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# 1 **Atmospheric infrasound generation by ocean waves in finite** 2 **depth: unified theory and application to radiation patterns**

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## 5 **SUMMARY**

6 Between 0.1 and 0.5 Hz, infrasound signals recorded in the atmosphere are dominated by  
7 ocean-generated noise called microbaroms. Microbaroms propagate through the atmosphere  
8 over thousands of kilometers due to low absorption and efficient ducting between the ground  
9 and the stratopause. Different theoretical models have been developed to characterize the  
10 source of microbaroms, all based on the second-order non-linear interaction of ocean waves.  
11 While early theories considered an infinite ocean depth and a source radiation depending on  
12 the acoustic wave elevation angle, other works have approximated the radiation pattern as a  
13 monopole, and found a considerable effect of the water depth. This paper reviews these mod-  
14 els and extends the previous theories to the combined effects of both finite depth ocean and  
15 source directivity in both elevation and azimuth angles. It is found that the water depth has  
16 a negligible effect for the near-horizontally propagating acoustic waves that should dominate  
17 the measured microbarom records. Another important result is that the microbarom azimuthal  
18 variation can be highly directive locally, but it generally becomes isotropic when integrated  
19 over a realistic source region.

20 **Key words:** Infrasound – Interface waves – Wave propagation

21 **1 INTRODUCTION**

22 Continuous oscillations of the ground displacement and atmospheric pressure, named respectively  
23 secondary microseisms and microbaroms, are measured worldwide by seismological and infra-  
24 sound networks with a dominant frequency around 0.2 Hz (Benioff & Gutenberg, 1939). They  
25 are generated by second-order non linear interaction of ocean gravity waves of similar frequency  
26 propagating in almost opposite directions (Longuet-Higgins, 1950; Hasselmann, 1963).

27 Microbarom propagate through the atmosphere over large distances due to low absorption rates  
28 and efficient atmospheric ducting between the ground and the stratopause (Drob, 2019; Waxler &  
29 Assink, 2019). Studying microbaroms recorded for four years at Palisades, New York, (Donn &  
30 Rind, 1971) have revealed the importance of winds in the higher atmosphere for their propaga-  
31 tion, pointing to the capability of ground-based measurements to probe the higher atmosphere, for  
32 which very few other observations are available.

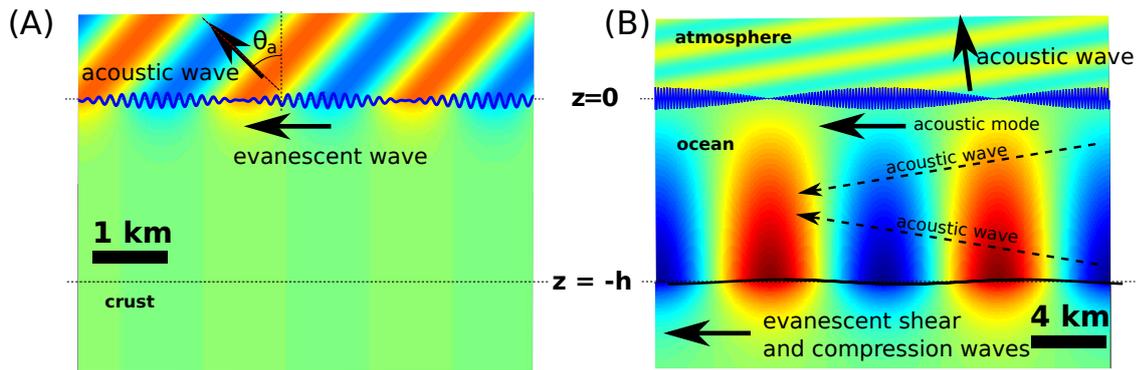
33 Recent developments of infrasound networks at global and continental scales facilitate the  
34 analysis of acoustic waves for probing unresolved atmospheric structures in the middle atmo-  
35 sphere (Marty, 2019; Blanc et al., 2018). This has motivated mathematical developments of geo-  
36 physical inverse problems using infrasound from well identified sources (Drob et al., 2010; Assink  
37 et al., 2014). Ducting of infrasound depends on the 3-D wind and temperature fields and is most  
38 efficient if the propagation direction coincides with the polar vortex at mid-latitude regions. In par-  
39 ticular, the main characteristics of Sudden Stratospheric Warming events have been successfully  
40 derived from directional microbarom amplitude variations resulting from changes in stratospheric  
41 and thermospheric propagation conditions (e.g. Garcés et al., 2002; Landès et al., 2010; Smets &  
42 Evers, 2014). Such studies demonstrate the advantage of an infrastructure that integrates indepen-  
43 dent middle atmospheric measurement techniques currently not assimilated in numerical weather  
44 prediction models (NWP) and provides quantitative understanding of stratosphere-troposphere dy-  
45 namical coupling useful for NWP applications (Le Pichon et al., 2015).

46 So far, microbarom studies have used qualitative comparisons between source models and  
47 received signals, with difficulties of interpretation associated with uncertainties in the measure-  
48 ments and in the propagation. Thanks to novel measurements from a stratospheric balloon fitted

49 with microbarometers, Bowman & Lees (2018) were able to verify quantitatively the predictions  
50 based on numerical ocean wave models and the microbarom source theory of Waxler et al. (2007)  
51 using measured sound spectra over the Southern Ocean. Their further interpretation of micro-  
52 baroms as a major heat source for the thermosphere, well above their measurement altitude, relies  
53 on the monopolar radiation pattern predicted by Waxler & Gilbert (2006).

54 Following the work of Longuet-Higgins (1950) on microseisms, a first theory of microbarom  
55 generation was proposed by Posmentier (1967), with the atmospheric motion coming from the  
56 continuity of the velocity field at the air-sea interface, and no feedback of the atmosphere on the  
57 pressure field in the ocean. A more complete theory for random waves, consistent with the mi-  
58 croseism generation theory of Hasselmann (1963) and an accurate treatment of the air-sea bound-  
59 ary condition is given by Brekhovskikh et al. (1973), leading to significant differences for near-  
60 horizontal propagation. However, that work only considered an ocean of infinite depth. The effect  
61 of the ocean depth, with the amplification of particular frequencies corresponding to an ‘organ  
62 pipe resonance’ of the water column, was later considered by Waxler et al. (2007), extending the  
63 work done by Waxler & Gilbert (2006). The major difference between that work and the earlier  
64 analysis of Brekhovskikh et al. (1973) is the monopole radiation pattern that, as we show here,  
65 comes from an assumption on the coherence of the source over only very short scales whereas  
66 Brekhovskikh et al. (1973) did not introduce this assumption. A unified theory is thus necessary  
67 for further quantitative analysis of microbarom records and the analysis of their impact in regions  
68 where no measurement is available, such as the thermosphere.

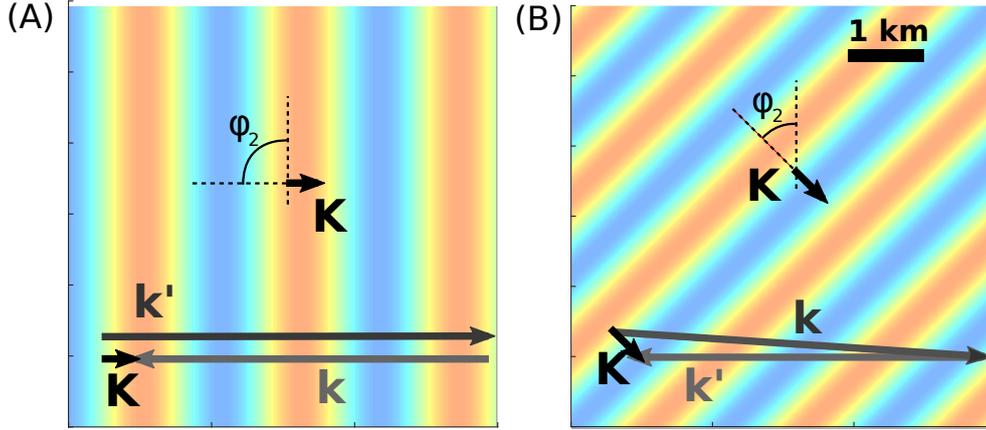
69 Given that microbaroms and microseisms are related (Donn & Naini, 1973) it is interesting to  
70 discuss microseisms for which more quantitative analyses are available. Microseisms have their  
71 most energetic sources associated to severe ocean storms but not necessarily co-located with the  
72 storm due to the propagation of ocean waves as swells (Obrebski et al., 2012). In particular, mea-  
73 surements at seismic stations near coasts can be dominated by the interaction of storm waves with  
74 their reflection from the coast (Bromirski et al., 1999; Ardhuin et al., 2011). In general, the sources  
75 at a frequency  $f_s$  correspond to the interaction of waves with similar frequencies  $f = f_s/2$  and  
76 nearly opposite directions, as illustrated in Fig. 1.



**Figure 1.** Example of vertical cross-section of the pressure pattern, in colors, radiated by a pair of interacting ocean wave trains of period around 10 s. The sea surface elevation, plotted as a blue line around  $z = 0$  is the sum of the elevations of the two wave trains and has a wave group structure. (A) The periods of the two wave trains are 10 s and 9.66 s (B) 10 s and 9.94 s. Note that the vertical displacement of the sea surface is strongly exaggerated in order to make waves visible. A realistic ocean wave field includes many wave trains and thus all possible pairs of interactions radiating acoustic waves in all directions  $\theta_a$ . As the two periods of the wave train get closer, from (A) to (B), the lengths of the groups get larger and the angle  $\theta_a$  becomes smaller. This paper focuses on the radiated power as a function of  $\theta_a$ .

77 The generation of acoustic and seismic modes in the ocean and solid Earth with horizontal  
 78 propagation speeds that exceed 1500 m/s, much more than the typical ocean waves phase speeds  
 79 around 15 m/s, was explained quantitatively by Hasselmann (1963) as the result of an interference  
 80 of pairs of ocean wave trains of wavenumbers  $\mathbf{k}$  and  $\mathbf{k}'$ , giving seismic or acoustic waves at the  
 81 wavenumber  $\mathbf{K} = \mathbf{k} + \mathbf{k}'$ . The microseism generation theory is one particular example of the gen-  
 82 eral theory of wave-wave interactions developed by Hasselmann (1966). The horizontal radiation  
 83 pattern of a single pair of ocean waves gives a single sinusoidal pressure field propagating in the  
 84 direction  $\varphi_2$  of the wavenumber vector  $\mathbf{K}$ . For microseisms  $K = |\mathbf{K}|$  is generally much smaller  
 85 than the width of the ocean wave spectrum, so that the combination of all pairs of ocean waves  
 86 gives an isotropic source.

87 In the case of microbaroms of frequency 0.2 Hz, the sound speed in the air is only 20 times the  
 88 phase speed of the ocean waves, so that  $K$  is comparable to the ocean wave spectrum width, and  
 89 there may be a preferential radiation in some directions  $\varphi_2$ . Applications and further analysis of  
 90 microbaroms require a knowledge of the source magnitude and variability. In recent work, Waxler  
 91 et al. (2007) investigated the influence of the water depth on the source magnitude, similar to what



**Figure 2.** Example of horizontal pressure patterns, in colors, radiated by a pair of interacting ocean wave trains of periods 10 s and 9.66 s with (A) opposing directions (B) not exactly opposing directions. For acoustic waves, the maximum magnitude  $K$  of the wavenumber vector  $\mathbf{K}$  corresponds to horizontal propagation with  $K = \Omega/\alpha_a$  and can be in any azimuth  $\varphi_2$  depending on the exact ocean wave wavenumbers  $\mathbf{k}$  and  $\mathbf{k}'$ . Modes with larger values of  $K$  decay exponentially over the vertical and are not relevant for microbaroms measured on land.

92 is found for microseisms. The conclusion of the present paper is that the effect of water depth  
 93 depends on the angle  $\theta_a$  of the sound propagation relative to the vertical direction, and that the  
 94 pressure field of microbaroms over the ocean generally contains a wide range of angles. Indeed, the  
 95 coupling of the ocean and atmosphere strongly depends on  $\theta_a$ , as demonstrated by Brekhovskikh  
 96 et al. (1973), but neglected by Waxler & Gilbert (2006).

97 This importance of  $\theta_a$  is now well known for microseisms and is easy to understand in relation  
 98 to the physical properties of the solid Earth, ocean, and atmosphere. A usual approximation of the  
 99 propagating medium is a stack of uniform horizontal layers  $l$  characterized by different veloci-  
 100 ties of propagation  $\alpha_l$  and  $\beta_l$  for compression and shear waves. Close to the source, microseisms  
 101 are dominated by Rayleigh waves that correspond to relatively slow components, with horizontal  
 102 propagation speeds between the sound speed in water  $\alpha_w$ , and the shear wave speed in the crust  $\beta_c$ .  
 103 These Rayleigh modes combine motions that decay exponentially with depth in the solid Earth,  
 104 with propagating acoustic waves in the ocean, and their acoustic propagation angles in the water  
 105  $\theta_w$  are larger than  $\arcsin(\alpha_w/\beta_c) \simeq 30^\circ$  (Ardhuin et al., 2019). For very large distances, seismic  
 106 body waves may dominate the signal because of their weaker attenuation with distance, and these  
 107 are due to the ocean acoustic noise that is more nearly vertical, allowing propagation in the crust,

108 with  $\theta_w < \arcsin(\alpha_w/\alpha_c) \simeq 16^\circ$ . The water depth effect is clearly different for Rayleigh and body  
 109 waves, as predicted by Ardhuin & Herbers (2013) and Gualtieri et al. (2014), and demonstrated  
 110 by (Obrebski et al., 2013) and Meschede et al. (2017).

111 Now looking at microbaroms, we expect similar dependencies on  $\theta_a$  because different com-  
 112 ponents of the ocean wave forcing field, with wavenumbers  $\mathbf{K} = \mathbf{k} + \mathbf{k}'$  give acoustic modes that  
 113 have different apparent horizontal speeds  $C = 2\pi f_s/|\mathbf{K}|$ . Speed faster than the compression speed  
 114 in the crust  $C > \alpha_c$  leads to the generation of compression P-waves in the solid Earth, correspond-  
 115 ing to nearly vertical propagation in the ocean layer and even more vertical propagation in the  
 116 atmosphere, given by Snell's law. The limit  $C \rightarrow \infty$  corresponding to vertical propagation and  
 117  $|\mathbf{K}| = 0$ , which are the exactly standing waves of Longuet-Higgins. At the other extreme,  $C < \alpha_w$   
 118 gives evanescent waves in the water layer that correspond to acoustic-gravity (AG) modes that  
 119 dominate the pressure field measured in the top 300 m of the ocean, as observed by Cox & Jacobs  
 120 (1989) and Ardhuin et al. (2013). For these AG modes, we expect no influence of water depths  
 121 larger than 300 m on the pressure at the ocean surface. These AG waves are coupled to atmospheric  
 122 acoustic waves that have propagation angles larger than  $\theta_{a0} = \arcsin \alpha_a/\alpha_w \simeq 12^\circ$ .

123 The difference in water depth effects between body waves, Rayleigh waves and AG modes  
 124 should influence the amplitude of acoustic waves radiated in the atmosphere, and the amplitude of  
 125 microbaroms should strongly depend on the direction of propagation  $\theta_a$  relative to the vertical. The  
 126 decomposition of the ocean wave forcing in different horizontal wavenumbers  $\mathbf{K} = K (\cos \varphi_2, \sin \varphi_2)$   
 127 allows to consider separately the different acoustic wave components and how they may contribute  
 128 to different acoustic modes. Each  $K$  corresponds to a propagation angle such that  $\sin \theta_l = K \alpha_l/(2\pi f_s)$   
 129 where  $\alpha_l$  is the sound speed in the layer  $l$ , which is related to Snell's law.

130 Because the ocean wave spectra are relatively broad, they contain a wide range of pairs  $\mathbf{k}$   
 131 and  $\mathbf{k}'$  so that all possible  $\mathbf{K}$  are excited simultaneously. For microseisms, this produces a spec-  
 132 trum of the excitation that is white in wavenumber, and thus equivalent to a point force. In the  
 133 case of microbaroms, the conversion from wave motion to acoustic pressure is a function of the  
 134 wavenumber  $K$ , first given by Brekhovskikh et al. (1973), which determines the radiation pattern  
 135 in the atmosphere.

136 Hence the previous works either lack the possible important effect of finite depth in the ocean,  
 137 or important aspects of the radiation pattern in the atmosphere. In order to progress towards a  
 138 quantitative understanding of microbarom signals it is thus necessary to have both effects in the  
 139 same theory, and this is the objective of the present paper. For the sake of simplicity, we only  
 140 consider the case of a homogeneous atmosphere, and extend the theory of Brekhovskikh et al.  
 141 (1973) to take into account a finite water depth. The theoretical formulation and the main results  
 142 are given in section 2, with details of the derivation in the Supporting Information. These results  
 143 are interpreted in section 3 and conclusions follow in section 4.

## 144 **2 A GENERAL THEORY OF MICROBAROM SOURCES**

145 In order to facilitate the translation between the different papers we have listed in table 1 the cor-  
 146 respondence of the main symbols used. We have also included Longuet-Higgins (1950) because it  
 147 treats almost the same physical problem, with a focus on the water layer, and the same decompo-  
 148 sition in particular and homogeneous solutions of the forced wave equation.

149 As detailed in the Supporting information, which follows the method of Brekhovskikh et al.  
 150 (1973), the basis of microseism and microbaroms generation theory is the coupling of motions in  
 151 different layers, with a forcing coming from nonlinear ocean wave effects, in which the nonlin-  
 152 earity is necessary to allow the generation of waves with long wavelenghts  $2\pi/|\mathbf{k} + \mathbf{k}'|$  from the  
 153 interference of shorter ocean waves with wavelenghts  $2\pi/|\mathbf{k}|$  and  $2\pi/|\mathbf{k}'|$ . The velocity potential  
 154  $\phi$  in layer  $l$  is solution of a wave equation (Brekhovskikh et al., 1973),

$$155 \quad \frac{\partial^2 \phi}{\partial t^2} - \alpha_l^2 \nabla^2 \phi = -\frac{\partial}{\partial t} (\nabla \phi)^2 - g \frac{\partial \phi}{\partial z}, \quad (1)$$

156 where  $\nabla^2$  is the 3-dimensional Laplace operator. The two terms on the right hand side can be  
 157 neglected in the water layer (Longuet-Higgins, 1950), but are generally significant in the air. The  
 158 first term corresponds to the effect of compressibility. It adds one particular solution  $\phi_p$  that is  
 159 zero away from the boundary but modifies the homogeneous solution via the boundary condition  
 160 at the air-sea interface. The second term is the effect of gravity, which gives a weak additional  
 161  $\exp(gz/2\alpha_l^2)$  vertical decay, with a half-decay distance of 15 km in the atmosphere and 300 km

**Table 1.** Notations used in different papers: LH50 stands for Longuet-Higgins (1950), BGKN73 stands for Brekhovskikh et al. (1973), WG06 stands for Waxler & Gilbert (2006) and AH13 stands for Ardhuin & Herbers (2013).

quantity	this paper	LH50	BGKN73	WG06	AH13
vertical coordinate	$z$	$-z$	$z$	$z$	$z$
angle relative to vertical	$\theta_a$ or $\theta_w$	—	$\theta_1$ or $\theta_2$	—	—
surface elevation	$\zeta$	$\zeta$	$\zeta$	$\xi$	$\zeta$
azimuth of spectrum	$\varphi$	$\theta$	$\varphi$	$\theta$	$\theta$
azimuth of acoustic signal	$\varphi_2$	—	$\varphi_a$	—	—
velocity potential	$\phi$	$-\phi$	$-\varphi$	$\phi$	$\phi$
layer index	$l$	—	$j$	$\sigma$	—
sound speed	$\alpha_l$	$c$	$c_j$	$c_\sigma$	$\alpha$
density ratio	$m$	—	$m$	—	—
horizontal wavenumber	$\mathbf{K}$	—	$\mathbf{q}$	—	$\mathbf{K}$
radian frequency	$\Omega$	—	$\Omega$	—	$2\pi f_s$
horizontal wavenumbers	$\mathbf{k}, \mathbf{k}'$	$(-uk, -vk)$	$\varkappa, \varkappa_1$	$\mathbf{k}, \mathbf{q}$	$\mathbf{k}, \mathbf{k}'$
radian frequencies	$\sigma, \sigma'$	$\sigma$	$\omega(\varkappa), \omega(\varkappa_1)$	$\omega(\mathbf{k}), \omega(\mathbf{q})$	$\sigma, \sigma'$
pressure	$p$	$p$	$\rho\mathcal{P}$	$p$	$p$
vertical wavenumbers	$\nu, \mu$	$-, \alpha$	$\lambda_1, \lambda_2$	—	$l_a, l$
upward amplification	$g/2\alpha_l$	$\gamma$	—	—	—

162 in the ocean. That second effect was considered by Brekhovskikh et al. (1973) but neglected in  
 163 Waxler & Gilbert (2006).

