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► **To cite this version:**

Archontia C Giannopoulou, O-Joung Kwon, Jean-Florent Raymond, Dimitrios M. Thilikos. A Menger-like property of tree-cut width. *Journal of Combinatorial Theory, Series B*, 2021, 148, pp.1-22. 10.1016/j.jctb.2020.12.005 . hal-03094586

HAL Id: hal-03094586

<https://hal.science/hal-03094586>

Submitted on 4 Jan 2021

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A Menger-like property of tree-cut width*

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Abstract

In 1990, Thomas proved that every graph admits a tree decomposition of minimum width that additionally satisfies a certain vertex-connectivity condition called *leanness* [A Menger-like property of tree-width: The finite case. *Journal of Combinatorial Theory, Series B*, 48(1):67 – 76, 1990]. This result had many uses and has been extended to several other decompositions. In this paper, we consider tree-cut decompositions, that have been introduced by Wollan as a possible edge-version of tree decompositions [The structure of graphs not admitting a fixed immersion. *Journal of Combinatorial Theory, Series B*, 110:47 – 66, 2015]. We show that every graph admits a tree-cut decomposition of minimum width that additionally satisfies an edge-connectivity condition analogous to Thomas’ leanness.

1 Introduction

The notion of treewidth is a cornerstone of the theory of graph minors of Robertson and Seymour. Formally, a *tree decomposition* of a graph G is a pair (T, \mathcal{X}) where T is a tree and $\mathcal{X} = \{X_t \subseteq V(G), t \in V(T)\}$ is a collection of vertex sets, called *bags*, with the following properties:

1. $\bigcup_{t \in V(T)} X_t = V(G)$;

*A preliminary version of this paper appeared in the proceedings of the 36th Symposium on Theoretical Aspects of Computer Science (STACS ’19) [GKRT19].

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[¶]Supported by the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (No. NRF-2018R1D1A1B07050294).

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^{**}Supported by the projects “ESIGMA” (ANR-17-CE23-0010) and “DEMOGRAPH” (ANR-16-CE40-0028) and by the Research Council of Norway and the French Ministry of Europe and Foreign Affairs, via the Franco-Norwegian project PHC AURORA 2019.

2. every edge of G belongs to some bag in \mathcal{X} ; and
3. for every $u \in V(G)$, the set $\{t \in V(T), u \in X_t\}$ induces a connected subgraph of T .

The *width* of (T, \mathcal{X}) is $\max_{t \in V(T)} |X_t| - 1$ and the *treewidth* of G , that we denote by $\text{tw}(G)$, is defined as the minimum width of a tree decomposition of G .

When writing [RS90] (see [Tho90, Theorem 2] and the introduction of [RS90, Section 5]), Robertson and Seymour proved that there exists tree decompositions of “small” width that satisfy a certain connectivity condition here called *linkedness property*.

Theorem 1.1 ([RS90]). *Every graph G admits a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ of width less than $3 \cdot 2^{\text{tw}(G)}$ such that the following holds:*

Linkedness property: for every $k \in \mathbb{N}$ and $a, b \in V(T)$, either there are k disjoint paths linking X_a to X_b , or there is a node c on the path of T between a and b such that $|X_c| < k$.

That is, while Menger’s theorem [Men27] states the existence of a set of less than k vertices disconnecting X_a from X_b , **Theorem 1.1** additionally guarantees that such vertices that can be found as a bag of size less than k of the tree decomposition. The exponential bound in **Theorem 1.1** has been subsequently improved by Thomas to its optimal value. In fact, Thomas showed a stronger property called *leanness*.

Theorem 1.2 ([Tho90]). *Every graph G admits a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ of width $\text{tw}(G)$ such that the following holds:*

Leanness property: for every $k \in \mathbb{N}$, $a, b \in V(T)$, $A \subseteq X_a$, and $B \subseteq X_b$ such that $|A| = |B| = k$, either there are k disjoint paths linking A to B , or there is a node c on the path of T between a and b such that $|X_c| < k$.

A simplified proof of **Theorem 1.2** was then found by Diestel and Bellenbaum [BD02]. The aim of Robertson and Seymour was to use **Theorem 1.1** as an ingredient in their proof that graphs of bounded treewidth are well-quasi-ordered by the minor relation [RS90]. Since then, the notions of leanness and linkedness have been extensively studied and extended to several different width parameters such as θ -tree-width [CDHH14, GJ16], path-width [Lag98], directed path-width [KS15], DAG-width [Kin14], rank-width [Oum05], linear-rankwidth [KK14], profile- and block-width [Erd18], matroid treewidth [GGW02a, Azz11, Erd18] and matroid branchwidth [GGW02a]. They have important applications, for instance in order to bound the size of obstructions for certain classes of graphs [Sey93, Lag98, KK14, GW02, GPR⁺18], in well-quasi-ordering proofs [Oum08, Liu14, GGW02b], in extremal graph theory [OOT93, CRS11], and for algorithmic purpose [CKL⁺18]. We refer to [Erd18] for an unified introduction to lean decompositions.

In this paper, we show that a similar leanness property holds for *tree-cut width*. Tree-cut width is a graph invariant introduced by Wollan in [Wol15] and defined via graph decompositions called *tree-cut decompositions*. Several results are supporting the claim that tree-cut width would be the *right* parameter for studying graph immersions. For instance, there is an analog to the Grid-minor Exclusion Theorem of Robertson and Seymour [RS86] in the

setting of immersions [Wol15]. Also, tree-cut decompositions can be used for dynamic programming in the same way as tree decompositions do, for certain algorithmic problems that cannot be tackled under the bounded-treewidth framework [GKS15, KOP⁺18, GPR⁺17]. Therefore, we expect that this invariant will play a central role in the flourishing theory of graph immersions.

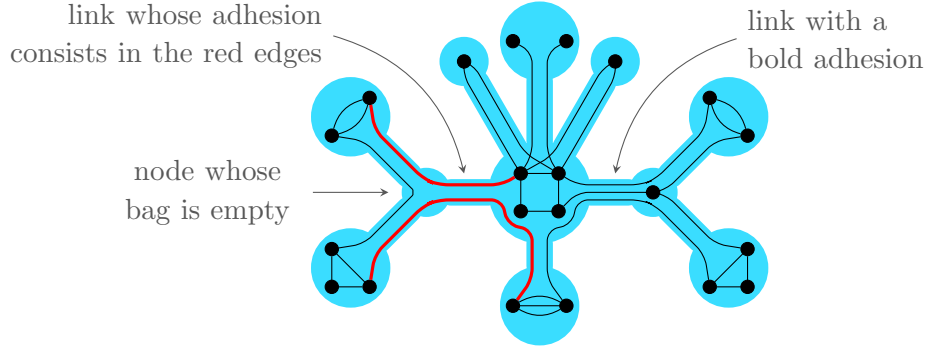


Figure 1: Representation of a tree-cut decomposition. The tree of the decomposition is depicted in blue and the graph is drawn in black on top of it to specify which vertices (resp. edges) belong to which bags (resp. adhesions).

Formally, a *tree-cut decomposition* of a graph G is a pair $\mathcal{D} = (T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_t \subseteq V(G), t \in V(T)\}$ is a collection of disjoint vertex sets, called *bags*, with the property that $\bigcup_{t \in V(T)} X_t = V(G)$.¹ See Figure 1 for an example. To avoid confusion with the vertices or edges of G , we respectively use the synonyms *nodes* and *links* when we refer to the vertices and edges of the tree of a tree-cut decomposition. Let uv be a link of T , let T_{uv} and T_{vu} be the two connected components of $T - uv$, let $X_{uv}^T = \bigcup_{t \in V(T_{uv})} X_t$ and symmetrically for X_{vu}^T . The *adhesion* $\text{adh}_{\mathcal{D}}(uv)$ of the link uv is defined as the set of edges of G with one endpoint in X_{uv}^T and the other one in X_{vu}^T . We drop the subscript when it is clear from the context. We say that an adhesion is *bold* if it has size more than two. Then the width of the decomposition \mathcal{D} is defined as:

$$\text{width}(\mathcal{D}) = \max \left\{ \max_{e \in E(T)} |\text{adh}(e)|, \max_{t \in V(T)} (|X_t| + |\{t' \in N_T(t), \text{adh}(tt') \text{ is bold}\}|) \right\},$$

where $N_T(t)$ denotes the set of nodes of T that are adjacent to t . The tree-cut width of G is defined as the minimum width of a tree-cut decomposition of it. We note that this definition differs from the original definition of Wollan in [Wol15], however the two definitions have been proved to be equivalent in [GPR⁺17]. Our definition of leanness for tree-cut decompositions is a transposition to the edge setting of the leanness notion of Thomas.

