{\partial F_j(w)}{\partial x_j}=Q(w). \tag{1}$$

Here the unknown $w=w(t,x)$ with $t\in\mathbb{R}^+$ and $x\in\mathbb{R}^d$ is valued in an open convex subset $\mathcal{O}_w$ of $\mathbb{R}^n$ and $Q,F_j:\mathbb{R}^n\to\mathcal{O}_w$ are given n-vector valued smooth functions on $\mathcal{O}_w$.

It is well known that classical systems of conservation laws (that is with $Q(w)=0$) supplemented with smooth data admit local-in-time strong solutions that may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution (see for instance the works by Majda in [23] and Serre in [27]). A sufficient condition for global existence for small perturbations of a constant solution $\bar w$ of (1) is the *total dissipation hypothesis*, namely the damping (or dissipation) term $Q(w)$ acts directly on each component of the system, making the whole solution to tend to $\bar w$ exponentially fast. However, in most evolutionary systems coming from physics, that condition is not verified, and even though global-in-time strong solutions do exist, exponential decay is very unlikely. A more reasonable assumption is that dissipation acts only on some components of the system. After suitable change of coordinates, we may write:

$$Q(w)=\begin{pmatrix}0\\ q(w)\end{pmatrix} \tag{2}$$

where $0\in\mathbb{R}^{n_1}, q(w)\in\mathbb{R}^{n_2}$, $n_1,n_2\in\mathbb{N}$ and $n_1+n_2=n$. This so-called *partial dissipation hypothesis* arises in many applications such as gas dynamics or numerical simulation of conservation laws by relaxation scheme. A well known example is the damped compressible Euler system for isentropic flows that we will be investigated at the end of the paper. For this system,

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the works by Wang and Yang [31] and Sideris, Thomases and Wang [29] pointed out that the dissipative mechanism, albeit only present in the velocity equation, can prevent the formation of singularities that would occur if $Q \equiv 0$.

Looking for conditions on the systems of the form (1)-(2) guaranteeing global existence of strong solutions for small perturbations of a constant solution $\bar{w}$ goes back to the paper of Shizuta and Kawashima [28], the thesis of Kawashima [20] and, more recently, to the paper of Yong [37]. Their researches reveal the importance of a rather explicit linear stability criterion, that is nowadays called the (SK) (for *Shizuta-Kawashima*) condition and of the existence of an entropy that provides a suitable symmetrisation of the system. Roughly speaking, (SK) condition ensures that the partial damping provided by (2) acts on all the components of the solution, although indirectly, so that all the solutions of (1) emanating from small perturbations of $\bar{w}$ eventually tend to $\bar{w}$, while the paper by Yong provides tools to get quantitative estimates on the solutions when $Q(\bar{w}) = 0$. As observed by Bianchini, Hanouzet and Natalini [4], in many situations, a careful analysis of the Green kernel of the linearized system about $\bar{w}$ allows to get explicit (and optimal) algebraic rates of convergence in L^p of smooth global solutions to $\bar{w}$. Let us finally mention that a more general approach has been proposed by Beauchard and Zuazua in [2], that allows to handle partially dissipative systems that need not satisfy the (SK) condition.

Recently, Kawashima and Xu in [34] and [35] extended all the prior works on partially dissipative hyperbolic systems satisfying the (SK) and entropy conditions (including the compressible Euler system with a damping term) to 'critical' non-homogeneous Besov spaces of L^2 type. To obtain their results, they symmetrized the system thanks to the entropy hypothesis, applied a frequency localization argument relying on the Littlewood-Paley decomposition and used new properties concerning Chemin-Lerner's spaces. They took advantage of the equivalence between the condition (SK) and the existence of a compensating function so as to to exhibit global-in-time L^2 integrability properties of all the components of the system.

The present paper focuses on the particular situation where the space dimension is $d = 1$ and the number of components is $n = 2$ (the multi-dimensional case will be investigated in a forthcoming paper [8] for the whole class of partially dissipative systems verifying the (SK) condition). Our goal is to propose a method and a functional framework with *different* regularities for low and high frequencies. For the high frequencies, we do not really have the choice as it is known that the optimal regularity for local well-posedness in the context of general quasilinear hyperbolic systems, is given by the 'critical' Besov space $\mathbb{B}^{\frac{3}{2}}_{2,1}$. The novelty here is that we propose to look at the low frequencies of the solution in another space, not necessarily related to L^2. The advantage is not only that we will be able to consider a larger class of initial data that may be less decaying at infinity, but also that one can easily keep track of the dependency of the solution with respect to the dissipation coefficient, and thus have some informations on the large dissipation asymptotics. Various considerations lead us to think that a suitable space for low frequencies is the homogeneous Besov space $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ (with, possibly, $p > 2$) that corresponds to the critical embedding in L^∞.

For expository purpose, we spend most of the paper implementing our method on a simple 'toy model' that may be seen as a simplification of the one-dimensional compressible Euler system with damping, and pressure law $P(\rho) = \frac{1}{2}\rho^2$, namely

$$(TM_\lambda) \qquad \begin{cases} \partial_t u + v\partial_x u + \partial_x v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ \partial_t v + v\partial_x v + \partial_x u + \lambda v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ (u,v)|_{t=0} = (u_0, v_0). & \text{on} \quad \mathbb{R} \end{cases}$$

Above, the unknown u may be seen as the discrepancy to the reference density normalized to 1, (then, the first equation is a simplification of the mass balance), while the unknown v stands for

the velocity of the fluid, and the second equation corresponds to the evolution of velocity with a friction term of magnitude $\lambda > 0$ (which could also be interpreted as a relaxation parameter).

In order to have a robust method that can be adapted to more involved systems, we shall not compute explicitly the solution of the linearized system (TM_λ) about $(0,0)$, but rather use modified energy arguments (different from those of S. Kawashima in his thesis [20]) and suitable change of unknowns. More specifically, we will introduce a 'modified' velocity that plays the same role as the 'viscous effective flux' in the works of Hoff [18] and, more recently, of Haspot [17] dedicated to the compressible Navier-Stokes equations.

Our approach will enable us to obtain more accurate estimates and a weaker smallness condition than in prior works (in particular [20, 37, 2, 36]). We will see that it is enough to assume that the low frequencies of the data have Besov regularity for some Lebesgue index that may be greater than 2. Also, we will improve the decay that was obtained for the compressible Euler system with damping in [36] and, adapting an idea from Xu and Xin in [32] for the compressible Navier-Stokes system will enable us to discard the additional smallness assumption on the low frequencies that is usually required to obtain the decay estimates.

The rest of the paper unfolds as follow. In Section 1, we present our main results for (TM_λ), namely the global existence of a solution corresponding to small data with optimal estimates with respect to the dissipation coefficient, and time decay estimates. In the next section, we focus on the particular case of data with regularity in Besov spaces built on L^2, and prove global existence in this case, as well as the time decay estimates. The method we here propose is different than the one for the general case, and is more easily extendable to the multi-dimensional setting. In Section 3, we propose another method that allows to get our global existence result for a larger class of data, not necessarily in L^2 type spaces. The next two sections are devoted to adapting our results, first for the isentropic Euler system with damping, and next for a general class of one-dimensional systems of two conservations laws, with partial damping. Some technical lemmas are proved in Appendix.

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1. Main results

Before stating the main results, we need to introduce a few notations. First, throughout the paper, we fix a homogeneous Littlewood-Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$ that is defined by

$$\dot\Delta_j \triangleq \varphi(2^{-j}D) \quad \text{with} \quad \varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$$

where χ stands for a smooth function with range in $[0,1]$, supported in $]-4/3, 4/3[$ and such that $\chi \equiv 1$ on $[-3/4, 3/4]$. We further set

$$\dot S_j \triangleq \chi(2^{-j}D) \quad \text{for all} \quad j \in \mathbb{Z}$$

and define $\mathcal{S}'_h$ to be the set of tempered distributions z such that $\dot S_j z \to 0$ uniformly when $j \to -\infty$.

Following [1], we introduce the homogeneous Besov semi-norms:

$$\|z\|_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \big\|2^{js}\|\dot\Delta_j z\|_{L^p(\mathbb{R})}\big\|_{\ell^r(\mathbb{Z})},$$

then define the homogeneous Besov spaces $\dot{\mathbb{B}}^s_{p,r}$ (for any $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$) to be the subset of z in $\mathcal{S}'_h$ such that $\|z\|_{\dot{\mathbb{B}}^s_{p,r}}$ is finite.

To any element z of $\mathcal{S}'_h$, we associate its low and high frequency parts with respect to some fixed threshold $J_0 \in \mathbb{Z}$, through

$$z^\ell \triangleq \sum_{j\leq J_0} \dot\Delta_j z = \dot S_{J_0+1} z \quad \text{and} \quad z^h \triangleq \sum_{j>J_0} \dot\Delta_j z = (\mathrm{Id} - \dot S_{J_0+1})z.$$

In order to emphasize the dependency of the notation with respect to the threshold parameter J_0, we use sometimes the notation z^{ℓ,J_0} and z^{h,J_0}. Likewise, we set[1] if $r<\infty$

$$\|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \Big(\sum_{j\leq J_0}(2^{js}\left\|\dot{\Delta}_j z\right\|_{L^p})^r\Big)^{\frac{1}{r}} \quad\text{and}\quad \|z\|^{h,J_0}_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \Big(\sum_{j\geq J_0}(2^{js}\left\|\dot{\Delta}_j z\right\|_{L^p})^r\Big)^{\frac{1}{r}}.$$

Whenever the value of J_0 is clear from the context, we shall only write $\|z\|^{\ell}_{\dot{\mathbb{B}}^s_{p,r}}$

For any Banach space X, index ρ in $[1,\infty]$ and time $T\in[0,\infty]$, we use the notation $\|z\|_{L^\rho_T(X)}\triangleq\big\|\|z\|_X\big\|_{L^\rho(0,T)}$. If $T=+\infty$, then we just write $\|z\|_{L^\rho(X)}$. In the case where z has n components z_j in X, we slightly abusively keep the notation $\|z\|$ to mean $\sum_{j\in\{1,\cdots,n\}}\|z_j\|_X$.

Throughout the paper, $C>0$ designates a generic harmless constant, the value of which depends on the context. We use the notation p' for the conjugate Lebesgue exponent of p. Finally, we denote by $(c_j)_{j\in\mathbb{Z}}$ nonnegative sequences such that $\sum_{j\in\mathbb{Z}}c_j=1$.

We can now state our main global existence result for (TM_λ).

Theorem 1.1. *Let $2\leq p\leq 4$. There exist $k=k(p)\in\mathbb{Z}$ and $c_0=c_0(p)>0$ such that for $J_\lambda\triangleq\lfloor\log_2\lambda\rfloor+k$, if we assume that $u_0^{\ell,J_\lambda},v_0^{\ell,J_\lambda}\in\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^{h,J_\lambda},v_0^{h,J_\lambda}\in\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$ with*

$$\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\lambda^{-1}\|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\leq c_0,$$

then System (TM_λ) admits a unique global solution (u,v) in the space $E_p^{J_\lambda}$ defined by

$$\begin{aligned}
&u^{\ell,J_\lambda}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})\cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}),\quad u^{h,J_\lambda}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}),\\
&v^{\ell,J_\lambda}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})\cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}),\quad v^{h,J_\lambda}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\\
&\lambda v+\partial_x u\in L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})\quad\textit{and}\quad v\in L^2(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}).
\end{aligned}$$

Moreover we have the following a priori estimate:

$$X_{p,\lambda}(t)\lesssim\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\lambda^{-1}\|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\quad\textit{for all } t\geq 0,$$

where

$$\begin{aligned}
X_{p,\lambda}(t)\triangleq\|(u,v)\|^{\ell,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}+\lambda^{-1}\|(u,v)\|^{h,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}+\lambda^{-1}\|u\|^{\ell,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})}+\|(u,v)\|^{h,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}\\
+\|\lambda v+\partial_x u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}+\lambda^{\frac{1}{2}}\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}.
\end{aligned}$$

Remark 1.1. *Somehow, the function $\lambda v+\partial_x u$ may be seen as a damped mode of the system, which explains its better time integrability. This is actually the key to closing the estimates globally in time, and this enables us to prove similar results for more general systems (see Sections 4 and 5).*

Remark 1.2. *Kawashima and Xu in* [34] *obtained a result in critical nonhomogeneous Besov spaces built on L^2 for a class of system containing (TM_λ). In their functional setting however, it seems difficult to track the exact dependency of the smallness condition and of the estimates with respect to the damping parameter λ. Furthermore, whether a L^p approach may be performed for the whole classe of systems that is considered therein, is unclear.*

Remark 1.3. *In the L^2 case, the method we here propose is robust enough to be adapted to higher dimension and to systems with more components, see* [5] *and* [8].

[1]For technical reasons, we need a small overlap between low and high frequencies.

Remark 1.4. *In Section 4 a statement similar to the above one is obtained for the isentropic compressible Euler system with a damping term in the velocity equation. To our knowledge, it is the first result (partially) in the L^p setting for this system. Obtaining a similar result in higher dimension is a work in progress.*

The above theorem gives us for free some insight on the diffusive relaxation limit of (TM_λ) in the case of fixed initial data[2].

Corollary 1.1. *Under the hypotheses of Theorem 1.1, we have $u \to u_0$ and $v \to 0$ when λ goes to infinity. More precisely,*

$$\|v\|_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \leq Cc_0\lambda^{-1/2} \quad \textit{and} \quad \|u(t)-u_0\|_{\dot{\mathbb{B}}^0_{p,1}} \leq Cc_0\biggl(\frac{t}{\lambda}\biggr)^{\frac{1}{2p}}.$$

Proof. The first inequality follows from the estimate for $X_{p,\lambda}$ in Theorem 1.1. For the second inequality, we observe that by interpolation in Besov spaces and Hölder inequality,

$$\|\partial_x v\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|\partial_x v\|^{1-\frac{1}{p}}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|\partial_x v\|^{\frac{1}{p}}_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1})} \quad \text{with} \quad \frac{1}{r} \triangleq 1-\frac{1}{2p}\cdot$$

Since

$$\|\partial_x v\|_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^{\ell}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})} + \|v\|^{h}_{L^1(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})},$$

Theorem 1.1 gives us

$$\|\partial_x v\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \leq Cc_0\lambda^{-\frac{1}{2p}}.$$

Similarly, we have

$$\|\partial_x u\|_{L^{\widetilde{r}}(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|\partial_x u\|^{\frac{1}{2}-\frac{1}{2p}}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})} \|\partial_x u\|^{\frac{1}{2}+\frac{1}{2p}}_{L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1})} \quad \text{with} \quad \frac{1}{\widetilde{r}} \triangleq \frac{1}{2}-\frac{1}{2p}\cdot$$

Hence, using that the product maps $\dot{\mathbb{B}}^0_{p,1} \times \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ to $\dot{\mathbb{B}}^0_{p,1}$ and Theorem 1.1, we deduce that

$$\|v\partial_x u\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|v\|_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|\partial_x u\|_{L^{\widetilde{r}}(\dot{\mathbb{B}}^0_{p,1})} \leq Cc_0\lambda^{-\frac{1}{2p}}.$$

Since $\partial_t u = -\partial_x v - v\partial_x u$, we get the desired inequality for $u(t)-u_0$, by time integration and Hölder inequality. □

Our second main result concerns the optimal decay estimates of the solution constructed in the first theorem. For now, we only consider the case $p=2$.

Theorem 1.2. *Under the hypotheses of Theorem 1.1 with $p=2$, there exists a Lyapunov functional associated to the solution (u,v) constructed there, which is equivalent to $\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$.*

If, additionally, $(u_0,v_0) \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \bigl(-\frac{1}{2},\frac{1}{2}\bigr]$ then, there exists a constant C depending only on σ_1 and such that

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

Furthermore, there exists a constant κ_0 depending only on σ_1, λ and on the norm of the data (and that may be taken equal to one in certain regimes, see the remark below) such that, if

$$\langle t\rangle \triangleq 1+\kappa_0 t, \quad \alpha_2 \triangleq \sigma_1+\frac{1}{2} \quad \textit{and} \quad C_{0,\lambda} \triangleq \lambda^{1+\alpha_2}\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

[2]We actually expect our method to be appropriate for investigating the connections between the compressible Euler system and the porous media equation, in the spirit of [19, 22, 33]. This is a work in progress.

then we have the following decay estimates:

$$\lambda^{\frac{3}{2}-\sigma}\,\|\langle\lambda t\rangle^{\alpha}(u,v)(t)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq CC_{0,\lambda},\quad \sigma\in[-\sigma_1,1/2],\quad \alpha\triangleq\frac{\sigma+\sigma_1}{2},$$
$$\|\langle\lambda t\rangle^{\alpha_2}(u,v)(t)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,\infty}}\leq CC_{0,\lambda},$$
$$\lambda^{\sigma_1+\frac{3}{2}}\,\|\langle\lambda t\rangle^{\alpha_1}v\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\leq CC_{0,\lambda},\qquad \alpha_1\triangleq\frac{1}{2}\Big(\frac{1}{2}+\sigma_1\Big)\cdot$$

Remark 1.5. *Our proof reveals that*[3] $\kappa_0\approx 1$ *whenever the first term of* $C_{0,\lambda}$ *is controlled by the second one (which amounts to saying that the low frequencies of the data are dominated by the high frequencies).*

Remark 1.6. *The fact that* v *undergoes direct dissipation and not* u *explains why the decay of the low frequencies of* v *is stronger than that of* u*.*

Remark 1.7. *In light of the embedding* $L^1\hookrightarrow\dot{\mathbb{B}}^{-\frac{1}{2}}_{2,\infty}$*, the above statement with* $\sigma_1=1/2$ *encompasses the classical* L^1 *condition of* [24]*. Actually, choosing suitable exponents allows to recover all the conditions used in* [4] *for getting decay estimates.*

2. The case $p=2$

The present section is dedicated to the case $p=2$ in Theorem 1.1 and to the proof of Theorem 1.2. The reason for looking first at $p=2$ is that one can exhibit a Lyapunov functional for (TM_λ) that allows to treat the low and high frequencies of the solution together. Throughout this section, we focus on the proof of a priori estimates for smooth solutions to (TM_λ), the reader being referred to the next section for the rigorous proof of existence and uniqueness, in the general case.