164 Neglecting these two terms for the water layer, solutions that are periodic in time and space  
 165 take the following homogeneous form, with  $\Omega = 2\pi f_s$  the radian frequency and  $\kappa_a = \nu_+$  and  
 166  $\kappa_w = \mu_-$  the vertical wave-numbers (going upwards in the air and downwards in the water)

$$\begin{aligned}
 167 \quad \phi &= \sum_{\mathbf{k}} \Phi_l \exp [i (\mathbf{K} \cdot \mathbf{x} + \kappa_l z - \Omega t)] + \text{c.c.} \\
 168 \quad &= \sum_{\mathbf{k}, s} \Phi_l \exp [i (\mathbf{K} \cdot \mathbf{x} + \kappa_l z - s\Omega t)], \tag{2}
 \end{aligned}$$

169 where c.c. stands for the complex conjugate and  $s = 1$  or  $s = -1$  is a sign index. Neglecting the  
 170 right hand side of (1), one gets

$$171 \quad \kappa_l^2 + K^2 = \Omega^2 / \alpha_l^2. \tag{3}$$

172 With  $N$  the number of unknown potential amplitudes  $(\Phi_i)_{1 \leq i \leq N}$ , there are  $N$  continuity condi-  
 173 tions for stresses and displacements at the layer interfaces, linking the  $N$  amplitudes of velocity  
 174 potentials.

175 All the variables, the pressure  $p$ , the density  $\rho$ , the velocity potential  $\phi$  and the sea surface ele-  
 176 vation  $\zeta$  are expanded in powers of  $\varepsilon = ak$  that is the product of a typical ocean wave wavenumber  
 177  $k$  and surface elevation amplitude  $a$ ,

$$178 \quad p = p_0 + p_1 + p_2 + \dots \quad (4)$$

$$180 \quad \rho = \rho_0 + \rho_1 + \rho_2 + \dots \quad (5)$$

$$182 \quad \phi = \phi_0 + \phi_1 + \phi_2 + \dots \quad (6)$$

$$184 \quad \zeta = \zeta_0 + \zeta_1 + \zeta_2 + \dots \quad (7)$$

185 In addition to the wave slope  $\varepsilon$ , two other small parameters are defined, the ratio between the  
 186 air and water densities  $m = \rho_a/\rho_w$  and  $\delta_l = \sigma/k\alpha_l$  the ratio between the speed of surface waves  
 187 and the speed of sound in the air or water.

188 Collecting the terms of same order, we obtain at each order a system of  $N$  equations for  $N$   
 189 unknowns with a detailed derivation in Supporting Information. At order  $\varepsilon^0$ , the solution is the  
 190 hydrostatic equilibrium of pressure and gravity. The first order solution corresponds to Airy waves,  
 191 which are linear gravity waves, with negligible  $O(\delta_w^2)$  and  $O(m)$  corrections due to the presence of  
 192 air and the compressibility of air and water, as given by Brekhovskikh et al. (1973, eqs. 11 and 12.  
 193 See also Supporting information). Namely the surface elevation is given by Hasselmann (1962),

$$194 \quad \zeta_1(\mathbf{x}, t) = \sum_{\mathbf{k}, s} Z_{1,\mathbf{k}}^s \exp [i(\mathbf{k} \cdot \mathbf{x} - s\sigma t)], \quad (8)$$

195 where  $Z_{1,\mathbf{k}}^s$  is the amplitude of the first order sea surface elevation for wavenumber  $\mathbf{k}$  and propa-  
 196 gation direction  $s$  and  $s = \pm 1$  is a sign index that gives the direction of propagation relative the  
 197 direction of the wave vector  $\mathbf{k}$ . The velocity potential in the water is given by,

$$198 \quad \phi_w(\mathbf{x}, z, t) = \sum_{\mathbf{k}, s} \Phi_{1,\mathbf{k}}^s \frac{\cosh(kz + kh)}{\cosh(kh)} \exp [i(\mathbf{k} \cdot \mathbf{x} - s\sigma t)] \quad (9)$$

with

$$\Phi_{1,\mathbf{k}}^s = \frac{g}{i\sigma} Z_{1,\mathbf{k}}^s. \quad (10)$$

In the air, the effects of gravity and compressibility (i.e. the right hand side terms in eq. 1), are less negligible, and we have,

$$\phi_a(\mathbf{x}, z, t) = \sum_{\mathbf{k},s} \frac{k}{k_a} \Phi_{1,\mathbf{k}}^s \exp[-k_a z + i(\mathbf{k} \cdot \mathbf{x} - s\sigma t)] \quad (11)$$

with eq. 12 in Brekhovskikh et al. (1973),

$$k_a = k \left( \sqrt{1 - \delta_a^2 + \delta_a^4/2} - \delta_a^2/2 \right) \simeq k (1 - \delta_a^2) \quad (12)$$

where  $\delta_a = \sqrt{g/k}/\alpha_a$ . We note that half of this correction to  $k_a$  comes from the  $\delta_a^2/2$  that is due to gravity, and the other half comes from the air compressibility.

Finally, in the  $\varepsilon^2$  system, the wave spectrum acts as a forcing, coming through either the particular solutions that satisfy the wave equation with the right hand side, or from the boundary conditions between the different layers. In other words the wave forcing  $\Lambda = (\Lambda_i)_{1 \leq i \leq N}$  is a vector on the right hand side of a matrix equation

$$\mathbf{M}\Phi = \Lambda. \quad (13)$$

The only differences between all the theories discussed here are in the approximations of the boundary conditions between ocean and atmosphere and ocean and solid Earth. Mathematically, different terms are neglected in the coefficients of the matrix  $\mathbf{M}$  or in the forcing vector  $\Lambda$ , as detailed below.

Further extensions to multiple layers in the atmosphere and solid Earth give rise to different horizontally propagating modes, which correspond to zeros of the determinant of  $\mathbf{M}$ , for which a growth rate of the energy can be computed as done for seismic Rayleigh waves by Hasselmann (1963). The size of the matrix  $\mathbf{M}$  grows by two lines and columns for each extra fluid layer, for which the two unknowns are one upward and one downward propagating potential amplitudes. For a solid layer there are four unknowns due to the presence of both compression and shear motions (see Hasselmann, 1963, eq. 1.4). The important difference with Hasselmann (1963) is that  $\Lambda_i$  was non-zero only for the sea surface pressure continuity equation in Hasselmann's case, whereas in

our case we will consider forcing in both the pressure and velocity equations. We also note that for finite water depth there is also a forcing term in the boundary condition for the ocean bottom pressure coming into  $\Lambda_3$  (Ardhuin & Herbers, 2013).

## 2.1 Existing solutions

### 2.1.1 Case of infinite water depth - Brekhovskikh et al. (1973)

Brekhovskikh et al. (1973) considered only two layers, air and water, that are half spaces. As a result, for each frequency  $\Omega$  and wavenumber vector  $\mathbf{K}$ , there are only two unknowns, one amplitude  $A$  for the velocity potential of upgoing acoustic waves in the atmosphere, and one amplitude  $W$  for downgoing acoustic waves or evanescent modes in the ocean. The approximation  $m = \rho_a/\rho_w \ll 1$  removes the feedback of the atmospheric pressure on the air pressure, so that the atmosphere is only driven by the continuity of vertical velocities at the interface.

The coupling of air and water layers at  $z = 0$  by the continuity of pressure and velocity gives a 2 by 2 matrix  $\mathbf{M}$ , with one line for the continuity of vertical velocity  $w = \partial\phi/\partial z$  and the other for the continuity of pressure  $p$ ,

$$M_{1,1}A + M_{1,2}W = \Lambda_1 \tag{14}$$

$$M_{2,1}A + M_{2,2}W = \Lambda_2 \tag{15}$$

Following Brekhovskikh et al. (1973), we introduce the small parameters

$$\delta_a^2 = g/(k\alpha_a^2) \tag{16}$$

$$m = \rho_a/\rho_w \tag{17}$$

$$n = \alpha_a/\alpha_w \tag{18}$$

and we note that  $|K/k| < 2\delta_a$ . We now keep only the lowest order terms in  $\delta_a$  and  $m$ , giving the

246 following matrix coefficients and right hand side (see Supporting information for details),

$$247 \quad M_{1,1} = \nu_+ = i \frac{\Omega}{\alpha_a} \cos \theta_a + \frac{k}{2} \delta_a^2 \quad (19)$$

$$248 \quad M_{1,2} = -\mu_- = 2k\delta_a l - \frac{\delta_a^2}{2} kn^2 \quad (20)$$

$$249 \quad M_{2,1} = -m\Omega^2 - gi \frac{\Omega}{\alpha_a} \cos \theta_a \quad (21)$$

$$250 \quad M_{2,2} = \Omega^2 \quad (22)$$

$$251 \quad \Lambda_1 = is\Omega k \delta_a^2 \left( 2 - 2 \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) + n^2 \right) \quad (23)$$

$$253 \quad \Lambda_2 = \frac{is\Omega}{\rho_w} p_{\text{surf}}^s(\mathbf{K}, \Omega) + o(\delta_a^2) \quad (24)$$

254 where we have defined

$$255 \quad l = (\sin^2 \theta_a - n^2)^{1/2} = n \cos \theta_w \quad \text{if} \quad \theta_a < \theta_{a0}. \quad (25)$$

256 Following Hasselmann (1963), we define the amplitude of the equivalent surface pressure in-  
257 duced at second order by the wave motion. Assuming that  $kh \gg 1$  we take the following definition,

$$258 \quad p_{\text{surf}}^s(\mathbf{K}, \Omega) = -2\rho_w \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} \sigma \sigma' Z_{1,\mathbf{k}}^s Z_{1,\mathbf{k}'}^s. \quad (26)$$

259 In the following we will write  $p_{\text{surf}}^s$  instead of  $p_{\text{surf}}^s(\mathbf{K}, \Omega)$ .

260 The solution of the matrix equations eq. (14)–(15) is given by Cramer's rule,

$$261 \quad A = \frac{\Lambda_1 M_{2,2} - \Lambda_2 M_{1,2}}{\det(\mathbf{M})}, \quad \text{and} \quad W = \frac{\Lambda_2 M_{1,1} - \Lambda_1 M_{2,1}}{\det(\mathbf{M})}. \quad (27)$$

262 Following details in Supporting material section S4.1, We find the amplitudes of the velocity  
263 potentials at the air-sea interface to be, for the water and air respectively,

$$264 \quad W \simeq \frac{i}{\rho_w} \frac{1}{2\sigma'} p_{\text{surf}}^s \quad (28)$$

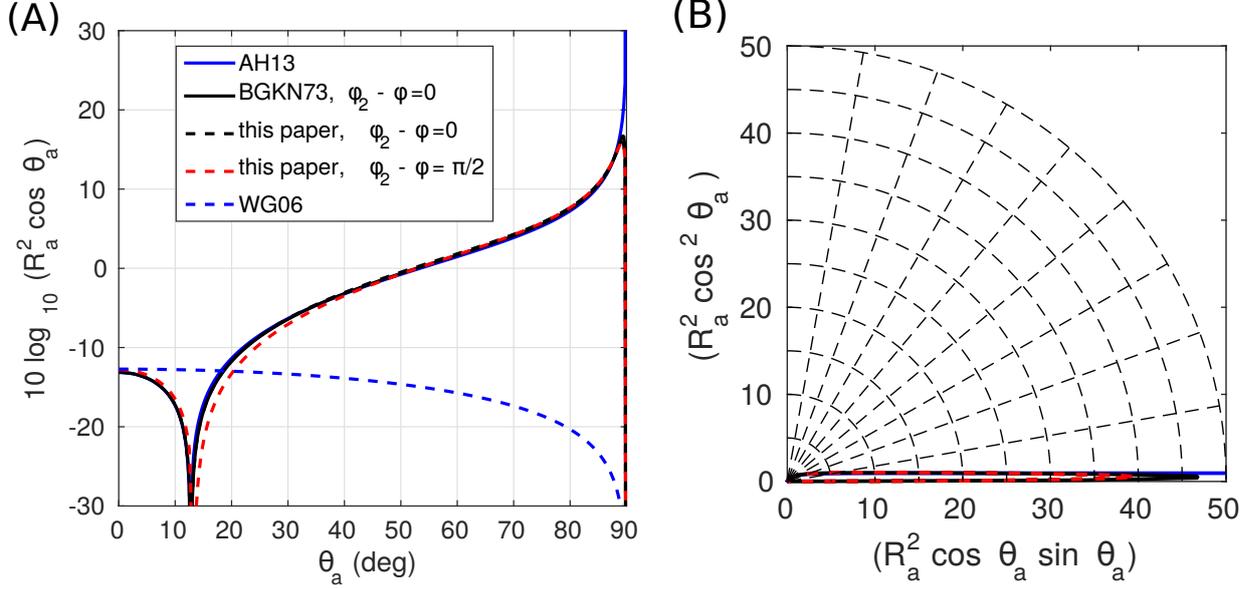
$$265 \quad A \simeq \frac{R_a}{\rho_w} \frac{1}{2\sigma'} p_{\text{surf}}^s \quad (29)$$

266 with

$$267 \quad R_a = \frac{l - 2\delta_a \left[ 1 - \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) + \frac{5}{8} n^2 \right]}{\cos \theta_a (1 - \delta_a l/2) - i(\delta_a/4 + ml)}. \quad (30)$$

268 This form of  $R_a$  is identical to Brekhovskikh et al. (1973, eq. 22), except for the addition  
269 of one extra term  $\cos \theta_a \delta_a/2$  in the denominator, and a change in the sign of the denominator

270 term  $ml$ . As shown in Fig. 3, these two terms have a negligible impact on the solution, except for  
 271  $\theta_a > 89.5^\circ$ , with less than 1% change in the total radiated acoustic power.



**Figure 3.** Patterns of acoustic pressure variance as a function of the elevation angle  $\theta_a$  for an ocean wave period of 10 s, given by the different theories without ocean bottom, in cartesian (A), and polar (B) representation. Note that when the radiated power is considered, these patterns must be multiplied by  $\sin \theta_a$  before integration over  $\theta_a$ , as given by eq. (52). In general, as given by eq. (30) the radiated power is also a function of the relative azimuth of the first ocean wave train  $\varphi$  and the azimuth of the radiated acoustic power  $\varphi_2$ .

### 272 2.1.2 A simplified case - Ardhuin & Herbers (2013)

273 The solution given by Ardhuin et al. (2013) corresponds to the simplified solution given by eq.  
 274 (21) in Brekhovskikh et al. (1973), with  $m$  and  $\delta_a$  terms neglected, corresponding to an absence of  
 275 feedback from the atmospheric pressure on the oceanic pressure, i.e.  $M_{1,1} = 0$ , and neglecting the  
 276 right hand side of the acoustic wave equation in the air, i.e.  $\Lambda_1 = 0$ , giving

$$277 \quad R_a = \frac{l}{\cos \theta_a} = \frac{i(n^2 - \sin^2 \theta_a)^{1/2}}{\cos \theta_a} \quad (31)$$

278 where we recall that  $l$  is imaginary for  $\theta_a < \theta_{a0}$ .

279 This simplified solution corresponds to infinite water depth. It presents a singularity for hori-

280 zontal acoustic propagation as  $\cos \theta_a$  goes to zero. That singularity is removed when the feedback  
 281 of the air on the water motion is taken into account.

### 282 2.1.3 Theory by Waxler and Gilbert (2006)

283 Following Brekhovskikh et al. (1973), Waxler & Gilbert (2006) showed that microbarom signals  
 284 are due both to ocean radiation and to compression of the air by the surface motion, but Waxler &  
 285 Gilbert (2006) neglected the effect of gravity in the air. As detailed in the Supporting information,  
 286 accounting for gravity in the air changes their term  $3\delta_a^2/2$  in their eq. (57) to  $2\delta_a$  in eq. (32). we  
 287 also note a change of sign from -2 to 1.5,

$$288 R_a^{\text{WG06}} = in + 1.5\delta_a. \quad (32)$$

289 The particularity of the derivation by Waxler & Gilbert (2006) is the fact that they neglect  
 290 the phase shift in the Green's function within the source region. They justified that approxima-  
 291 tion by assuming that the coherence length scale in the acoustic source is small compared to the  
 292 acoustic wavelength. Here we do not use such an approximation, as detailed in the Appendix A,  
 293 as the correlation function of the source is given by the pressure spectrum  $p_{\text{surf}}^s(\mathbf{K}, \Omega)$  (our eq. 26)  
 294 that overlaps with the acoustic wavelengths. The assumption in Waxler & Gilbert (2006) comes  
 295 between their equations (50) and (51) and simplifies the expression of the radiation pattern in  
 296 the atmosphere to a monopolar radiation pattern. It also reduces all expressions to their values  
 297 for  $\mathbf{K} = 0$ , corresponding to strictly opposing wave trains, so that the evanescent ocean motions  
 298 that correspond to  $\theta_a > \theta_{a0}$  are not properly represented. Without this assumption, (Brekhovskikh  
 299 et al., 1973) found that the radiation pattern is very different from a monopole, with an overwhelm-  
 300 ing radiation at very grazing angles, as illustrated in Fig. 1.A.

## 301 2.2 Generalization of Brekhovskikh et al. (1973) to a finite ocean depth

302 The first discussion of water depth effects on microbarom sources is due to Waxler et al. (2007).  
 303 That work extended the analysis by Waxler & Gilbert (2006), limiting the water depth to  $h$  and

304 including an interface with a solid half space below. The velocity potential in the water is now

$$305 \quad \phi_{w,2} = \sum (W_+ e^{\mu_+ z} + W_- e^{\mu_- z}) e^{i\Theta}. \quad (33)$$

306 with the complex wavenumber  $\mu_{\pm} = g/(2\alpha_w) \pm \sqrt{K^2 - \Omega^2/\alpha_w^2} \simeq \pm\mu = \pm\sqrt{K^2 - \Omega^2/\alpha_w^2}$ .