Definition 1.3 (leanness property for tree-cut decompositions). A tree-cut decomposition (T, \mathcal{X}) is said to be *lean* if for every $k \in \mathbb{N}$, every $a, b \in E(T)$, and every $A \subseteq \text{adh}(a), B \subseteq \text{adh}(b)$ such that $|A| = |B| = k$, one of the following holds:

- there are k edge-disjoint paths linking A to B ; or

¹In other words, $\{X_t, t \in V(T)\}$ is a partition of $V(G)$ plus possibly some empty sets.

- there is an link c on the path of T between a and b such that $|\text{adh}(c)| < k$.

Notice that Thomas' notion of leanness for tree decompositions relates vertex-disjoint paths to vertex-separators given by the decomposition, while ours links edge-disjoint paths to edge-separators. A related notion of linkedness has been previously studied in [GPR⁺18] in the simpler setting of cutwidth orderings. Our main result is the following.

Theorem 1.4. *Every graph G admits a tree-cut decomposition of width $\text{tcw}(G)$ that is lean.*

This result can be used to give explicit upper-bounds on the size of the immersion-obstructions of graphs of bounded tree-cut width², a result that we postpone to a future paper (see [GKRT19] for an extended abstract containing both results).

2 Preliminaries

Given two integers a, b we denote by $[a, b]$ the set $\{a, \dots, b\}$ and by $[a]$ the set $\{1, \dots, a\}$.

Graphs. Unless otherwise specified, we follow standard graph theory terminology; see e.g. [Die05]. All graphs considered in this paper are finite, undirected, without loops, and may have multiple edges. The vertex sets of a graph G is denoted by $V(G)$ and its multiset of edges by $E(G)$. For a subset of vertices $S \subseteq V(G)$, $G - S$ is the induced subgraph $G[V(G) \setminus S]$. For a subset $F \subseteq E(G)$ of edges, $G - F$ is the subgraph $(V(G), E(G) \setminus F)$. The subgraph of G *induced* by F has the set of endpoints of edges in F as vertex set and F as edge set.

A path of G *links* two edges of G if it starts with one and ends with the other. Given two sets A and B of edges of G , we say that a path P *links* A and B if it starts with an edge of A , ends with an edge of B , and none of its internal edges belong to $A \cup B$. In particular, the path reduced to a single edge $e \in A \cap B$ links A and B .

A *cut* in a graph G is a set $F \subseteq E(G)$ such that $G - F$ has more connected components than G . If $A, B \subseteq E(G)$, we say that F is an (A, B) -cut if no path links A and B in $G - F$. In particular, $A \cup B$ is an (A, B) -cut.

For two subsets $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of all edges $xy \in E(G)$ for which $x \in X$ and $y \in Y$. For $k \in \mathbb{N}$, a graph is said to be *k -edge-connected* if it has no cut on (strictly) less than k edges.

Tree-cut decompositions. Let G be a graph and let $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$ be a tree-cut decomposition of G , as defined in the introduction. For any nodes $u, v \in V(T)$, we denote by uTv the (unique) path of T with endpoints u and v . Similarly, if $e, f \in E(T)$, we denote by eTf the (unique) path of T starting with e and ending with f . Notice that if G is 3-edge-connected, then every link of T has a bold adhesion. In this case the definition of the width of (T, \mathcal{X}) can be simplified to

$$\max \left\{ \max_{e \in E(T)} |\text{adh}(e)|, \max_{t \in V(T)} (|X_t| + \deg_T(t)) \right\}. \quad (1)$$

²We refer here, for every $k \in \mathbb{N}$, to the immersion-minimal graphs that have tree-cut width more than k .

When a tree-cut decomposition is not lean (see [Definition 1.3](#)), this is witnessed by what we call a non-leanness certificate, defined as follows.

Definition 2.1 (non-leanness certificate). Let (T, \mathcal{X}) be a tree-cut decomposition of a graph G . A *non-leanness certificate* for (T, \mathcal{X}) is a quintuple (k, a, b, A, B) where $k \in \mathbb{N}_{\geq 1}$, a and b are links of T and A and B are sets of edges of G where $A \subseteq \text{adh}(a)$, $B \subseteq \text{adh}(b)$ and $|A| = |B| = k$, such that the following two conditions hold:

- (A) there is no collection of k edge-disjoint paths linking A to B and
- (B) every link e in aTb satisfies $|\text{adh}(e)| \geq k$.

A *minimal non-leanness certificate* of G is a non-leanness certificate of the form (k, a, b, A, B) (for some k, a, b, A, B as above) such that, among all non-leanness certificates of G , the value of k is minimum and, subject to that, the distance between a and b is minimum (possibly $a = b$).

In [Section 3](#) we show how a non-leanness certificate can be used in order to gradually improve a tree-cut decomposition towards leanness. We now define the operation that we use for these improvement steps.

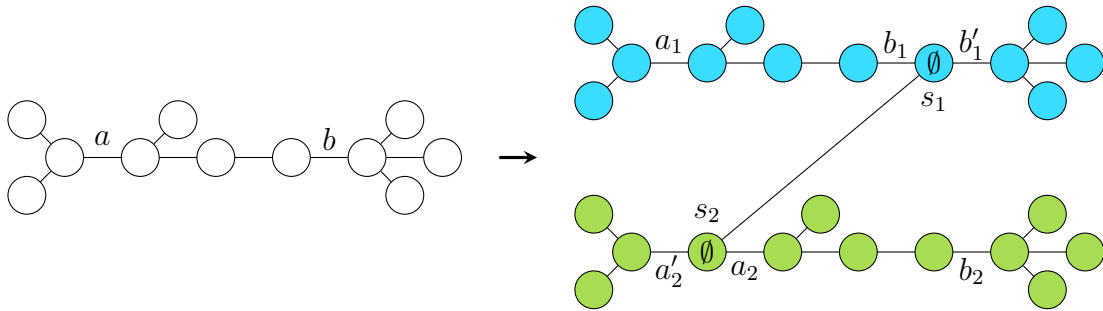


Figure 2: A tree-cut decomposition of a graph G (left) and its (a, b, V_1, V_2) -segregation (right), for some partition (V_1, V_2) of its vertex set. The vertices of V_1 and V_2 respectively lie in blue and green bags. Newly introduced bags, corresponding to nodes s_1, s_2 , are empty. The adhesion of s_1s_2 is exactly $E_G(V_1, V_2)$.

Definition 2.2 (segregation of a tree-cut decomposition). Let (T, \mathcal{X}) be a tree-cut decomposition of a graph G , let $a, b \in E(T)$, and let (V_1, V_2) be a partition of $V(G)$. We define the (a, b, V_1, V_2) -segregation of (T, \mathcal{X}) as the pair (U, \mathcal{Y}) obtained as follows:

1. consider a first copy U_1 of T , subdivide once the link corresponding to b , call s_1 the subdivision node, and call the two created links b_1 and b'_1 , with the convention that (the copy of) a is closer to b_1 in U_1 (if $a = b$, choose arbitrarily);
2. symmetrically, consider a second copy U_2 of T , subdivide once the link corresponding to a , call s_2 the subdivision node, and call the two created links a_2 and a'_2 , with the convention that (the copy of) b is closer to a_2 in U_2 , or, if $a = b$, coherently as the previous step;

3. in the disjoint union of U_1 and U_2 , add a link joining s_1 and s_2 : this gives U ;

$$4. \text{ for every } t \in V(U), \text{ let } Y_t = \begin{cases} X_t \cap V_1 & \text{if } t \in V(T_1) \setminus \{s_1\} \\ X_t \cap V_2 & \text{if } t \in V(T_2) \setminus \{s_2\}, \text{ and} \\ \emptyset & \text{if } t \in \{s_1, s_2\}. \end{cases}$$

An example of a segregation is presented in [Figure 2](#). The following remark follows from the definition of a segregation.