Before starting, let us observe that (u,v) is a solution to (TM_λ) if and only if the couple $(\tilde u,\tilde v)$ defined by

$$(u,v)(t,x)\triangleq(\tilde u,\tilde v)(\lambda t,\lambda x)\tag{3}$$

satisfies (TM_1). Therefore, it suffices to establish Theorems 1.1 and 1.2 for $\lambda=1$, scaling back giving the desired inequalities, owing to the use of *homogeneous* Besov norms.

In the rest of this section, and in the following one, we shall use the short notation (TM) to designate (TM_1).

2.1. Global a priori estimates for the linearized toy model.

Here we are concerned with the proof of a priori estimates for the following linearization of (TM):

$$(LTM)\qquad\begin{cases}\partial_t u+w\partial_x u+\partial_x v=0 & \text{in}\quad \mathbb{R}^+\times\mathbb{R},\\ \partial_t v+w\partial_x v+\partial_x u+v=0 & \text{in}\quad \mathbb{R}^+\times\mathbb{R},\\ (u,v)|_{t=0}=(u_0,v_0) & \text{on}\quad \mathbb{R},\end{cases}$$

where the given function $w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is smooth.

In the following computations, we assume that we are given a smooth solution (u,v) of (LTM) on $[0,T]\times\mathbb{R}$, and denote, for all $j\in\mathbb{Z}$,

$$u_j\triangleq\dot\Delta_j u\quad\text{and}\quad v_j\triangleq\dot\Delta_j v.\tag{4}$$

Inspired by the work of the second author in [9, 13], we consider the following functional:

$$\mathcal{L}_j\triangleq\sqrt{\|u_j\|_{L^2}^2+\|v_j\|_{L^2}^2+\|\partial_x u_j\|_{L^2}^2+\|\partial_x v_j\|_{L^2}^2+\int_{\mathbb{R}}v_j\,\partial_x u_j}.\tag{5}$$

[3]The exact value is $\kappa_0=\Big(\frac{\lambda\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{1/2}_{2,1}}+\|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{3/2}_{2,1}}}{\lambda^{1+\alpha_2}\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}+\|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{3/2}_{2,1}}}\Big)^{\frac{2}{\sigma_1+1/2}}$.

Applying operator $\dot{\Delta}_j$ to (LTM), simple computations lead to

$$\frac{1}{2}\frac{d}{dt}\|(u_j,v_j)\|_{L^2}^2+\|v_j\|_{L^2}^2+\int_{\mathbb{R}}\Big(\dot{\Delta}_j(w\partial_x u)\,\dot{\Delta}_j u+\dot{\Delta}_j(w\partial_x v)\,\dot{\Delta}_j v\Big)=0,$$
$$\frac{1}{2}\frac{d}{dt}\|(\partial_x u_j,\partial_x v_j)\|_{L^2}^2+\|\partial_x v_j\|_{L^2}^2+\int_{\mathbb{R}}\big(\dot{\Delta}_j\partial_x(w\partial_x u)\big)\,\dot{\Delta}_j\partial_x u+\dot{\Delta}_j(\partial_x(w\partial_x v))\,\dot{\Delta}_j\partial_x v\Big)=0,$$

$$\frac{d}{dt}\int_{\mathbb{R}}v_j\,\partial_x u_j+\int_{\mathbb{R}}v_j\,\partial_x u_j+\|\partial_x u_j\|_{L^2}^2-\|\partial_x v_j\|_{L^2}^2+\int_{\mathbb{R}}\Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v+\dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big)=0.$$

Using the fact that

$$\dot{\Delta}_j(w\partial_x z)=w\partial_x z_j+[\dot{\Delta}_j,w]\partial_x z\quad\text{for}\quad z=u,v \tag{6}$$

and integrating by parts, we see that

$$\int_{\mathbb{R}}\dot{\Delta}_j(w\partial_x z)\,\dot{\Delta}_j z=-\frac{1}{2}\int_{\mathbb{R}}\partial_x w\,|z_j|^2+\int_{\mathbb{R}}[\dot{\Delta}_j,w]\partial_x z\,z_j.$$

Hence, using the classical commutator estimate (79) recalled in the Appendix and the embedding $\dot{\mathbb{B}}^{\frac12}_{2,1}\hookrightarrow L^\infty$, we get an absolute constant $C>0$ such that for all $j\in\mathbb{Z}$,

$$\Big|\int_{\mathbb{R}}\dot{\Delta}_j(w\partial_x z)\,\dot{\Delta}_j z\Big|\le Cc_j2^{-\frac{j}{2}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\dot{\Delta}_j z\|_{L^2}\quad\text{with}\quad\sum_{j\in\mathbb{Z}}c_j=1.$$

Likewise, we have, thanks to an integration by parts,

$$\int_{\mathbb{R}}\dot{\Delta}_j\partial_x(w\partial_x z)\,\dot{\Delta}_j\partial_x z=\frac{1}{2}\int_{\mathbb{R}}\partial_x w\,(\partial_x z_j)^2+\int_{\mathbb{R}}\partial_x[\dot{\Delta}_j,w]\partial_x z\,\partial_x z_j.$$

Hence, using (81) and $\dot{\mathbb{B}}^{\frac12}_{2,1}\hookrightarrow L^\infty$,

$$\Big|\int_{\mathbb{R}}\dot{\Delta}_j\partial_x(w\partial_x z)\,\dot{\Delta}_j\partial_x z\Big|\le Cc_j2^{-\frac{j}{2}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x z_j\|_{L^2}.$$

Finally, integrating by parts reveals that

$$\int_{\mathbb{R}}\Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v+\dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big)=\int_{\mathbb{R}}[\dot{\Delta}_j,w]\partial_x v\,\partial_x u_j-\int_{\mathbb{R}}[\dot{\Delta}_j,w]\partial_x u\,\partial_x v_j.$$

Hence, using (79),

$$\Big|\int_{\mathbb{R}}\Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v+\dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big)\Big|$$
$$\le Cc_j2^{-\frac{j}{2}}\big(\|u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x v_j\|_{L^2}+\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x u_j\|_{L^2}\big)\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot$$

In order to conclude the proof of estimates for $\mathcal{L}_j$, one can observe that there exist two absolute constants C and c such that

$$C^{-1}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2\le\mathcal{L}_j^2\le C\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2 \tag{7}$$
$$\text{and}\quad\|v_j\|_{L^2}^2+\tfrac12\Big(\|\partial_x u_j\|_{L^2}^2+\|\partial_x v_j\|_{L^2}^2+\textstyle\int_{\mathbb{R}}v_j\,\partial_x u_j\,dx\Big)\ge c\min(1,2^{2j})\mathcal{L}_j^2.$$

Consequently, putting together with the above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+c\min(1,2^{2j})\mathcal{L}_j^2\le Cc_j2^{-\frac{j}{2}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\mathcal{L}_j. \tag{8}$$

Then, integrating on [0,t] for $t\in[0,T]$ and using Lemma A.1 yields

$$2^{\frac{j}{2}}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)(t)\|_{L^2}+\min(1,2^{2j})2^{\frac{j}{2}}\int_0^t\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}\\ \leq C\biggl(2^{\frac{j}{2}}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)(0)\|_{L^2}+\int_0^t c_j\|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\biggr)\cdot \tag{9}$$

Since (direct) damping is present in the equation for v, one can expect v to have better decay and time integrability properties than u. In fact, as explained at the beginning of Section 3, it is even better to consider the function $z\triangleq v+\partial_x u$ that satisfies:

$$\partial_t z+z+w\partial_x z=-\partial_{xx}^2 v-\partial_x w\,\partial_x u.$$

Now, applying Operator $\dot\Delta_j$ to the above equation, then using the basic energy method gives:

$$\frac12\frac{d}{dt}\|z_j\|_{L^2}^2+\|z_j\|_{L^2}^2=-\int_{\mathbb{R}}z_j\,\partial_{xx}^2 v_j-\int_{\mathbb{R}}\dot\Delta_j(w\partial_x z)\,z_j-\int_{\mathbb{R}}\dot\Delta_j(\partial_x w\,\partial_x u)z_j.$$

The last term may be handled thanks to the decomposition (6), integration by parts (as above) and Inequality (79) with $s=1/2$. This gives

$$\frac12\frac{d}{dt}\|z_j\|_{L^2}^2+\|z_j\|_{L^2}^2\leq\|z_j\|_{L^2}\bigl(\|\partial_{xx}^2 v_j\|_{L^2}+C2^{-\frac{j}{2}}c_j\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\dot\Delta_j(\partial_x w\,\partial_x u)\|_{L^2}\bigr)\cdot$$

After time integration (use Lemma A.1), we end up for all $t\in[0,T]$ with

$$2^{\frac{j}{2}}\|z_j(t)\|_{L^2}+2^{\frac{j}{2}}\int_0^t\|z_j\|_{L^2}\leq 2^{\frac{j}{2}}\|z_j(0)\|_{L^2}\\ +2^{\frac{j}{2}}\int_0^t\|\partial_{xx}^2 v_j\|_{L^2}+C\int_0^t c_j\bigl(\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_x w\,\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\bigr)\cdot$$

Hence, summing up on $j\leq 0$ and using the stability of the space $\dot{\mathbb{B}}^{\frac12}_{2,1}$ by product yields

$$\|z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\int_0^t\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\leq\|z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+C\int_0^t\|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac52}_{2,1}}+C\int_0^t\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\bigl(\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\bigr)\cdot \tag{10}$$

Let us sum (9) on $j\in\mathbb{Z}$, then add (10) multiplied by a small enough constant. Using (7) and denoting $X(t)\triangleq\|(u,v,\partial_x u,\partial_x v)(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$, we eventually get

$$X(t)+\int_0^t\bigl(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac52}_{2,1}}+\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac32}_{2,1}}+\|v+\partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\bigr)\leq C\biggl(X(0)+\int_0^t\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}X\biggr)\cdot \tag{11}$$

Let us revert to our toy model, assuming that $w=v$. Then, denoting by $Y(t)$ the left-hand side of (11), we get

$$Y(t)\leq C\bigl(X(0)+Y^2(t)\bigr).$$

As $Y(0)=X(0)$, a continuity argument ensures that there exists $c_0>0$ such that if

$$X(0)\approx\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\leq c_0, \tag{12}$$

then we have

$$Y(t)\leq 2CX(0)\quad\text{for all}\quad t\in[0,T].$$

2.2. **Proof of Theorem 1.2.** For getting decay estimates *without any additional smallness condition*, the first step is to prove that the extra negative regularity for low frequencies is preserved through the time evolution. This is stated in the following lemma which is an adaptation to our setting of a result first proved by J. Xu and Z. Xin in [32] for the compressible Navier-Stokes system.

Lemma 2.1. *Let $\sigma_1 \in]-\frac{3}{2},\frac{1}{2}]$. If, in addition to the hypotheses of Theorem 1.1, $\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}$ is bounded then, for all $t\geq 0$, we have*

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\,.$$

Proof. Applying $\dot{\Delta}_j$ to (TM) yields

$$\begin{cases} \partial_t u_j + \partial_x v_j = -v\partial_x u_j + [v,\dot{\Delta}_j]\partial_x u,\\ \partial_t v_j + \partial_x u_j + v_j = -v\partial_x v_j + [v,\dot{\Delta}_j]\partial_x v. \end{cases}$$

Hence, an energy method, followed by time integration (use Lemma A.1) gives

$$\|(u_j,v_j)(t)\|_{L^2} + \int_0^t \|v_j\|_{L^2} \leq \|(u_j,v_j)(0)\|_{L^2} \\ +\frac{1}{2}\int_0^t \|\partial_x v\|_{L^\infty}\|(u_j,v_j)\|_{L^2} + \int_0^t \|[v,\dot{\Delta}_j]\partial_x u\|_{L^2} + \int_0^t \|[v,\dot{\Delta}_j]\partial_x v\|_{L^2}.$$

Omitting the second term in the left-hand side, and using the commutator estimate (80) that is valid provided $-1/2\leq -\sigma_1<3/2$, we get

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Hence, by Gronwall lemma,

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \exp\biggl(C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\biggr).$$

Since the term in the exponential is small (as $X(0)$ is small), we get the lemma. □

The second ingredient is that one can work out from the computations we did in the previous paragraph, a *Lyapunov functional* that is equivalent to the norm of (u,v) in $\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}$. To proceed, observe that, on the one hand, Inequality (8) implies that for all $t\geq 0$,

$$\mathcal{L}(t) + c\int_0^t \mathcal{H} \leq \mathcal{L}(0) + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\mathcal{L}$$

$$\text{with}\quad \mathcal{L}\triangleq\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\mathcal{L}_j \quad\text{and}\quad \mathcal{H}\triangleq\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\min(1,2^{2j})\mathcal{L}_j$$

and that, on the other hand, (10) gives us

$$\|(v+\partial_x u)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|v+\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \leq \|v_0+\partial_x u_0\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|v\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\mathcal{L}.$$

Hence, there exist $\eta>0$ and $c'>0$ such that, denoting $\widetilde{\mathcal{L}}\triangleq\mathcal{L}+\eta\|v+\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$, we have

$$\widetilde{\mathcal{L}}(t) + c'\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\mathcal{L} \quad\text{with}\quad \widetilde{\mathcal{H}}\triangleq\mathcal{H}+\eta\|v+\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Observe that $\widetilde{\mathcal{H}} \gtrsim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$. Since the previous step ensures that $\mathcal{L} \lesssim X(0)$, one can conclude that the last term of the above inequality may be absorbed by the second term of the left-hand side provided $X(0)$ is small enough. So finally, taking c' smaller if need be, we discover that

$$\widetilde{\mathcal{L}}(t) + c' \int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0).$$

Clearly, one can start the proof from any time $t_0 \geq 0$ and get in a similar way:

$$\widetilde{\mathcal{L}}(t_0 + h) + c' \int_{t_0}^{t_0+h} \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(t_0), \quad h \geq 0.$$

This of course ensures that $\widetilde{\mathcal{L}}$ is nonincreasing on $\mathbb{R}^+$ (hence differentiable almost everywhere) and that for all $t_0 \geq 0$ and $h > 0$,

$$\frac{\widetilde{\mathcal{L}}(t_0 + h) - \widetilde{\mathcal{L}}(t_0)}{h} + c' \frac{1}{h} \int_{t_0}^{t_0+h} \widetilde{\mathcal{H}} \leq 0.$$

Consequently, passing to the limit $h \to 0$ gives

$$\frac{d}{dt}\widetilde{\mathcal{L}} + c'\widetilde{\mathcal{H}} \leq 0 \quad \text{a. e. on } \mathbb{R}^+. \tag{13}$$

We thus come to the conclusion that:

Lemma 2.2. *There exist two functionals $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}$ satisfying*

$$\widetilde{\mathcal{L}} \simeq \|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \quad \text{and} \quad \widetilde{\mathcal{D}} \simeq \|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} + \|u\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|v\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

and such that if $\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$ is small enough then (13) *is satisfied.*

One can now tackle the proof of decay estimates. Let us denote

$$C_0 \triangleq \|(u_0,v_0)\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

As a first, observe that interpolation for homogeneous Besov norms gives us:

$$\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \Big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\Big)^{\theta_0} \Big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}}\Big)^{1-\theta_0} \quad \text{with} \quad \theta_0 \triangleq \frac{2}{5/2+\sigma_1}\cdot$$

Therefore, owing to Lemma 2.1, there exists $c > 0$ such that

$$\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} \geq c\, C_0^{-\frac{\theta_0}{1-\theta_0}} \big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)^{\frac{1}{1-\theta_0}}.$$