307 Taking an acoustic reflection coefficient  $r = \rho_w \alpha_w / (\rho_s \alpha_s)$ , Waxler et al. (2007) found the  
308 reflection condition for the down-going acoustic waves, with potential  $\phi_w^-$  at the ocean bottom

$$309 \quad W^+ = \frac{1+r}{1-r} e^{2\mu h} W^-. \quad (34)$$

310 This gives,

$$311 \quad R_a^{\text{W07}} \simeq i n \frac{r \cos(\Omega h / \alpha_w) + i \sin(\Omega h / \alpha_w)}{\cos(\Omega h / \alpha_w) + i r \sin(\Omega h / \alpha_w)} + 1.5 \delta_a. \quad (35)$$

312 In order to properly consider the effect of the propagation angles, we can go back to the deriva-  
313 tion of Brekhovskikh et al. (1973), now including an upgoing acoustic wave in the water layer. For  
314 oblique incidence we have to consider the contribution of compression waves and shear waves in  
315 the crust with velocities  $\alpha_s$  and  $\beta$ . Defining the vertical wavenumbers in the the crust

$$316 \quad \chi_p = \sqrt{K^2 - \frac{\Omega^2}{\alpha_s^2}}, \quad \text{and} \quad \chi_s = \sqrt{K^2 - \frac{\Omega^2}{\beta^2}}. \quad (36)$$

317 The reflection at the bottom generalizes to

$$318 \quad r \simeq \frac{\Omega^4 \rho_w \chi_p}{\rho_s \mu [(\Omega^2 - 2K^2 \beta^2)^2 - 4\beta^4 K^2 \chi_p \chi_s]}. \quad (37)$$

319 This is obtained by eliminating the potentials of the compression and shear motions in the crust,  
320 using the continuity of velocity and a zero tangential stress (see also Ardhuin & Herbers, 2013;  
321 Gualtieri et al., 2014).

322 When only the dominant terms are kept we find,

$$323 \quad R_a = \frac{l}{\cos \theta_a} \frac{R}{Q} \quad (38)$$

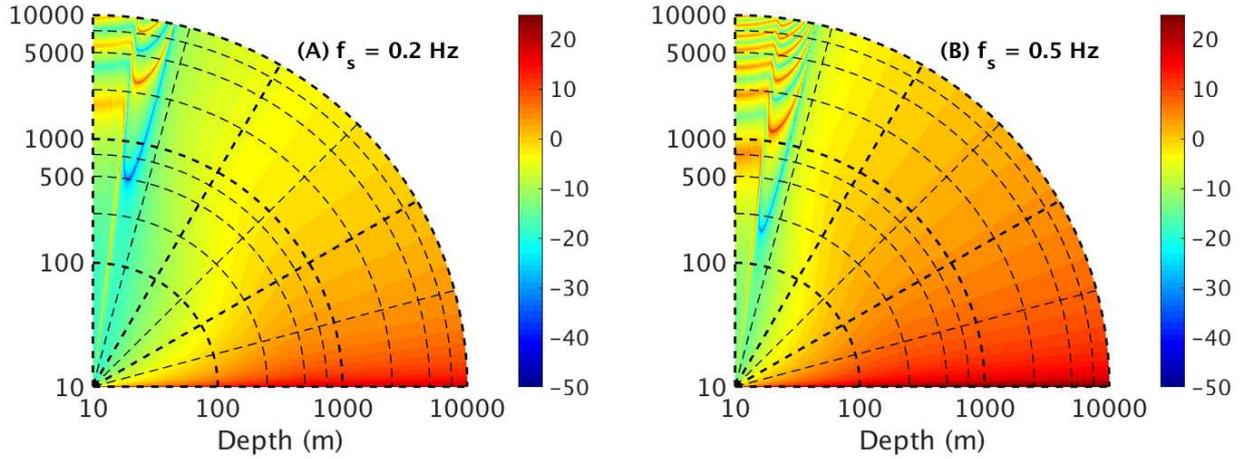
324 with

$$325 \quad R = \sinh(\mu h) + r \cosh(\mu h), \quad (39)$$

$$326 \quad Q = \cosh(\mu h) + r \sinh(\mu h). \quad (40)$$

327 In the case  $\theta_a = 0$ , eq. (38) corresponds to the first term of eq. (35).

328 When going to first order in  $\delta_a$  and  $m$ , the problem can be simplified by neglecting the effect  
 329 of gravity in the water layer, which contributed to the  $n^2$  term in the numerator of eq. (30). This  
 330 gives  $-\mu_- = \mu_+ = \mu$ . It is then more simple to eliminate the amplitude  $W^-$  by using the bottom  
 331 boundary condition on the vertical velocity. This amounts to replacing  $-\mu_-$  by  $-2\mu R e^{-\mu h}/(1+r)$   
 332 in  $M_{1,2}$  and  $\Omega^2$  by  $2Q\Omega^2 e^{-\mu h}/(1+r)$  in  $M_{2,2}$ , giving,



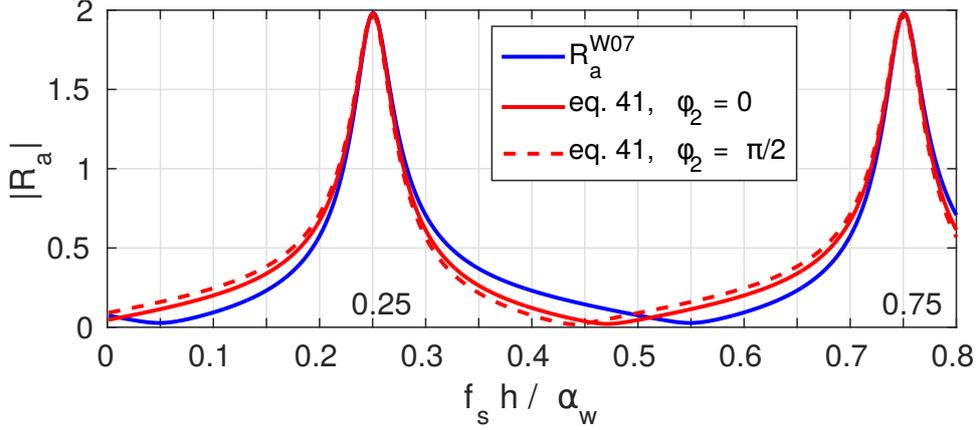
**Figure 4.** Radiation patterns of  $10\log_{10}|R_a|$  according to eq. (41) for (A)  $f_s = 0.2$  Hz , (B)  $f_s = 0.5$  Hz with  $f_s = \Omega/2\pi$  - polar representation against the angle  $\theta_a$  and the depth  $h$ .

333

$$R_a = \frac{lR - 2\delta_a Q [1 - \sin^2 \theta_a (1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi))]}{\cos \theta_a (Q + R\delta_a l/2) - i(Q\delta_a/4 + Rml)} \quad (41)$$

**Table 2.** Summary of differences between models, with  $R_a$  defined on eq. (29)

Model	depends on $\theta_a$	depth	compressible	gravity	$R_a$
BGKN73	✓	$\infty$	✓	✓	eq. (30)
AH13	✓	$\infty$	×	×	eq. (31)
WG06	×	$\infty$	✓	×	eq. (32)
W07	×	any	✓	×	eq. (35)
this paper	✓	any	✓	✓	eq. (41)



**Figure 5.** Magnitude of the velocity potential amplification from the water to the air,  $R_a$ , as a function of the ratio of the water depth and acoustic wavelength, in the case of vertical propagation, i. e.  $\theta_a = 0$ . For  $f_s = 0.2$  Hz, the two peaks corresponds to depths of 1900 and 5600 m. Here we have used  $\alpha_s = 5540$  m/s,  $\beta = \alpha_s/\sqrt{3}$ ,  $\rho_s = 2500\text{kg/m}^3$ .

### 335 2.3 Radiated acoustic power as a function of elevation and azimuth

336 We introduce the spectral density of the homogeneous (propagating) pressure field at  $z = 0$ , in  
 337 the three spectral dimensions  $(K_x, K_y, f_s)$  using the Fourier amplitude of  $p_{a,2}$  at  $z = 0$ , obtained  
 338 from the average over realizations of the sea state, represented by angular brackets, of the pressure  
 339 amplitude squared,

$$340 \quad F_{p,2h}(\mathbf{K}, f_s) = 2 \lim_{\Delta K_x \rightarrow 0, \Delta K_y \rightarrow 0, \Delta f_s \rightarrow 0} \frac{\langle |P_{2,h}^+|^2(\mathbf{K}, f_s) \rangle}{\Delta K_x \Delta K_y \Delta f_s} \quad (42)$$

341 Given the dispersion relation of ocean surface gravity waves in deep water, the Jacobian of the  
 342 transformation from  $(k_x, k_y)$  to  $(f, \varphi)$  is  $2\pi k / (\partial\sigma/\partial k) = 4\pi\sigma^3/g^2$ .

343 We now define the ocean wave spectrum as

$$344 \quad E(f, \varphi) = \frac{4\pi\sigma^3}{g^2} E(k_x, k_y) = 2 \lim_{\Delta k_x \rightarrow 0, \Delta k_y \rightarrow 0} \frac{|Z_{\mathbf{k}}^+|^2}{\Delta K_x \Delta K_y} \frac{4\pi\sigma^3}{g^2}$$

$$345 \quad = 2 \lim_{\Delta f \rightarrow 0, \Delta\varphi \rightarrow 0} \frac{|Z_{\mathbf{k}}^+|^2}{\Delta f \Delta\varphi}. \quad (43)$$

346 We use eq. (26) and replace the amplitude  $P_{2,h}^+$  by  $i(\sigma + \sigma')\rho_a A$ , namely,

$$347 \quad P_{2,h}^+ = \frac{i\rho_a(\sigma + \sigma')}{\rho_w 2\sigma'} R_a p_{\text{surf}}^+ \quad (44)$$

348 This gives,

$$349 \quad 2 |P_{2,h}^+|^2 = 2\rho_a^2 |R_a|^2 \sigma^2 (\sigma + \sigma')^2 \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} Z_{1,\mathbf{k}} Z_{1,\mathbf{k}'} \right|^2 \quad (45)$$

$$350 \quad = \sigma^2 (\sigma + \sigma')^2 \rho_a^2 |R_a|^2 \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} 2 |Z_{1,\mathbf{k}}^+|^2 2 |Z_{1,\mathbf{k}'}^+|^2, \quad (46)$$

351 where the last equality is obtained by considering that each pair of wavenumbers  $(\mathbf{k}_1, \mathbf{k}_2)$  is  
352 counted twice, a first time when  $\mathbf{k} = \mathbf{k}_1$  and  $\mathbf{k}' = \mathbf{k}_2$  and a second time when  $\mathbf{k} = \mathbf{k}_2$  and  $\mathbf{k}' = \mathbf{k}_1$ .

353 This is well understood when considering the simplest form with the ocean wave field consisting  
354 of only two cosine waves (See Supporting information, eqs. S124–S128).

355 Taking the limit to continuous sums and using a change of variable from  $(k_x, k_y, k'_x, k'_y)$  to  
356  $(f_s, \varphi, K_x, K_y)$ , with  $K_x = k_x + k'_x$ ,  $K_y = k_y + k'_y$  and  $f_s = (\sqrt{gk} + \sqrt{gk'})/(2\pi)$  the Jacobian of  
357 the coordinate transform is

$$358 \quad \det \left( \frac{\partial f_s \partial \varphi \partial K_x \partial K_y}{\partial k_x \partial k_y \partial k'_x \partial k'_y} \right) = \begin{vmatrix} g \cos \varphi / (4\pi \sigma) & -\sin \varphi / k & 1 & 0 \\ g \sin \varphi / (4\pi \sigma) & \cos \varphi / k & 0 & 1 \\ g \cos \varphi' / (4\pi \sigma') & 0 & 1 & 0 \\ g \sin \varphi' / (4\pi \sigma') & 0 & 0 & 1 \end{vmatrix}$$

$$359 \quad = \frac{g^2}{4\pi \sigma^3 \sigma'} [\sigma' - \sigma \cos(\varphi - \varphi')], \quad (47)$$

360 and gives

$$361 \quad \int F_{p,2h}(\mathbf{K}, f_s) dK_x dK_y df_s = \rho_a^2 \int \sigma^2 (\sigma + \sigma')^2 |R_a|^2 E(k_x, k_y) E(k_x, k_y) dk_x dk_y dk'_x dk'_y$$

$$362 \quad = \rho_a^2 \int \sigma^2 (\sigma + \sigma')^2 |R_a|^2 \frac{E(k_x, k_y) E(k'_x, k'_y) 4\pi \sigma^3 \sigma'}{g^2 [\sigma' - \sigma \cos(\varphi - \varphi')]} df_s d\varphi dK_x dK_y.$$

363 Now we use the unicity of the Fourier transform to identify the spectral density in the left and right  
364 hand sides, and using eq. (43) gives

$$365 \quad F_{p,2h}(\mathbf{K}, f_s) = \frac{1}{2} \rho_a^2 g^2 f_s \int_0^{2\pi} \frac{\sigma^2 (\sigma + \sigma')}{\sigma'^2} |R_a|^2 \frac{E(f, \varphi) E(f', \varphi')}{\sigma' - \sigma \cos(\varphi - \varphi')} d\varphi. \quad (48)$$

366 We note that the form of the acoustic power given by eq. (48) is generally a function of the direction  
367  $\varphi_2$  of the horizontal wave vector  $\mathbf{K}$  of the acoustic waves.

368 In the limit  $\delta_a \ll 1$ , this simplifies to a horizontally isotropic form

$$369 \quad F_{p,2h}(\mathbf{K}, f_s) \simeq \frac{1}{2} \rho_a^2 g^2 f_s |R_a|^2 \int_0^{2\pi} E(f, \varphi) E(f, \varphi + \pi) d\varphi. \quad (49)$$

370 This last expression, with  $|R_a|^2 \simeq |(\sin^2 \theta_a - n^2)| / \cos^2 \theta_a$ , is the one used by Ardhuin & Herbers  
 371 (2013).

372 The pressure spectrum can be re-written as a directional spectrum, with the proper change of  
 373 coordinate this gives,

$$374 F_{p,2h}(\theta_a, \varphi_2, f_s) = \frac{4\pi^2 f_s^2 \cos \theta_a \sin \theta_a}{\alpha_a^2} F_{p,2h}(\mathbf{K}, f_s). \quad (50)$$

375 When  $\delta_a$  terms are kept with eq. (48), the acoustic power radiated by the ocean surface in  
 376 direction  $\varphi_2$  can be integrated in any range of incidence angles  $\theta_{a,1}$  to  $\theta_{a,2}$ ,

$$377 P(\theta_{a,1}, \theta_{a,2}, f_s, \varphi_2) = \frac{2\pi^2 \rho_a g^2}{\alpha_a^3} f_s^3 \int_{\theta_{a,1}}^{\theta_{a,2}} \sin \theta_a \cos \theta_a \\ 378 \times \int_0^{2\pi} \frac{\sigma^2 (\sigma + \sigma') |R_a|^2 E(f, \varphi) E(f, \varphi')}{\sigma'^2 [\sigma' - \sigma \cos(\varphi - \varphi')]} d\varphi d\theta_a, \quad (51)$$

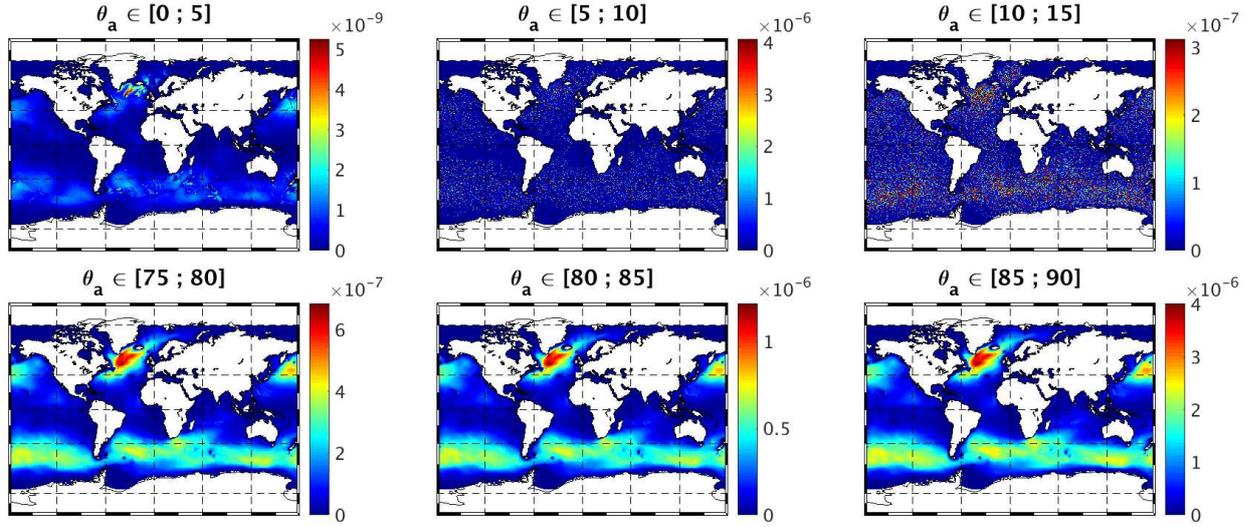
379 Taking the isotropic form (49) the radiated acoustic power becomes isotropic and the sum over  
 380 all directions is  $2\pi$  times eq. (51), giving

$$381 2\pi P(\theta_{a,1}, \theta_{a,2}, f_s, \varphi_2) = \frac{4\pi^3 \rho_a g^2}{\alpha_a^3} f_s^3 H(f_s/2) \int_{\theta_{a,1}}^{\theta_{a,2}} \sin \theta_a \cos \theta_a |R_a|^2 d\theta_a, \quad (52)$$

382 with units of  $\text{W/m}^2/\text{Hz}$ , where the so-called ‘Hasselmann integral’ can be defined from the ‘overlap  
 383 integral’ (Farrell & Munk, 2008) and the wave spectrum in frequency,

$$384 H(f) = [E(f)]^2 I(f) = \int_0^{2\pi} E(f, \varphi) E(f, \varphi + \pi) d\varphi. \quad (53)$$

385 The total integrated radiated power, with units of  $\text{W/m}^2$ , is obtained by integrating eq. (52)  
 386 across acoustic frequencies  $f_s$ . In Fig. 6, the mean acoustic intensity over the year 2018 is rep-  
 387 resented for six ranges of vertical incidence angles. The distribution pattern of sources for the  
 388 vertical angles  $[0^\circ, 5^\circ]$  shows the effect of bathymetry, unlike the near-horizontal angles with simi-  
 389 lar patterns depending almost solely on the Hasselman integral. For near-vertical angles -  $[5^\circ, 10^\circ]$   
 390 and  $[10^\circ, 15^\circ]$  - the distribution pattern is not continuous, there are resonant points all over the  
 391 globe. The acoustic intensity is higher for near-horizontal angles, as predicted in Fig. 4.