Remark 2.3. Any segregation of a tree-cut decomposition of a graph is a tree-cut decomposition of this graph.

In order to ensure that the aforementioned improvement steps eventually lead to a lean tree-cut decomposition, we use the following notion.

Definition 2.4 (fatness). Let G be a graph on m edges and let (T, \mathcal{X}) be a tree-cut decomposition of G . For every $i \in [m]$, we denote by $T^{\geq i}$ the subgraph of T induced by the links that have an adhesion of size at least i . The *fatness* of (T, \mathcal{X}) is defined as the $(2m)$ -tuple

$$(\alpha_m, -\beta_m, \alpha_{m-1}, -\beta_{m-1}, \dots, \alpha_1, -\beta_1),$$

where α_i is the number of links of $T^{\geq i}$ and β_i is the number of connected components of $T^{\geq i}$. We order fatnesses by lexicographic order.

The following is a slight variant of Menger's Theorem that we use in the next section.

Lemma 2.5. *Let G be a graph, $A, B \subseteq E(G)$, and $k \in \mathbb{N}$. Then either there is a set of k pairwise edge-disjoint paths of linking A and B in G , or G has an (A, B) -cut of size less than k .*

Proof. Suppose that there is no set of k pairwise edge-disjoint paths linking A and B in G . We create a new graph G' by subdividing every edge $e \in A \cup B$ into a new vertex v_e . Define $V_A = \{v_e, e \in A\}$ and symmetrically for V_B . Observe then that G' has k pairwise edge-disjoint paths linking vertex sets V_A and V_B if and only if G has k pairwise edge-disjoint paths linking edge sets A and B . So, by the edge version of Menger's Theorem, G' has an edge cut F' of size less than k separating V_A from V_B . Let F be obtained from F' by replacing every edge of the form v_e for $e \in A \cup B$ (if any) with e . It follows that $|F| < k$ and F separates A from B in G . \square

3 The proof of [Theorem 1.4](#)

In order to prove [Theorem 1.4](#), we use the aforementioned notion of fatness as a potential that we aim to minimize. The strategy we follow is to show that if a tree-cut decomposition of a graph is not lean, then it can be modified into a tree-cut decomposition of smaller fatness, without increasing width ([Lemma 3.1](#)). As there is no infinite decreasing sequence of fatnesses of tree-cut decompositions of a given graph, this process will eventually result in a lean tree-cut decomposition. Starting from a tree-cut decomposition of minimum width, we will therefore obtain a lean tree-cut decomposition of the same width, as desired ([Lemma 3.15](#)).

We first focus on the case where the considered graph is 3-edge-connected, which is the crux of the proof. The reduction from the general case is given at the end of the section.

Lemma 3.1. *Let $w \in \mathbb{N}$ and let (T, \mathcal{X}) be a tree-cut decomposition of width w of a 3-edge connected graph G . If (T, \mathcal{X}) is not lean, then G admits a tree-cut decomposition of width at most w that has smaller fatness than (T, \mathcal{X}) .*

Proof. Every edge e of G defines two (possibly equal) nodes of T , those indexing the bags that contain its endpoints. The T -path of e is defined as the path of T linking these vertices. For every $e \in E(G)$, we define $d_{a,b}(e)$ as 0 if the T -path of e shares a link with aTb and $1+d$ otherwise, where d denotes the minimal distance between a node of aTb and a node of the T -path of e . For every set $F \subseteq E(G)$ we set $d_{a,b}(F) = \sum_{e \in F} d_{a,b}(e)$.

Let us fix a minimal non-lean certificate (k, a, b, A, B) of (T, \mathcal{X}) . We set $m = |E(G)|$. We associate to (k, a, b, A, B) an (A, B) -cut F as follows. By [Lemma 2.5](#) (the variant of Menger's theorem for edge sets), there is in G a cut of size strictly smaller than k that separates A from B . Let F be such an (A, B) -cut of minimal size that, additionally, minimizes $d_{a,b}(F)$. Notice that none of $A \subseteq F$ and $B \subseteq F$ is possible, since $|A| = |B| = k$ and $|F| < k$. By the minimality of the size of F , the graph $G - F$ has exactly two connected components. We call them G_A and G_B , with the convention that

$$A \subseteq E(G_A) \cup F \quad \text{and} \quad B \subseteq E(G_B) \cup F. \quad (2)$$

We denote by (U, \mathcal{Y}) the $(a, b, V(G_A), V(G_B))$ -segregation of G . Recall that the tree U is obtained from two copies U_1 and U_2 of T . For every node t of T and $i \in [2]$, we denote by t_i the copy of t in U_i . Similarly, for every link e of T and $i \in [2]$, we denote by e_i the copy of e in U_i (except for a and b where the corresponding subdivided links have already been named in the definition of a segregation). Notice that we can unambiguously use **adh** without specifying the tree-cut decomposition it refers to as $E(U) \cap E(T) = \emptyset$. For every link $e \in E(U) \setminus \{a'_2, b'_1\}$, we denote by \hat{e} the corresponding link of T , that is, the only link such that $e \in \{\hat{e}_1, \hat{e}_2\}$. For the special cases $e = a'_2$ and $e = b'_1$ we respectively set $\hat{e} = a$ and $\hat{e} = b$.

In what follows we prove that $\text{width}(U, \mathcal{Y}) \leq \text{width}(T, \mathcal{X})$ ([Sublemma 3.4](#)) and that the fatness of (U, \mathcal{Y}) is (strictly) smaller than that of (T, \mathcal{X}) ([Sublemma 3.14](#)). The proof is split in a series of sublemmas. The end of the proof of each sublemma is marked with a “■”. When a sublemma contains a claim, we use the symbol “◇” to mark the end of its proof. We start with a series of sublemmas related to properties of adhesions.

Sublemma 3.2. *For every $e \in E(T)$ and $i \in [2]$,*

$$(i) \quad |\text{adh}(e_i)| \leq |\text{adh}(e)|;$$

$$(ii) \quad |\text{adh}(a'_2)| \leq |\text{adh}(a)| \quad \text{and} \quad |\text{adh}(b'_1)| \leq |\text{adh}(b)|;$$

$$(iii) \quad \text{if } |\text{adh}(e_i)| = |\text{adh}(e)| \text{ then } \text{adh}(e_{3-i}) \subseteq F;$$

$$(iv) \quad \text{if } |\text{adh}(a_1)| = |\text{adh}(a)| \text{ then } \text{adh}(a'_2) \subseteq F \text{ and if } |\text{adh}(b_2)| = |\text{adh}(b)| \text{ then } \text{adh}(b'_1) \subseteq F.$$

Proof of [Sublemma 3.2](#). We assume that $i = 1$. The proof for the case $i = 2$ is symmetric. Before proving the desired inequalities and inclusions, we give some definitions and prove a claim. Let $e \in E(T)$ and let T_1 and T_2 be the two connected components of $T - \{e\}$, named as follows:

- if $e = a = b$, then T_1 , corresponds (via the isomorphism from T to U_1) to the connected component of $U_1 - \{s_1\}$ that is incident with a_1 , and then T_2 to that that is incident with b'_1 (recall that in this case, the edge $a = b$ is replaced during the construction of U_1 by the two edges $a_1 = b_1$ and b'_1 , both incident to s_1);
- otherwise, if $e \notin E(aTb)$ or $e \in \{a, b\}$, we define T_1 and T_2 so that aTb is disjoint from T_1 ;
- in the remaining case, $e \in E(aTb) \setminus \{a, b\}$ and we choose them so that $a \in V(T_1)$ (and then $b \in V(T_2)$).

We define $C = \bigcup_{t \in V(T_1)} X_t$ and $D = \bigcup_{t \in V(T_2)} X_t$.