Note that our definition of C_0 and the estimates we proved for (u,v) in the previous paragraph also ensure that

$$\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \gtrsim C_0^{-\frac{\theta_0}{1-\theta_0}} \big(\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)^{\frac{1}{1-\theta_0}}.$$

Hence, thanks to the above lemma, we have,

$$\frac{d}{dt}\widetilde{\mathcal{L}} + cC_0^{-\frac{\theta_0}{1-\theta_0}} \widetilde{\mathcal{L}}^{\frac{1}{1-\theta_0}} \leq 0 \quad \text{with} \quad \widetilde{\mathcal{L}} \simeq \|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

Integrating, this gives us

$$\widetilde{\mathcal{L}}(t) \leq \big(1 + \kappa_0\, t\big)^{1-\frac{1}{\theta_0}} \widetilde{\mathcal{L}}(0) \quad \text{with} \quad \kappa_0 \triangleq c\frac{\theta_0}{1-\theta_0}\left(\frac{\widetilde{\mathcal{L}}(0)}{C_0}\right)^{\frac{\theta_0}{1-\theta_0}}.$$

Rewriting θ_0 in terms of σ_1 and using that $\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\lesssim C_0$, we get

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\leq C(1+t)^{-\alpha_1}\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\quad\text{with}\quad\alpha_1\triangleq\frac12\Big(\sigma_1+\frac12\Big)\cdotp \tag{14}$$

In order to get the decay rate in $\dot{\mathbb{B}}^{\sigma}_{2,1}$ for all $\sigma\in(-\sigma_1,1/2)$, we just need the interpolation inequality

$$\|(u,v)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq\|(u,v)\|^{\theta_1}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\,\|(u,v)\|^{1-\theta_1}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\quad\text{with}\quad\theta_1\triangleq\frac{1/2-\sigma}{1/2+\sigma_1}\in(0,1).$$

In the end, we get (since $\kappa_0\lesssim1$):

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq C(1+\kappa_0 t)^{-\frac{\sigma+\sigma_1}{2}}\|(u_0,v_0)\|^{\theta_1}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|(u_0,v_0)\|^{1-\theta_1}_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\cdotp$$

In order to improve the decay for the damped component v, let us start from

$$\partial_t v+v=-\frac12\partial_x(v^2)-\partial_x u.$$

As v_0^ℓ is in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1\in]-\frac12,\frac12]$, we get

$$\|v(t)\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\leq e^{-t}\,\|v_0\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}+\int_0^t e^{-(t-\tau)}\,\big\|(\partial_x v^2,\partial_x u)\big\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\,d\tau. \tag{15}$$

It is important to observe that, as $1-\sigma_1\geq1/2$,

$$\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\lesssim\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\cdotp \tag{16}$$

Hence, multiplying (15) by $\langle t\rangle^{\alpha_1}$ and using the product laws in Besov spaces recalled in Proposition A.3 yields:

$$\|\langle t\rangle^{\alpha_1}v(t)\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\leq\|v_0\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}+\int_0^t\langle t\rangle^{\alpha_1}e^{-(t-\tau)}\,\|u\|^\ell_{\dot{\mathbb{B}}^{1-\sigma_1}_{2,\infty}}\,d\tau+\int_0^t\langle t\rangle^{\alpha_1}e^{-(t-\tau)}\,\|v\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,d\tau,$$

and one can conclude as above that

$$\langle t\rangle^{\alpha_1}\|v(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\lesssim\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}+\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}\cdotp$$

Let us finally exhibit the (optimal) decay rate of the high frequencies for the norm in $\dot{\mathbb{B}}^{\frac32}_{2,1}$. Recall that for $j\geq0$, we have

$$\frac{d}{dt}\mathcal{L}_j^2+c\mathcal{L}_j^2\lesssim\mathcal{L}_j^2\,\|\partial_x v\|_{L^\infty}+c_j2^{-\frac{j}{2}}X\,\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\mathcal{L}_j.$$

Hence, using Lemma A.1, multiplying by $2^{\frac{j}{2}}$, summing up on $j\geq0$ and remembering that

$$\sum_{j\geq0}2^{\frac{j}{2}}\mathcal{L}_j\simeq\|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}},$$

we get

$$\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\lesssim e^{-ct}\,\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}+\int_0^t e^{-c(t-\tau)}\,\|v\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\,\|(u,v)\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\cdotp \tag{17}$$

Multiplying both sides by $\langle t\rangle^{2\alpha_1}$, we get

$$\big\|\langle t\rangle^{2\alpha_1}(u,v)(t)\big\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\lesssim\langle t\rangle^{2\alpha_1}e^{-ct}\,\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}+\int_0^t\left(\frac{\langle t\rangle}{\langle\tau\rangle}\right)^{2\alpha_1}e^{-c(t-\tau)}\big(\langle\tau\rangle^{\alpha_1}\,\|v\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\big)\big(\langle\tau\rangle^{\alpha_1}\,\|(u,v)\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\big).$$

Taking advantage of (14) for bounding the norms in the time integral, one ends up with the desired decay estimate for $\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$. This completes the proof of Theorem 1.2.

Remark 2.1. *In the same way, making the slightly stronger assumption that* $v_0^\ell \in \dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}$, *we get*

$$\langle t\rangle^{\alpha_1}\|v(t)\|_{\dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}} \lesssim \|v_0\|_{\dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}} + \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\cdot$$

3. Proof of Theorem 1.1

An explicit computation in the Fourier space of the solution to (LTM) with $w=0$ reveals that:

- In low frequencies, the matrix of the system corresponding to frequency ξ has two real eigenvalues that tend to be equal to 1 and to ξ^2, for ξ going to 0;
- In high frequencies, two complex conjugated eigenvalues coexist, that are, asymptotically, equal to $\frac{1}{2}(\xi^2 \pm i\xi)$.

Consequently, one can expect that the low frequency part of System (TM) is solvable in some L^p type functional framework with, possibly, $p\neq 2$, whereas going beyond the L^2 framework in high frequency is bound to fail. A similar dichotomy has been observed for the compressible Navier-Stokes equations (see in particular [6, 7, 17]) but the behavior of the low and high frequencies in our situation is exchanged.

In order to extend the results of the previous section to the L^p framework for low frequencies, we shall adapt [17] to our setting, introducing an 'effective velocity' that reads $z = v+\partial_x u$ and may be seen as an approximate dissipative eigenmode of the system, in the low frequency regime.

The bulk of the proof consists in establishing estimates in the functional framework of Theorem 1.1 for (LTM). This will be carried out in the first two subsections of this part. Then, we will prove the existence part of the theorem and, finally, the uniqueness of a solution.

3.1. Low frequencies estimates in L^p.

The main result of this section reads as follows.

Proposition 3.1. *Let* (u,v) *be a smooth solution of* (LTM) *on* $[0,T]$. *Then, for all* $1\leq p<\infty$, *we have*

$$\|(u,v)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|v+\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \leq C\biggl(\|(u_0,v_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|(u,v,\partial_x u)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\,\|w\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\biggr). \tag{18}$$

Proof. Let us set $z \triangleq v+\partial_x u$. We observe that the couple (u,z) satisfies

$$\begin{cases} \partial_t u - \partial^2_{xx}u + w\partial_x u = -\partial_x z,\\ \partial_t z + z + w\partial_x z = \partial^3_{xxx}u - \partial^2_{xx}z - \partial_x w\,\partial_x u. \end{cases} \tag{19}$$

In low frequencies, we expect the linear terms of the right-hand side to be negligible, so that we will look at the first equation as a heat equation with a convection term, and at the second one as a damped transport equation.

Now, applying $\dot{\Delta}_j$ to the first equation of (19) yields

$$\begin{aligned}\partial_t u_j - \partial^2_{xx}u_j &= -\dot{\Delta}_j(w\partial_x u) - \partial_x z_j\\ &= -w\partial_x u_j - \partial_x z_j + [w,\dot{\Delta}_j]\partial_x u.\end{aligned}$$

Multiplying by $|u_j|^{p-2}u_j$ and integrating in space, we get

$$\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p - \int_{\mathbb{R}} \partial_{xx}^2 u_j |u_j|^{p-2}u_j = -\int_{\mathbb{R}} \partial_x z_j |u_j|^{p-2}u_j - \int_{\mathbb{R}} w\partial_x u_j\,|u_j|^{p-2}u_j + \int_{\mathbb{R}} [w,\dot\Delta_j]\partial_x u\,|u_j|^{p-2}u_j.$$

Hence, integrating by parts, using Cauchy-Schwarz inequality and Proposition A.1 gives

$$\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p + c_p 2^{2j}\|u_j\|_{L^p}^p \le \frac{1}{p}\|\partial_x w\|_{L^\infty}\|u_j\|_{L^p}^p + \Big(\|\partial_x z_j\|_{L^p} + \Big\|[w,\dot\Delta_j]\partial_x u\Big\|_{L^p}\Big)\|u_j\|_{L^p}^{p-1}.$$

Multiplying by $2^{\frac{j}{p}}$, summing up on $j\le J_0$ and using Lemma A.1, we obtain

$$\|u(t)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + c_p\int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p+2}_{p,1}} \le \|u_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}} + \frac{1}{p}\int_0^t \|\partial_x w\|_{L^\infty}\|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \sum_{j\le J_0} 2^{\frac{j}{p}}\int_0^t \Big\|[w,\dot\Delta_j]\partial_x u\Big\|_{L^p}.$$

The commutator term may be bounded according to Inequality (79) with $s = 1/p$. Hence, remembering that $\dot{\mathbb{B}}^{\frac1p}_{p,1}\hookrightarrow L^\infty$, we end up with

$$\|u(t)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + c_p\int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p+2}_{p,1}} \le \|u_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}} + C\int_0^t \|w\|_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\|u\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}. \tag{20}$$

Let us next look at the second equation of (19). We have for all $j\in\mathbb{Z}$,

$$\partial_t z_j + z_j + w\partial_x z_j = \partial^3_{xxx}u_j - \partial^2_{xx}z_j - \dot\Delta_j(\partial_x w\,\partial_x u) + [w,\dot\Delta_j]\partial_x z.$$

Multiplying by $z_j|z_j|^{p-2}$ and adapting what we did for the for the first equation of (LTM), we obtain

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} \le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p+2}_{p,1}} + \int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p+3}_{p,1}} + \frac{1}{p}\int_0^t \|\partial_x w\|_{L^\infty}\|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \sum_{j\le J_0} 2^{\frac{j}{p}}\Big\|[w,\dot\Delta_j]\partial_x z\Big\|_{L^p} + \int_0^t \|\partial_x w\,\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}.$$

Combining Proposition A.2, the commutator estimate (79), the embedding $\dot{\mathbb{B}}^{\frac1p}_{p,1}\hookrightarrow L^\infty$ and Proposition A.3, we discover that

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} \le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p+2}_{p,1}} + \int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p+3}_{p,1}} + \int_0^t \|w\|_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\|z\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}. \tag{21}$$

At this stage, the key observation is that, owing to Bernstein inequality, there exists an absolute constant C such that for any couple $(\sigma,\sigma')\in\mathbb{R}^2$ with $\sigma\le\sigma'$, we have

$$\|f\|^\ell_{\dot{\mathbb{B}}^{\sigma'}_{p,1}} \le C2^{J_0(\sigma'-\sigma)}\|f\|^\ell_{\dot{\mathbb{B}}^{\sigma}_{p,1}}. \tag{22}$$

Consequently, if J_0 is chosen small enough, then after adding up (20) and (21), we just get

$$\|(u,z)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \big(\|u\|^\ell_{\dot{\mathbb{B}}^{\frac1p+2}_{p,1}} + \|z\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\big) \lesssim \|(u_0,z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \int_0^t \|w\|_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\big(\|u\|_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}} + \|u\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|z\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\big)\cdot$$

Because

$$\|z\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \lesssim \|u\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + \|v\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \quad \text{and} \quad \|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \lesssim \|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}},$$

we conclude to the desired inequality. □

3.2. **High frequencies estimates in L^2.** Our second task is to bound the high frequencies of the solution of (LTM). Although the functional framework for high frequencies is the same as before, one cannot repeat exactly the computations therein since the terms $(w\partial_x u)^h$ and $(w\partial_x v)^h$ contain a little amount of low frequencies of w, u and v, that are only in spaces of the type $\dot{\mathbb{B}}^s_{p,1}$ with $p>2$ (and thus not in some $\dot{\mathbb{B}}^{s'}_{2,1}$). To overcome the difficulty, we have to study more carefully the commutators coming into play in the proof (see Lemma A.3).

Proposition 3.2. *Let (u,v) be of solution of (LTM) with $u_0^\ell, v_0^\ell \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^h, v_0^h \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$ for some $2\leq p\leq 4$. Define p^* by the relation $1/p+1/p^*=1/2$. Then, the following a priori estimate holds for some constant C depending only on J_0:*

$$\begin{aligned}\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \Big(\|\partial_x w\|_{L^\infty}\|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &+ \|w\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|(\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|(\partial_x u,\partial_x v)\|_{L^\infty}\|w\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|(\partial_x u,\partial_x v)\|^{\ell}_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x w\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big)\cdot\end{aligned}$$

Proof. We localize System (LTM) by means of $\dot{\Delta}_j$, getting

$$\begin{cases} \partial_t u_j + \partial_x v_j + \dot{S}_{j-1}w\,\partial_x u_j = R_j^1 \\ \partial_t v_j + \partial_x u_j + v_j + \dot{S}_{j-1}w\,\partial_x v_j = R_j^2 \end{cases}$$

with

$$R_j^1 \triangleq \dot{S}_{j-1}w\,\partial_x u_j - \dot{\Delta}_j(w\,\partial_x u) \quad \text{and} \quad R_j^2 \triangleq \dot{S}_{j-1}w\,\partial_x v_j - \dot{\Delta}_j(w\,\partial_x v).$$

The remainder terms R_j^1 and R_j^2 will be bounded according to Lemma A.3. To handle the left-hand side of the above localized system, we introduce the following functional, designed for high frequencies:

$$\widetilde{\mathcal{L}}_j^2 \triangleq \|(\partial_x u_j, \partial_x v_j)\|_{L^2}^2 + \int_{\mathbb{R}} v_j\,\partial_x u_j,$$

and get

$$\begin{aligned}&\frac{1}{2}\Big(\frac{d}{dt}\widetilde{\mathcal{L}}_j^2 + \widetilde{\mathcal{L}}_j^2\Big) + \int_{\mathbb{R}} \partial_x\big((\dot{S}_{j-1}w\,\partial_x u_j)\partial_x u_j + \partial_x(\dot{S}_{j-1}w\,\partial_x v_j)\partial_x v_j\big) \\ &+ \int_{\mathbb{R}}\big(\dot{S}_{j-1}w\partial_x v_j\partial_x u_j + \partial_x(\dot{S}_{j-1}w\partial_x u_j)v_j\big) = \int_{\mathbb{R}}\big(\partial_x R_j^1\partial_x u_j + \partial_x R_j^2\partial_x v_j + \frac{1}{2}\big(\partial_x R_j^1 v_j + R_j^2\partial_x u_j\big)\big)\cdot\end{aligned}$$

Using integration by parts and multiplying by 2 then yields

$$\frac{d}{dt}\widetilde{\mathcal{L}}_j^2 + \widetilde{\mathcal{L}}_j^2 + \int_{\mathbb{R}} \partial_x\dot{S}_{j-1}w\,\big((\partial_x u_j)^2 + (\partial_x v_j)^2\big) = \int_{\mathbb{R}}\big(R_j^2\,\partial_x u_j - R_j^1\,\partial_x v_j - 2R_j^1\,\partial_x^2 u_j - 2R_j^2\,\partial_x^2 v_j\big)\cdot$$

From this, using Cauchy-Schwarz inequality, that $\widetilde{\mathcal{L}}_j \simeq \|(\partial_x u_j, \partial_x v_j)\|_{L^2}$ and Lemma A.1, we get for all $t\geq 0$ and $j\geq J_0$,

$$\widetilde{\mathcal{L}}_j(t) + \int_0^t \widetilde{\mathcal{L}}_j \leq C\Big(\int_0^t \|\partial_x w\|_{L^\infty}\widetilde{\mathcal{L}}_j + 2^j\int_0^t \|R_j^1, R_j^2\|_{L^2}\Big),$$

with C depending only on J_0. Hence, multiplying by $2^{\frac{j}{2}}$ and summing up on $j \geq J_0$,

$$\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|\partial_x w\|_{L^\infty} \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \sum_{j\geq J_0} 2^{\frac{3}{2}j} \big\|(R^1_j, R^2_j)\big\|_{L^2}. \tag{23}$$

At this stage, taking advantage of Lemma A.3 with $s = 3/2$ to bound the sum, we conclude to the desired inequality. □

3.3. **Global a priori estimates for the toy model.** We are now ready to establish the following proposition which is the key to the proof of the existence part of the theorem.