**Figure 6.** Acoustic intensity in  $\text{W/m}^2$  - computed by integrating eq. 52 over  $f_s$  - for 6 ranges of incidence angles : A)  $[\theta_{a,1}, \theta_{a,2}] = [0^\circ, 5^\circ]$ , B)  $[\theta_{a,1}, \theta_{a,2}] = [5^\circ, 10^\circ]$ , C)  $[\theta_{a,1}, \theta_{a,2}] = [10^\circ, 15^\circ]$ , D)  $[\theta_{a,1}, \theta_{a,2}] = [75^\circ, 80^\circ]$ , E)  $[\theta_{a,1}, \theta_{a,2}] = [80^\circ, 85^\circ]$ , F)  $[\theta_{a,1}, \theta_{a,2}] = [85^\circ, 90^\circ]$ . Mean of a 3-hourly model over 2018. These were computed from numerical ocean wave model output already described by Rasche & Ardhuin (2013) and available at (<ftp://ftp.ifremer.fr/ifremer/ww3/>), using the WAVEWATCH III<sup>®</sup> code (The WAVEWATCH III<sup>®</sup> Development Group, 2016)

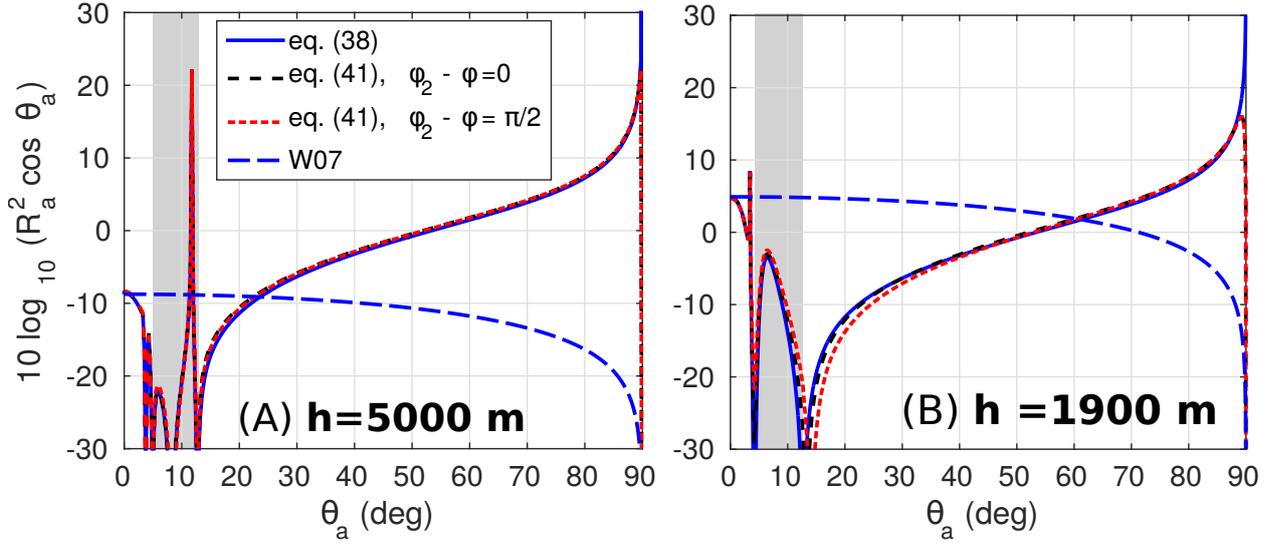
### 392 3 PRACTICAL IMPLICATIONS AND DISCUSSION

#### 393 3.1 Near-vertical propagation and Rayleigh wave overestimation

394 We note that for vertical propagation ( $\theta_a = 0$ ) the effect of the finite depth changes the amplitude  
 395 by a factor that ranges from 0.125 to 8.5 with sharp maxima corresponding to the organ pipe  
 396 resonance at  $h = (0.25 + n/2)\alpha_w/f_s$ , as illustrated in Fig. 5, which is similar to fig. 11 in Ardhuin  
 397 & Herbers (2013).

398 The behaviour at other angles is very interesting as  $\theta_a$  goes through the different regimes of  
 399 associated seismic waves, from  $P$ -waves for  $\sin \theta_a < \alpha_a/\alpha_s$  - i.e.  $\theta_a < 3.41^\circ$  for our choice  
 400 of parameters - to Rayleigh waves, which usually contain most of the microseism signal, with  
 401  $\arcsin \alpha_a/\beta < \theta_a < \arcsin \alpha_a/\alpha_w$  corresponding to a range of  $5.9$  to  $12.7^\circ$  which is shaded in Fig.  
 402 7.

403 The two maxima that appear in the Rayleigh domain in Fig. 7.A are the two modes that can  
 404 exist at 5000 m depth, whereas only one mode can exist at 1900 m depth. As discussed by Ard-

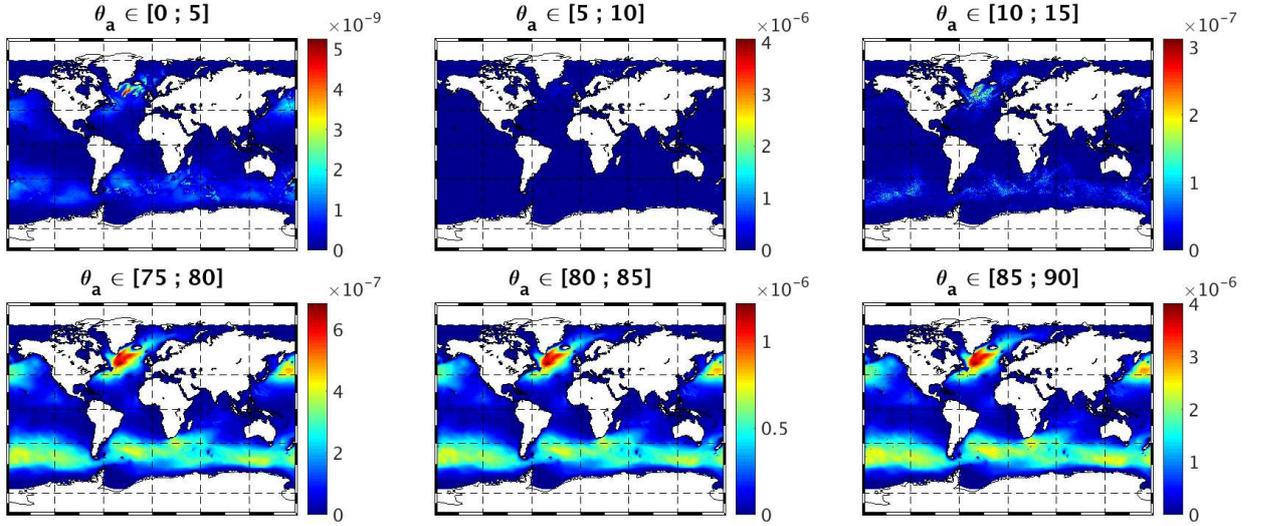


**Figure 7.** Radiation patterns for an ocean wave period of 10 s, given by the different theories with an ocean bottom at (A)  $h = 5000$  m, (B)  $h = 1900$  m. Note that when the radiated power is considered, these patterns must be multiplied by  $\sin \theta_a$  before integration over  $\theta_a$ , as given by eq. 52. The shaded region corresponds to the domain of seismic Rayleigh waves with  $\arcsin(\alpha_a/\beta) < \theta_a < \arcsin(\alpha_a/\alpha_w)$ .

405 huin & Herbers (2013) the depth and frequency at which the amplification is maximum is shifted  
 406 compared to Fig. 5. This is because the acoustic water component of these Rayleigh modes do  
 407 not propagate vertically but obliquely at an angle  $\theta_w$ , as shown in Fig. 1.B, giving a resonance at  
 408  $h = (0.25 + n/2)\alpha_w/(f_s \cos \theta_w)$ .

409 Our model certainly overestimates the amplitude of these Rayleigh waves and associated mi-  
 410 crobaroms because we looked for solutions that are homogeneous in space and time. In the solution  
 411 given above, the leak of energy to the atmosphere is the only loss of energy of the Rayleigh waves  
 412 and it compensates the source of energy from the local waves. In reality, two important effects  
 413 limit the microbarom amplitude to a much lower level. First, the ocean storm area of microbarom  
 414 generation may not be large enough to reach the stationary solution, and second there is a much  
 415 larger (6 to 1000 times or more) sink of seismic energy, due to the presence of fluid in the crust,  
 416 varying with the age of the crust (Stutzmann et al., 2012). The dissipation rate of the energy  $E$   
 417 is generally parameterized as proportional to  $dE/dt = -\Omega E/Q_R$  where  $Q_R$  is of the order of  
 418 200 to 1000. Including this effect in the present paper amounts to replacing  $\Omega$  by  $\Omega(1 - i/(2Q_R))$   
 419 which is done in Fig. 8. In other words, the Rayleigh wave energy is overestimated when dissi-

420 pation in the crust is neglected because in that case, the leakage of Rayleigh wave energy to the  
 421 atmosphere is the loss of seismic energy. Hence, our calculation has a seismic attenuation with a  
 422 very large quality factor  $Q_{\max} = \Omega E/S$ , where  $S$  is the source of seismic energy that equals the  
 423 radiated power given by eq. (52):  $P(\arcsin(\alpha_a/\beta), \arcsin(\alpha_a/\alpha_w), f_s)$ . Taking twice the kinetic  
 424 energy in the water column as a lower bound for the total energy we find that, for  $h = 5000$  m  
 425 and  $f_s = 0.2$  Hz,  $Q_{\max} > 10^6$  (Supporting information, section S6.3), meaning that the present  
 426 solution overestimates the real microbarom amplitude by a factor that exceeds  $Q_{\max}/Q_R > 1000$ .



**Figure 8.** Same as Fig. 6 replacing  $\Omega$  by  $\Omega(1 - i/(2Q_R))$  with  $Q_R = 200$  in order not to overestimate Rayleigh wave energy.

427 Alternatively, instead of looking for the homogeneous solution to the atmosphere-ocean-crust  
 428 problem, we can use solutions for the ocean-crust problem with an energy that grows over the  
 429 source region (Hasselmann, 1963; Arduin & Herbers, 2013), and compute the microbaroms ra-  
 430 diated by microseisms (free Rayleigh waves). These microbaroms are radiated both in the source  
 431 region of microseism but also all along the propagation path of the Rayleigh waves, even on land.  
 432 For example, a huge microseism with amplitude  $a = 10$  micrometers vertical displacement of  
 433 the sea or land surface corresponds to an energy flux  $\rho_a \alpha_a \Omega^2 a^2 / 2$  of only  $3 \times 10^{-8} \text{ W/m}^2$  for  
 434  $f_s = 0.2$  Hz. This is 2000 times smaller than the peak power measured by Bowman & Lees  
 435 (2018). It is thus unlikely that these measurements are dominated by near-vertical propagating  
 436 sound waves. Indeed, the near-horizontal energy level is usually much stronger.

### 3.2 Radiation patterns as a function of azimuth

The variation of the radiated acoustic power with azimuth  $\varphi_2$  has not been described before as most studies focused on near-vertical radiation ( $K = 0$ ). In eq. (51) there are two reasons why the radiated power varies with  $\varphi_2$ .

As expressed by the general form given in eq. (51), the radiated power varies with azimuth  $\varphi_2$  due to the general form of  $R_a$ , as shown in figures 3 and 5 but that variation is limited to a few percent. More important is the fact that the Hasselmann integral  $H(f)$  is modified by the interaction of  $E(f, \varphi)$  with  $E(f', \varphi')$  instead of  $E(f, \varphi + \pi)$ , and should be replaced by,

$$H(f, \varphi_2) = \int_0^{2\pi} \frac{\sigma^2(\sigma + \sigma')E(f, \varphi)E(f, \varphi')}{\sigma'^2 [\sigma' - \sigma \cos(\varphi - \varphi')]} d\varphi. \quad (54)$$

Following the classification in (Ardhuin et al., 2011), the oceanic conditions in which the Hasselmann integral takes the largest values correspond to ‘class III’ events, with two narrow swells propagating in opposite directions. These are also conditions in which  $E(f', \varphi')$  may be most different from  $E(f, \varphi + \pi)$ .

#### 3.2.1 Case of analytical ocean wave spectra

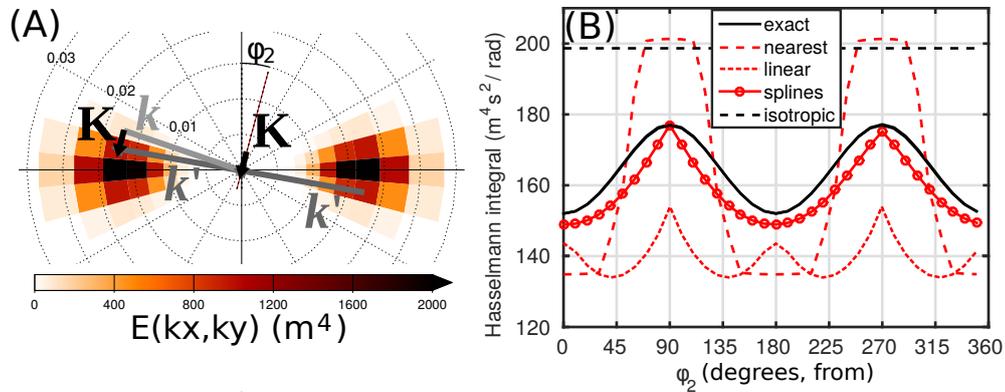
To illustrate this effect, we take an example of a family of wave spectra adapted from the case discussed in (Obrebski et al., 2012). This family of spectra is defined by the analytic expression,

$$E(f, \varphi) = E_0 \left[ e^{-\frac{(f-f_1)^2}{2f_2^2} - \frac{(\varphi-\varphi_1)^2}{2\varphi_3^2}} + e^{-\frac{(f-f_3)^2}{2f_4^2} - \frac{(\varphi-\varphi_4)^2}{2\varphi_5^2}} \right], \quad (55)$$

where the  $f_1, f_2, f_3, f_4$  parameters define the peak frequency and width for the two swell trains, and  $\varphi_1, \varphi_3, \varphi_4, \varphi_5$  define the mean direction and width. The present theory is not restricted to this family of spectra and generally applies to any wave spectrum. Such an analytical form is particularly useful for testing the influence of the discretization when the spectrum is given by a numerical model.

Fig. 9.A shows this spectrum transformed to wavenumber space  $E(k_x, k_y) = g^2 E(f, \varphi) / 4\pi\sigma^3$ , with the mean frequency and direction set to  $f_1 = 0.066$  Hz,  $f_3 = 0.066$  Hz,  $\varphi_1 = 90^\circ$ ,  $\varphi_3 = 270^\circ$ ,

462 the widths  $f_2 = 0.007$  Hz,  $f_2 = 0.007$  Hz,  $\varphi_4 = 8^\circ$ ,  $\varphi_5 = 8^\circ$ , and the normalization factor  
 463  $E_0 = 20\text{m}^2/\text{Hz}$ , giving a significant wave height of 2 m.

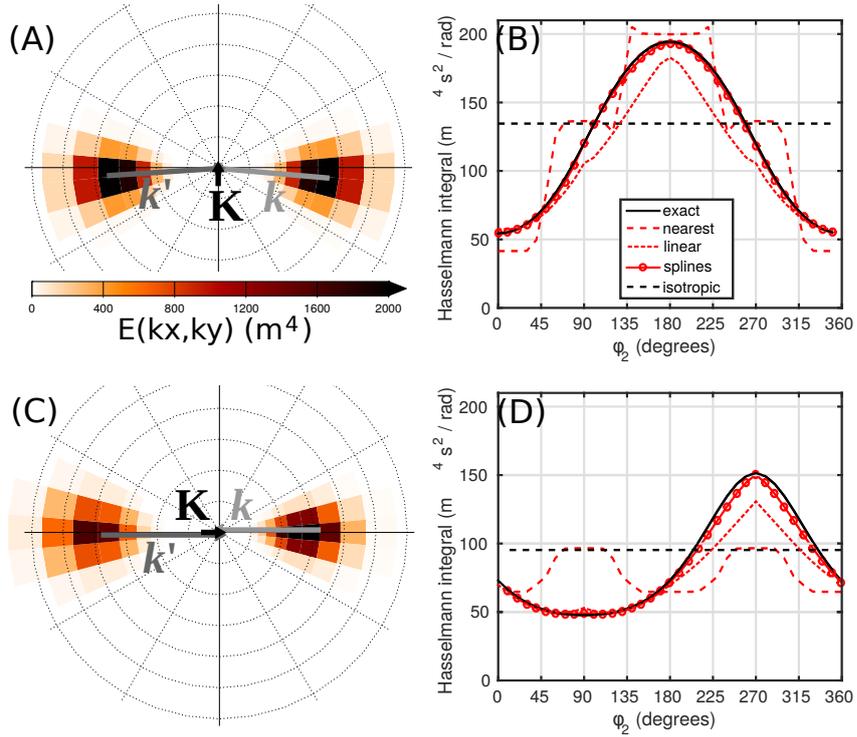


**Figure 9.** (A) Example spectrum discretized with 36 directions and frequency exponentially spaced with an 1.1 increment from one frequency to the next, typically used in numerical wave models. The spectral density of the ocean waves is shown in colors, in directions from where the waves come. (B) Resulting integrals for the acoustic frequency  $f_s = 0.13$  Hz, and horizontal propagation ( $\theta_a = 90^\circ$ ,  $K = 0.0025$  rad/m). Three methods were used to compute the integral: the analytic spectral expression or interpolation of the discrete spectrum using nearest neighbor or linear interpolation.

464 The azimuthal variation of the generalized Hasselmann integral, as given by eq. (54) is illus-  
 465 trated in Fig. 9.B.

466 The exact calculation uses the analytic expression of the spectrum, and exhibits variations of  
 467 7% of the radiated acoustic power as a function of  $\varphi_2$ , with a maximum in the direction of the  
 468 waves because the spectrum, in  $k$ -space is more narrow in the  $k_y$  than in the  $k_x$  direction and when  
 469  $K$  is aligned with the  $y$ -axis, as shown, the wavenumber vector  $\mathbf{k}'$  falls away from the peak faster  
 470 than when  $K$  is aligned with the  $x$ -axis. For the present example the azimuthal variation goes away  
 471 when the directional spread is increased from 8 to  $12^\circ$  and it has a maximum at 0 and  $180^\circ$  for  
 472 wider directional spectra.