Also, we set:

$$C_A = C \cap V(G_A), \quad D_A = D \cap V(G_A), \quad (3)$$

$$C_B = C \cap V(G_B), \quad D_B = D \cap V(G_B). \quad (4)$$

See [Figure 3](#) for an example. By the choice of T_1 and T_2 , every edge in B has an endpoint in D . This, together with the second statement of [\(2\)](#) ensures that:

$$\text{Every edge in } B \text{ either has an endpoint in } D_B \text{ or is an edge of } E(D_A, C_B). \quad (5)$$

Recall that F consists of all edges with one endpoint in $C_A \cup D_A$ and the other in $C_B \cup D_B$. This means that:

$$F = E(C_A, C_B) \cup E(C_A, D_B) \cup E(D_A, C_B) \cup E(D_A, D_B). \quad (6)$$

Also, the edges of $\text{adh}(e)$ are those with one endpoint in C and the other in D . This implies that:

$$\text{adh}(e) = E(C_A, D_A) \cup E(C_A, D_B) \cup E(C_B, D_A) \cup E(C_B, D_B) \quad \text{and} \quad (7)$$

$$\text{adh}(e_1) = E(C_A, D_A) \cup E(C_A, C_B) \cup E(C_A, D_B). \quad (8)$$

We now set:

$$F' = (F \setminus E(C_A, C_B)) \cup E(C_B, D_B). \quad (9)$$

Claim 3.3. F' is an (A, B) -cut and also an $(E(C_A, D_A), B)$ -cut.

(The second statement will be useful when proving [\(iii\)](#).)

Proof. Looking for a contradiction, let us assume that there is a path P linking an edge of $A \cup E(C_A, D_A)$ and an edge of B in $G - F'$. We denote by e_A and e_B the edges of P that are incident to its endpoints, with $e_A \in A \cup E(C_A, D_A)$ and $e_B \in B$. From [\(5\)](#), either e_B has an endpoint in D_B or $e_B \in E(D_A, C_B)$. We first exclude the case where $e_B \in E(D_A, C_B)$. Indeed, if this is the case then, using [\(6\)](#), we obtain that $e_B \in F \setminus E(C_A, C_B) \subseteq F'$ a contradiction. We conclude that e_B has an endpoint, say y , where $y \in D_B$.

As P links an edge of $F \cup E(G_A)$ to an edge of $F \cup E(G_B)$, P contains at least one edge of F . Since P does not contain edges of F' , we deduce that this edge belongs to $E(C_A, C_B)$.

Among all edges of P that belong to $E(C_A, C_B)$ let g be the one that is closer to e_B in P and let x_g be the endpoint of g that belongs to C_B . By the choice of g , we know that the subpath P' of P that is between x_g and y is a subgraph of G_B . As $x_g \in C_B$ and $y \in D_B$, we have that P' (and therefore P as well) contains an edge $f \in E(C_B, D_B)$. However $E(C_B, D_B) \subseteq F'$, a contradiction. Therefore F' is indeed an (A, B) -cut and a $(E(C_A, D_A), B)$ -cut. The claim follows. \diamond

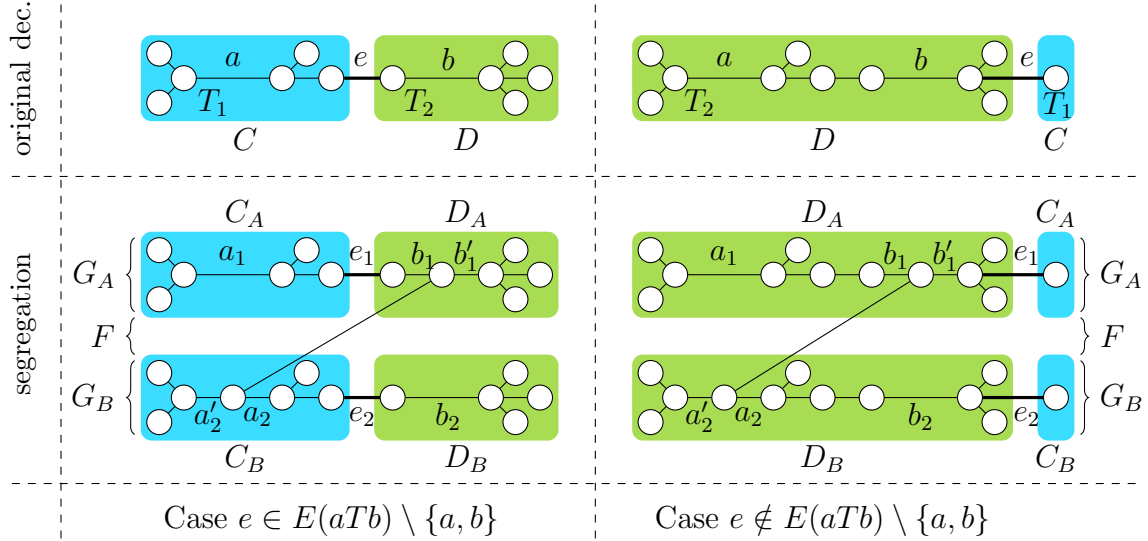


Figure 3: The sets C , D , C_A , D_A , C_B , and D_B in the original decomposition and in the segregation considered in the proof of [Sublemma 3.2](#), depending on the position of e . Note that e could also be an edge of $E(T) \setminus E(aTb)$ such that its closest vertex on aTb is an internal vertex.

Now we prove (i)–(iv).

Proof of (i). Assume for contradiction that $|\text{adh}(e_1)| > |\text{adh}(e)|$. Comparing (7) with (8), we get $|E(C_A, C_B)| > |E(C_B, D_A)| + |E(C_B, D_B)|$ which implies:

$$|E(C_A, C_B)| > |E(C_B, D_B)|. \quad (10)$$

In particular $E(C_A, C_B)$ is non-empty. From (6), $E(C_A, C_B) \subseteq F$. Using (10) and the definition of F' in (9), we deduce that $|F'| < |F|$, a contradiction to the minimality of $|F|$. This proves (i).

Proof of (ii). The proof is identical to the proof of (i), using $e = a$, $i = 2$, $e_2 = a'_2$ (resp. $e = b$, $i = 1$, $e_1 = b'_1$) to get the first (resp. second) inequality.

Proof of (iii). Let us assume that $|\text{adh}(e_1)| = |\text{adh}(e)|$ (the case where $|\text{adh}(e_2)| = |\text{adh}(e)|$ is symmetric). As we assume $|\text{adh}(e_1)| = |\text{adh}(e)|$, using (7) and (8), we get

$$|E(C_A, C_B)| \geq |E(C_B, D_B)|, \quad (11)$$

Considering F' as defined in (9), we deduce from (11) that $|F'| \leq |F|$, which we will use later. Towards a contradiction with (iii), let us assume that $\text{adh}(e_2) \not\subseteq F$ or, equivalently, that $|\text{adh}(e_2) \setminus F| > 0$. We consider two different cases.

First case: $e \notin E(aTb)$ or $e \in \{a, b\}$. We notice the following equality:

$$\text{adh}(e_2) = E(C_A, C_B) \cup E(C_B, D_A) \cup E(C_B, D_B). \quad (12)$$

Eq. (12) together with (6), implies that $\text{adh}(e_2) \setminus F = E(C_B, D_B)$ and we deduce $|E(C_B, D_B)| > 0$. With (11) this implies that $E(C_A, C_B)$ is non-empty. Also, from **Claim 3.3** we have that F' is an (A, B) -cut and we proved above that it is not larger than F . Notice that the T -path of any edge of $E(C_B, D_B)$ contains e . On the other hand,

- when $e \in \{a, b\}$, no T -path of an edge of $E(C_A, C_B)$ does contain e , and
- when $e \notin E(aTb)$, no T -path of an edge of $E(C_A, C_B)$ does contain the endpoint t_e of e that is the closest to a node of aTb .

Therefore, for every $f \in E(C_A, C_B)$ and $f' \in E(C_B, D_B)$,

- either $e \in \{a, b\}$, then $d_{a,b}(f') = 0$ (because the T -path of f' contains an edge of aTb , which is e) and $d_{a,b}(f) > 0$ (for the opposite reason);
- or $e \notin E(aTb)$, then $d_{a,b}(f') \leq \text{dist}_T(V(aTb), t_e)$ (because the T -path of f' contains t_e) and $d_{a,b}(f) \geq \text{dist}_T(V(aTb), t_e) + 1$ (for the opposite reason, and by definition of t_e).

In both cases we have $d_{a,b}(f') < d_{a,b}(f)$. The fact that $E(C_A, C_B)$ is non-empty, together with (11), imply that $d_{a,b}(E(C_A, C_B)) > d_{a,b}(E(C_B, D_B)) \geq 0$, hence $d_{a,b}(F') < d_{a,b}(F)$. This contradicts the choice of F , thus this case is not possible.