Proposition 3.3. *Let (u,v) be a smooth solution of (TM) on $[0,T]$. Then, still assuming that $2 \leq p \leq 4$, there exists a constant C and an integer J_0 (corresponding to the threshold between low and high frequencies) such that for all $t \in [0,T]$, we have*

$$X_p(t) \leq C\big(X_{p,0} + X_p^2(t)\big)$$

with $X_{p,0} \triangleq \|(u_0,v_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$ *and*

$$X_p(t) \triangleq \|(u,v)\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|(u,v)\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|u\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|v+\partial_x u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|(u,v)\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}.$$

Proof. As a first, let us observe that $\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ is dominated by the other terms of $X_p(t)$ (let us denote them by $\widetilde{X}_p(t)$). This is clearly the case of the high frequency part since, by Bernstein inequality, Hölder inequality and the embedding $\dot{\mathbb{B}}^s_{2,1} \hookrightarrow \dot{\mathbb{B}}^{s+\frac{1}{p}-\frac{1}{2}}_{p,1}$ with $s = 3/2$,

$$\|v\|^h_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^2_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \lesssim \sqrt{\|v\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \|v\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}}.$$

For the low frequency part, we write that

$$\|v\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \leq \|\partial_x u\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|z\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \quad \text{with} \quad z \triangleq v + \partial_x u.$$

By Hölder inequality and interpolation, we have

$$\|\partial_x u\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \Big(\|u\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|u\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})}\Big)^{\frac{1}{2}} \quad \text{and} \quad \|z\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \Big(\|z\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|z\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}\Big)^{\frac{1}{2}}.$$

As the low frequencies of z in $L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ may be bounded by $\widetilde{X}_p(t)$, we proved that

$$\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \widetilde{X}_p(t) \quad \text{for all} \quad t \in \mathbb{R}^+. \tag{24}$$

Let us also notice that by Sobolev embedding and Bernstein inequality,

$$\|v+\partial_x u\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|\partial_x u\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})}.$$

Therefore, adding up the inequalities from Propositions 3.1 and 3.2 for $w = v$ and observing that $2 \leq p \leq p^*$, we get

$$X_p(t) \lesssim X_{p,0} + \int_0^t \big(\|(u,v)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)\big(\|v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + \|v\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big) + \int_0^t \|v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|(\partial_x u, \partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}.$$

Since

$$\|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \leq \|v+\partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + C\|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}},$$

we conclude that the inequality of Proposition 3.3 is satisfied. □

3.4. **Proof of the existence part of Theorem 1.1.** The proof relies on the following classical result about the local existence of strong solutions for hyperbolic symmetric systems of type

$$(QS) \qquad \begin{cases} \partial_t U + \sum_{k=1}^d A_k(U)\partial_k U + A_0(U) = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where A_k, $k=0,\dots,d$ are smooth functions from $\mathbb{R}^n$ to the space of $n\times n$ matrices, that are symmetric if $k\neq 0$.

Theorem 3.1. [1, Chap. 4] *Let U_0 be in the nonhomogeneous Besov space $\mathbb{B}^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d;\mathbb{R}^n)$. Then, (QS) admits a unique maximal solution U in $\mathcal{C}([0,T^*[;\mathbb{B}^{\frac{d}{2}+1}_{2,1})\cap\mathcal{C}^1([0,T^*[;\mathbb{B}^{\frac{d}{2}}_{2,1})$, and there exists a positive constant c such that*

$$T^* \geq \frac{c}{\|U_0\|_{\mathbb{B}^{\frac{d}{2}+1}_{2,1}}}.$$

Furthermore,

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla U\|_{L^\infty} = \infty.$$

The proof of the existence part of Theorem 1.1 is structured as follows. First, we multiply the low frequencies of the data by a cut-off function in order to have data in $\mathbb{B}^{\frac{3}{2}}_{2,1}$. One can then use the above theorem to construct a sequence of approximate solutions, that are shown to be global. We prove uniform estimates in the space E_p for those solutions, pass to the limit up to subsequence by means of compactness arguments, and finally check that the limit is a solution of (TM) with the required properties.

First step. Construction of approximate solutions. Let (u_0,v_0) be such that $u_0^\ell, v_0^\ell \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^h, v_0^h \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$. Since (u_0,v_0) need not be in $\mathbb{B}^{\frac{3}{2}}_{2,1}$, we set for all $n\geq 1$,

$$u_0^n \triangleq \chi_n\, \dot{S}_{J_0-5}u_0 + (\mathrm{Id}-\dot{S}_{J_0-5})u_0 \quad \text{and} \quad v_0^n \triangleq \chi_n\, \dot{S}_{J_0-5}v_0 + (\mathrm{Id}-\dot{S}_{J_0-5})v_0$$

with $\chi_n \triangleq \chi(n^{-1}\cdot)$, where χ stands (for instance) for the bump function of Section 1.

It is obvious that the sequences $(u_0^n)_{n\in\mathbb{N}}$ and $(v_0^n)_{n\in\mathbb{N}}$ tend to u_0 and v_0 in the sense of distributions, when n tends to infinity. Moreover, as u_0^ℓ and v_0^ℓ are in $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$, the low frequencies of the data are in L^∞, and the spatial truncation thus guarantees that $u_0^n, v_0^n \in \mathbb{B}^{\frac{3}{2}}_{2,1}$. Hence, Theorem 3.1 provides us with a unique maximal solution $(u^n,v^n)\in\mathcal{C}([0,T_n[;\mathbb{B}^{\frac{3}{2}}_{2,1})\cap\mathcal{C}^1([0,T_n[;\mathbb{B}^{\frac{1}{2}}_{2,1})$.

We claim that we have for $z_0 = u_0, v_0$,

$$\|z_0^n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0^n\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}. \tag{25}$$

Indeed, since $\|\chi_n\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} \simeq \|\chi\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} < \infty$, owing to the invariance of the norm in $\dot{\mathbb{B}}^{\frac1p}_{p,1}$ by spatial dilation (see e.g. [1, Rem. 2.19]), we may write

$$\begin{aligned}\|z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} &\leq \Big\|\chi_n\,\dot S_{J_0-5}z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \Big\|(\mathrm{Id}-\dot S_{J_0-5})z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\\ &\lesssim \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\,\|\chi_n\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|z_0\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\\ &\lesssim \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|z_0\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\cdot\end{aligned}$$

Next, we see that

$$\|z_0^n\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \leq \|\chi_n\,\dot S_{J_0-5}z_0\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|(\mathrm{Id}-\dot S_{J_0-5})z_0^h\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}.$$

It is obvious that the last term may be bounded by $\|z_0\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}$. For the other term, the important observation is that for $j\geq J_0$, we have

$$\dot\Delta_j\big(\chi_n\,\dot S_{J_0-5}z_0\big) = \sum_{j'\geq j-3}\dot\Delta_j\big(\dot S_{j'+2}\dot S_{J_0-5}z_0\,\dot\Delta_{j'}\chi_n\big).$$

Hence, owing to the scaling properties of the space $\dot{\mathbb{B}}^{\frac32}_{2,1}$,

$$\|\chi_n\,\dot S_{J_0-5}z_0\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \lesssim \|\dot S_{J_0-5}z_0\|_{L^\infty}\|\chi_n\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \lesssim n^{-1}\|z_0\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}},$$

which eventually yields (25).

Second step. Uniform estimates. Since, for all $T>0$, the space $\mathcal{C}([0,T];\dot{\mathbb{B}}^{\frac32}_{2,1})\cap\mathcal{C}^1([0,T];\dot{\mathbb{B}}^{\frac12}_{2,1})$ is included in our 'solution space' $E_p(T)$ (that is, E_p restricted to $[0,T]$), one can take advantage of Proposition 3.3 for bounding our sequence. From it and (25), we get, denoting X_p^n the function X_p pertaining to (u^n,v^n),

$$X_p^n \leq C\big(X_{p,0}+(X_p^n)^2\big). \tag{26}$$

It is clear that if

$$2CX_p^n(t)\leq 1, \tag{27}$$

then Inequality (26) implies that

$$X_p^n(t)\leq 2CX_{p,0}.$$

Then, thanks to a classical bootstrap argument, we can conclude that if $X_{p,0}$ is small enough then (27) is true as long as the solution exists. Hence, there exists a constant C such that

$$X_p^n(t)\leq CX_{p,0}\quad\text{for all}\quad n\geq 1\quad\text{and}\quad t\in[0,T_n[. \tag{28}$$

In order to show that the above inequality implies that the solution is global (namely that $T_n=\infty$), one can argue by contradiction, assuming that $T_n<\infty$, and use the blow-up criterion of Theorem 3.1. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}^{\frac32}_{2,1}$ of the solution is under control *up to time* T_n. Now, applying the standard energy method to (TM) yields for all $t<T_n$,

$$\|(u^n,v^n)(t)\|^2_{L^2}\leq\|(u_0^n,v_0^n)\|^2_{L^2}+\int_0^t\|\partial_xv^n\|_{L^\infty}\|(u^n,v^n)\|^2_{L^2}.$$

Since (28) and the embedding of $\dot{\mathbb{B}}^{\frac1p}_{p,1}$ and $\dot{\mathbb{B}}^{\frac12}_{2,1}$ in L^∞ ensure that ∂_xv^n is in $L^1_{T_n}(L^\infty)$, using Gronwall lemma gives that (u^n,v^n) is in $L^\infty_{T_n}(L^2)$, and thus in $L^\infty_{T_n}(\mathbb{B}^{\frac32}_{2,1})$ owing, again, to (28).

It is now easy to conclude : for all $t_{0,n} \in [0, T_n[$, Theorem 3.1 provides us with a solution of (TM) with the initial data $(u(t_{0,n}), v(t_{0,n}))$, on $[t_{0,n}, T + t_{0,n}]$ for some T that may be bounded from below independently of $t_{0,n}$. Consequently, choosing $t_{0,n}$ such that $t_{0,n} > T_n - T$, we see that the solution (u^n, v^n) can be extended beyond T_n, which contradicts the maximality of T_n. Hence $T_n = +\infty$ and the solution corresponding to the initial data (u_0^n, v_0^n) is global in time and satisfies (28) for all time.

Third step. Convergence. We have to show that $(u^n, v^n)_{n\in\mathbb{N}}$ tends, up to subsequence, to some $(u, v) \in E_p$ in the sense of distribution, that satisfies (TM).

The proof that we here propose rests on Ascoli Theorem and suitable compact embeddings. Let us explain how it goes for $(u^n)_{n\in\mathbb{N}}$, the convergence of $(v^n)_{n\in\mathbb{N}}$ being similar. From (28) and elementary embedding, we know that :

- $(\partial_x u^n)_{n\in\mathbb{N}}$ and $(\partial_x v^n)_{n\in\mathbb{N}}$ are bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$,
- $(v^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$.

Hence, both $(v^n\partial_x u^n)_{n\in\mathbb{N}}$ and $(\partial_x v^n)_{n\in\mathbb{N}}$ are bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$, which implies that $(\partial_t u^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. This means that $(u^n)_{n\in\mathbb{N}}$ viewed as a sequence of functions valued in $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ is locally equicontinuous on $\mathbb{R}^+$.

Moreover $(u^{n,h})_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$, $(u^{n,\ell})_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ and we know, thanks to a result of [1, Chap. 2], that the embedding from $F = \{u \in \mathcal{S}',\ u^\ell \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \text{ and } u^h \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}\}$ to $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ is locally compact. Therefore, one can combine Ascoli Theorem and the Cantor diagonal extraction process to deduce that there exists a distribution u such that, up to subsequence $(\phi u^n)_{n\in\mathbb{N}}$ converges to ϕu in $\mathcal{C}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ for all function ϕ compactly supported in $\mathbb{R}^+ \times \mathbb{R}^n$. Then, using the Fatou property (cf. [1], chapter 2) we obtain that $u^\ell \in L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})$ and $u^h \in L^\infty(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$, with norms bounded by the right-hand side of (28). One can argue similarly for establishing the weak convergence of $(v^n)_{n\in\mathbb{N}}$ to some distribution v fulfilling the desired properties of regularity up to time continuity.

Finally, passing to the limit in (TM) is not an issue, since we have strong convergence (after localization) in norms with positive indices of regularity.

Last step. Proving that $(u, v) \in E_p$. The only property that misses is the time continuity. It may be obtained by looking at u and v as solutions of transport equations. Indeed, by construction, we have

$$\partial_t u + v\partial_x u = -\partial_x v \quad \text{and} \quad \partial_t v + v\partial_x v + v = -\partial_x u.$$

The properties we proved so far for u and v ensure that $\partial_x u$ and $\partial_x v$ belong to $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. Hence, the standard properties for the transport equation (see e.g. [1, Chap. 3]) give us that $(u, v) \in \mathcal{C}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$.

To show that $(u, v)^h \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})$, one can mimic the proof for general symmetric hyperbolic systems, summing up only on high frequencies, as presented at [1, p.196] for instance.

In the end, we thus have proved that (u, v) is a global solution of (TM), that verifies the desired properties of regularity and $X_p(t) \leq CX_{p,0}$ for all $t \in \mathbb{R}^+$.

3.5. **Proof of uniqueness.** Consider two solutions (u_1, v_1) and (u_2, v_2) of (TM) (not necessarily small) in the space E_p, that correspond to the same initial data (u_0, v_0). The proof of uniqueness will follow from stability estimates involving suitable norms. The difficulty is that our functional framework is not the standard one for the low frequency of the solution, so that

one cannot follow the classical proof for hyperbolic symmetric systems. Here we shall estimate $(\delta u, \delta v) := (u_2 - u_1, v_2 - v_1)$ in the space

$$F_p(T) \triangleq \Big\{ z \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}) \,:\, z^h \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}) \Big\}. \tag{29}$$

The reason for this choice is the usual loss of one derivative when proving stability estimates for quasilinear hyperbolic systems (hence the exponent $1/2$ for high frequencies). The exponent for low frequencies looks to be the best one for controlling the nonlinearities. Before starting the proof, we introduce the notation

$$\delta\!\mathcal{U}(t) \triangleq \|(\delta u, \delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|(\delta u, \delta v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Step 1. Proving that $(\delta u, \delta v) \in F_p(T)$. Remember that $\partial_t u_i = -\partial_x v_i - v_i \partial_x u_i$ for $i = 1, 2$. By interpolation in Besov spaces and Hölder inequality with respect to the time variable, since $\partial_x u_i^\ell$ and $\partial_x v_i^\ell$ are in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})$, we get

$$\partial_x u_i^\ell, \partial_x v_i^\ell \in L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}) \quad \text{with} \quad \frac{1}{r} \triangleq \frac{1}{4} + \frac{1}{2p}. \tag{30}$$

It is clear that the same property holds for the high frequencies of $\partial_x u_i$ and $\partial_x v_i$, since they belong to $L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. We also know that v_i belongs to $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. Therefore, from the product laws in Besov spaces that have been recalled in Proposition A.3, we gather that $\partial_x v_i$ and $v_i \partial_x u_i$ are in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$. Hence, $\partial_t u_i$ is in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$, and thus

$$(u_i - u_0) \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}). \tag{31}$$

Proving the result for v_i is almost the same, except that we have to handle the damping term. To overcome it, we notice that

$$\partial_t(e^t v_i) = -e^t v_i\, \partial_x v_i - e^t \partial_x u_i.$$

Arguing as above, we get that $\partial_t(e^t v_i) \in L^r_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$, whence

$$(e^t v_i - v_0) \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}). \tag{32}$$

From (31) and (32), we conclude that $(\delta u, \delta v) \in F_p(T)$ for all finite T.