473 The practical estimation of the integral is very sensitive to the discretization used, which is not  
 474 an issue when  $K$  is much smaller than the discretization of the spectrum and the isotropic form  
 475 can be used. Numerical wave models that typically use 10% increments from one frequency to  
 476 the next and 24 or 36 directions do not resolve very well the narrow swell peaks such as those in



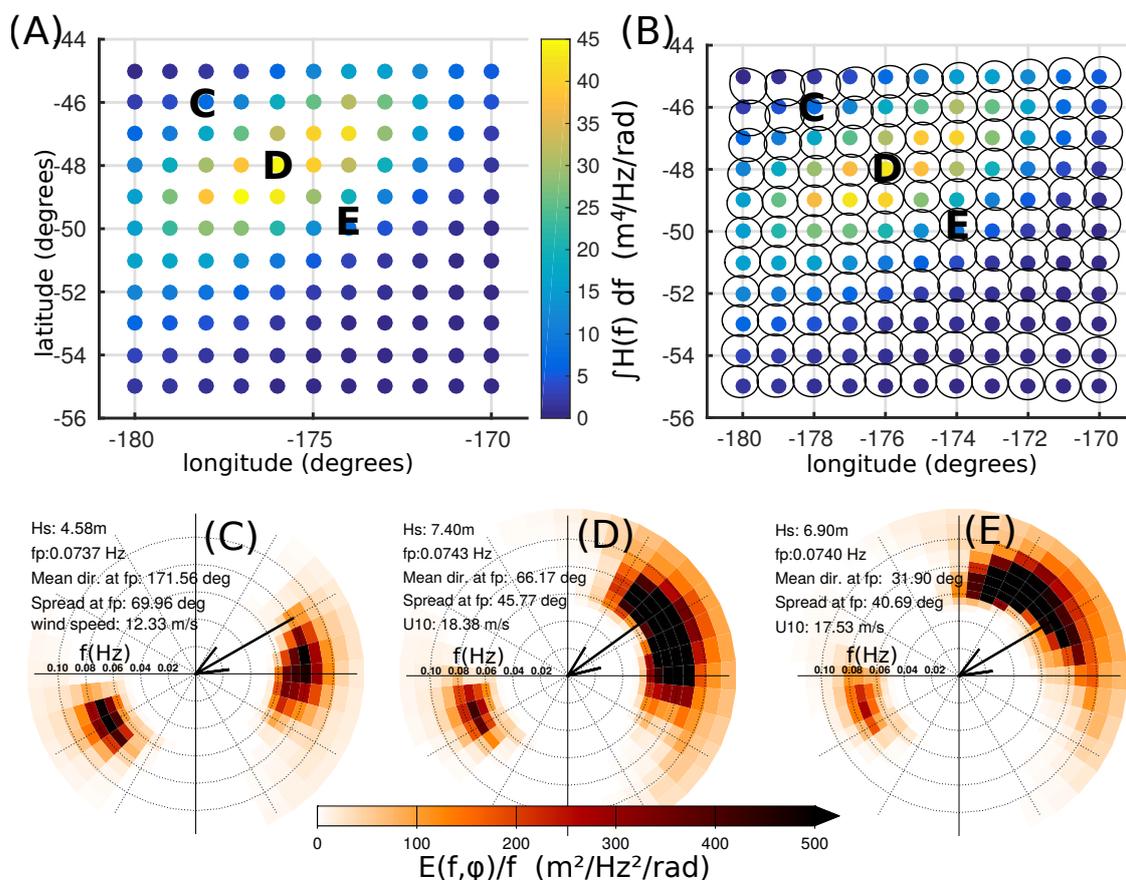
**Figure 10.** Same as Fig. 9 with different ocean wave spectra (A) that have mean directions shifted by 5 degrees to  $\varphi_1 = 95^\circ$  and  $\varphi_3 = 265^\circ$  producing the Hasselmann integral in (B). The wave spectrum in produces the Hasselmann integral in (D), with a peak frequencies shifted by 0.006 Hz to  $f_1 = 0.072$  Hz and  $f_2 = 0.060$  Hz. For both cases, the configuration of  $\mathbf{k}$  and  $\mathbf{k}'$  that gives the largest contribution to the Hasselmann integral in indicated in (A) and (C) with arrows.

477 Fig. 9.A. As a result, a linear interpolation underestimates the integral because the peak appears  
 478 narrower. On average this is corrected by using the nearest point, but that approach can exaggerate  
 479 the anisotropy of the acoustic source.

480 Besides causing anisotropic sources when two swell peaks are exactly opposed, the generalized  
 481 integral may broaden the region where sources are significant, as shown in Fig. 10, because it  
 482 allows a wider range of directions and frequencies to interact compared to  $\mathbf{k} = \mathbf{k}'$  in the simplified  
 483 form.

### 484 3.2.2 General case using numerical wave model output

485 In order to test this idea, we have computed the Hasselmann integral from wave-model output  
 486 which are 'real' spectrum (non idealised spectrum) with the usual form and its generalization in  
 487 the case of the event discussed by Bowman & Lees (2018). Fig. 11 shows modeled maps of wave



**Figure 11.** Maps of the Hasselmann Integral for (A) the isotropic and (B) non-isotropic expressions. In (B) the black curve around each location represents the Hasselmann integral estimated as a function of the acoustic propagation azimuth, and plotted in direction from where the acoustic waves are coming and normalized so that the average radius is half the distance between neighboring locations. (C), (D) and (E) are showing modeled directional spectra at the C, D, and E locations mentioned in (A) and (B). The large arrows indicate the wind direction, from the north-east, associated to a broad windsea spectrum which opposes a more narrow swell spectrum from the south-west.

488 source magnitude, in color, using the isotropic or azimuth-dependent form of the Hasselmann  
 489 integral over a 10 degree by 10 degree region located to the south-east of New Zealand, valid for  
 490 May 2016, at 06:00 UTC. The wave model used here is very similar to the one used in Ardhuin  
 491 et al. (2015), with a number of discrete directions increased from 24 to 36. This strong microbarom  
 492 source is associated to a strong local wind, up to 18 m/s, blowing against swell coming from a  
 493 remote storm, typical of a class-III event described by Ardhuin et al. (2011).

494 The first striking result is that the colors are very similar, with a correlation of 0.9998, meaning  
 495 that the simplified isotropic form is a good approximation of the total radiated power, at least for

496 this example. We also note that the highest sources are most isotropic, such as at location D in  
497 Fig. 11.B. Where sources are more strongly radiated in one direction, such as at locations C and  
498 E, this is due to a gradual shift in the direction of both interacting wave trains, the swell from the  
499 south-west has a local direction that is close to  $240^\circ$  at C, and  $255^\circ$  at E, and the time evolution of  
500 the local wind means that the wind sea is rather from the East at C and the North-East at E. In this  
501 particular example this gives a dominant radiation from the south-east at C and the north at E.

502 When averaged over the entire area, the difference between the radiated power in any given  
503 direction and the isotropic solution is less than 15%, suggesting that the isotropic approximation  
504 may be accurate enough for most applications.

#### 505 **4 CONCLUSIONS**

506 In this paper we have reviewed and unified the microbarom source theories developed by Brekhovskikh  
507 et al. (1973) and Waxler et al. (2007). A prominent feature of Brekhovskikh et al. (1973), that was  
508 not taken into account by Waxler et al. (2007), is the radiation pattern as a function of the eleva-  
509 tion angle  $\theta_a$ . In Waxler et al. (2007), the radiation pattern is monopolar due to an assumed lack  
510 of coherence of the sources at scales comparable to the acoustic wavelength. In Brekhovskikh  
511 et al. (1973) the acoustic power at near-horizontal incidence that up to 1000 times larger because  
512 the main coupling of water and air via the vertical velocity of the air-sea interface gives a much  
513 stronger amplification for grazing angles. Including finite depth ocean effects in Brekhovskikh  
514 et al. (1973) model is only relevant for near-vertical propagation, and has almost no effect on  
515 the predicted dominant near-horizontal propagation of infrasound. For shallow propagation an-  
516 gles that generally correspond to ground-based measurements (i.e.  $\theta_a > 40^\circ$ , relative to verti-  
517 cal) Brekhovskikh et al. (1973)'s formulation is compared to the more complete depth-dependent  
518 model presented here. We find that in regions of water depths under 1000 m, which cover 10%  
519 of the total ocean surface, - Brekhovskikh et al. (1973)'s formulation overestimates the source  
520 amplitude and underestimates it for deeper waters. On average there is a 7% understimation and  
521 Brekhovskikh et al. (1973) gives overall acceptable results, with a negligible effect of the water  
522 depth for near-horizontal propagation angles.

523 For near-vertical propagation, the generation of microbaroms involve both a source mecha-  
524 nism similar to the one of microseism dominated by the non-linear interaction of near-opposing  
525 waves, and Rayleigh waves propagating away from microseism sources. However, the associated  
526 acoustic power is at least 2 orders of magnitude lower than the near-horizontal radiated power.  
527 These vertical angles can be of interest for altitude measurements such as balloon measurements  
528 (Bowman & Lees, 2018); in such situation, the bathymetry can still impact the received signal.  
529 The horizontal anisotropy of the source has also been investigated, leading to the conclusion that,  
530 for computational applications, the isotropic approximation may be accurate enough and could  
531 then be used. Hence, the discretization of the wave spectrum might not be an issue.

532 Beyond theoretical issues, efforts should be pursued to validate the proposed model by con-  
533 sidering available observations of infrasound ambient noise as recorded by the global infrasound  
534 network of the International Monitoring System (IMS) (Ceranna et al., 2019). For practical appli-  
535 cations, further developments of a numerical model are needed to propagate microbarom signals  
536 over large distances through a realistic atmosphere. The implementation of this source model,  
537 based on a state-of-the-art numerical wave model (Ardhuin, 2019) should help building a global  
538 and time-dependent reference database. Exploiting this database of oceanic noise sources will be  
539 useful for developing middle-atmospheric remote sensing methods. The evaluation of infrasound  
540 ambient noise model is essential in the context of the future verification of the Comprehensive nu-  
541 clear Test Ban Treaty (CTBT), as accurate atmospheric models are basic prerequisite to assess the  
542 IMS network performance in higher resolution, reduce source location errors, and improve source  
543 characterization methods.

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651 **APPENDIX A: GREEN FUNCTION PHASE, SPATIAL CORRELATIONS AND**  
 652 **TREATMENT BY WAXLER AND GILBERT, 2006**

653 In order to clarify the difference between our derivation and the derivation by (Waxler & Gilbert,  
 654 2006), we go back to their expression of the atmospheric pressure spectrum as recorded at the  
 655 horizontal position  $\mathbf{x}_H$  and vertical altitude  $z$ , from a collection of sources over positions  $y$  and  $y'$   
 656 in domain  $S$ . The variance of pressure is given by their eq. (49) with  $\mathbf{x}'_H = \mathbf{x}_H$ ,  $z' = z$  and  $\tau = 0$

$$\langle P^2 \rangle = \rho^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_S \int_S \langle G(x, y, -\tau_1) G^*(x, y', \tau - \tau_2) \rangle \left\langle \frac{\partial v(y, \tau_1)}{\partial \tau_1} \frac{\partial v(y', \tau_2)}{\partial \tau_2} \right\rangle_S d^2 y d^2 y' d\tau_1 d\tau_2 \quad (\text{A.1})$$

Following their derivation, we use the Fourier in time  $\widehat{G}$  of the Green's function, representing the propagation of acoustic waves in a layered medium as a sum of discrete modes. As given by Waxler & Gilbert (2006)[, eq. 65] the Fourier transform of the Green function for mode  $j$  and

frequency  $\nu$  is given by,

$$\widehat{G}(\mathbf{x}_H, z, \mathbf{y}_H, \nu) = f(r, z) \sum_j \frac{e^{ik_j |\mathbf{x}_H - \mathbf{y}_H|}}{\sqrt{k_j}} \psi_j(z) \psi_j(z'), \quad (\text{A.2})$$

657 where  $r = |\mathbf{x}_H - \mathbf{y}_H|$  is the horizontal distance, the  $k_j$  are the horizontal wavenumbers pointing  
658 in the direction of  $\widehat{\mathbf{k}} = (\mathbf{x}_H - \mathbf{y}_H)/r$ , and  $\psi_j$  are normal modes satisfying the bi-orthogonality  
659 condition and their phases are assumed to be uniformly distributed and statistically independent.

Considering that for each  $y$  in the source  $|y'_H - y_0|/|\mathbf{x}_H - \mathbf{y}_0| \ll 1$  with  $y_0$  the center of the source (i.e. the receptor is far from the source), and defining the vector  $\mathbf{k}_j = k_j \widehat{\mathbf{k}}$ , the product of the Green's function for  $\mathbf{y}_H$  and its complex conjugate for  $\mathbf{y}'_H$  writes

$$\widehat{G}(\mathbf{x}_H, z, \mathbf{y}_H, \nu) \widehat{G}(\mathbf{x}_H, z, \mathbf{y}'_H, \nu)^* = |f(r_0, z)|^2 \sum_j e^{i\mathbf{k}_j \cdot (\mathbf{y}' - \mathbf{y})} \frac{|\psi_j(z)|^2 |\psi_j(z')|^2}{k_j} \quad (\text{A.3})$$

660 The phase of this product then contains the phase shift of the propagation from  $\mathbf{y}$  to  $\mathbf{y}'$  with a  
661 wavenumber  $k_j$ . When correlated with the source structure, that contains a phase  $\exp[i(\mathbf{k} + \mathbf{q}) \cdot$   
662  $(\mathbf{y}' - \mathbf{y})]$  it gives,

$$\begin{aligned} \langle P^2 \rangle &= \rho^2 \int_S \int_S \left( |f(r_0, z)|^2 \sum_j \frac{e^{i\mathbf{k}_j \cdot (\mathbf{y}' - \mathbf{y})}}{|k_j|} |\psi_j(z)|^2 |\psi_j(z')|^2 \right) \\ &\quad \times \left( 2 \int \int \nu^2 F(\mathbf{k}) F(\mathbf{q}) |C^+|^2 |\Omega|^2 e^{i(\mathbf{k} + \mathbf{q}) \cdot (\mathbf{y} - \mathbf{y}')} d^2 \mathbf{k} d^2 \mathbf{q} \right) d^2 y d^2 y' \\ &= \rho^2 \int_S \int_S |f(r_0, z)|^2 \sum_j \int \int \left[ \frac{|\psi_j(z)|^2 |\psi_j(z')|^2}{|k_j|} e^{i(\mathbf{k}_j - (\mathbf{k} + \mathbf{q})) \cdot (\mathbf{y}' - \mathbf{y})} \right. \\ &\quad \left. \times 2 \nu^2 F(\mathbf{k}) F(\mathbf{q}) |C^+|^2 |\Omega|^2 \right] d^2 \mathbf{k} d^2 \mathbf{q} d^2 y d^2 y' \end{aligned}$$

This can be simplified as the integral over  $\mathbf{y}'$  in the the source area  $S$  gives a term proportional to

$$\int_S \int_S e^{i[(\mathbf{k}_j - (\mathbf{k} + \mathbf{q})) \cdot (\mathbf{y}_H - \mathbf{y}'_H)]} d^2 y d^2 y' = (2\pi)^2 \delta(\mathbf{k}_j - (\mathbf{k} + \mathbf{q})) S \quad (\text{A.4})$$

663 instead of the  $(2\pi)^2 \delta(\mathbf{k} + \mathbf{q})$  term found by Waxler & Gilbert (2006).

664 The pressure variance becomes

$$\langle P^2 \rangle = \rho^2 (2\pi)^2 S |f(r_0, z)|^2 \sum_j \int_{\mathbf{k} + \mathbf{q} = \mathbf{k}_j} \frac{|\psi_j(z)|^2 |\psi_j(z')|^2}{|k_j|} 2 \nu^2 F(\mathbf{k}) F(\mathbf{q}) |C^+|^2 |\Omega|^2 d^2 \mathbf{k} d^2 \mathbf{q}.$$

665 This is the same as eq. (51) in Waxler & Gilbert (2006), except for the fact that we did not replace  
666  $\mathbf{q}$  with  $-\mathbf{k}$ , we thus need to compute the the air pressure over the source for all  $\mathbf{K} = \mathbf{k}_j$ . Although

<sup>667</sup>  $|\mathbf{K}| \ll |\mathbf{k}|$ , the approximation  $|\mathbf{K}| = 0$ , which is only strictly valid for an azimuth angle  $\theta_a = 0$ ,  
<sup>668</sup> leads to very large differences in the source amplitude, up to 30 dB as shown in figure 3.

1 **Supporting Information for ”Atmospheric infrasound**  
2 **radiation from ocean waves in finite depth: a unified generation**  
3 **theory and application to radiation patterns”**

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6 This document presents all the details of derivations necessary to support the paper ”Atmo-  
7 spheric infrasound radiation from ocean waves in finite depth: a unified generation theory and  
8 application to radiation patterns”. It follow Brekhovskikh et al. (1973, hereinafter BGKN73), as  
9 much as possible. Because we use the more common convention that the velocity vector is  $\mathbf{v} = \nabla\phi$   
10 this leads to changes in signs that are highlighted in red. A notable difference with Waxler &  
11 Gilbert (2006, hereinafter WG06) is the non-zero value of  $\mathbf{k}\cdot\mathbf{k}' + kk'$  and similar terms. Some of  
12 these were obtained by WG06 using the divergence equation, but not all of them, which misses  
13 the azimuthal dependence of the solution.

14 For convenience we repeat in table 1 the list of notations from the paper, including a few more  
15 symbols that were not used in the paper.

16 **S1 EQUATIONS UP TO EQ. (9) IN BGKN73**

17 We start with the Euler equations for a perfect fluid (no viscosity),

18 
$$\rho_j \left( \frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \right) = -\nabla p_j - g \rho_j \nabla z \quad (\text{S1})$$

**Table S1.** Notations used in different papers: LH50 stands for (Longuet-Higgins, 1950), BGKN73 stands for (Brekhovskikh et al., 1973), WG06 stands for (Waxler & Gilbert, 2006) and AH13 stands for (Ardhuin & Herbers, 2013).

quantity	this paper	LH50	BGKN73	WG06	AH13
vertical coordinate	$z$	$-z$	$z$	$z$	$z$
angle relative to vertical	$\theta_a$ or $\theta_w$	—	$\theta$	—	—
surface elevation	$\zeta$	$\zeta$	$\zeta$	$\xi$	$\zeta$
azimuth of spectrum	$\varphi$	$\theta$	$\varphi$	$\theta$	$\theta$
azimuth of acoustic signal	$\theta_2$	—	$\varphi_a$	—	—
velocity potential	$\phi$	$-\phi$	$-\varphi$	$\phi$	$\phi$
layer index	$l$	—	$j$	$\sigma$	—
sound speed	$\alpha_l$	$c$	$c_j$	$c_\sigma$	$\alpha$
density ratio	$m$	—	$m$	—	—
horizontal wavenumber	$\mathbf{K}$	—	$\mathbf{q}$	—	$\mathbf{K}$
radian frequency	$\Omega$	—	$\Omega$	—	$2\pi f_s$
horizontal wavenumbers	$\mathbf{k}, \mathbf{k}'$	$(-uk, -vk)$	$\varkappa, \varkappa_1$	$\mathbf{k}, \mathbf{q}$	$\mathbf{k}, \mathbf{k}'$
radian frequencies	$\sigma\sigma'$	$\sigma$	$\omega(\varkappa), \omega(\varkappa_1)$	$\omega(\mathbf{k}), \omega(\mathbf{q})$	$\sigma\sigma'$
pressure	$p$	$p$	$\rho\mathcal{P}$	$p$	$p$
vertical wavenumbers	$\nu_\pm, \mu_\pm$	—, $\alpha$	$\lambda_1, \lambda_2$	—	$l_a, l$
upward amplification	$g/2\alpha_l$	$\gamma$	—	—	—

19 We then use the compressible form of the Bernoulli Equation. And the mass conservation.

$$20 \quad \frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0 \quad (\text{S2})$$

21 We do not repeat here the equations (1) to (6) from BGKN73.

### 22 **S1.1 About equation (1) in BGKN73**

23 Eq. (S1) can be rewritten as

$$24 \quad (\rho_0 + s\rho_1 + s^2\rho_2) \left( \frac{\partial s\mathbf{v}_1 + s^2\mathbf{v}_2}{\partial t} + (s\mathbf{v}_1 + s^2\mathbf{v}_2) \cdot \nabla (s\mathbf{v}_1 + s^2\mathbf{v}_2) \right) \\ 25 \quad = -\nabla \cdot (p_0 + sp_1 + s^2p_2) - g(\rho_0 + s\rho_1 + s^2\rho_2)\nabla z. \quad (\text{S3})$$

26 Its truncation at the different orders in wave slope give,

27 • Order 0,

$$28 \quad -\nabla p_0 = g\rho_0\nabla z \quad (\text{S4})$$

29 • Order 1,

$$30 \quad \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + 0 = -\nabla p_1 - g\rho_1\nabla z$$

31 However (eq 6) :  $\nabla p_1 = \nabla(\rho_0 \mathcal{P}_1) = \mathcal{P}_1 \nabla(\rho_0) + \rho_0 \nabla(\mathcal{P}_1) = \mathcal{P}_1 \rho_0 \cdot \frac{-g}{\alpha^2} \nabla z + \rho_0 \nabla(\mathcal{P}_1) =$   
 32  $\frac{\alpha^2 \rho_1}{\rho_0} \cdot \rho_0 \cdot \frac{-g}{\alpha^2} \nabla z + \rho_0 \nabla(\mathcal{P}_1).$