Second case: $e \in E(aTb) \setminus \{a, b\}$. Recall (**Claim 3.3**) that F' is a $(E(C_A, D_A), B)$ -cut. Notice that because of (6) and (9), it follows that $\text{adh}(e) \setminus F' = E(C_A, D_A)$. We deduce that F' , in fact, a $(\text{adh}(e), B)$ -cut.

We choose a subset F'' of $\text{adh}(e)$ such that $|F''| = k$. (Because $e \in aTb$ and of the definition of a, b , $|\text{adh}(e)| \geq k$ and such a subset always exists.) We claim that the quintuple (k, e, b, F'', B) satisfies conditions (A) and (B) of **Definition 2.1** (non-leanness certificate). Since eTb is a subpath of aTb , we have $|\text{adh}(e')| \geq k$ for every $e' \in E(eTb)$ and thus Condition (B) holds. For Condition (A) observe that F' separates $\text{adh}(e)$ from B and $|F'| \leq |F| < k$, therefore there are no k edge-disjoint paths linking $\text{adh}(e)$ to B .

Besides Conditions (A) and (B), $e \in E(aTb) \setminus \{a, b\}$, and therefore eTb is shorter than aTb . This contradicts the minimality of the distance between a and b that we assumed. Therefore, this case is not possible either and we have in both cases that $\text{adh}(e_2) \subseteq F$.

Proof of (iv). The proof follows the very same steps as the proof of (iii) (first case) using $e = a$, $i = 1$, $e_2 = a'_2$ (resp. $e = b$, $i = 2$, $e_1 = b'_1$). \blacksquare

We can now complete the first goal of this proof.

Sublemma 3.4. $\text{width}(U, \mathcal{Y}) \leq \text{width}(T, \mathcal{X})$

Proof. From Sublemma 3.2.(i) and Sublemma 3.2.(ii) we obtain that

$$\max_{g \in E(U)} |\mathbf{adh}(g)| \leq \max_{g \in E(T)} |\mathbf{adh}(g)|.$$

By definition of a segregation, for every $i \in [2]$ and $t \in V(T)$, we have $Y_{t_i} \subseteq X_t$ and $\deg_U(t_i) = \deg_T(t)$, hence $|X_t| + \deg_T(t) \geq |Y_{t_i}| + \deg_U(t_i)$. For $i \in [2]$ we also have $Y_{s_i} = \emptyset$, and $\deg_U(s_i) = 3$. As G is 3-edge-connected and $\mathbf{adh}(a)$ is not empty, $|\mathbf{adh}(a)| \geq 3$ so in particular, $|\mathbf{adh}(a)| \geq |Y_{s_i}| + \deg_U(s_i)$. Using the simplified definition of width for tree-cut decompositions of 3-edge-connected graphs (1), we conclude that $\mathbf{width}(U, \mathcal{Y}) \leq \mathbf{width}(T, \mathcal{X})$. \blacksquare

In the rest of the proof we focus on the second goal, i.e. showing that the fatness of (U, \mathcal{Y}) is smaller than that of (T, \mathcal{X}) . Let $(\alpha_m, -\beta_m, \alpha_{m-1}, -\beta_{m-1}, \dots, \alpha_1, -\beta_1)$ be the fatness of (T, \mathcal{X}) and let $(\alpha'_m, -\beta'_m, \alpha'_{m-1}, -\beta'_{m-1}, \dots, \alpha'_1, -\beta'_1)$ be that of (U, \mathcal{Y}) , as defined in Definition 2.4 (recall that $m = |E(G)|$).

Sublemma 3.5. *For every $e \in E(T)$,*

- *either $|\mathbf{adh}(e)| > |\mathbf{adh}(e_1)|$ and $|\mathbf{adh}(e)| > |\mathbf{adh}(e_2)|$;*
- *or there is some $i \in [2]$ such that $|\mathbf{adh}(e)| = |\mathbf{adh}(e_i)|$ and $\mathbf{adh}(e_{3-i}) \subseteq F$.*

Proof of Sublemma 3.5. This sublemma is a direct corollary of Sublemma 3.2.(i) and Sublemma 3.2.(iii). \blacksquare

Sublemma 3.6. *$A \subseteq \mathbf{adh}(a_1) \cup F$ and $B \subseteq \mathbf{adh}(b_2) \cup F$.*

Proof of Sublemma 3.6. We only prove the first statement as the proof of the second one is symmetric. We define C_A, C_B, D_A , and D_B as in (3) and (4) in the proof of Sublemma 3.2 for the case where $e = a$. Under this setting, (6), (7), and (8) are still valid; we restate them below for clarity.

$$F = E(C_A, C_B) \cup E(C_A, D_B) \cup E(D_A, C_B) \cup E(D_A, D_B), \quad (13)$$

$$\mathbf{adh}(a_1) = E(C_A, D_A) \cup E(C_A, C_B) \cup E(C_A, D_B), \text{ and} \quad (14)$$

$$\mathbf{adh}(a) = E(C_A, D_A) \cup E(C_A, D_B) \cup E(C_B, D_A) \cup E(C_B, D_B). \quad (15)$$

Recall that $A \subseteq E(G_A) \cup F$ (see (2)). Also, from (4), we obtain that $E(C_B, D_B) \subseteq E(G_B)$. These two relations imply that $A \cap E(C_B, D_B) = \emptyset$. Combining this last relation with (15), we get:

$$A \subseteq E(C_A, D_A) \cup E(C_A, D_B) \cup E(C_B, D_A). \quad (16)$$

As each of the terms of the right side of (16) appears on the right side of either (13) or (14), we conclude that $A \subseteq F \cup \mathbf{adh}(a_1)$ as required. \blacksquare

Given an integer p , we say that a link $e \in E(T)$ is *p-excessive* if

$$|\mathbf{adh}(e)| \geq p, \quad |\mathbf{adh}(e)| > |\mathbf{adh}(e_1)|, \quad \text{and} \quad |\mathbf{adh}(e)| > |\mathbf{adh}(e_2)|. \quad (17)$$

Sublemma 3.7. *Let l be an integer such that $l \geq k$ and none of the links of T of adhesion more than l is k -excessive. Then*

(i) $\alpha'_l \leq \alpha_l$ and $\beta'_l \geq \beta_l$, and

(ii) for every $j \in [l + 1, m]$, $\alpha'_j = \alpha_j$ and $\beta'_j \geq \beta_j$.

Proof of Sublemma 3.7. Let $j \in [l, m]$. Recall that we denote by $U^{\geq j}$ the subgraph of U induced by links that have an adhesion of size at least j . We need first the following claim:

Claim 3.8. *If f is a link of T such that f_i belongs to $U^{\geq j}$ for some $i \in [2]$, then $|\text{adh}(f)| = |\text{adh}(f_i)|$ and $\text{adh}(f_{3-i}) \subseteq F$.*

Proof. We first prove that it is not possible that $|\text{adh}(f)| > |\text{adh}(f_1)|$ and $|\text{adh}(f)| > |\text{adh}(f_2)|$. Towards a contradiction, let us suppose that it holds. If $|\text{adh}(f)| > l$, then f is a k -excessive link of adhesion greater than l , a contradiction to the hypothesis of the lemma. If $|\text{adh}(f)| \leq l$, then for every $i \in [2]$ we have $|\text{adh}(f_i)| < l \leq j$, hence $f_i \notin E(U^{\geq j})$, a contradiction. By **Sublemma 3.5**, there is some $i' \in [2]$ such that $|\text{adh}(f)| = |\text{adh}(f_{i'})|$ and $\text{adh}(f_{3-i'}) \subseteq F$. As $|\text{adh}(f_{3-i'})| \leq |F| < k \leq j \leq |\text{adh}(f_i)|$, we have that $i = i'$, and therefore $|\text{adh}(f)| = |\text{adh}(f_i)|$ and $\text{adh}(f_{3-i}) \subseteq F$, as desired. \diamond

Let $f \in E(T)$. From the above claim we have the following:

$$\text{If } f_i \in E(U^{\geq j}) \text{ for some } i \in [2], \text{ then } f \in E(T^{\geq j}) \text{ and } f_{3-i} \notin E(U^{\geq j}). \quad (18)$$

We next claim that if $f \in \{a, b\}$ and f_i belongs to $U^{\geq j}$ for some $i \in [2]$, then $i = 1$ in case $f = a$ and $i = 2$ in case $f = b$. We present the proof of this claim for the case where $f = a$ (the case $f = b$ is symmetric). Assume to the contrary that $i = 2$. Then, from the above claim, $\text{adh}(a_1) \subseteq F$. Recall that $A \subseteq \text{adh}(a_1) \cup F$, according to **Sublemma 3.6**. We conclude that $A \subseteq F$, a contradiction as $|A| = k$ and $|F| < k$. Thus, the claim holds.