Step 2. Estimates for the low frequencies. The system satisfied by $(\delta u, \delta v)$ reads:

$$\begin{cases} \partial_t \delta u + \partial_x \delta v = -\delta v\, \partial_x u_1 - v_2\, \partial_x \delta u, \\ \partial_t \delta v + \delta v + \partial_x \delta u = -\delta v\, \partial_x v_1 - v_2\, \partial_x \delta v. \end{cases} \tag{33}$$

Then, we follow the computations leading to Proposition 3.1 with $w = 0$, looking at $-\delta v\, \partial_x u_1 - v_2\, \partial_x \delta u$ and $-\delta v\, \partial_x v_1 - v_2\, \partial_x \delta v$ as source terms, and working at the level of regularity $2/p - 1/2$ instead of $1/p$ (since the left-hand side of (33) is linear with constant coefficients, this shift of regularity does not change the proof). Omitting the time integral in the left-hand side of the Inequality given by Proposition 3.1, we find that

$$\|(\delta u, \delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \int_0^t \big(\|\delta v\, \partial_x u_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|v_2\, \partial_x \delta u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|\delta v\, \partial_x v_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|v_2\, \partial_x \delta v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \big).$$

In order to bound the right-hand side, we may use that Proposition A.4 yields

$$\|a\, b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\, \|b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \tag{34}$$

as well as the following inequality that is a consequence of (84) in the appendix (take $s=\frac{2}{p}-\frac{1}{2}$ therein):

$$\|a\,b\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\lesssim\|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}\cap\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\|b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{3}{2}}_{p,1}}\cdot \tag{35}$$

In the end, choosing $a=v_2$ and $b=\partial_x\delta u$ or $\partial_x\delta v$, we get

$$\|(\delta u,\delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\lesssim\int_0^t\Big(\|(\partial_x u_1,\partial_x v_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\|\delta v\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|v_2\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}\cap\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\|(\delta u,\delta v)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\Big)\cdot \tag{36}$$

Step 3. Estimates for the high frequencies. We look at the system satisfied by $(\delta u,\delta v)$ under the form:

$$\begin{cases}\partial_t\delta u+v_2\,\partial_x\delta u+\partial_x\delta v=-\delta v\,\partial_x u_1,\\ \partial_t\delta v+\delta v+v_2\,\partial_x\delta v+\partial_x\delta u=-\delta v\,\partial_x v_1.\end{cases}$$

This is System (LTM) except for the source terms in the right-hand side. Clearly, following the computations leading to (23), but using the index $1/2$ instead of $3/2$ gives

$$\begin{aligned}\|(\delta u,\delta v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}&\lesssim\int_0^t\|\partial_x v_2\|_{L^\infty}\|(\delta u,\delta v)\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\&+\int_0^t\sum_{j\geq J_0}2^{\frac{j}{2}}\big(\|[v_2,\dot\Delta_j]\partial_x\delta u\|_{L^2}+\|[v_2,\dot\Delta_j]\partial_x\delta v\|_{L^2}\big)+\int_0^t\big(\|\delta v\,\partial_x u_1\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\delta v\,\partial_x v_1\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot\end{aligned} \tag{37}$$

Let $p^*\triangleq 2p/(p-2)$. Lemma A.3 tells us that, for $z=\delta u,\delta v$,

$$\begin{aligned}\sum_{j\geq J_0}2^{\frac{j}{2}}\Big\|[v_2,\dot\Delta_j]\partial_x z\Big\|_{L^2}\lesssim\|\partial_x v_2\|_{L^\infty}\|z\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\\+\|\partial_x z\|_{\dot{\mathbb{B}}^{-1}_{\infty,\infty}}\|v_2\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p}}_{p,1}}\|\partial_x v_2\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\cdot\end{aligned}$$

Hence, using obvious embedding and the fact that

$$\|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p}}_{p,1}}\lesssim\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}},\quad\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\lesssim\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\quad\text{and}\quad\|\partial_x v_2\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\lesssim\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}$$

yields for $z=\delta u,\,\delta v$,

$$\sum_{j\geq J_0}2^{\frac{j}{2}}\Big\|[v_2,\dot\Delta_j]\partial_x z\Big\|_{L^2}\lesssim\big(\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|v_2\|^{h}_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\big)\big(\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}+\|z\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot \tag{38}$$

The last two terms of (37) may be bounded thanks to Inequality (85): we get for $z=u_1,\,v_1$,

$$\|\delta v\,\partial_x z\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\lesssim\big(\|\delta v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}+\|\delta v\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\big(\|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}+\|\partial_x z\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot \tag{39}$$

Plugging (38) and (39) in (37), and using also the fact that

$$\|\partial_x v_2\|_{L^\infty}\lesssim\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|v_2\|^{h}_{\dot{\mathbb{B}}^{\frac32}_{2,1}},$$

we end up with

$$\|(\delta u,\delta v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\lesssim\int_0^t\big(\|(u_1,v_1,v_2)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|(u_1,v_1,v_2)\|^{h}_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\big)\mathscr{U}. \tag{40}$$

Step 4. Conclusion. Putting (36) and (40) together yields

$$\mathscr{U}(t) \leq C \int_0^t \big(\|(u_1,v_1,v_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_1,v_1,v_2)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|(\partial_x u_1, \partial_x v_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \big) \mathscr{U}\, d\tau.$$

Knowing that (u_1,v_1) and (u_2,v_2) are in $E_p(T)$ and remembering (30), we get,

$$\int_0^T \big(\|(u_1,v_1,v_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_1,v_1,v_2)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \big) < \infty.$$

Hence, applying Gronwall lemma allows to conclude that $\mathscr{U} \equiv 0$ on $[0,T]$. In other words, (u_1,v_1) and (u_2,v_2) coincide on $[0,T]\times\mathbb{R}$. Since T is arbitrary, uniqueness is proved. □

4. Compressible Euler system with damping

As pointed out in the introduction, System (TM) may be seen as an approximation of the damped isentropic compressible Euler system with pressure law $P(\rho) = \frac{\rho^2}{2}\cdot$ Here we want to adapt the method of the previous sections to the true damped compressible Euler system :

$$(E) \qquad \begin{cases} \partial_t \rho + \partial_x(V\rho) = 0, \\ \partial_t(\rho V) + \partial_x(\rho V^2) + \partial_x(P(\rho)) + \lambda \rho V = 0, \end{cases}$$

supplemented with initial data (ρ_0, V_0) that is a perturbation of some constant state $(\bar\rho, 0)$ with $\bar\rho > 0$. The (given) pressure function P is assumed to be smooth and such that :

$$(41) \qquad \begin{cases} \text{Case } 2 < p \leq 4 : P(\rho) = a\rho^\gamma \text{ for some positive } a \text{ and } \gamma \text{ in a neighborhood of } \bar\rho. \\ \text{Case } p = 2 : \text{just } \ P'(\bar\rho) > 0. \end{cases}$$

Note that, performing a suitable normalization reduces the study to the case $\bar\rho = P'(\bar\rho) = 1$ (hence $a = 1/\gamma$ in the first case), which will be assumed from now on.

Consider the new unknown

$$n(\rho) \triangleq \int_1^\rho \frac{P'(s)}{s}\, ds.$$

Since our assumptions on the pressure guarantee that $\rho \mapsto n(\rho)$ is a smooth diffeomorphism from a neighborhood of 1 to a neighborhood of 0, one can rewrite (E) under the form

$$(42) \qquad \begin{cases} \partial_t n + V\partial_x n + \partial_x V + G(n)\partial_x V = 0, \\ \partial_t V + V\partial_x V + \partial_x n + \lambda V = 0, \end{cases}$$

where $G(n)$ is defined by the relation[4] $G(n(\rho)) \triangleq P'(\rho) - 1$.

Theorem 4.1. *Under hypothesis* (41)*, there exist $k = k(p) \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that for $J_\lambda \triangleq \lfloor \log_2 \lambda \rfloor + k$, if we assume that (n_0, V_0) fulfills the same conditions as in Theorem 1.1, then System* (42) *admits a unique global solution (n,V) verifying the same properties as the solution therein. Furthermore, Corollary 1.1 and Theorem 1.2 hold true (with the same decay rate).*

Proof. Performing the rescaling (3) reduces the proof to the case $\lambda = 1$, and we are thus left with bounding for all $t \geq 0$, the functional

$$X_p(t) \triangleq \|(n,V)\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|(n,V)\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|n\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|(n,V)\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|V + \partial_x n\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$$

[4] In what follows, we shall only use that G is a smooth function vanishing at 0.

in terms of

$$X_{p,0} \triangleq \|(n_0,V_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(n_0,V_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

Remember that $\|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ and $\|V+\partial_x n\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ are bounded by the first four terms of X_p (see (24)).

Low frequencies estimates. We follow the method we used for (TM), looking at $G(n)\partial_x V$ as a source term. Owing to Propositions A.3 and A.4, we have

$$\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \lesssim \|n\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}.$$

Therefore, mimicking the proof of Proposition 3.1, we end up again for all $t \geq 0$ with

$$\|(n,V)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|V+\partial_x n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \leq C\biggl(\|(n_0,V_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|(n,V,\partial_x n)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\biggr). \tag{43}$$

High frequencies estimates. One has the following proposition:

Proposition 4.1. *Let (n,V) be a smooth solution of (42) on the interval $[0,T]$, under assumption (41). Define p^* by the relation $1/p+1/p^*=1/2$. There exists a constant C depending only on the threshold J_0 between the low and high frequencies such that for all $t\in[0,T]$,*

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n,\partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &\quad + \int_0^t \biggl(\|V\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|(\partial_x n,\partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|(\partial_x n,\partial_x V)\|^{\ell}_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x V\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\biggr) \\ &\quad + \int_0^t \|\partial_x V\|_{L^\infty} \|G(n)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \biggl(\|(n,G(n))\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|\partial_x V\|^{\ell}_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x G(n)\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\biggr).\end{aligned}$$

Proof. We localize System (42) by means of $\dot{\Delta}_j$, getting

$$\begin{cases} \partial_t n_j + \dot{S}_{j-1}V\partial_x n_j + \partial_x V_j + \dot{S}_{j-1}G(n)\partial_x V_j = R^1_j + R'^1_j, \\ \partial_t V_j + \dot{S}_{j-1}V\partial_x V_j + \partial_x n_j + V_j = R^2_j \end{cases} \tag{44}$$

where

$$R^1_j \triangleq \dot{S}_{j-1}V\partial_x n_j - \dot{\Delta}_j(V\partial_x n), \qquad R'^1_j \triangleq \dot{S}_{j-1}G(n)\partial_x V_j - \dot{\Delta}_j(G(n)\partial_x V)$$
$$\text{and} \quad R^2_j \triangleq \dot{S}_{j-1}V\partial_x V_j - \dot{\Delta}_j(V\partial_x V).$$

The only difference with (TM) is the appearance of $\dot{S}_{j-1}G(n)\partial_x V_j$ in the first equation and of the commutator R'^1_j. To handle the former term, one has to add a suitable weight in the definition of the functional we used for (TM): for $j\geq J_0$ and $\eta=\eta(J_0)>0$ (to be chosen small enough), we set

$$\widetilde{\mathcal{L}}^2_j \triangleq \int_{\mathbb{R}} (\partial_x n_j)^2 + (1+\dot{S}_{j-1}G(n))(\partial_x V_j)^2 + \eta\int_{\mathbb{R}} V_j\partial_x n_j.$$

Differentiating in time this quantity and performing several integration by parts yields:

$$\frac{d}{dt}\widetilde{\mathcal{L}}_j^2+\widetilde{\mathcal{H}}_j^2+\int_{\mathbb{R}}\partial_x\dot{S}_{j-1}V\left((\partial_x n_j)^2+(1+\dot{S}_{j-1}G(n))(\partial_x V_j)^2\right)$$
$$-\int_{\mathbb{R}}\partial_x\dot{S}_{j-1}G(n)\,S_{j-1}V\,(\partial_x V_j)^2=\int_{\mathbb{R}}(\partial_x V_j)^2\,\partial_t\dot{S}_{j-1}G(n)$$
$$+2\int_{\mathbb{R}}\big(\partial_x(R_j^1+R_j'^1)\,\partial_x n_j+(1+S_{j-1}G(n))\partial_x R_j^2\,\partial_x V_j\big)+\eta\int_{\mathbb{R}}\big(\partial_x(R_j^1+R_j'^1)\,V_j+R_j^2\,\partial_x n_j\big)\cdot \tag{45}$$

with $\widetilde{\mathcal{H}}_j^2\triangleq\eta\|\partial_x n_j\|_{L^2}^2+(2-\eta)\displaystyle\int_{\mathbb{R}}(1+\dot{S}_{j-1}G(n))(\partial_x V_j)^2+\eta\int_{\mathbb{R}}V_j\partial_x n_j$.

To continue, let us assume that

$$\|n\|_{L^\infty}+\|V\|_{L^\infty}\ll 1\quad\text{on}\quad[0,T]. \tag{46}$$

Then, since $G(0)=0$, we have, using the mean value theorem and the uniform boundedness of operator $\dot{S}_{j-1}$ in all Lebesgue spaces:

$$\|\dot{S}_{j-1}G(n)\|_{L^\infty}\lesssim\|n\|_{L^\infty}\ll 1,$$

and thus, if η is small enough,

$$\widetilde{\mathcal{L}}_j^2\simeq\|(\partial_x n_j,\partial_x V_j)\|_{L^2}^2\quad\text{and}\quad\widetilde{\mathcal{H}}_j^2\simeq\|(\partial_x n_j,\partial_x V_j)\|_{L^2}^2\quad\text{for all}\quad j\geq J_0. \tag{47}$$

Let us also observe that

$$\begin{aligned}\partial_t G(n)&=G'(n)\partial_t n\\&=-G'(n)(V\partial_x n+(1+G(n))\partial_x V).\end{aligned}$$

Owing to assumption (46) and to the mean value theorem, we thus get

$$\|\partial_t G(n)\|_{L^\infty}\lesssim\|\partial_x V\|_{L^\infty}+\|V\|_{L^\infty}\|\partial_x n\|_{L^\infty}. \tag{48}$$

Proceeding analogously, we obtain

$$\|\partial_x G(n)\|_{L^\infty}\lesssim\|\partial_x n\|_{L^\infty}. \tag{49}$$

Hence, from inequality (45) and (47), we get for some small enough c and large enough C,

$$\frac{d}{dt}\widetilde{\mathcal{L}}_j^2+c\widetilde{\mathcal{L}}_j^2\leq C\big(\|(\partial_x V,\partial_x n)\|_{L^\infty}\widetilde{\mathcal{L}}_j+2^j\|(R_j^1,R_j'^1,R_j^2)\|_{L^2}\big)\widetilde{\mathcal{L}}_j\quad\text{for all}\quad j\geq J_0. \tag{50}$$

At this point, taking advantage of Lemma A.1 yields

$$\widetilde{\mathcal{L}}_j(t)+c\int_0^t\widetilde{\mathcal{L}}_j\leq\widetilde{\mathcal{L}}_j(0)+C\int_0^t\|(\partial_x V,\partial_x n)\|_{L^\infty}\,\mathcal{L}_j+C2^j\int_0^t\left\|(R_j^1,R_j'^1,R_j^2)\right\|_{L^2}. \tag{51}$$

Now, multiplying (51) by $2^{\frac{j}{2}}$, using (47) and summing up on $j\geq J_0$ gives us

$$\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\lesssim\|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$$
$$+\int_0^t\|(\partial_x V,\partial_x n)\|_{L^\infty}\|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\sum_{j\geq J_0}2^{\frac{3j}{2}}\left\|(R_j^1,R_j'^1,R_j^2)\right\|_{L^2}. \tag{52}$$

The terms R_j^1 and R_j^2 may be bounded exactly as in the proof Proposition 3.2. As regards $R_j'^1$, Lemma A.3 gives us

$$\sum_{j\geq J_0}2^{\frac{3}{2}j}\|R_j'^1\|_{L^2}\lesssim\|\partial_x G(n)\|_{L^\infty}\|\partial_x V\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|G(n)\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x V\|_{L^\infty}\|G(n)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}}\|\partial_x G(n)\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}.$$

Using (49) completes the proof of the proposition. □

Global-in-time a priori estimate. We claim that granted with Inequalities (43) and the above proposition, we have, whenever Condition (46) is satisfied on $[0,T]$,

$$X_p(t) \lesssim X_{p,0} + X_p^2(t) \quad \text{for all} \quad t \in [0,T]. \tag{53}$$

Inequality (43) is exactly the same as for (TM). Hence, the terms in $X_p(t)$ corresponding to the low frequencies of (n,V) are bounded by $X_p^2(t)$. Note also that $\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ may be bounded according to (24), and thus eventually by $X_p^2(t)$.

In order to handle the high frequencies, we shall proceed differently depending on whether $P(\rho)=\rho^\gamma/\gamma$ or P is a general pressure law with $P'(1)=1$. In fact, to handle the latter case, we need to assume that $p=2$.

1. Case $P(\rho)=\rho^\gamma/\gamma$ with $\gamma>0$. Then, $G(n)=(\gamma-1)n$ and the inequality of Proposition 4.1 reduces to

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n,\partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &+ \int_0^t \biggl(\|V\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|(\partial_x n,\partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|(\partial_x n,\partial_x V)\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \biggr) \\ &+ \int_0^t \biggl(\|n\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x n\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \biggr).\end{aligned}$$

Compared to our study of (TM), only the last line is new. However, one can use the fact that

$$\begin{aligned}&\int_0^t \|n\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} \lesssim \|n\|^\ell_{L^2_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \\ &\int_0^t \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x n\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \lesssim \|V\|^\ell_{L^1_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \|n\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \qquad \text{as} \quad 1-\frac{1}{p^*} \geq \frac{1}{p}.\end{aligned}$$

The terms on the right may be bounded by $X_p^2(t)$. Hence we have (53).

2. Case of a general pressure law with $P'(1)=1$. For $p=2$, Proposition 4.1 together with the embeddings $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow \dot{\mathbb{B}}^0_{\infty,1} \hookrightarrow L^\infty$ and $\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1} \hookrightarrow \dot{\mathbb{B}}^1_{\infty,1}$ give us

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n,\partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &+ \int_0^t \|V\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|(n,V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|G(n)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,G(n))\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.\end{aligned}$$

Since, by Proposition A.4 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|G(n)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|\partial_x V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})} \|n\|_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}, \\ &\int_0^t \|(n,G(n))\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|n\|_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})},\end{aligned}$$

one can conclude that (53) is satisfied.