33 Equation (S3) for order 1 becomes

$$34 \quad \frac{\partial \mathbf{v}_1}{\partial t} + \nabla \mathcal{P}_1 = 0. \quad (\text{S5})$$

35 • Order 2:

$$36 \quad \rho_0 \frac{\partial \mathbf{v}_2}{\partial t} + \rho_1 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_0 \mathbf{v}_1 \nabla \mathbf{v}_1 = -\nabla p_2 - g\rho_2 \nabla z \quad (\text{S6})$$

We similarly obtain

$$\begin{aligned} -\nabla p_2 - g\rho_2 \nabla z &= -\nabla(\rho_0 \mathcal{P}_2) - g\rho_2 \nabla z = -\rho_0 \nabla(\mathcal{P}_2) - \alpha^2 \frac{\rho_2}{\rho_0} \nabla(\rho_0) - g\rho_2 \nabla z \\ &= -\rho_0 \nabla(\mathcal{P}_2) - \alpha^2 \frac{\rho_2}{\rho_0} \cdot \frac{-g}{\alpha^2} \nabla z - g\rho_2 \nabla z = -\rho_0 \nabla(\mathcal{P}_2). \end{aligned}$$

37 Then,

$$38 \quad \frac{\partial \mathbf{v}_2}{\partial t} + \frac{\rho_1}{\rho_0} \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \nabla \mathbf{v}_1 = -\nabla \mathcal{P}_2.$$

Remembering from order 1 that  $\frac{\partial \mathbf{v}_1}{\partial t} = -\nabla \mathcal{P}_1$ , one obtains

$$\begin{aligned} \frac{\partial \mathbf{v}_2}{\partial t} + \nabla \mathcal{P}_2 &= -\frac{\rho_1}{\rho_0} \frac{\partial \mathbf{v}_1}{\partial t} - \mathbf{v}_1 \nabla \mathbf{v}_1 = \frac{\rho_1 \alpha^2}{\rho_0 \alpha^2} \nabla \mathcal{P}_1 - \mathbf{v}_1 \nabla \mathbf{v}_1 \\ &= \frac{1}{\alpha^2} \mathcal{P}_1 \nabla \mathcal{P}_1 - \mathbf{v}_1 \nabla \mathbf{v}_1 = \frac{1}{2} \nabla \left( \frac{\mathcal{P}_1^2}{\alpha^2} - \mathbf{v}_1^2 \right) + \mathbf{v}_1 \times \text{rot} \mathbf{v}_1 \end{aligned}$$

39 Because the velocity field is irrotational, it can be expressed as the gradient of a potential velocity

40  $\phi$ ,

$$41 \quad \mathbf{v}_i = +\nabla \phi. \quad (\text{S7})$$



64  $\frac{\partial}{\partial t}(\rho_0 \cdot (7)) - \alpha^2 \cdot (10)$  leads to:

$$\begin{aligned}
 65 \quad & \frac{\partial \rho_0 \mathcal{P}_2}{\partial t} - \alpha^2 \frac{\partial \rho_2}{\partial t} - \alpha^2 \rho_0 \Delta \phi_2 - \alpha^2 \nabla \phi_2 \nabla \rho_0 - \alpha^2 \rho_1 \Delta \phi_1 - \alpha^2 \nabla \phi_1 \nabla \rho_1 = -\frac{\partial^2 \rho_0 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 66 \quad & \iff \frac{\partial p_2}{\partial t} - \alpha^2 \frac{\partial \alpha^{-2} p_2}{\partial t} - \alpha^2 \rho_0 \Delta \phi_2 + \alpha^2 \rho_0 \frac{g}{\alpha^2} \frac{\partial \phi_2}{\partial z} - p_1 \Delta \phi_1 - \nabla \phi_1 \nabla p_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 67 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \alpha^2 \rho_0 \frac{g}{\alpha^2} \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \Delta \phi_1 - \mathcal{P}_1 \nabla \phi_1 \nabla \rho_0 - \rho_0 \nabla \phi_1 \nabla \mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 68 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \Delta \phi_1 + \mathcal{P}_1 \rho_0 \frac{g}{\alpha^2} \frac{\partial \phi_1}{\partial z} - \rho_0 \nabla \phi_1 \nabla \mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 69 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \left( \Delta \phi_1 - \frac{g}{\alpha^2} \frac{\partial \phi_1}{\partial z} \right) - \rho_0 \nabla \phi_1 \nabla \mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 70 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \frac{1}{\alpha^2} \frac{\partial^2 \phi_1}{\partial t^2} - \rho_0 \nabla \phi_1 \nabla \mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 71 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \mathcal{P}_1 \frac{1}{\alpha^2} \frac{\partial \mathcal{P}_1}{\partial t} - \rho_0 \nabla \phi_1 \nabla \mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \left( \frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla \phi_1)^2}{2} \right) \\
 72 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \nabla \phi_1 \nabla \frac{\partial \phi_1}{\partial t} = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_0 \frac{\partial}{\partial t} \frac{(\nabla \phi_1)^2}{2} \\
 73 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \frac{\partial}{\partial t} \frac{(\nabla \phi_1)^2}{2} = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_0 \frac{\partial}{\partial t} \frac{(\nabla \phi_1)^2}{2} \\
 74 \quad & \iff -\alpha^2 \rho_0 \Delta \phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \frac{\partial^2 \phi_2}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t} (\nabla \phi_1)^2 \\
 75 \quad & \iff \Delta \phi_2 - \frac{g}{\alpha^2} \frac{\partial \phi_2}{\partial z} - \frac{1}{\alpha^2} \frac{\partial^2 \phi_2}{\partial t^2} = + \frac{1}{\alpha^2} \frac{\partial}{\partial t} (\nabla \phi_1)^2 \tag{S13}
 \end{aligned}$$

### 77 S1.3 About Boundary conditions

78 We use the same boundary conditions as in BGKN73 for  $z = 0$  for velocity,

$$79 \quad - \frac{\partial \phi_1}{\partial z} \Big|_{z=0} + \frac{\partial \zeta_1}{\partial t} = 0 \tag{S14}$$

$$80 \quad - \frac{\partial \phi_2}{\partial z} \Big|_{z=0} + \frac{\partial \zeta_2}{\partial t} = - \left( - \frac{\partial^2 \phi_1}{\partial z^2} \Big|_{z=0} \zeta_1 + \nabla \phi_1 \Big|_{z=0} \nabla \zeta_1 \right) \tag{S15}$$

82 And for pressure,

$$83 \quad (\mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_{z=0} - g(1-m)\zeta_1 = 0 \tag{S16}$$

$$\begin{aligned}
 84 \quad & (\mathcal{P}_{w,2} - m\mathcal{P}_{a,2})_{z=0} - g(1-m)\zeta_2 = - \left( \frac{\partial \mathcal{P}_{w,1}}{\partial z} - m \frac{\partial \mathcal{P}_{a,1}}{\partial z} \right)_0 \zeta_1 \\
 85 \quad & + \frac{g}{\alpha_a^2} (n^2 \mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_0 \zeta_1 - \frac{g^2}{2\alpha_a^2} (n^2 - m^2) \zeta_1^2 \tag{S17}
 \end{aligned}$$

86 Here is a summary of the system of equation that corresponds to eq. (9) in BGKN73

$$\Delta\phi_{j,i} - \frac{g}{\alpha_j^2} \frac{\partial\phi_{j,i}}{\partial z} - \frac{1}{\alpha_j^2} \frac{\partial^2\phi_{j,i}}{\partial t^2} = S_{j,i} \quad \mathcal{P}_{j,i} = -\frac{\partial\phi_{j,i}}{\partial t} + F_{j,i}$$

$$-\frac{\partial\phi_{j,i}}{\partial z}\Big|_{z=0} + \frac{\partial\zeta_i}{\partial t} = Q_{j,i} \quad (\mathcal{P}_{w,i} - m\mathcal{P}_{a,i})_{z=0} - g(1-m)\zeta_i = R_i$$

where,

$$F_{j,1} = S_{j,1} = Q_{j,1} = R_1 = 0$$

$$F_{j,2} = \frac{\mathcal{P}_{j,1}^2}{2\alpha_j^2} - \frac{(\nabla\phi_{j,1})^2}{2}, \quad S_{j,2} = +\frac{1}{\alpha_j^2} \frac{\partial}{\partial t} (\nabla\phi_{j,1})^2$$

$$Q_{j,2} = -\nabla\phi_{j,1}|_0 \nabla\zeta_1 + \frac{\partial^2\phi_{j,1}}{\partial z^2}\Big|_{z=0} \zeta_1$$

$$R_2 = -\left(\frac{\partial\mathcal{P}_{w,1}}{\partial z} - m\frac{\partial\mathcal{P}_{a,1}}{\partial z}\right)_0 \zeta_1 + \frac{g}{\alpha_a^2} (n^2\mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_0 \zeta_1 - \frac{g^2}{2\alpha_a^2} (n^2 - m^2) \zeta_1^2$$

$$m = \rho_{a,0}/\rho_{w,0}, \quad n = \alpha_a/\alpha_w, \quad \delta_a = \left(\frac{g}{\alpha_a^2 k}\right)^{1/2} = \frac{\sigma}{\alpha_a k}, \quad \delta_w = n\delta_a$$

## S2 SOLVING FOR FIRST ORDER AND EXPRESSING THE SECOND ORDER PROBLEM

### S2.1 First order

By Fourier transform in horizontal space and time we can take

$$\phi_{j,1} = -is\sigma \sum \Phi_{j,1}(z) Z e^{i(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \quad (\text{S18})$$

The boundary condition in  $z = 0$  leads to

$$\Phi_{j,1}(z = 0) = 1. \quad (\text{S19})$$

Assuming  $\Phi_{j,1}(z) = f_j(z) e^{\gamma_j z}$  with  $\gamma_j = g/2\alpha_j^2$  one obtains,

- for the air

$$\phi_{a,1} = \sum i \frac{s\sigma}{k_a} e^{-k_a z} Z e^{i(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \quad (\text{S20})$$

with

$$\begin{aligned}
 k_a &= -\gamma_a + k_{a0} \\
 &= -\frac{g}{2\alpha_a^2} + \sqrt{k^2 - \gamma_a^2 + \frac{g\gamma_a}{\alpha_a^2} - \frac{\sigma^2}{\alpha_a^2}} \\
 &= -\frac{g}{2\alpha_a^2} + k\left(1 - \frac{g^2}{4\alpha_a^4 k^2} + \frac{g^2}{2\alpha_a^4 k^2} - \frac{\sigma^2}{k^2 \alpha_a^2}\right) \\
 &= -k\frac{g}{2k\alpha_a^2} + k\left(1 - \frac{\delta_a^4}{4} + \frac{\delta_a^4}{2} - \delta_a^2\right)^{1/2} \\
 &= -k\frac{\delta_a^2}{2} + k\left(1 - \frac{\delta_a^2}{2}\right) \\
 &= k(1 - \delta_a^2)
 \end{aligned}$$

• for the water :

$$\phi_{w,1} = \sum -is\sigma \frac{k_{w0} \cosh(k_{w0}(z+h)) - \gamma_w \sinh(k_{w0}(z+h))}{k_w^2 \sinh(k_{w0}h)} e^{\gamma_w z} Z e^{i(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \quad (\text{S21})$$

with  $k_w^2 = k_{w0}^2 - \gamma_w^2 = k^2(1 - 2\delta_w^2)$

If we consider  $\delta_w^2$  to be negligible ( $\delta_w = n^2\delta_a^2 \simeq 0.05\delta_a^2$ ) we obtain :

$$\phi_{w,1} = \sum -is\sigma \frac{\cosh(k_{w0}(z+h))}{k \sinh(k_{w0}h)} e^{\gamma_w z} Z e^{i(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \quad (\text{S22})$$

For simplicity we will now write that :

$$\phi_{w,1} = \sum -is\sigma f_{w,k}(z) e^{\gamma_w z} Z e^{i(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \quad (\text{S23})$$

## S2.2 Second order

At second order, the effects of waves comes into the pressure and velocity boundary conditions at the interfaces, but also as forcing terms on the right hand side of the wave equation. All these different terms take different forms, in particular for waves in intermediate or shallow water (Ardhuin & Herbers, 2013). In the limit of deep water waves,  $kh \gg 1$ , and neglecting  $\delta_w^2$  terms, all the wave forcing terms can be expressed as a function of  $\hat{p}_{2,u}$ , defined as

$$\hat{p}_{2,u}(x, y, z) = \rho_w |\nabla \phi_1|^2 = \frac{\rho_w g^2}{s\sigma s'\sigma'} \sum (\mathbf{k}\cdot\mathbf{k}' - kk') Z Z' e^{(k+k')z} e^{i\Theta} \quad (\text{S24})$$

112 with  $\Theta = \mathbf{K} \cdot \mathbf{x} - \Omega t$ ,  $\mathbf{K} = \mathbf{k} + \mathbf{k}'$ , and  $\Omega = s\sigma + s'\sigma'$ . At the surface,  $z = 0$ , this equivalent pressure,  
 113 correspond to the pressure that drives microseisms as given by Hasselmann (1963, eq. 2.12).

114 In the following, we will neglect all the short wavelength components that correspond to the  
 115 middle line of eq. (2.13) of (Hasselmann, 1963), keeping only the large wavelengths that excite  
 116 microbaroms, and for which  $|\mathbf{k} + \mathbf{k}'| \ll |\mathbf{k}|$ .

117 Given that acoustic waves in the atmosphere are much slower than those in water, we will retain  
 118  $\delta_a^2$  terms. As a result, following (Brekhovskikh et al., 1973), we cannot use the approximation  
 119  $\mathbf{k} \cdot \mathbf{k}' \simeq 0$ , but instead, using  $\mathbf{k} \cdot \mathbf{k}' < 0$  for those components that produce microseisms, we can use

$$120 \quad \frac{K}{k} = \frac{K\alpha_a}{2\sigma} \frac{2\sigma}{k\alpha_a} = 2 \sin \theta_a \delta_a \quad (\text{S25})$$

121 and the law of cosine in triangles,

$$122 \quad 2\mathbf{k} \cdot \mathbf{K} = k^2 + K^2 - k'^2 \quad (\text{S26})$$

123 this gives,

$$\begin{aligned} 124 \quad kk' + \mathbf{k} \cdot \mathbf{k}' &= kk' \left[ 1 - \left( \left( \frac{-\mathbf{k} \cdot \mathbf{k}'}{kk'} \right)^2 \right)^{1/2} \right] = kk' \left[ 1 - \left( \left( \frac{\mathbf{K} \cdot \mathbf{k}' - \mathbf{k}' \cdot \mathbf{k}'}{kk'} \right) \left( \frac{\mathbf{k} \cdot \mathbf{K} - \mathbf{k} \cdot \mathbf{k}}{kk'} \right) \right)^{1/2} \right] \\ 125 &= kk' \left[ 1 - \left( \left( \frac{-\mathbf{K} \cdot \mathbf{k} + K^2 - k'^2}{kk'} \right) \left( \frac{\mathbf{k} \cdot \mathbf{K} - k^2}{kk'} \right) \right)^{1/2} \right] \\ 126 &= kk' \left[ 1 - \left( \frac{k^2 k'^2 - (\mathbf{K} \cdot \mathbf{k})^2 + \mathbf{K} \cdot \mathbf{k} (k^2 + K^2 - k'^2) - k^2 K^2}{k^2 k'^2} \right)^{1/2} \right] \\ 127 &= kk' \left[ 1 - \left( 1 + \frac{-(\mathbf{K} \cdot \mathbf{k})^2 + 2\mathbf{K} \cdot \mathbf{k}(\mathbf{K} \cdot \mathbf{k}) - k^2 K^2}{k^2 k'^2} \right)^{1/2} \right] \\ 128 &= kk' \left[ 1 - \left( 1 + \frac{(\mathbf{K} \cdot \mathbf{k})^2 - k^2 K^2}{k^2 k'^2} \right)^{1/2} \right] \\ 129 &\simeq kk' \left[ -\frac{1}{2} \left( \frac{(\mathbf{K} \cdot \mathbf{k})^2}{k^2 k'^2} - \frac{K^2}{k'^2} \right) \right] \simeq kk' \frac{1}{2} \frac{K^2}{k'^2} \left[ 1 - \left( \frac{(\mathbf{K} \cdot \mathbf{k})^2}{k^2 K^2} \right) \right] \\ 130 &\simeq 2kk' \sin^2 \theta_a \delta_a^2 \left[ 1 - \left( \frac{(\mathbf{k} \cdot \mathbf{K})^2}{k^2 K^2} \right) \right] = 2kk' \sin^2 \theta_a \delta_a^2 [1 - \cos^2(\varphi_2 - \varphi)] \quad (\text{S27}) \end{aligned}$$

131 which is a function of the azimuth  $\varphi_2$  of the acoustic wave propagation, with  $\cos(\varphi_2 - \varphi) =$   
 132  $\mathbf{k} \cdot \mathbf{K} / (kK)$ .