By **Claim 3.8**, if $a_1 \in E(U^{\geq j})$ then $|\text{adh}(a_1)| = |\text{adh}(a)|$ and $|\text{adh}(a_2)| \leq |F| < k \leq j$. Therefore $a \in E(T^{\geq j})$ and $a_2 \notin E(U^{\geq j})$. Moreover, the fact that $|\text{adh}(a_1)| = |\text{adh}(a)|$ together with the first statement of **Sublemma 3.2(iv)** implies that $\text{adh}(a'_2) \subseteq F$. This implies that $|\text{adh}(a'_2)| \leq |F| < k \leq j$, therefore $a'_2 \notin E(U^{\geq j})$. We resume these observations, along with the symmetric observations for the case where $b_2 \in E(U^{\geq j})$, to the following statements:

$$\text{If } a_1 \in E(U^{\geq j}), \text{ then } a \in E(T^{\geq j}), a_2 \notin E(U^{\geq j}), \text{ and } a'_2 \notin E(U^{\geq j}). \quad (19)$$

$$\text{If } b_2 \in E(U^{\geq j}), \text{ then } b \in E(T^{\geq j}), b_1 \notin E(U^{\geq j}), \text{ and } b'_1 \notin E(U^{\geq j}). \quad (20)$$

Let $j \in [l, m]$ and let us define now the function $\varphi: E(U^{\geq j}) \rightarrow E(T^{\geq j})$ so that $\varphi(e) = \hat{e}$ for every $e \in E(U^{\geq j})$. (Recall that \hat{e} is the edge of T from which e has been copied, see the paragraph following (2) for the definition.) According to (18), (19), and (20), the function φ is injective. Hence $|E(U^{\geq j})| \leq |E(T^{\geq j})|$, i.e., $\alpha'_j \leq \alpha_j$. This proves the first half of (i).

When $j \in [l + 1, m]$, the function φ is even surjective: by definition of l , every link $f \in E(T^{\geq j})$ satisfies $|\text{adh}(f)| = |\text{adh}(f_i)|$ for some $i \in [2]$, therefore $f_i \in U^{\geq j}$ is the preimage of f by φ . As a consequence, for every $j \in [l + 1, m]$, we have $|E(U^{\geq j})| = |E(T^{\geq j})|$, that is, $\alpha'_j = \alpha_j$ and the first part of (ii) holds.

We now deal with the second parts of (i) and (ii). Let $j \in [l, m]$; we will prove that $\beta'_j \geq \beta_j$. Recall that s_1s_2 is the link joining the nodes s_1 and s_2 in U and $\text{adh}(s_1s_2) = F$. As $|F| < k \leq l \leq j$, we have that $s_1s_2 \notin E(U^{\geq j})$. This means that none of the connected components of $U^{\geq j}$ contains s_1s_2 . Let Q be a connected component of $U^{\geq j}$. Then, from (18), for every $f \in E(Q)$ it holds that $\hat{f} \in E(T^{\geq j})$. Therefore, if Q is a connected component of $U^{\geq j}$, then the subgraph T_Q of T with link set $\{\hat{f} : f \in E(Q)\}$ is a (connected) subtree of $T^{\geq j}$. Let ψ be the function that maps every connected component C of $U^{\geq j}$ to the connected component of $T^{\geq j}$ that contains the subgraph T_C , defined as above. Recall that by definition of l , if f is a link of $T^{\geq j}$ then f is not k -excessive and, thus, f_i is a link in $U^{\geq j}$, for some $i \in [2]$. Therefore, the connected component of $T^{\geq j}$ containing f is the image by ψ of the connected component of $U^{\geq j}$ containing f_i . This proves that ψ is surjective. The forest $U^{\geq j}$ then has at least as many connected components as $T^{\geq j}$ or, in other words, $\beta'_j \geq \beta_j$. This proves the second part of (ii) and concludes the proof. \blacksquare

Sublemma 3.9. *If T has a k -excessive link, then there is an integer $l \geq k$ such that*

(i) $\alpha'_l < \alpha_l$ and

(ii) for every $j \in [l + 1, m]$, $\alpha'_j = \alpha_j$ and $\beta'_j \geq \beta_j$.

Proof of Sublemma 3.9. Let g be an k -excessive link of maximum adhesion and let $l = |\text{adh}(g)|$. By definition of k -excessive (see (17)), we have $l \geq k$. By the choice of g , the integer l satisfies the requirements of Sublemma 3.7. Item (ii) then directly follows. Let us consider the same function φ as in the proof of Sublemma 3.7 (i.e., we set $\varphi(e) = \hat{e}$). By definition of g , we have $g \in E(T^{\geq l})$ whereas $g_1, g_2 \notin E(U^{\geq l})$. Therefore g has no preimage in $E(T^{\geq l})$ by φ : this function is not surjective. Thus, $|E(U^{\geq l})| < |E(T^{\geq l})|$, or, equivalently, $\alpha'_l < \alpha_l$. \blacksquare

Sublemma 3.10. *If a and b are distinct and not incident in T , then aTb has at least one k -excessive link.*

Proof of Sublemma 3.10. Towards a contradiction, let us assume the opposite statement: for every link $e \in E(aTb)$, (at least) one of the following holds: $|\text{adh}(e)| \leq |\text{adh}(e_1)|$ or $|\text{adh}(e)| \leq |\text{adh}(e_2)|$ (the case where $|\text{adh}(e)| < k$ is excluded because $e \in E(aTb)$ and Condition (B) holds). Our aim is to find a non-leanness certificate for (T, \mathcal{X}) that contradicts the minimality of (k, a, b, A, B) .

Let $e \in E(aTb)$ and $i \in [2]$ such that $|\text{adh}(e)| \leq |\text{adh}(e_i)|$. By Sublemma 3.2.(i)-(iii), we in fact have $|\text{adh}(e)| = |\text{adh}(e_i)|$ and $\text{adh}(e_{3-i}) \subseteq F$. In particular $|\text{adh}(e_{3-i})| < |\text{adh}(e)|$, because $|\text{adh}(e)| \geq k$, as noted above, while $|F| < k$. We deduce:

$$\forall e \in E(aTb), \exists i \in [2], \begin{cases} |\text{adh}(e_i)| = |\text{adh}(e)| \text{ and} \\ |\text{adh}(e_{3-i})| < |\text{adh}(e)|. \end{cases} \quad (21)$$

For the case where $e = a$ in (21), we claim that $i = 1$, i.e.

$$|\text{adh}(a_1)| = |\text{adh}(a)| \quad \text{and} \quad |\text{adh}(a_2)| < |\text{adh}(a)|. \quad (22)$$

If this claim was not correct, then we would have $|\text{adh}(a_2)| = |\text{adh}(a)|$ and by applying Sublemma 3.2.(iii) for $e = a$ we would get $\text{adh}(a_1) \subseteq F$. Together with $A \subseteq \text{adh}(a_1) \cup F$ (from [Sublemma 3.6](#)), this would imply $A \subseteq F$. Hence $k = |A| \leq |F| < k$, a contradiction.

By replacing a, A, a_1 by b, B, b_2 in the argument above, we can similarly show

$$|\text{adh}(b_1)| < |\text{adh}(b)| \quad \text{and} \quad |\text{adh}(b_2)| = |\text{adh}(b)|. \quad (23)$$

Claim 3.11. *There are two incident links $e, f \in E(aTb)$ and $i \in [2]$ such that*

1. $\{e, f\} \neq \{a, b\}$,
2. $e \in E(aTf)$, and
3. $|\text{adh}(e)| = |\text{adh}(e_1)|$ and $|\text{adh}(f)| = |\text{adh}(f_2)|$.