Uniqueness. As for (TM), we look at the system satisfied by the difference $(\delta n, \delta V) := (n_2 - n_1, V_2 - V_1)$ between two solutions, namely:

$$\left\{\begin{array}{l} \partial_t\delta n + \partial_x\delta V + V_2\,\partial_x\delta n + G(n_2)\partial_x\delta V = -\delta V\,\partial_x n_1 - (G(n_2)-G(n_1))\partial_x V_1,\\ \partial_t\delta V + \delta V + \partial_x\delta n + V_2\,\partial_x\delta V = -\delta V\,\partial_x V_1, \end{array}\right. \tag{54}$$

and estimate $(\delta n, \delta V)$ for all $T>0$ in the space $F_p(T)$ defined in (29). Compared to the proof of uniqueness for (TM) we have to handle the two terms containing the function G.

Let us first explain how to estimate the low frequencies. We have to bound the additional terms $G(n_2)\partial_x\delta V$ and $(G(n_2)-G(n_1))\partial_x V_1$ in $\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}$. Now, according to (34) and (35), we have

$$\begin{aligned} \|(G(n_2)-G(n_1))\partial_x V_1\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}} &\lesssim \|G(n_2)-G(n_1)\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}\,\|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}},\\ \|G(n_2)\partial_x\delta V\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}} &\lesssim \|n_2\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}\cap\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\|\partial_x\delta V\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac32}_{p,1}}. \end{aligned}$$

From the relation

$$G(n_2)-G(n_1) = \delta n\int_0^1 G'(n_1+\tau\delta n)\,d\tau, \tag{55}$$

and Propositions A.3 and A.4, we find out:

$$\|G(n_2)-G(n_1)\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} \lesssim \|\delta n\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}}.$$

Therefore, we eventually have

$$\begin{aligned} \|(\delta n,\delta V)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}} \lesssim \int_0^t \Big(\|(\partial_x n_1,\partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}\|\delta V\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|V_2\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}\cap\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\|(\delta n,\delta V)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}\Big)\\ + \int_0^t \Big(\|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}\|\delta n\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|n_2\|_{\dot{\mathbb{B}}^{\frac1p}_{p,1}\cap\dot{\mathbb{B}}^{\frac1p+1}_{p,1}}\|\delta V\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1}}\Big). \end{aligned} \tag{56}$$

Let us next estimate the high frequencies of $(\delta n,\delta V)$ in $\dot{\mathbb{B}}^{\frac12}_{2,1}$. Applying operator $\dot\Delta_j$ to (54), we get for all $j\geq J_0$,

$$\left\{\begin{array}{l} \partial_t\delta n_j + \partial_x\delta V_j + \dot S_{j-1}V_2\,\partial_x\delta n_j + \dot S_{j-1}G(n_2)\partial_x\delta V_j\\ \qquad\qquad = -\dot\Delta_j\big(\delta V\,\partial_x n_1 + (G(n_2)-G(n_1))\partial_x V_1\big) + \delta R^1_j + \delta R'^1_j,\\ \partial_t\delta V_j + \delta V_j + \partial_x\delta n_j + \dot S_{j-1}V_2\,\partial_x\delta V_j = -\dot\Delta_j(\delta V\,\partial_x V_1) + \delta R^2_j, \end{array}\right.$$

with $\delta R^1_j \triangleq \dot S_{j-1}V_2\partial_x\delta n_j - \dot\Delta_j(V_2\partial_x\delta n)$, $\delta R'^1_j \triangleq \dot S_{j-1}G(n_2)\partial_x\delta V_j - \dot\Delta_j(G(n_2)\partial_x\delta V)$ and $\delta R^2_j \triangleq \dot S_{j-1}V_2\partial_x\delta V_j - \dot\Delta_j(V_2\partial_x\delta V)$.

Arguing as in the proof of Proposition 4.1, we consider the functional

$$\int_{\mathbb{R}}(\partial_x\delta n_j)^2 + (1+\dot S_{j-1}G(n_2))(\partial_x\delta V_j)^2 + \eta\int_{\mathbb{R}}\delta V_j\,\partial_x\delta n_j$$

and follow the computations therein, with regularity exponent $1/2$ instead of $3/2$. We get

$$\begin{aligned} \|(\delta n,\delta V)(t)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \int_0^t \|(\partial_x n_2,\partial_x V_2)\|_{L^\infty}\|(\delta n,\delta V)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t\sum_{j\geq J_0}2^{\frac j2}\big(\|\delta R^1_j\|_{L^2}+\|\delta R'^1_j\|_{L^2}+\|\delta R^2_j\|_{L^2}\big)\\ + \int_0^t\big(\|\delta V\,\partial_x n_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\delta V\,\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|(G(n_2)-G(n_1))\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big). \end{aligned}$$

The terms with δR_j^1 and δR_j^2 may be bounded as in the proof of uniqueness for (TM). Regarding $\delta R_j'^1$, we use Lemma A.3 with $s=1/2$, and get

$$\sum_{j\geq J_0} 2^{\frac{j}{2}}\|\delta R_j'^1\|_{L^2} \lesssim \|\partial_x G(n_2)\|_{L^\infty}\|\delta V\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x\delta V\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}}\|G(n_2)\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x\delta V\|_{L^\infty}\|G(n_2)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x\delta V\|^\ell_{\dot{\mathbb{B}}^{1-\frac1p}_{p,1}}\|\partial_x G(n_2)\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}.$$

To continue the proof, we have two distinguish two cases depending on whether $P(\rho)=\rho^\gamma/\gamma$ and $2\leq p\leq 4$, or P is a general pressure law with $P'(1)=1$, and $p=2$. In the first case, we have $G(n)=(\gamma-1)n$, so that $G(n_2)-G(n_1)=(\gamma-1)\delta n$. Now, in light of (85), we have

$$\|\delta n\,\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \big(\|\delta n\|^\ell_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}} + \|\delta n\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\big(\|\partial_x V_1\|^\ell_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}} + \|\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big).$$

As all the terms with $G(n_2)$ in the estimate for $R_j'^1$ are proportional to n_2, we arrive at

$$\|(\delta n,\delta V)(t)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \int_0^t \big(\|(n_1,n_2,V_1,V_2)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|(n_1,n_1,V_1,V_2)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|(\partial_x n_1,\partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}}\big)\|(\delta n,\delta V)\|_{F_p}. \tag{57}$$

In the case $p=2$ with $P'(1)=1$, then one may proceed essentially as in the proof of Proposition 4.1 to bound the terms with $G(n_2)$ in the estimate for $R_j'^1$, and one can use Proposition A.4 combined with product laws and Relation (55) to eventually arrive at

$$\|(G(n_2)-G(n_1))\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|\delta n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Consequently, (57) still holds true.

In all cases, putting (56) and (57) together yields

$$\|(\delta n,\delta V)\|_{F_p(t)} \leq C\int_0^t \big(\|(n_1,n_2,V_1,V_2)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|(n_1,n_2,V_1,V_2)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|(\partial_x n_1,\partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}}\big)\|(\delta n,\delta V)\|_{F_p},$$

and using Gronwall lemma completes the proof of uniqueness.

Decay estimates. Here, we assume that $p=2$ and follow the same approach as for (TM).

Step 1: estimating the solution in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$. This is only a matter of handling the additional term $G(n)\partial_x v$. Applying $\dot\Delta_j$ to the system satisfied by (n,V) yields

$$\begin{cases} \partial_t n_j + \partial_x V_j = -v\partial_x n_j - G(n)\partial_x V + [V,\dot\Delta_j]\partial_x n,\\ \partial_t V_j + \partial_x n_j + V_j = -V\partial_x V_j + [v,\dot\Delta_j]\partial_x V.\end{cases}$$

So, considering $G(n)\partial_x V$ as a source term, we get

$$\|(n_j,V_j)(t)\|_{L^2} + \int_0^t\|V_j\|_{L^2} \leq \|(n_j,V_j)(0)\|_{L^2} + \int_0^t\|\partial_x V\|_{L^\infty}\|(n_j,V_j)\|_{L^2}$$
$$+\int_0^t\|[V,\dot\Delta_j]\partial_x n\|_{L^2} + \int_0^t\|[V,\dot\Delta_j]\partial_x V\|_{L^2} + \Big\|\dot\Delta_j(G(n)\partial_x V)\Big\|_{L^2}\|n_j\|_{L^2}.$$

We have

$$\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|G(n)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

In order to bound $G(n)$ in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$, one cannot readily use Proposition A.4 since $-\sigma_1$ may be negative. However, from Taylor formula, we know that there exists a smooth function H vanishing at 0 such that

$$G(n) = G'(0)\,n + H(n)\,n.$$

Hence, combining product and composition estimates gives

$$\|G(n)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|n\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \bigl(1 + \|n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\bigr).$$

In the regime we consider, $\|n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$ is small. Hence we conclude that

$$\|(n,V)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(n_0,V_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|(n,V)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}},$$

which ensures after using Gronwall lemma and the bound of $\|\partial_x V\|_{L^1(\dot{\mathbb{B}}^{\frac12}_{2,1})}$ in terms of $X_{2,0}$, that

$$\forall t\in\mathbb{R}^+,\ \|(n,V)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\|(n_0,V_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Step 2: Lyapunov functional. We aim at exhibiting a Lyapunov functional that is equivalent to $\|(n,V)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}$. The high frequency part of the solution has already been treated efficiently with $\widetilde{\mathcal{L}}_j$. To bound the low frequency part, consider the evolution equation for $z \triangleq V + \partial_x n$:

$$\partial_t z + V\partial_x z + z = -\partial^2_{xx}V - \partial_x V\partial_x n - \partial_x(G(n)\partial_x V).$$

Following the computations we did for (TM) leads to

$$\begin{aligned}\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\leq \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial^2_{xx}V\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &\quad + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial_x(G(n)\partial_x V)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.\end{aligned}$$

The last term may be bounded. by $\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$. Then, using Propositions A.3 and A.4, one ends up with

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \leq \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial^2_{xx}V\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|(z,n,\partial_x n)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}. \tag{58}$$

Next, using the fact that

$$\partial_t n + V\partial_x n - \partial^2_{xx}n = -G(n)\partial_x V - \partial_x z,$$

we get

$$\|n(t)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|n\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} \leq \|n_0\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}. \tag{59}$$

The high frequency part of the solution may be bounded according to (51). Hence, setting

$$\widetilde{\mathcal{L}} \triangleq \sum_{j\leq J_0} 2^{\frac{j}{2}}\|(\dot\Delta_j n, \dot\Delta_j z)\|_{L^2} + \sum_{j\geq J_0} 2^{\frac{j}{2}}\widetilde{\mathcal{L}}_j \quad\text{and}\quad \widetilde{\mathcal{H}} \triangleq \|V+\partial_x n\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|V\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|n\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + \|n\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}$$

and bounding R^1_j, R'^1_j and R^2_j as in the proof of Proposition 4.1, we discover that if taking J_0 negative enough, then all the linear terms in (58) and (59) may be absorbed by $\widetilde{\mathcal{H}}$, so that we have for some suitably small positive c,

$$\widetilde{\mathcal{L}}(t) + c\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\mathcal{L} + C\int_0^t \|n\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}. \tag{60}$$

Above, we used that $\widetilde{\mathcal{L}} \simeq \|(n,V)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}}$ and that $\widetilde{\mathcal{H}} \gtrsim \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$. Now, since furthermore $\widetilde{\mathcal{H}} \gtrsim \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$ and $\widetilde{\mathcal{L}} \gtrsim \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$, one may write

$$\begin{aligned}\|n\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\|V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\leq \|n\|^2_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|n\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &\lesssim \|n\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|n\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + (\|n\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}})^2 + \|n\|^\ell_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|n\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &\lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{H}}\widetilde{\mathcal{L}}.\end{aligned}$$

Hence, if $\widetilde{\mathcal{L}}(0)$ is small enough then, combining (60) with a bootstrap argument yields

$$\widetilde{\mathcal{L}}(t) + \frac{c}{2}\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) \quad \text{for all} \quad t \geq 0.$$

Step 3: Proof of decay estimates. From this point, one can repeat word for word the proof of decay estimates for the low frequencies of the solutions to (TM).

For the high frequencies, starting from (50), using Lemma A.3 and integrating gives

$$\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \lesssim e^{-ct}\|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \int_0^t e^{-c(t-\tau)}\big(\|V\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\|(n,V)\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|V\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|n\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\big)\cdot$$

Compared to (17), there is one more term. However, as for (TM), Steps 1 and 2 together imply that

$$\|(n,V)(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}\cap\dot{\mathbb{B}}^{\frac32}_{2,1}} \lesssim \langle t\rangle^{-\alpha_1}.$$

Hence, one may easily conclude that

$$\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \lesssim \langle t\rangle^{-2\alpha_1}.$$

This completes the proof of the theorem (up to the proof of existence, which is totally analogous as for (TM)). □

5. A more general 1D model

In this section, we consider a more general class of one dimensional systems, namely

$$\begin{cases}\partial_t u + \alpha\partial_x v + V^1\partial_x u + W^1\partial_x v = 0,\\ \partial_t v + \beta\partial_x u + V^2\partial_x u + W^2\partial_x v + \lambda v + \kappa\lambda v^q = 0\end{cases} \tag{61}$$

where[5] κ is a real parameter, $q \geq 2$, an integer, $V^1 = V^1(v)$ and $V^2 = V^2(v)$ are smooth functions vanishing at 0, $W^1 = W^1(u,v)$ and $W^2 = W^2(u,v)$ are smooth functions vanishing at $(0,0)$, and α, β, λ are strictly positive constants.

Theorem 5.1. *Let the data (u_0,v_0) satisfy the assumptions of Theorem 1.1 with $J_\lambda \triangleq \lfloor \log_2\lambda\rfloor$ and $p = 2$. Then, System (61) admits a unique global solution (u,v) verifying the same properties as the solution therein. Furthermore, Corollary 1.1 and Theorem 1.2 hold true.*

Remark 5.1. *If V^1, V^2, W^1 and W^2 are 'general' smooth functions, then it is unlikely that a L^p theory may be worked out. We need a very specific structure of the nonlinear terms in order that the L^p estimates of the low frequencies fit with the L^2 regularity of the high frequencies.*

Remark 5.2. *We do not know how to handle terms like $u\partial_x u$ in any equations of the system (this is the reason why we assumed that V^1 and V^2 only depend on v). In fact, although the system is locally well-posed if V^1 and V^2 also depend on u, the time integrability of u is not good enough for global estimates.*

[5] In the case $q = 3$ and $\kappa > 0$, κv^q is a classical representation of a drag term.

Elements of proof. We just explain how to find a Lyapunov function and to control the norm in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ of a smooth solution (u,v) of (61) on $[0,T]$, in terms of the data. Proving existence and uniqueness is essentially the same as for the systems we treated before (uniqueness is easier somehow since we assumed $p=2$). Although the system under consideration is no longer symmetric if $\alpha+W^1\neq\beta+V^2$, it is symmetrizable (see [3, Chap. 10]).

Note that performing a suitable rescaling reduces our problem to the case

$$\alpha=\beta=\lambda=1. \tag{62}$$

Indeed, if we set

$$(u,v)(t,x)=(\sqrt{\alpha}\,\tilde{u},\sqrt{\beta}\,\tilde{v})\Big(\lambda t,\frac{\lambda}{\sqrt{\alpha\beta}}\,x\Big),$$

then (u,v) satisfies (61) if and only if $(\tilde{u},\tilde{v})$ satisfies a similar system with (62), parameter $\kappa\beta^{\frac{q-1}{2}}$ and slightly modified functions V_1, V_2, W_1 and W_2 (the modification depending only on α and β). So we will assume (62) in the rest of this section.

A priori estimates. We adapt the method we used for (TM) *in the case* $p=2$. The terms $V^1\partial_x u$ and $W^2\partial_x v$ are a slight generalization of $v\partial_x u$ and $v\partial_x v$ and may be treated similarly. To handle $W^1\partial_x v$ and $V^2\partial_x u$, we need to introduce suitable weights in the definition of the Lyapunov. Finally, v^q may be seen as a harmless nonlinear source term.