133 Then,

$$134 \quad \mathbf{k} \cdot \mathbf{k}' - kk' \simeq -2kk' (1 - \sin^2 \theta_a \delta_a^2 [1 - \cos^2(\varphi_2 - \varphi)]) \quad (\text{S28})$$

135 This gives,

$$136 \quad \hat{p}_{2,u}(x, y, z) \simeq -2\rho_w \sigma \sigma' \sum (1 - \sin^2 \theta_a \delta_a^2 [1 - \cos^2(\varphi_2 - \varphi)]) ZZ' e^{(k+k')z} e^{i\Theta} \quad (\text{S29})$$

137 Other similar terms have more simple forms with no azimuthal dependency

$$138 \quad \frac{1}{2} (k^2 + \mathbf{k} \cdot \mathbf{k}' + k'^2 + \mathbf{k}' \cdot \mathbf{k}) = \frac{1}{2} (\mathbf{K} \cdot \mathbf{K} + \mathbf{k}' \cdot \mathbf{K}) = \frac{1}{2} K^2 \simeq 2k^2 \sin^2 \theta_a \delta_a^2. \quad (\text{S30})$$

### 139 S3 SECOND ORDER SOLUTION

#### 140 S3.1 General form of the solution in the water layer

141 The homogeneous solution is obtained for  $S_{w,2} = 0$ ,

$$142 \quad \phi_{w,2,h}(x, y, z, t) = \sum \Phi_{w,2,h} e^{i\Theta}, \quad \text{with} \quad \Theta = \mathbf{K} \cdot \mathbf{x} - \Omega t, \quad \mathbf{K} = \mathbf{k} + \mathbf{k}', \quad \Omega = s\sigma + s'\sigma' \quad (\text{S31})$$

143 Assuming a  $e^{i\mu z}$  variation over the vertical and replacing in the homogeneous equation (S13) gives,

$$144 \quad \mu^2 + i \frac{g}{\alpha_w^2} \mu + (K^2 - \Omega^2 / \alpha_w^2) = 0 \quad (\text{S32})$$

145 with solutions,

$$146 \quad \mu_{\pm} = -i \frac{g}{\alpha_w^2} \pm \sqrt{\frac{g^2}{2\alpha_w^4} + (\Omega^2 / \alpha_w^2 - K^2)} \simeq \pm k_{w2,0} (1 + O(\delta_w^2)) \quad (\text{S33})$$

147 with the complex wavenumber  $k_{w2,0} = \sqrt{\Omega^2 / \alpha_w^2 - K^2}$  so that the homogeneous solution is

$$148 \quad \Phi_{w,2,h} = W_+ e^{i\mu_+ z} + W_- e^{i\mu_- z}. \quad (\text{S34})$$

149 We recall that the wave equation is forced by,

$$150 \quad S_{w,2} = + \frac{1}{\alpha_w^2} \frac{\partial}{\partial t} (\nabla \phi_1)^2 = + \frac{1}{\rho_w \alpha^2} \frac{\partial \hat{p}_{2,u}}{\partial t} \quad (\text{S35})$$

151 This forcing adds a particular solution of order  $\delta_w^2$  that could be neglected here but we will only  
 152 keep the lowest order term to be consistent with BGKN73. This is also discussed by (Longuet-  
 153 Higgins, 1950) and (Waxler & Gilbert, 2006). We will only give its expression in the limit of deep  
 154 water, i.e.  $kh \gg 1$ .

we recall the right hand side,

$$S_{w,2}(x, z, t) \simeq + \frac{1}{\rho_w \alpha_w^2} \frac{\partial \widehat{p}_{2,u}(x, y, z, t)}{\partial t} = - \frac{g^2}{\alpha_a^2} \sum i \frac{s\sigma + s\sigma'}{s\sigma s\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k k') Z Z' e^{(k_w + k'_w)z} e^{i\Theta}. \quad (\text{S36})$$

Looking for a solution of the form

$$\phi_{w,2,p} = \sum \Phi_{w,2,p} e^{i\Theta}. \quad (\text{S37})$$

we replace in the wave equation (S13) and find

$$\Phi_{w,2,p} \simeq -i \frac{g^2}{u} \cdot \frac{s\sigma + s\sigma'}{s\sigma s\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k k') Z Z' e^{(k_w + k'_w)z}. \quad (\text{S38})$$

with the denominator defined by

$$u = \alpha_w^2 \left[ -K^2 + \frac{\Omega^2}{\alpha_w^2} + (k_w + k'_w)^2 \right] + g(k_w + k'_w) \simeq \alpha_w^2 (k_w + k'_w)^2 \simeq 4\alpha_w^2 k^2. \quad (\text{S39})$$

Of particular interest is the long-wavelength part – with  $s = s'$  – of the vertical derivative of

$\phi_{w,2,p}$ , given by,

$$\begin{aligned} \frac{\partial \phi_{w,2,p}}{\partial z} &\simeq \sum -i s \frac{g^2}{4\alpha_w^2 k^2} \frac{2\sigma(k_w + k'_w)}{\sigma\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k k') Z Z' e^{(k_w + k'_w)z} e^{i\Theta}, \\ &\simeq + \sum i s \delta_w^2 \frac{g}{\sigma} 2k^2 Z Z' e^{(k_w + k'_w)z} e^{i\Theta}. \end{aligned} \quad (\text{S40})$$

### S3.2 General form of the solution in the air layer

For the air, we only consider here acoustic waves radiating upward, giving the homogeneous solution,

$$\phi_{a,2,h}(x, y, z, t) = \sum s A_+ Z Z' e^{\nu_+ z} e^{i\Theta}, \quad (\text{S41})$$

where

$$\nu_+ = \frac{g}{\alpha_a^2} + i \sqrt{\frac{g^2}{2\alpha_a^4} + (\Omega^2/\alpha_w^2 - K^2)}. \quad (\text{S42})$$

For the particular solution, we recall the right hand side,

$$S_{a,2}(x, z, t) \simeq + \frac{1}{\rho_w \alpha_a^2} \frac{\partial \widehat{p}_{2,u}(x, y, -z, t)}{\partial t} = - \frac{g^2}{\alpha_a^2} \sum i \frac{s\sigma + s\sigma'}{s\sigma s\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k_a k'_a) Z Z' e^{-(k_a + k'_a)z} e^{i\Theta}. \quad (\text{S43})$$

175 Looking for a solution of the form

$$176 \quad \phi_{a,2,p} = \sum \Phi_{a,2,p} e^{i\Theta}. \quad (\text{S44})$$

177 we replace in the wave equation (S13) and find

$$178 \quad \Phi_{a,2,p} \simeq -i \frac{g^2 s\sigma + s\sigma'}{u s\sigma s\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k_a k'_a) Z Z' e^{-(k_a + k'_a)z}. \quad (\text{S45})$$

179 with the denominator defined by

$$180 \quad u = \alpha_a^2 \left[ -K^2 + \frac{\Omega^2}{\alpha_a^2} + (k_a + k'_a)^2 \right] + g(k_a + k'_a) \simeq \alpha_a^2 (k_a + k'_a)^2 \simeq 4\alpha_a^2 k^2. \quad (\text{S46})$$

$$181 \quad u = \alpha_a^2 \left[ -K^2 + \frac{\Omega^2}{\alpha_a^2} + (k_a + k'_a)^2 \right] + g(k_a + k'_a)$$

$$182 \quad \simeq \alpha_a^2 k^2 \left( 4\delta_a^2 \cos^2 \theta_a + \left( 1 + \frac{k'}{k} \right)^2 (1 - 2\delta_a^2) + \frac{g}{\alpha_a^2 k^2} (k_a + k'_a) \right)$$

$$183 \quad \simeq \alpha_a^2 k^2 (4\delta_a^2 \cos^2 \theta_a + 4(1 - 2\delta_a^2) + 2\delta_a^2(1 - \delta_a^2))$$

$$184 \quad \simeq 4\alpha_a^2 k^2 (1 - \delta_a^2 (\sin^2 \theta_a + \frac{1}{2})) \simeq 4\alpha_a^2 k^2 (1 + O(\delta_a^2)) \quad (\text{S47})$$

186 Of particular interest is the long-wavelength part – with  $s = s'$  – of the vertical derivative of

187  $\phi_{a,2,p}$ , given by,

$$188 \quad \frac{\partial \phi_{a,2,p}}{\partial z} \simeq + \sum \text{is} \frac{g^2}{4\alpha_a^2 k^2} (1 - O(\delta_a^2)) \frac{2\sigma(k_a + k'_a)}{\sigma\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k_a k'_a) Z Z' e^{-(k_a + k'_a)z} e^{i\Theta},$$

$$189 \quad \simeq + \sum \text{is} \frac{g^2}{4\alpha_a^2 k^2} (1 - O(\delta_a^2)) \frac{2\sigma(k + k')(1 - \delta_a^2)}{\sigma\sigma'} (\mathbf{k} \cdot \mathbf{k}' - k k' - 2k k' \delta_a^2) Z Z' e^{-(k_a + k'_a)z} e^{i\Theta},$$

$$190 \quad \simeq + \sum \text{is} \delta_a^2 \frac{g}{\sigma} (\mathbf{k} \cdot \mathbf{k}' - k k') (1 + O(\delta_a^2)) Z Z' e^{-(k_a + k'_a)z} e^{i\Theta},$$

$$191 \quad \simeq - \sum \text{is} \delta_a^2 2\sigma' k Z Z' e^{-(k_a + k'_a)z} e^{i\Theta}. \quad (\text{S48})$$

### 192 **S3.3 The BGKN terms - $F_{j,2}$ , $Q_{j,2}$ , $R_2$**

#### 193 *S3.3.1 In the water layer*

194 To simplify the calculation of these terms we use  $kh \gg 1$  for waves in deep water, and  $k_{w0} \simeq k$ ,

195 we may also use eq. (S27) and eq. (S28). These simplifications lead to :

$$196 \quad \phi_{w,1} = \sum -i \frac{s\sigma}{k} e^{k_{w0}z} e^{\gamma_w z} Z e^{i(\mathbf{k} \cdot \mathbf{x} - s\sigma t)} \quad (\text{S49})$$

And then we obtain the second order terms :

$$\begin{aligned}
 F_{w,2}(z=0) &= \frac{\mathcal{P}_{w,1}^2}{2\alpha_w^2} \Big|_0 - \frac{(\nabla\phi_{w,1})^2}{2} \Big|_0 \\
 &= \sum \frac{ss'\sigma\sigma'}{2kk'} \left[ \left( \frac{s\sigma s'\sigma'}{\alpha_w^2} - \mathbf{k}\mathbf{k}' + kk' \right) \right] ZZ'e^{i\Theta} \\
 &\simeq \sum \sigma^2 \left[ 1 + \delta_a^2 \left( \frac{n^2}{2} - \sin^2\theta_a [1 - \cos^2(\varphi_2 - \varphi)] \right) \right] ZZ'e^{i\Theta} \quad (\text{S50})
 \end{aligned}$$

Using the law of cosines in a triangle,

$$k'^2 = k^2 + K^2 - 2\mathbf{k}\cdot\mathbf{K} \quad (\text{S51})$$

so that

$$\sqrt{k'} = \sqrt{k} \left( 1 + \frac{K^2 - 2\mathbf{k}\cdot\mathbf{K}}{k^2} \right)^{1/4} \simeq \sqrt{k} \left( 1 + \frac{1}{4} \frac{K^2 - 2\mathbf{k}\cdot\mathbf{K}}{k^2} \right) \quad (\text{S52})$$

we get

$$\begin{aligned}
 Q_{w,2}|_{z=0} &= + \frac{\partial^2\phi_{w,1}}{\partial z^2} \Big|_0 \zeta_1 - \nabla\phi_{w,1}|_0 \cdot \nabla\zeta_1 \\
 &\simeq -i \sum \left[ \frac{s\sigma k + s'\sigma'k'}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k} + \frac{s'\sigma'}{k'} \right) \right] ZZ'e^{i\Theta} \\
 &= -i \sum \left[ s\sigma \frac{1}{2k} (k^2 + \mathbf{k}\cdot\mathbf{k}') + s' \frac{\sigma'}{2k'} (k'^2 + \mathbf{k}\cdot\mathbf{k}') \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum \left[ s\sigma \frac{\mathbf{k}\cdot(\mathbf{k} + \mathbf{k}')}{2k} + s'\sigma' \frac{\mathbf{k}'\cdot(\mathbf{k}' + \mathbf{k})}{2k'} \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum \left[ s\sigma \frac{\mathbf{k}\cdot\mathbf{K}}{2k} + s'\sigma' \frac{\mathbf{k}'\cdot\mathbf{K}}{2k'} \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum s \left[ \frac{\sigma}{2k} (\mathbf{k}\cdot\mathbf{K} + \mathbf{k}'\cdot\mathbf{K}) + \frac{\sigma'k - \sigma k'}{2k'k} (\mathbf{k}'\cdot\mathbf{K}) \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum s\sigma k \left[ \frac{K^2}{2k^2} + \sqrt{g} \frac{\sqrt{k} - \sqrt{k'}}{2k\sigma\sqrt{kk'}} (-\mathbf{k}\cdot\mathbf{K} + K^2) \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum s\sigma k \left[ 2\delta_a^2 \sin^2\theta_a + \frac{1}{4k^2} (2\mathbf{k}\cdot\mathbf{K} - K^2) \frac{1}{2k^2} (-\mathbf{k}\cdot\mathbf{K} + K^2) \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum s\sigma k \left[ 2\delta_a^2 \sin^2\theta_a - \frac{K^2}{4k^2} \left( \frac{\mathbf{k}\cdot\mathbf{K}}{kK} \right)^2 \right] ZZ'e^{i\Theta} \\
 &\simeq -i \sum s\sigma k \left[ 2\delta_a^2 \sin^2\theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right] ZZ'e^{i\Theta}. \quad (\text{S53})
 \end{aligned}$$

### S3.3.2 *In the air*

In a similar way we obtain :

Then,

$$\begin{aligned}
 F_{a,2}(z=0) &= \frac{\mathcal{P}_{w,1}^2}{2\alpha_w^2} \Big|_0 - \frac{(\nabla\phi_{w,1})^2}{2} \Big|_0 \\
 &= \sum \frac{ss'\sigma\sigma'}{2k_a k'_a} \left[ \frac{s\sigma s'\sigma'}{\alpha_a^2} - \mathbf{k}\mathbf{k}' + k_a k'_a \right] ZZ' e^{i\Theta} \\
 &= \sum ss'\sigma\sigma' (1 + 2\delta_a^2) \left[ \frac{s\sigma s'\sigma'}{2kk'\alpha_a^2} - \frac{\mathbf{k}\mathbf{k}' - kk' + 2\delta_a^2 kk'}{2kk'} \right] ZZ' e^{i\Theta} \\
 &\simeq \sum \sigma^2 \left[ 1 + \delta_a^2 \left( \frac{3}{2} - \sin^2 \theta_a [1 - \cos^2(\varphi_2 - \varphi)] \right) \right] ZZ' e^{i\Theta} \quad (\text{S54})
 \end{aligned}$$

And using :

$$\begin{aligned}
 \frac{s\sigma k + s'\sigma' k'}{2} - \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k} + \frac{s'\sigma'}{k'} \right) &= s\sigma \frac{1}{2k} (k^2 - \mathbf{k}\cdot\mathbf{k}') + s' \frac{\sigma'}{2k'} (k'^2 - \mathbf{k}\cdot\mathbf{k}') \\
 &= s\sigma \frac{1}{2k} (2k^2 - \mathbf{k}\cdot\mathbf{K}) + s' \frac{\sigma'}{2k'} (2k'^2 - \mathbf{k}'\cdot\mathbf{K}) \\
 &= s\sigma k + s'\sigma' k' - \left[ s\sigma \frac{\mathbf{k}\cdot\mathbf{K}}{2k} + s'\sigma' \frac{\mathbf{k}'\cdot\mathbf{K}}{2k'} \right] \\
 &\simeq s\sigma k \left[ 2 - 2\delta_a^2 \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right] \quad (\text{S55})
 \end{aligned}$$

one gets :

$$\begin{aligned}
 Q_{a,2}|_{z=0} &= + \frac{\partial^2 \phi_{a,1}}{\partial z^2} \Big|_0 \zeta_1 - \nabla\phi_{a,1}|_0 \cdot \nabla\zeta_1 \\
 &= +i \sum \left[ \frac{s\sigma k_a + s'\sigma' k'_a}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k_a} + \frac{s'\sigma'}{k'_a} \right) \right] ZZ' e^{i\Theta} \\
 &= +i \sum \left[ \frac{s\sigma k + s'\sigma' k'}{2} (1 - \delta_a^2) + \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k} + \frac{s'\sigma'}{k'} \right) (1 + \delta_a^2) \right] ZZ' e^{i\Theta} \\
 &= +i \sum \left[ \frac{s\sigma k + s'\sigma' k'}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k} + \frac{s'\sigma'}{k'} \right) - \delta_a^2 \cdot \left( \frac{s\sigma k + s'\sigma' k'}{2} - \frac{\mathbf{k}\cdot\mathbf{k}'}{2} \left( \frac{s\sigma}{k} + \frac{s'\sigma'}{k'} \right) \right) \right] ZZ' e^{i\Theta} \\
 &\simeq +i \sum s\sigma k 2\delta_a^2 \left[ \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) - \left[ 1 - \delta_a^2 \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right] \right] ZZ' e^{i\Theta} \\
 &\simeq -i \sum s\sigma k 2\delta_a^2 \left[ 1 - \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) (1 + \delta_a^2) \right] ZZ' e^{i\Theta} \quad (\text{S56})
 \end{aligned}$$

240 *S3.3.3 R<sub>2</sub> coefficient*

$$\begin{aligned}
 241 \quad R_2 &= - \left( \frac{\partial \mathcal{P}_{w,1}}{\partial z} - m \frac{\partial \mathcal{P}_{a,1}}{\partial z} \right)_0 \zeta_1 + \frac{g}{\alpha_a^2} (n^2 \mathcal{P}_{w,1} - m \mathcal{P}_{a,1})_0 \zeta_1 - \frac{g^2}{2\alpha_a^2} (n^2 - m^2) \zeta_1^2 \\
 242 \quad &= \sum \left[ \frac{\sigma^2 + \sigma'^2}{2} \cdot (1 - m) - \delta_a^2 \frac{\sigma^2 + \sigma'^2}{2} (n^2 + m(1 + \delta_a^2)) - \frac{\delta_a^2}{2} \frac{\sigma^2 + \sigma'^2}{2} (n^2 - m^2) \right] ZZ' e^{i\Theta} \\
 243 \quad &\simeq \sum \left[ \sigma^2 \cdot \left( 1 - m - \delta_a^2 \left( \frac{3n^2}{2} + m \left( 1 - \frac{m}{2} + \delta_a^2 \right) \right) \right) \right] ZZ' e^{i\Theta} \tag{S57}
 \end{aligned}$$

245 **S4 MATRIX PROBLEM FOR THE SECOND ORDER AMPLITUDES**

 246 • Velocity continuity at  $z = 0$ 

247 
$$\left. \frac{\partial \phi_{a,2}}{\partial z} \right|_0 + Q_{a,2} = \left. \frac{\partial \phi_{w,2}}{\partial z} \right|_0 + Q_{w,2} \quad (\text{S58})$$

248 
$$\iff \nu_+ A_+ - \mu_- W_- - \mu_+ W_+ = \left. \frac{\partial \Phi_{w,2,p}}{\partial z} \right|_0 - \left. \frac{\partial \Phi_{a,2,p}}{\partial z} \right|_0 - Q_{a,2} + Q_{w,2} \quad (\text{S59})$$

 250 • Pressure continuity at  $z = 0$ 

251 
$$\left( \frac{\partial \mathcal{P}_{w,2}}{\partial t} - m \frac{\partial \mathcal{P}_{a,2}}{\partial t} \right)_0 - g(1-m) \frac{\partial \zeta_2}{\partial t} = \frac{\partial R_2}{\partial t}$$
 252 
$$\iff \left( \frac{\partial \mathcal{P}_{w,2}}{\partial t} - m \frac{\partial \mathcal{P}_{a,2}}{\partial t} \right)_0 - g(1-m) \left. \frac{\partial \phi_{a,2}}{\partial z} \right|_0 - g(1-m) Q_{a,2} = \frac{\partial R_2}{\partial t}$$
 253 
$$\iff - \left. \frac{\partial^2 \phi_{w,2}}{\partial t^2} \right|_0 + \left. \frac{\partial F_{w,2}}{\partial t} \right|_0 + m \left. \frac{\partial^2 \phi_{a,2}}{\partial t^2} \right|_0 - m \left. \frac{\partial F_{a,2}}{\partial t} \right|_0 - g(1-m) \left. \frac{\partial \phi_{a,2}}{\partial z} \right|_0 - g(1-m) Q_{a,2} = \frac{\partial R_2}{\partial t}$$
 254 
$$\iff -\Omega^2 (-\phi_{w,2,p}(0) - W_+ - W_- + m\phi_{a,2,p}(0) + mA_+) + i\Omega(mF_{a,2}(0) - F_{w,2}(0))$$
 255 
$$-g(1-m)(\phi'_{a,2,p}(0) + \nu A_+ + Q_{a,2}) = \frac{\partial R_2}{\partial t} \quad (\text{S60})$$
 256

 257 • Boundary condition at  $z = -h$ 

 258 This boundary condition is given as an example, a more realistic boundary condition will be de-  
 259 veloped further:

260 
$$\left. \frac{\partial \Phi_{w,2,p}}{\partial z} \right|_{-h} + \mu_- W_- e^{-\mu_- h} + \mu_+ W_+ e^{-\mu_+ h} = 0 \quad (\text{S61})$$