Proof. Let us color in blue every edge e of aTb such that $|\text{adh}(e)| = |\text{adh}(e_1)|$ and in red every edge such that $|\text{adh}(e)| = |\text{adh}(e_2)|$. By the virtue of (21), every edge receives exactly one color. By (22) and (23), a is colored blue and b is colored red. Let f be the first red edge met when following aTb from a and let e be the edge met just before. This choice ensures the two last desired properties. We assumed that a and b are not incident, so $\{e, f\} \neq \{a, b\}$. \diamond

Claim 3.12. *Let $e, f \in E(T)$ be links satisfying the conditions of [Claim 3.11](#). Then F separates $\text{adh}(e)$ from $\text{adh}(f)$.*

Proof. The third condition of [Claim 3.11](#) along with Sublemma 3.2.(iii), implies that

$$\text{adh}(e_2) \subseteq F \quad \text{and} \quad \text{adh}(f_1) \subseteq F. \quad (24)$$

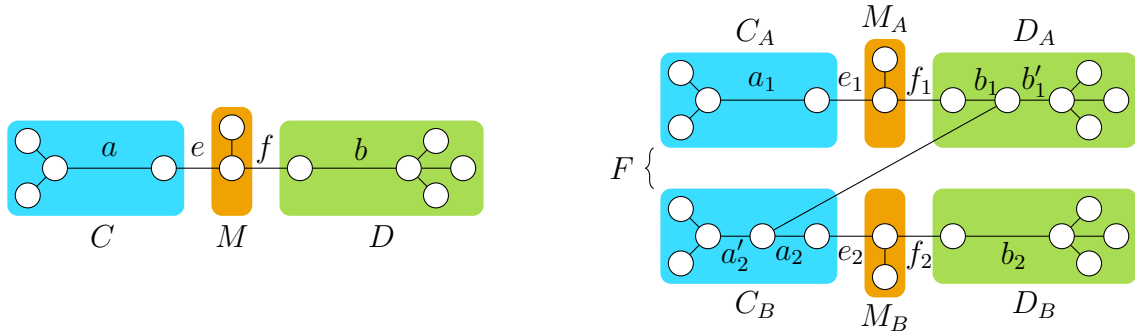


Figure 4: The sets $C, D, C_A, M_A, D_A, C_B, M_B,$ and D_B in the original decomposition (left) and in the segregation that we consider in the proof of [Claim 3.12](#) (right).

Let us call T_C, T_M and T_D the connected components of $T - \{e, f\}$ that contain, respectively, one endpoint of e and none of f , both one endpoint of e one of f , and one endpoint of f but none of e . As in the proof of [Sublemma 3.2](#), we set $C = \bigcup_{t \in V(T_C)} X_t$, $M = \bigcup_{t \in V(T_M)} X_t$, and $D = \bigcup_{t \in V(T_D)} X_t$, and for every $i \in [2]$ we define

$$\begin{aligned} C_A &= C \cap V(G_A) & M_A &= M \cap V(G_A) & D_A &= D \cap V(G_A). \\ C_B &= C \cap V(G_B) & M_B &= M \cap V(G_B) & D_B &= D \cap V(G_B). \end{aligned}$$

These sets are depicted in [Figure 4](#) on an example of a tree-cut decomposition. Notice that F contains all edges that have the one endpoint in $C_A \cup M_A \cup D_A$ and the other in $C_B \cup M_B \cup D_B$. In other words:

$$F = E(C_A, C_B) \cup E(C_A, M_B \cup D_B) \cup E(C_B, M_A \cup D_A) \cup E(M_A \cup D_A, M_B \cup D_B), \quad (25)$$

$$F = E(D_A, D_B) \cup E(D_A, M_B \cup C_B) \cup E(D_B, M_A \cup C_A) \cup E(M_A \cup C_A, M_B \cup C_B). \quad (26)$$

On the other hand, we have

$$E(C_B, M_B \cup D_B) \subseteq \text{adh}(e_2), \quad (27)$$

$$E(D_A, M_A \cup C_A) \subseteq \text{adh}(f_1), \quad (28)$$

and

$$\begin{aligned} \text{adh}(e) = & E(C_A, M_A \cup D_A) \cup E(C_A, M_B \cup D_B) \\ & \cup E(C_B, M_A \cup D_A) \cup E(C_B, M_B \cup D_B), \end{aligned} \quad (29)$$

$$\begin{aligned} \text{adh}(f) = & E(D_A, M_A \cup C_A) \cup E(D_A, M_B \cup C_B) \\ & \cup E(D_B, M_A \cup C_A) \cup E(D_B, M_B \cup C_B). \end{aligned} \quad (30)$$

From [\(27\)](#), [\(28\)](#), and [\(24\)](#) we have

$$E(C_B, M_B \cup D_B) \subseteq F, \quad (31)$$

$$E(D_A, M_A \cup C_A) \subseteq F. \quad (32)$$

Using [\(25\)](#), [\(29\)](#), and [\(31\)](#) and also [\(26\)](#), [\(30\)](#) and [\(32\)](#), we deduce:

$$\text{adh}(e) \setminus F = E(C_A, M_A \cup D_A), \quad \text{and} \quad (33)$$

$$\text{adh}(f) \setminus F = E(D_B, M_B \cup C_B). \quad (34)$$

In order to prove that F separates $\text{adh}(e)$ from $\text{adh}(f)$, let P be a path in G connecting an edge of $\text{adh}(e)$ and an edge of $\text{adh}(f)$. If this path contains an edge of F , then we are done. Otherwise, from [\(33\)](#) and [\(34\)](#), P should be a path from an edge from $E(C_A, M_A \cup D_A) \subseteq E(G_A)$ to an edge from $E(D_B, M_B \cup C_B) \subseteq E(G_B)$. Clearly, this path will have an edge in F and the claim follows. \diamond

Let $e, f \in E(T)$ be links satisfying the conditions of [Claim 3.11](#). We now claim that the quintuple $(k, e, f, \text{adh}(e), \text{adh}(f))$ is a non-leanness certificate for (T, \mathcal{X}) . Condition [\(A\)](#) follows as, from [Claim 3.12](#), F separates $\text{adh}(e)$ from $\text{adh}(f)$. Condition [\(B\)](#) holds because eTf is a subpath of aTb . Notice now that e and f are incident while a and b are not. Therefore, the distance between e and f in T is smaller than that between a and b . This contradicts the minimality of the choice of (k, a, b, A, B) as a minimal non-leanness certificate for (T, \mathcal{X}) . [Sublemma 3.10](#) follows. \blacksquare

Sublemma 3.13. *If T does not contain any k -excessive link, then there is an integer $l \geq k$ such that*

- for every $j \in [l + 1, m]$, $\alpha'_j = \alpha_j$ and $\beta'_j \geq \beta_j$, while

- $\alpha'_l \leq \alpha_l$ and $\beta'_l > \beta_l$.

Proof of Sublemma 3.13. Let us assume that T has no k -excessive link. By Sublemma 3.10, a and b are either incident edges, or they are the same edge. We set $l = \min\{|\text{adh}(a)|, |\text{adh}(b)|\}$ and observe that $l \geq k$. Clearly, l satisfies the requirements of Sublemma 3.7, so we have $\alpha'_j = \alpha_j$ and $\beta'_j \geq \beta_j$ for every $j \in [l+1, m]$ and $\alpha'_l \leq \alpha_l$. Let ψ be the function that maps connected components of $U^{\geq l}$ to connected components of $T^{\geq l}$, as defined in the proof of Sublemma 3.7 (for $j = l$), where it is shown to be surjective.

As a is not k -excessive, the first statement of Sublemma 3.5 does not hold for $e = a$. We conclude that for some $i \in [2]$, $|\text{adh}(a)| = |\text{adh}(a_i)|$ and $\text{adh}(a_{3-i}) \subseteq F$. The case $i = 2$ is not possible because then $\text{adh}(a_1) \subseteq F$ which, together with $A \subseteq \text{adh}(a_1) \cup F$ from Sublemma 3.6, implies $k \leq |\text{adh}(a_1)| \leq |F| < k$, a contradiction. Hence $i = 1$ and we have that $|\text{adh}(a)| = |\text{adh}(a_1)|$. By definition $l \leq |\text{adh}(a)|$, hence a_1 belongs to $U^{\geq l}$. Symmetrically, we can show that b_2 belongs to $U^{\geq l}$.