Let us start the computations : we assume that we are given a smooth function (u,v) of (61) on some time interval $[0,T]$ such that for some suitably small $\eta>0$,

$$\sup_{t\in[0,T]}\|(u,v)(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\leq\eta, \tag{63}$$

and, still denoting $u_j=\dot{\Delta}_j u$ and $v_j=\dot{\Delta}_j v$, we set for all $j\in\mathbb{Z}$,

$$\mathcal{L}_j\triangleq\biggl(\|(u_j,v_j)\|_{L^2}^2+\int_{\mathbb{R}}v_j\partial_x u_j+\int_{\mathbb{R}}(1+V^2)(\partial_x u_j)^2+\int_{\mathbb{R}}(1+W^1)(\partial_x v_j)^2\biggr)^{1/2}\cdot$$

We shall use repeatedly that (63) implies that

$$\sup_{t\in[0,T]}\max\bigl(\|u(t)\|_{L^\infty},\,\|v(t)\|_{L^\infty},\,\|V^1(t)\|_{L^\infty},\,\|V^2(t)\|_{L^\infty},\,\|W^1(t)\|_{L^\infty},\,\|W^2(t)\|_{L^\infty}\bigr)\ll 1, \tag{64}$$

which in particular entails that

$$\mathcal{L}_j\simeq\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}. \tag{65}$$

Now, applying $\dot{\Delta}_j$ to (61)-(62) yields for all $j\in\mathbb{Z}$,

$$\begin{cases}\partial_t u_j+(1+W^1)\partial_x v_j+V^1\partial_x u_j=R_j^1,\\ \partial_t v_j+(1+V^2)\partial_x u_j+W^2\partial_x v_j+v_j=R_j^2-\kappa\dot{\Delta}_j(v^q)\end{cases} \tag{66}$$

with

$$R_j^1\triangleq[V^1,\dot{\Delta}_j]\partial_x u+[W^1,\dot{\Delta}_j]\partial_x v\quad\text{and}\quad R_j^2\triangleq[V^2,\dot{\Delta}_j]\partial_x u+[W^2,\dot{\Delta}_j]\partial_x v.$$

In order to compute the time derivative of $\mathcal{L}_j^2$, we need the following obvious identities:

$$\frac12\frac{d}{dt}\|(u_j,v_j)\|_{L^2}^2+\|v_j\|_{L^2}^2-\frac12\int_{\mathbb{R}}\bigl((u_j)^2\partial_x V^1+(v_j)^2\partial_x W^2\bigr)+\int_{\mathbb{R}}\bigl(W^1u_j\partial_x v_j+V^2v_j\partial_x u_j\bigr)$$
$$=\int_{\mathbb{R}}\bigl(R_j^1u_j+R_j^2v_j-\kappa\bigl(\dot{\Delta}_j v^q\bigr)v_j\bigr),$$

$$\frac{d}{dt}\int_{\mathbb{R}}v_j\partial_x u_j+\|\partial_x u_j\|_{L^2}^2-\|\partial_x v_j\|_{L^2}^2+\int_{\mathbb{R}}v_j\partial_x u_j$$
$$+\int_{\mathbb{R}}\bigl((W^2-V^1)\partial_x u_j\,\partial_x v_j+V^2(\partial_x u_j)^2-W^1(\partial_x v_j)^2\bigr)=\int_{\mathbb{R}}\bigl(R_j^2\partial_x u_j-R_j^1\partial_x v_j-\kappa\bigl(\dot{\Delta}_j v^q\bigr)\partial_x u_j\bigr)\cdot$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(1+V^2)(\partial_x u_j)^2+\int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x(V^1\partial_x u_j)+\int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x((1+W^1)\partial_x v_j)$$
$$=\int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x R_j^1+\frac{1}{2}\int_{\mathbb{R}}\partial_t V^2(\partial_x u_j)^2.$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(1+W^1)(\partial_x v_j)^2+\int_{\mathbb{R}}(1+W^1)\partial_x v_j\partial_x(W^2\partial_x v_j)+\int_{\mathbb{R}}(1+W^1)\partial_x v_j\partial_x((1+V^2)\partial_x u_j)$$
$$+\int_{\mathbb{R}}(1+W^1)(\partial_x v_j)^2=\int_{\mathbb{R}}(1+W^1)\partial_x v_j\big(\partial_x R_j^2-\kappa\partial_x\dot\Delta_j v^q\big)+\frac{1}{2}\int_{\mathbb{R}}\partial_t W^1(\partial_x v_j)^2.$$

The fundamental observation that justifies our using those very weights in the definition of $\mathcal{L}_j$ is that the third integrals in the last two relations compensate. Consequently, denoting

$$\mathcal{H}_j^2\triangleq\|v_j\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}}v_j\partial_x u_j+\frac{1}{2}\|\partial_x u_j\|_{L^2}^2+\int_{\mathbb{R}}\Big(W^1+\frac{1}{2}\Big)(\partial_x v_j)^2,$$

and using the fact that

$$\int_{\mathbb{R}}V^2v_j\partial_x u_j=-\int_{\mathbb{R}}V^2u_j\partial_x v_j-\int_{\mathbb{R}}u_jv_j\partial_x V^2,$$

we arrive at

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+\mathcal{H}_j^2=\frac{1}{2}\int((v_j)^2\partial_x W^2+(u_j)^2\partial_x V^1)+\int_{\mathbb{R}}u_jv_j\partial_x V^2+\int_{\mathbb{R}}(V^2-W^1)u_j\partial_x v_j$$
$$+\frac{1}{2}\int_{\mathbb{R}}\big((V^1-W^2)\partial_x u_j\,\partial_x v_j-V^2(\partial_x u_j)^2+W^1(\partial_x v_j)^2\big)+\frac{1}{2}\int_{\mathbb{R}}(\partial_x u_j)^2\big(V^1\partial_x V^2-(1+V^2)\partial_x V^1\big)$$
$$+\frac{1}{2}\int_{\mathbb{R}}(\partial_x v_j)^2\big(W^2\partial_x W^1-(1+W^1)\partial_x W^2\big)+\frac{1}{2}\int_{\mathbb{R}}\big((\partial_x u_j)^2\partial_t V^2+(\partial_x v_j)^2\partial_t W^1\big)$$
$$+\int_{\mathbb{R}}\big(u_j-\frac{1}{2}\partial_x v_j\big)R_j^1+\int_{\mathbb{R}}\big(v_j+\frac{1}{2}\partial_x u_j\big)R_j^2+\int_{\mathbb{R}}\big((1+V^2)\partial_x u_j\partial_x R_j^1+(1+W^1)\partial_x v_j\partial_x R_j^2\big)$$
$$-\kappa\int_{\mathbb{R}}\Big(\big(v_j+\frac{1}{2}\partial_x u_j\big)\dot\Delta_j v^q+(1+W^1)\partial_x\dot\Delta_j v^q\partial_x v_j\Big).$$

Since

$$\partial_t V^2=-(V^2)'\big((1+V^2)\partial_x u+W^2\partial_x v+v+\kappa v^q\big),$$

remembering (64), we have

$$\|\partial_t V^2\|_{L^\infty}\lesssim\|\partial_x u\|_{L^\infty}+\|(u,v)\|_{L^\infty}\|\partial_x v\|_{L^\infty}+\|v\|_{L^\infty}$$

and, similarly,

$$\|\partial_t W^1\|_{L^\infty}\lesssim\|\partial_x u\|_{L^\infty}+\|\partial_x v\|_{L^\infty}+\|v\|_{L^\infty}.$$

Observe also that

$$\|\partial_x V^i\|_{L^\infty}\lesssim\|\partial_x v\|_{L^\infty}\quad\text{and}\quad\|\partial_x W^i\|_{L^\infty}\lesssim\|(\partial_x u,\partial_x v)\|_{L^\infty}\quad\text{for}\quad i=1,2,$$

whence, in particular

$$\int_{\mathbb{R}}u_jv_j\partial_x V^2\lesssim\|\partial_x v\|_{L^\infty}\|u_j\|_{L^2}\|v_j\|_{L^2}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+\mathcal{H}_j^2\lesssim\|(u,v)\|_{L^\infty}\|\partial_x v_j\|_{L^2}\|(u_j,\partial_x u_j,\partial_x v_j)\|_{L^2}+\|v\|_{L^\infty}\|\partial_x u_j\|_{L^2}^2$$
$$+\|\partial_x u\|_{L^\infty}\|(v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2+\|\partial_x v\|_{L^\infty}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2$$
$$+\|(R_j^1,R_j^2)\|_{L^2}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}+\|(\partial_x R_j^1,\partial_x R_j^2)\|_{L^2}\|(\partial_x u_j,\partial_x v_j)\|_{L^2}$$
$$+\|(v_j,\partial_x u_j)\|_{L^2}\|\dot\Delta_j v^q\|_{L^2}+\|\partial_x v_j\|_{L^2}\|\partial_x\dot\Delta_j v^q\|_{L^2}.$$

Then, remembering (65) and using lemma A.1, we discover that for all $j\in\mathbb{Z}$ and $t\in[0,T]$,

$$\mathcal{L}_j(t)+c\min(1,2^{2j})\int_0^t\mathcal{L}_j\leq\mathcal{L}_j(0) \tag{67}$$
$$+C\int_0^t\big(\|\partial_xv\|_{L^\infty}\mathcal{L}_j+\|\partial_xu\|_{L^\infty}\|v_j\|_{L^2}+\|(v,\partial_xu)\|_{L^\infty}\|\partial_xu_j\|_{L^2}+\|(u,v,\partial_xu)\|_{L^\infty}\|\partial_xv_j\|_{L^2}\big)$$
$$+\int_0^t\|(\dot\Delta_jv^q,\partial_x\dot\Delta_jv^q)\|_{L^2}+C\int_0^t\|(R_j^1,R_j^2,\partial_xR_j^1,\partial_xR_j^2)\|_{L^2}.$$

To bound the commutator terms, let us use (79) that yields

$$\|R_j^1\|_{L^2}\lesssim c_j2^{-\frac j2}\big(\|\partial_xV^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_xW^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\quad\text{for all}\quad j\in\mathbb{Z}.$$

Clearly, since v is small in $\dot{\mathbb{B}}^{\frac12}_{2,1}$, $V^1=V^1(v)$ and $V^1(0)=0$, Proposition A.4 entails that

$$\|\partial_xV^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\lesssim\|\partial_xv\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}. \tag{68}$$

In order to bound the term with W^1, we use the fact that there exist two smooth functions $G=G(u,v)$ and $H=H(u,v)$ vanishing at $(0,0)$ and such that

$$\partial_xW^1=\partial_uW^1(0,0)\partial_xu+\partial_vW^1(0,0)\partial_xv+G(u,v)\partial_xu+H(u,v)\partial_xv.$$

Consequently, using the stability of the space $\dot{\mathbb{B}}^{\frac12}_{2,1}$ by product and results in [26, Section 5.5] for bounding $G(u,v)$ and $H(u,v)$, we get

$$\|\partial_xW^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\lesssim\|(\partial_xu,\partial_xv)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big(1+\|(u,v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big). \tag{69}$$

So finally, remembering (63), we have

$$\|R_j^1\|_{L^2}\lesssim c_j2^{-\frac j2}\big(\|u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_xv\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|(\partial_xu,\partial_xv)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big). \tag{70}$$

Bounding R_j^2 works exactly the same. Next, in light of (81), we have

$$\|\partial_xR_j^1\|_{L^2}\lesssim c_j2^{-\frac j2}\big(\|\partial_xV^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_xu\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_xW^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_xv\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big),$$

and a similar inequality for $\partial_xR_j^2$. Hence repeating the above arguments for bounding ∂_xV^1, ∂_xV^2, ∂_xW^1 and ∂_xW^2, we end up with

$$\|\partial_xR_j^i\|_{L^2}\lesssim c_j2^{-\frac j2}\|\partial_xv\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|(\partial_xu,\partial_xv)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}},\quad i=1,2. \tag{71}$$

Now, reverting to (67), using the embedding $\dot{\mathbb{B}}^{\frac12}_{2,1}\hookrightarrow L^\infty$ and that

$$\mathcal{H}_j\simeq\|(v_j,\partial_xu_j,\partial_xv_j)\|_{L^2} \tag{72}$$

we get, denoting

$$\mathcal{L}\triangleq\sum_{j\in\mathbb{Z}}2^{\frac j2}\mathcal{L}_j,$$

two positive constants c and C such that

$$\mathcal{L}(t)+c\sum_j\min(1,2^{2j})2^{\frac j2}\int_0^t\mathcal{L}_j\leq\mathcal{L}(0)+C\int_0^t\|\partial_xv\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\mathcal{L} \tag{73}$$
$$+C\int_0^t\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_xu\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+C\int_0^t\|\partial_xu\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+C\int_0^t\|(v^q,\partial_xv^q)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

As for (TM), we need better properties of integrability for v in order to close the above estimate. The situation is a bit more complex since the second line above was not present. Nevertheless, it is still possible to exhibit a control of $z \triangleq v + \partial_x u$ in $L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac12}_{2,1})$ (which, as we saw in (24) yields a bound for v in $L^2(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac12}_{2,1})$). Indeed, we have

$$\partial_t z + z + V^1\partial_x z = (V^1 - W^2)\partial_x v - V^2\partial_x u - \partial_{xx}^2 v - \partial_x V^1\,\partial_x u - \partial_x(W^1\partial_x v) - \kappa v^q \tag{74}$$

which, as in the proof of (10) leads to

$$\begin{aligned}\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial_{xx}^2 v\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &+ \int_0^t \|(V^1 - W^2)\partial_x v + V^2\partial_x u + \partial_x V^1\,\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial_x(W^1\partial_x v)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \kappa\int_0^t \|v^q\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.\end{aligned}$$

Using product and composition estimates and remembering (63), we get

$$\begin{aligned}\|(V^1 - W^2)\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}},\\ \|V^2\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}},\\ \|\partial_x V^1\,\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.\end{aligned}$$

Since only low frequencies are involved, we have

$$\|\partial_x(W^1\partial_x v)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|W^1\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Hence, using also (68), we get

$$\begin{aligned}\|z(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\le \|z_0\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial_{xx}^2 v\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &+ C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|(u,v,z,\partial_x u)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|v^q\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.\end{aligned} \tag{75}$$

In order to close the estimates for the solution, it suffices to add up (73) to $\varepsilon\cdot$(75) with suitably small ε. More precisely, setting

$$\widetilde{\mathcal{L}} \triangleq \mathcal{L} + \varepsilon\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \quad\text{and}\quad \widetilde{\mathcal{H}} \triangleq c\sum_{j\in\mathbb{Z}} \min(1, 2^{2j}) 2^{\frac{j}{2}}\mathcal{L}_j + \varepsilon\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}},$$

we get for all $t \in [0,T]$ if ε has been chosen small enough,

$$\begin{aligned}\widetilde{\mathcal{L}}(t) + \frac12\int_0^t \widetilde{\mathcal{H}} \le \widetilde{\mathcal{L}}(0) &+ C\int_0^t \big(\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \widetilde{\mathcal{L}} + \|\partial_x u\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\\ &+ C\int_0^t \|(v^q, \partial_x v^q)\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}.\end{aligned} \tag{76}$$

Let us emphasize that

$$\widetilde{\mathcal{L}} \simeq \|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \quad\text{and}\quad \widetilde{\mathcal{H}} \simeq \|u\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + \|u\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|v + \partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|v\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}.$$

Hence in particular, we have $\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \widetilde{\mathcal{H}}$ and, as explained in the previous section (just replace n by u and V by v),

$$\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}}. \tag{77}$$

Furthermore,

$$\begin{aligned}\|\partial_x u\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim (\|\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}})^2 + (\|\partial_x u\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}})^2\\ &\lesssim \|u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\|u\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + (\|u\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}})^2 \lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}}.\end{aligned}$$

Finally, Lemma A.3 and (77) together ensure that

$$\begin{aligned}\|v^q\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|v\|^q_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &\lesssim \|v\|^{q-1}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big(\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\\ &\lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}^{q-2}\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\\ &\lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}}\end{aligned}$$

and

$$\|\partial_x v^q\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\|v\|^{q-1}_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}}.$$

Consequently, Inequality (76) reduces to

$$\widetilde{\mathcal{L}}(t) + \frac12\int_0^t \widetilde{\mathcal{H}} \le \widetilde{\mathcal{L}}(0) + C\int_0^t (\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}^{q-1})\widetilde{\mathcal{H}}.$$

Now, applying a bootstrap argument, one may conclude that there exists a small constant η such that if $\widetilde{\mathcal{L}}(0) \le \eta$, then

$$\forall t \in [0,T],\ \widetilde{\mathcal{L}}(t) + \frac14\int_0^t \widetilde{\mathcal{H}} \le \widetilde{\mathcal{L}}(0). \tag{78}$$

This gives the desired control on the norm of the solution and, in addition, that $\widetilde{\mathcal{L}}$ is a Lyapunov functional.

Decay estimates. Granted with a Lyapunov functional that has the same properties as in the previous sections, in order to get the whole family of decay estimates, it suffices to establish a uniform in time bound in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for the solution. The starting point is that, for all $j \in \mathbb{Z}$,

$$\begin{cases}\partial_t u_j + \partial_x v_j + V^1\partial_x u_j = [V^1,\dot\Delta_j]\partial_x - \dot\Delta_j(W^1\partial_x v),\\ \partial_t v_j + \partial_x u_j + v_j = -\dot\Delta_j(W^2\partial_x v) - \dot\Delta_j(V^2\partial_x u) - \kappa\dot\Delta_j v^q.\end{cases}$$

Applying an energy method, using Lemma A.1 and Inequality (80) eventually delivers:

$$\begin{aligned}\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \le \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &+ C\int_0^t \|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\,\|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\\ &+ \int_0^t \big(\|W^1\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|W^2\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|V^2\partial_x u\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \kappa\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\big)\cdot\end{aligned}$$

Using for $i=1,2$, the decomposition

$$W^i(u,v) = \big(\partial_u W^i(0,0) + G^i(u,v)\big)u + \big(\partial_v W^i(0,0) + H^i(u,v)\big)v$$

where G^i and H^i are smooth functions vanishing at $(0,0)$, we get thanks to results in [26, Section 5.5] and Proposition A.3

$$\|W^i\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Proposition A.3 also implies that

$$\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|v^2\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

In order to estimate the term with v^2, we use that $v = z - \partial_x u$ and get the decomposition:

$$v^2 = v^h(v + v^\ell) + z^\ell(v^\ell - \partial_x u^\ell) + (\partial_x u^\ell)^2.$$

By Proposition A.3 and interpolation, we thus have

$$\begin{aligned}\|v^2\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\lesssim \|v^h\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|v+v^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \big(\|v^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|\partial_x u^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\big)\|z^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x u^\ell\|^2_{\dot{\mathbb{B}}^{\frac12(\frac12-\sigma_1)}_{2,1}}\\ &\lesssim \|v\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} \|v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|u\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|u\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}}.\end{aligned}$$

Hence we have

$$\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \widetilde{\mathcal{H}}.$$

Finally, using the decomposition

$$V^2(v)\partial_x u = V^2(v)\partial_x u^h + V^2(z)\partial_x u^\ell - \bigg(\int_0^1 V^2(z-\tau\partial_x u)\bigg)\Big(\partial_x u^\ell \partial_x u^\ell + \partial_x u^\ell \partial_x u^h\Big),$$

we get by similar computations that

$$\|V^2\partial_x u\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \widetilde{\mathcal{H}}.$$

Therefore, in the end, we get

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \le \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \widetilde{\mathcal{H}}\big(1 + \|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}},$$

which, combined with (78) and Gronwall lemma implies that

$$\sup_{t\in[0,T]} \|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

At this stage, completing the proof of decay estimates is left to the reader. □

Appendix A.

Here we gather a few technical results that have been used repeatedly in the paper. The first one is a rather standard lemma pertaining to some differential inequality.

Lemma A.1. *Let $p \ge 1$ and $X : [0,T] \to \mathbb{R}^+$ be a continuous function such that X^p is differentiable almost everywhere. We assume that there exists a constant $B \ge 0$ and a measurable function $A : [0,T] \to \mathbb{R}^+$ such that*

$$\frac1p \frac{d}{dt} X^p + BX^p \le AX^{p-1} \quad \text{a.e. on } [0,T].$$

Then, for all $t \in [0,T]$, we have

$$X(t) + B\int_0^t X \le X_0 + \int_0^t A.$$

Proof. The case $p = 1$ being obvious, assume that $1 < p < \infty$. Then, we set $X_\varepsilon \triangleq (X^p + \varepsilon^p)^{1/p}$ for $\varepsilon > 0$, and observe that

$$\frac1p \frac{d}{dt} X_\varepsilon^p + BX_\varepsilon^p \le AX_\varepsilon^{p-1} + B\varepsilon^p \quad \text{a.e. on } [0,T].$$

Dividing both sides by the positive function X_ε^{p-1}, we get

$$\frac{d}{dt}X_\varepsilon + BX_\varepsilon \le A + B\varepsilon\left(\frac{\varepsilon}{X_\varepsilon}\right)^{p-1},$$

whence, as $\varepsilon/X_\varepsilon \le 1$,

$$\frac{d}{dt}X_\varepsilon + BX_\varepsilon \le A + B\varepsilon.$$

Then, integrating in time and taking the limit as ε tends to 0 yields the desired inequality.

The following result from [10] has been used in the proof of Proposition 3.1.

Proposition A.1. *If $Supp(\mathcal{F}f) \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\}$ for some $0 < R_1 < R_2$ then, there exists $c = c(d, R_1, R_2) > 0$ such that for all $p \in [2, \infty[$, we have*

$$c\lambda^2 \left(\frac{p-1}{p}\right) \int_{\mathbb{R}^d} |f|^p \leq (p-1) \int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} = -\int_{\mathbb{R}^d} f\Delta f |f|^{p-2}.$$

The proof of the following inequality may be found in e.g. [1, Chap. 2].

Lemma A.2. *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\cdot$ Let a be a function with gradient in L^p and b, a function in L^q. There exists a constant C such that*

$$\Big\| [\dot{\Delta}_j, a]b \Big\|_{L^r} \leq C2^{-j} \|\nabla a\|_{L^q} \|b\|_{L^p} \quad \text{for all} \quad j \in \mathbb{Z}.$$

The following estimates are proved in [1, Chap. 2] and [12], respectively.

Proposition A.2. *Assume that $d = 1$ and that $1 \leq p \leq \infty$. The following inequalities hold:*

- *If $s \in \Big]-\min\left(\frac{1}{p}, \frac{1}{p'}\right), \frac{1}{p}+1\Big]$, then*

$$2^{js} \Big\| [w, \dot{\Delta}_j]\partial_x v \Big\|_{L^p} \leq Cc_j \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|v\|_{\dot{\mathbb{B}}^s_{p,1}} \quad \text{with} \quad \sum_{j\in\mathbb{Z}} c_j = 1. \tag{79}$$

- *If $s \in \Big[-\min\left(\frac{1}{p}, \frac{1}{p'}\right), \frac{1}{p}+1\Big[$, then*

$$\sup_{j\in\mathbb{Z}} 2^{js} \|[w, \dot{\Delta}_j]\partial_x v\|_{L^p} \leq C\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|v\|_{\dot{\mathbb{B}}^s_{p,\infty}}. \tag{80}$$

- *If $s \in \Big]-1-\min(\frac{1}{p}, \frac{1}{p'}), \frac{1}{p}\Big]$, then we have*

$$\Big\| \partial_x([w, \dot{\Delta}_j]v) \Big\|_{L^p} \leq Cc_j 2^{-js} \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|v\|_{\dot{\mathbb{B}}^s_{p,1}} \quad \text{with} \quad \sum_{j\in\mathbb{Z}} c_j = 1. \tag{81}$$

The following product laws in Besov spaces have been used several times.

Proposition A.3. *Let $(s, p, r) \in]0, \infty[\times[1, \infty]^2$. Then, $\dot{\mathbb{B}}^s_{p,r} \cap L^\infty$ is an algebra and we have*

$$\|ab\|_{\dot{\mathbb{B}}^s_{p,r}} \leq C\big(\|a\|_{L^\infty} \|b\|_{\dot{\mathbb{B}}^s_{p,r}} + \|a\|_{\dot{\mathbb{B}}^s_{p,r}} \|b\|_{L^\infty}\big)\cdot \tag{82}$$

If, furthermore, $-\min(d/p, d/p') < s \leq d/p$, then the following inequality holds:

$$\|ab\|_{\dot{\mathbb{B}}^s_{p,1}} \leq C\|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|b\|_{\dot{\mathbb{B}}^s_{p,1}}. \tag{83}$$

We have, if $-\min(d/p, d/p') < s \leq d/p + 1$,

$$\|a\, b\|^\ell_{\dot{\mathbb{B}}^s_{p,1}} \lesssim \|a\|_{L^\infty \cap \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}}. \tag{84}$$

In the case $d = 1$ and $2 \leq p \leq 4$, we have

$$\|ab\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \big(\|a\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|a\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\big(\|b\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|b\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\cdot \tag{85}$$

Proof. The first two inequalities are direct consequences of the results stated in [1, Chap. 2]. To prove the third one, we need the following so-called Bony decomposition for the product of two tempered distributions a and b (whenever it is defined):

$$ab = T_a b + T'_b a \quad \text{with} \quad T_a b \triangleq \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} a\, \dot{\Delta}_j b \quad \text{and} \quad T'_b a \triangleq \sum_{j\in\mathbb{Z}} \dot{S}_{j+2} b\, \dot{\Delta}_j a.$$

Now, using Bernstein inequality and the results of continuity for T and T' stated in [1, Chap. 2], we may write:

$$\|T_a b\|^{\ell}_{\dot{\mathbb{B}}^{s}_{p,1}} \lesssim \|T_a b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}} \lesssim \|a\|_{L^\infty} \|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}}$$

and, provided, $s-1 \leq d/p$ and $s > -\min(d/p, d/p')$,

$$\|T'_b a\|_{\dot{\mathbb{B}}^{s}_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}}.$$

This gives (84).

For proving (85), we combine Bony's decomposition and decomposition of a and b in low and high frequencies, writing

$$ab = T'_a b^{\ell} + T'_a b^{h} + T_{b^{\ell}} a^{\ell} + T_b a^{h} + T_{b^{h}} a^{\ell}.$$

All the terms in the right-hand side, except for the last one, may be bounded by means of the standard results of continuity for operators T and T' (see again [1, Chap. 2]). Setting $p^* = 2p/(p-2)$, we get:

$$\begin{aligned}
\|T'_a b^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} \|b^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{p^*}+1}_{p^*,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} \|b\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}, \\
\|T'_a b^{h}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}, \\
\|T_{b^{\ell}} a^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|b^{\ell}\|_{L^{p^*}} \|a^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{p,1}}, \\
\|T_b a^{h}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|b\|_{L^\infty} \|a^{h}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.
\end{aligned}$$

Finally, since $a^{\ell} = \dot{S}_{J_0+1} a$ and $b^{h} = (\mathrm{Id} - \dot{S}_{J_0+1}) b$, we see that

$$T_{b^{h}} a^{\ell} = \dot{S}_{J_0} b^{h}\, \dot{\Delta}_{J_0+1} a^{\ell}.$$

Consequently,

$$\|T_{b^{h}} a^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|\dot{\Delta}_{J_0+1} a^{\ell}\|_{L^\infty} \|\dot{S}_{J_0} b^{h}\|_{L^2} \lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Adding up this latter inequality to the previous ones gives

$$\|ab\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} \|b\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + \|b\|_{L^\infty} \|a^{h}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|b^{\ell}\|_{L^{p^*}} \|a^{\ell}\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{p,1}}.$$

Then, using Bernstein inequality, $2/p - 1/2 \leq 1/p$ and the embeddings $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$ and $\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1} \hookrightarrow L^{p^*}$ completes the proof of (85). □

The following result for composition in Besov spaces may be found in [1].

Proposition A.4. *Let f be a function in $\mathcal{C}^\infty(\mathbb{R})$ such that $f(0) = 0$. Let $(s_1, s_2) \in]0, \infty[^2$ and $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$. We assume that $s_1 < \frac{d}{p_1}$ or that $s_1 = \frac{d}{p_1}$ and $r_1 = 1$.*

Then, for every real-valued function u in $\dot{\mathbb{B}}^{s_1}_{p_1,r_1} \cap \dot{\mathbb{B}}^{s_2}_{p_2,r_2} \cap L^\infty$, the function $f \circ u$ belongs to $\dot{\mathbb{B}}^{s_1}_{p_1,r_1} \cap \dot{\mathbb{B}}^{s_2}_{p_2,r_2} \cap L^\infty$, and we have in particular

$$\|f \circ u\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \leq C\left(f', \|u\|_{L^\infty}\right) \|u\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \quad \text{for } k \in \{1,2\}.$$

The following result is the key to Theorem 1.1 in the general case.

Lemma A.3. *Assume that $d = 1$. Let $p \in [2,4]$ and $s \in [1/2, 3/2[$. Define $p^* \triangleq 2p/(p-2)$. For all $j \in \mathbb{Z}$, denote $\mathcal{R}_j \triangleq \dot{S}_{j-1} w\, \partial_x \dot{\Delta}_j z - \dot{\Delta}_j (w\, \partial_x z)$.*

There exists a constant C depending only on the threshold number J_0 between low and high frequencies and on s, such that

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac{3}{2}}_{\infty,\infty}}\|w\|^h_{\dot{\mathbb{B}}^s_{2,1}}+\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{s-\frac12-\frac1p}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big).$$

In the case $s=3/2$, we have

$$\sum_{j\geq J_0}\left(2^{j\frac32}\left\|\mathcal{R}_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x z\|_{L^\infty}\|w\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}+\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{1-\frac1p}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big).$$

Proof. From Bony decomposition recalled above, we deduce that

$$\begin{aligned}\mathcal{R}_j&=-\dot\Delta_j(T'_{\partial_x z}w)-\sum_{|j'-j|\leq4}[\dot\Delta_j,\dot S_{j'-1}w]\partial_x\dot\Delta_{j'}z-\sum_{|j'-j|\leq1}\left(\dot S_{j'-1}w-\dot S_{j-1}w\right)\dot\Delta_j\dot\Delta_{j'}\partial_x z\\&=\mathcal{R}^1_j+\mathcal{R}^2_j+\mathcal{R}^3_j.\end{aligned}$$

To estimate $\mathcal{R}^1_j$, we decompose w into low and high frequencies, getting

$$T'_{\partial_x z}w=T'_{\partial_x z}w^\ell+T'_{\partial_x z}w^h.$$

Because $1/p+1/p^*=1/2$, the classical results of continuity for paraproduct and remainder operators (see e.g. [1, Chap. 2]) ensure that

$$\|T'_{\partial_x z}w^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\lesssim\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}}\|w^\ell\|_{\dot{\mathbb{B}}^{\frac{1}{p^*}+1}_{p^*,1}},$$

and we have

$$\|T'_{\partial_x z}w^h\|_{\dot{\mathbb{B}}^s_{2,1}}\lesssim\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac32}_{\infty,\infty}}\|w^h\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\ \text{ if } 0<s<3/2,\quad\text{and}\quad\|T'_{\partial_x z}w^h\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\lesssim\|\partial_x z\|_{L^\infty}\|w^h\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}.$$

Observing that $T'_{\partial_x z}w^\ell$ contains only low frequencies so that its norm in $\dot{\mathbb{B}}^s_{2,1}$ is controlled by its norm in $\dot{\mathbb{B}}^{\frac12}_{2,1}$ if $s\geq1/2$, we deduce that

$$\sum_{j\in\mathbb{Z}}\left(2^{js}\left\|\mathcal{R}^1_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}}\left\|w^\ell\right\|_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac32}_{\infty,\infty}}\left\|w^h\right\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\Big)\ \text{ if }\ 1/2\leq s<3/2, \tag{86}$$

$$\sum_{j\in\mathbb{Z}}\left(2^{\frac32 j}\left\|\mathcal{R}^1_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}}\left\|w^\ell\right\|_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}+\|\partial_x z\|_{L^\infty}\left\|w^h\right\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}\Big). \tag{87}$$

Next, taking advantage of Lemma A.2, we see that if $j'\geq J_0$ and $|j-j'|\leq4$, then we have

$$2^{js}\|[\dot\Delta_j,\dot S_{j'-1}w]\partial_x\dot\Delta_{j'}z\|_{L^2}\lesssim\|\partial_x\dot S_{j'-1}w\|_{L^\infty}\,2^{j'(s-1)}\|\partial_x\dot\Delta_{j'}z\|_{L^2}$$

while, if $j'<J_0$, $j\geq J_0$ and $|j-j'|\leq4$,

$$2^{js}\|[\dot\Delta_j,\dot S_{j'-1}w]\partial_x\dot\Delta_{j'}z\|_{L^2}\lesssim2^{-\frac{j'}{p^*}}\|\partial_x\dot S_{j'-1}w\|_{L^{p^*}}\,2^{j'(s-\frac12-\frac1p)}\|\partial_x\dot\Delta_{j'}z\|_{L^p}.$$

Therefore,

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}^2_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{s-\frac12-\frac1p}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big). \tag{88}$$

Finally, for all $j \geq J_0$ and $|j'-j| \leq 1$, we have

$$\begin{aligned} 2^{js}\|(\dot{S}_{j'-1}w - \dot{S}_{j-1}w)\dot{\Delta}_j\dot{\Delta}_{j'}\partial_x z\|_{L^2} &\leq 2^j\|\dot{\Delta}_{j\pm1}w\|_{L^\infty}\, 2^{j(s-1)}\|\partial_x\dot{\Delta}_{j'}\dot{\Delta}_j z\|_{L^2} \\ &\leq C\|\dot{\Delta}_{j\pm1}\partial_x w\|_{L^\infty}\, 2^{j(s-1)}\|\partial_x\dot{\Delta}_j z\|_{L^2}. \end{aligned}$$

Hence

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}_j^3\right\|_{L^2}\right) \leq C\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}. \tag{89}$$

Putting (86), (87), (88) and (89) together completes the proof. □

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