From the definition of l , both a and b belong to $T^{\geq l}$. As they are incident or equal we get that they belong to the same connected component of this graph. Besides, as noted above, both a_1 and b_2 belong to $U^{\geq l}$. However these links are separated in U by the link s_1s_2 , which has adhesion $|F| < k \leq l$. (Recall that s_1s_2 is the link added in the construction of U to join the two copies U_1 and U_2 of T ; see Definition 2.2 for a reminder.) Therefore, a_1 and b_2 do not belong to the same connected component of $U^{\geq l}$. This proves that ψ is not injective: $U^{\geq l}$ has more connected components than $T^{\geq l}$. Therefore, $\beta'_l > \beta_l$. ■

We are now in position to conclude the proof of Lemma 3.1.

Sublemma 3.14. *The fatness of (U, \mathcal{Y}) is smaller than that of (T, \mathcal{X}) .*

Proof. Recall that we respectively denote by

$$(\alpha_m, -\beta_m, \alpha_{m-1}, -\beta_{m-1}, \dots, \alpha_1, -\beta_1) \quad \text{and} \quad (\alpha'_m, -\beta'_m, \alpha'_{m-1}, -\beta'_{m-1}, \dots, \alpha'_1, -\beta'_1)$$

the fatnesses of (T, \mathcal{X}) and (U, \mathcal{Y}) . Notice that if the assumption of Sublemma 3.9 or of Sublemma 3.13 holds, then the fatness of (U, \mathcal{Y}) is (strictly) smaller than that of (T, \mathcal{X}) . As the assumptions of Sublemma 3.9 and Sublemma 3.13 are complementary, we are done. □

Sublemmas 3.4 and 3.14 show that (U, \mathcal{Y}) has the desired properties, so we are done. □

Lemma 3.15. *Every 3-edge connected graph G has a lean tree-cut decomposition of width $\text{tcw}(G)$.*

Proof. Recall that we order fatnesses by lexicographic order. The lemma follows from Lemma 3.1 and the fact that the set of fatnesses of tree-cut decompositions of a given graph does not contain an infinite decreasing sequence. □

Based on Lemma 3.15, we are now ready to give the proof of Theorem 1.4. The proof is essentially a reduction of the general case to that of 3-edge-connected graphs, that is handled by Lemma 3.15.

Proof of Theorem 1.4. For every $w \in \mathbb{N}$, we show that every graph G such that $\mathbf{tcw}(G) \leq w$ has a tree-cut decomposition of width at most w that is lean, by induction on the number of vertices of G .

Let $w \in \mathbb{N}$ and let G be a graph of tree-cut width at most w . If $|V(G)| \leq w$, then the tree-cut decomposition $((\{t\}, \emptyset), \{V(G)\})$ has width at most w and is trivially lean. Suppose now that $|V(G)| > w$ and that the statement holds for all graphs with less vertices than G (induction hypothesis).

Let F be a cut of G of minimum order and let $\{V_1, V_2\}$ be the corresponding partition of $V(G)$. (As we allow multiedges, it is possible that the edges in F share both endpoints.) If $|F| > 2$ then G is 3-edge-connected and the result follows because of [Lemma 3.15](#). Suppose now that $|F| \leq 2$. For every $i \in [2]$ we define G_i as follows. If $|F| = 2$ and the endpoints of F in V_i are distinct, we denote by G_i the graph obtained from $G[V_i]$ by adding an edge between these endpoints (or increasing the multiplicity by one if the edge already exists). In all the other cases, we set $G_i = G[V_i]$. Notice that G_1 and G_2 are both immersions of G , hence they have tree-cut-width at most w . Also, they have less vertices than G , so we can apply our induction hypothesis.

For every $i \in [2]$, let (T^i, \mathcal{X}^i) be a tree-cut decomposition of width $\mathbf{tcw}(G_i) \leq w$ of G_i , that is lean. Let x_i be an endpoint of F in V_i or, in the case $F = \emptyset$, any vertex of V_i . Let t_i be the node of T^i such that $x_i \in X_{t_i}^i$. We define T as the tree obtained from the disjoint union of T^1 and T^2 by adding the link $t_1 t_2$. We also set $\mathcal{X} = \mathcal{X}^1 \cup \mathcal{X}^2$. Clearly (T, \mathcal{X}) is a tree-cut decomposition of G . Notice that the adhesion of $t_1 t_2$ is F whose size is at most 2, hence it is not bold. Also, for every $i \in [2]$ and $e \in T^i$, if $\mathbf{adh}_{T^i}(e)$ contains the edge added in the construction of G_i (if any), then $\mathbf{adh}_T(e)$ contains instead one of the edges of F . Hence the size of the adhesion of e does not change from T^i to T . We can therefore express the width of (T, \mathcal{X}) in terms of (T^i, \mathcal{X}^i) and F :

$$\begin{aligned} \mathbf{width}(T, \mathcal{X}) &= \max \left\{ \max_{e \in E(T)} |\mathbf{adh}_{(T, \mathcal{X})}(e)|, \max_{t \in V(T)} (|X_t| + |\{t' \in N_T(t), \mathbf{adh}_{(T, \mathcal{X})}(tt') \text{ is bold}\}|) \right\} \\ &= \max \left\{ \max_{e \in E(T^1)} |\mathbf{adh}_{(T^1, \mathcal{X}^1)}(e)|, \max_{e \in E(T^2)} |\mathbf{adh}_{(T^2, \mathcal{X}^2)}(e)|, |F|, \right. \\ &\quad \max_{t \in V(T^1)} (|X_t| + |\{t' \in N_{T^1}(t), \mathbf{adh}_{(T^1, \mathcal{X}^1)}(tt') \text{ is bold}\}|), \\ &\quad \left. \max_{t \in V(T^2)} (|X_t| + |\{t' \in N_{T^2}(t), \mathbf{adh}_{(T^2, \mathcal{X}^2)}(tt') \text{ is bold}\}|) \right\} \\ &= \max \{ \mathbf{width}(T^1, \mathcal{X}^1), \mathbf{width}(T^2, \mathcal{X}^2), |F| \} \\ &\leq \max \{ w, |F| \} \end{aligned}$$

As $|V(G)| > w \geq \mathbf{tcw}(G)$, the tree of any minimum-width tree-cut decomposition of G has at least one link and the adhesion of this link has size at most w . Hence G has a cut of size at most w . By minimality of F , we deduce $|F| \leq w$, hence $\mathbf{width}(T, \mathcal{X}) \leq w$ from the inequalities above.

At this point of the proof we have constructed a tree-cut decomposition (T, \mathcal{X}) of G of width at most w . It remains to show that it is lean. For this, we consider some $a, b \in E(T)$ and subsets $A \subseteq \mathbf{adh}(a)$ and $B \subseteq \mathbf{adh}(b)$ of the same size k . It is enough in order to conclude

the proof to assume that there is no collection of k edge-disjoint paths linking A to B in G and show that aTb has a link whose adhesion has size less than k .

In the case where $a, b \in E(T^i)$ for some $i \in [2]$, we observe that, as it is an immersion of G , G_i does not contain k edge-disjoint paths linking A to B . Because (T^i, \mathcal{X}^i) is lean, there is a link e in aT^ib such that $|\text{adh}_{(T^i, \mathcal{X}^i)}(e)| < k$ (in particular, $a \neq b$). As noted above, $|\text{adh}_{(T^i, \mathcal{X}^i)}(e)| = |\text{adh}_{(T, \mathcal{X})}(e)|$ for every $e \in E(T^i)$, hence $|\text{adh}_{(T, \mathcal{X})}(e)| < k$ and we are done. It remains to consider the case where a and b do not belong to the same of T^1 and T^2 . By [Lemma 2.5](#) (the variant of Menger's Theorem), there is in G a cut of size strictly smaller than k that separates A from B . By minimality of F , this implies $|F| < k$. Observe then that t_1t_2 is an edge of aTb and $|\text{adh}(t_1t_2)| = |F| < k$, as desired. \square

Acknowledgments

The authors wish to thank Michał Pilipczuk and Marcin Wrochna for valuable discussions about lean tree-cut decompositions.

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