261 Then we can write the boundary conditions system as a matrix problem,

262 
$$\begin{pmatrix} \nu & -\mu_- & -\mu_+ \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 & \Omega^2 \\ 0 & \mu_- e^{-\mu_- h} & \mu_+ e^{-\mu_+ h} \end{pmatrix} \cdot \begin{pmatrix} A_+ \\ W_- \\ W_+ \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix} \quad (\text{S62})$$

 263 Because we have assumed  $kh \gg 1$  we can neglect the  $p_{bot}$  term of (Ardhuin & Herbers, 2013)

 264 in  $\Lambda_3$ , and the  $\Lambda$  forcing terms are,

265 
$$\Lambda_1 = \left. \frac{\partial \phi_{w,2,p}}{\partial z} \right|_0 - \left. \frac{\partial \phi_{a,2,p}}{\partial z} \right|_0 - Q_{a,2} + Q_{w,2} \quad (\text{S63})$$

266 
$$\Lambda_2 = -\Omega^2 (\phi_{w,2,p}(0) - m\phi_{a,2,p}(0)) - i\Omega(mF_{a,2}(0) - F_{w,2}(0)) + g(1-m)(\phi'_{a,2,p}(0) + Q_{a,2}) + \frac{\partial R_2}{\partial t}$$

267 
$$\Lambda_3 = - \left. \frac{\partial \phi_{w,2,p}}{\partial z} \right|_{-h} \quad (\text{S64})$$
 268

**SIMPLIFIED FORMS USED:**

$$F_{w,2}(z=0) \simeq \sum \sigma^2 [1 - \sin^2 \theta_a \delta_a^2 (1 - \cos^2(\varphi_2 - \varphi)) + \frac{\delta_a^2 n^2}{2}] Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma^2) + \mathbf{O}(\sigma^2 \delta_a^2 \sin^2 \theta_a)$$

$$Q_{w,2}(z=0) \simeq -i \sum s \sigma k 2 \sin^2 \theta_a \delta_a^2 (1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi)) Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2 \sin^2 \theta_a)$$

$$F_{a,2}(z=0) \simeq \sum \sigma^2 [1 - \sin^2 \theta_a \delta_a^2 (1 - \cos^2(\varphi_2 - \varphi)) + \frac{3}{2} \delta_a^2] Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma^2) + \mathbf{O}(\sigma^2 \delta_a^2 \sin^2 \theta_a)$$

$$Q_{a,2}(z=0) \simeq -i \sum s \sigma k 2 \delta_a^2 [1 - \sin^2 \theta_a (1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi))] Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2) + \mathbf{O}(\sigma \delta_a^2 \sin^2 \theta_a)$$

$$\phi_{w,2,p}(z=0) \simeq i \sum s \sigma \delta_a^2 n^2 Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2 n^2)$$

$$\left. \frac{\partial \phi_{w,2,p}}{\partial z} \right|_{z=0} \simeq i \sum s \sigma 2k \delta_a^2 n^2 Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2 n^2)$$

$$\phi_{a,2,p}(z=0) \simeq i \sum s \sigma \delta_a^2 Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2) + \mathbf{O}(\sigma \delta_a^2 \sin^2 \theta_a)$$

$$\left. \frac{\partial \phi_{a,2,p}}{\partial z} \right|_{z=0} \simeq -i \sum s \sigma 2k \delta_a^2 Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma \delta_a^2)$$

$$R_2 \simeq \sum -\sigma^2 (1 - m - \delta_a^2 (3n^2/2 + m)) Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma^2)$$

$$\frac{\partial R_2}{\partial t} \simeq i s \sum 2\sigma^3 (1 - m - \delta_a^2 (3n^2/2 + m)) Z Z' e^{i\Theta} \quad \mathbf{O}(\sigma^3)$$

$$\nu_{\pm} = 2i \delta_a k \left( \pm \cos \theta_a - i \frac{\delta_a}{4} \right)$$

$$\mu_{\pm} = 2i \delta_a k \left( \mp i l - i \frac{\delta_a}{4} n^2 \right)$$

$$\sin \theta_a = \frac{K \alpha_a}{\Omega},$$

$$\Omega \simeq 2\sigma, \quad n = \frac{\alpha_a}{\alpha_w}, \quad l = (\sin^2 \theta_a - n^2)^{1/2},$$

$$\delta_w = \left( \frac{g}{k \alpha_a^2} \right)^{1/2} \frac{\alpha_a}{\alpha_w} = \delta_a n$$

**S4.1 Matrix 2x2 : BGKN73**

When the ocean is assumed to have an infinite depth, we consider the atmosphere and ocean to be half spaces, with the continuity of velocity and pressure at  $z = 0$  giving a 2 by 2 matrix,

$$M = \begin{pmatrix} \nu & -\mu_- \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 \end{pmatrix} \quad (\text{S65})$$

274 The solution is given by Cramer's method

$$275 \quad A_+ = \frac{\det \mathbf{M}_1}{\det \mathbf{M}} \quad (\text{S66})$$

276 with

$$277 \quad \det \mathbf{M}_1 = \begin{vmatrix} \Lambda_1 & -\mu_- \\ \Lambda_2 & \Omega^2 \end{vmatrix} \quad (\text{S67})$$

$$278 \quad \det \mathbf{M} = \begin{vmatrix} \nu & -\mu_- \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 \end{vmatrix} = \nu\Omega^2 - m\mu_-\Omega^2 - g(1-m)\nu\mu_- \quad (\text{S68})$$

280 Here are the different pieces of  $\det \mathbf{M}$ ,

- $\nu\Omega^2$  :

$$\begin{aligned} \nu\Omega^2 &= 4\sigma^2 \left( \frac{g}{2\alpha_a^2} + i\frac{\Omega}{\alpha_a} \cos \theta_a \right) \\ &= i8\sigma^2 k \left( -i\frac{g}{4k\alpha_a^2} + \frac{\sigma}{k\alpha_a} \cos \theta_a \right) \\ &= 8i\sigma^2 k \delta_a \left( \cos \theta_a - i\frac{\delta_a}{4} \right) \end{aligned}$$

- $-m\mu_-\Omega^2$  :

$$\begin{aligned} -m\mu_-\Omega^2 &\simeq -m4\sigma^2 \cdot 2i\delta_a k \left[ il - i\frac{\delta_a}{2} n^2 \right] \\ &\simeq -8i\sigma^2 k \delta_a m [il] \end{aligned}$$

- $-g(1-m)\nu\mu_-$ ,

$$\begin{aligned} 282 \quad -g(1-m)\nu\mu_- &\simeq -g(1-m)2i\delta_a k \left( \cos \theta_a - i\frac{\delta_a}{4} \right) 2i\delta_a k \left( il + i\frac{\delta_a}{4} n^2 \right) \\ 283 &\simeq 4\sigma^2 \delta_a k \delta_a (il \cos \theta_a + O(\delta_a)) \\ 284 &\simeq 8i\sigma^2 k \delta_a \left[ \frac{1}{2} \delta_a l \cos \theta_a \right] \end{aligned}$$

285 This gives  $\det \mathbf{M}$ , keeping only  $\delta_a^2$  (one is in factor and should be remove when doing the  
286 ratio),

$$\begin{aligned} 287 \quad \det \mathbf{M} &= i8\sigma^2 \delta_a k \left[ -i\frac{\delta_a}{4} + \cos \theta_a - iml + \frac{\delta_a}{2} \cos \theta_a l \right] \\ 288 \quad \det \mathbf{M} &\simeq 8i\sigma^2 k \delta_a \left[ \cos \theta_a \left( 1 + \frac{\delta_a}{2} l \right) - i \left( \frac{\delta_a}{4} + ml \right) \right] \quad (\text{S69}) \\ 289 \end{aligned}$$

290 In *green* there is a difference with BGKN73 denominator, that is coming from the  $\partial\zeta_2/\partial t$  term in  
291 the Bernoulli equation for the pressure at  $z = 0$ .

292 Now the numerator is,

$$\begin{aligned}
 293 \det \mathbf{M}_1 &= \Lambda_1 \Omega^2 + \mu_- \Lambda_2 \\
 294 &= \Omega^2 \left( \frac{\partial \phi_{w,2,p}}{\partial z} \Big|_0 - \frac{\partial \phi_{a,2,p}}{\partial z} \Big|_0 - Q_{a,2} + Q_{w,2} \right) + \mu_- \left( -\Omega^2 (\Phi_{w,2,p}(0)) \right. \\
 295 &\quad \left. - m \Phi_{a,2,p}(0) - i s \Omega (m F_{a,2}(0) - F_{w,2}(0)) + g(1-m) (\Phi'_{a,2,p}(0) + Q_{a,2}) + \frac{\partial R_2}{\partial t} \right) \\
 296 &\simeq \Omega^2 \left( \frac{\partial \phi_{w,2,p}}{\partial z} \Big|_0 - \frac{\partial \phi_{a,2,p}}{\partial z} \Big|_0 - Q_{a,2} + Q_{w,2} \right) \\
 297 &\quad + \mu_- \left( i s \Omega F_{w,2}(0) + \frac{\partial R_2}{\partial t} \right) \tag{S70}
 \end{aligned}$$

298 where eqs. (S40), (S48), (S56), (S50) give

$$\begin{aligned}
 299 \Omega^2 \frac{\partial \phi_{w,2,p}}{\partial z} \Big|_0 &= 4\sigma^2 \cdot i s \delta_a^2 n^2 \sigma 2k \\
 300 &\simeq 8i\sigma^2 k \delta_a \sigma \delta_a n^2 \tag{S71}
 \end{aligned}$$

$$\begin{aligned}
 301 -\Omega^2 \frac{\partial \phi_{a,2,p}}{\partial z} \Big|_0 &= -4\sigma^2 \cdot (-i) s 2\sigma k \delta_a^2 \\
 302 &\simeq 8i\sigma^2 k \delta_a \sigma \delta_a \tag{S72}
 \end{aligned}$$

$$\begin{aligned}
 303 -\Omega^2 Q_{a,2} &\simeq -4\sigma^2 \cdot (-i) s 2\sigma k \delta_a^2 \left( 1 - \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right) \\
 304 &\simeq 8i\sigma^2 k \delta_a \sigma s \delta_a \left( 1 - \sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right) \tag{S73}
 \end{aligned}$$

$$\begin{aligned}
 305 +\Omega^2 Q_{w,2} &\simeq 4\sigma^2 \cdot i 2\sigma s k \delta_a^2 \left( -\sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right) \\
 306 &\simeq 8i\sigma^2 k \delta_a \sigma s \delta_a \left( -\sin^2 \theta_a \left( 1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi) \right) \right) \tag{S74}
 \end{aligned}$$

$$\begin{aligned}
 307 +i s \Omega \mu_- F_{w,2}(0) &\simeq i 2s \sigma \cdot 2\delta_a k \left( \frac{\delta_a}{4} n^2 - l \right) \cdot \sigma^2 [1 - \sin^2 \theta_a \delta_a^2 (1 - \cos^2(\varphi_2 - \varphi)) + \delta_a^2 n^2 / 2] \\
 308 &\simeq -8i\sigma^2 k \delta_a \sigma s \left[ \frac{l}{2} - \frac{\delta_a}{8} n^2 - \frac{l}{2} \sin^2 \theta_a \delta_a^2 [1 - \cos^2(\varphi_2 - \varphi)] \right] \tag{S75}
 \end{aligned}$$

$$\begin{aligned}
 309 \mu_- \frac{\partial R_2}{\partial t} &= 2\delta_a k \left( \frac{\delta_a}{4} n^2 - l \right) i s 2\sigma^3 (1 - m) \\
 310 &\simeq -8i\delta_a k \sigma^2 \sigma s \left[ \frac{l}{2} - \frac{\delta_a}{8} n^2 \right]
 \end{aligned}$$

311 Collecting all the terms we find,

$$\begin{aligned}
 \det \mathbf{M}_1 &\simeq -8i\sigma^2 k \delta_a \sigma s \left[ l - \frac{\delta_a}{4} n^2 - \delta_a + 2\delta_a \sin^2 \theta_a \left(1 - \frac{1}{2} \cos^2 \varphi_2\right) + \delta_a (n^2 - 1) \right] \\
 &\simeq -8i\sigma^2 k \delta_a \sigma s \left[ l - \delta_a \left[ 2 - 2 \sin^2 \theta_a \left(1 - \frac{1}{2} \cos^2 \varphi_2\right) + \frac{5}{4} n^2 \right] \right] \\
 &\simeq -8i\sigma^2 k \delta_a \sigma s \left[ l - 2\delta_a \left[ 1 - \sin^2 \theta_a \left(1 - \frac{1}{2} \cos^2 \varphi_2\right) + \frac{5}{8} n^2 \right] \right] \tag{S76}
 \end{aligned}$$

Then, we find the same expression as in BGKN73 numerator and the  $\delta_a$  term is larger than the one in WG06, with 2 instead of 3/2.

The main term arises from the pressure boundary condition and from the difference between the pressure and the temporal derivative of the potential velocity.

We recall that the homogeneous atmospheric potential that radiates from the surface is given by eq. (S41),

$$\phi_{a,h,2}(z) = \sum s A_+ Z Z' e^{\nu+z} e^{i\Theta}, \tag{S77}$$

with

$$\nu \simeq \frac{g}{2\alpha_a^2} + i \frac{\Omega}{\alpha_a} \cos \theta_a \tag{S78}$$

and

$$A_+ \simeq -\sigma \frac{l - 2\delta_a \left[ 1 - \sin^2 \theta_a \left(1 - \frac{1}{2} \cos^2(\varphi_2 - \varphi)\right) + \frac{5}{8} n^2 \right]}{\cos \theta_a \left(1 + \frac{\delta_a}{2} l\right) - i \left(\frac{\delta_a}{4} + ml\right)}. \tag{S79}$$

## S5 ADDING THE SOLID EARTH

The solid Earth is characterized by density  $\rho_s$ , compression velocity  $\alpha_s$  and shear velocity  $\beta$ .

Then the velocity potential writes,

$$\phi_{w,2} = \sum [(W_- e^{\mu-z} + W_+ e^{\mu+z}) Z Z' + \Phi_{w,2,p}] e^{i\Theta}, \quad \text{for } -h < z < \zeta \tag{S80}$$

$$\phi_{a,2} = \sum [s A_+ e^{\nu+z} Z Z' + \Phi_{w,2,p}] e^{i\Theta}, \quad \text{for } \zeta < z \tag{S81}$$

All the potentials share the same phase,  $\Theta = \mathbf{K} \cdot \mathbf{x} - \Omega t$ ,  $\Omega = s(\sigma + \sigma')$ , but they differ by their vertical structures and amplitudes.

The boundary conditions for ocean/atmosphere interfaces remain the same. For the ocean bottom, the motion in the crust is given by velocity potentials for compression and shear waves in the

336 solid Earth, we follow here the treatment in (Ardhuin & Herbers, 2013). Neglecting the effect of  
 337 gravity, crustal motions can be separated in an irrotational part with a velocity potential  $\phi_c$  and a  
 338 rotational part with a stream function  $\psi$ , both solutions to Laplace's equation.

$$339 \quad \phi_c = C_p e^{\chi_p(z+h)} e^{i\Theta}, \quad (\text{S82})$$

$$340 \quad \psi = C_s e^{\chi_s(z+h)} e^{i\Theta}, \quad (\text{S83})$$

341 with

$$342 \quad \chi_p = \sqrt{K^2 - \frac{\Omega^2}{\alpha_s^2}}, \quad \text{and} \quad \chi_s = \sqrt{K^2 - \frac{\Omega^2}{\beta^2}}. \quad (\text{S84})$$

343 where  $\alpha_s$  and  $\beta$  are respectively the compression and the shear wave speed in crust. Typically  $\beta$   
 344 ranges from 2800 to 3200 m s<sup>-1</sup>; and  $\alpha_s = \sqrt{3}\beta$ . And  $\rho_s \simeq 2500$  kg m<sup>-3</sup>. The constants  $C_p$  and  
 345  $C_s$  have dimensions of m<sup>2</sup>/s and are determined by the boundary conditions at the ocean bottom.

346 With  $\lambda_e$  and  $\mu_e$  the Lamé elasticity parameters of the crust, Hooke's law of elasticity gives

$$347 \quad \tau_{zz} = \lambda_e \left( \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_z}{\partial z} \right) + 2\mu_e \frac{\partial \xi_z}{\partial z}, \quad (\text{S85})$$

$$348 \quad \tau_{xz} = \mu_e \left( \frac{\partial \xi_x}{\partial z} + \frac{\partial \xi_z}{\partial x} \right). \quad (\text{S86})$$

349 We recall that the compression and shear velocity are related to the Lamé parameters,

$$350 \quad \alpha_c^2 = \frac{\lambda_e + 2\mu_e}{\rho_s}, \quad (\text{S87})$$

$$351 \quad \beta^2 = \frac{\mu_e}{\rho_s}. \quad (\text{S88})$$

352 The zero tangential stress on the ocean bottom  $\tau_{xz}(z = -h) = 0$  yields the following relation-  
 353 ship between  $C_p$  and  $C_s$ , which is typical of seismic Rayleigh waves (Stoneley, 1926),

$$354 \quad C_s = \frac{2iK\chi_p}{\chi_s^2 + K^2} C_p = \frac{2i\beta^2 K \chi_p}{2\beta^2 K^2 - \Omega^2} C_p. \quad (\text{S89})$$

355 We can now eliminate also  $C_p$  using the continuity of the vertical velocity at the bottom,

$$356 \quad \frac{\partial \phi_2}{\partial z} = \frac{\partial \phi_c}{\partial z} + \frac{\partial \psi}{\partial x} \quad \text{at} \quad z = -h \quad (\text{S90})$$

$$357 \quad W_{+\mu_+} e^{-\mu_+ h} + W_{-\mu_-} e^{-\mu_- h} = \chi_p C_p + iK C_s \quad (\text{S91})$$

$$358 \quad = \chi_p C_p + iK \frac{2i\beta^2 K \chi_p}{2\beta^2 K^2 - \Omega^2} C_p \quad (\text{S92})$$

$$359 \quad = \frac{\chi_p \Omega^2}{\Omega^2 - 2K^2 \beta^2} C_p \quad (\text{S93})$$

360 and the continuity of normal stresses, using the result from (Ardhuin et al., 2013) is

$$361 \quad -\rho_w \frac{\partial \phi_2}{\partial t} = \tau_{zz} \quad z = -h \quad (\text{S94})$$

$$362 \quad \rho_w \Omega i s e^{-\mu_+ h} W_+ + \rho_w \Omega i s W_- e^{-\mu_- h} = r_{AH} C_p \quad (\text{S95})$$

$$363 \quad (\text{S96})$$

364 where

$$365 \quad r_{AH} = \frac{i s}{\Omega} \rho_s \left[ -\frac{4\beta^4 K^2 \chi_p \chi_s}{\Omega^2 - 2K^2 \beta^2} + (\Omega^2 - 2K^2 \beta^2) \right]. \quad (\text{S97})$$

366 Defining

$$367 \quad r_{\pm} = \frac{i s \rho_w \Omega \frac{\chi_p \Omega^2}{\Omega^2 - 2K^2 \beta^2}}{\mu_{\pm} r_{AH}} \quad (\text{S98})$$

$$368 \quad = \frac{i s \rho_w \Omega \frac{\chi_p \Omega^2}{\Omega^2 - 2K^2}}{\frac{i}{\Omega} \mu_{\pm} \rho_s \left[ -\frac{4\beta^4 K^2 \chi_p \chi_s}{\Omega^2 - 2K^2 \beta^2} + (\Omega^2 - 2K^2 \beta^2) \right]}, \quad (\text{S99})$$

$$369 \quad = \frac{\rho_w \chi_p \Omega^4}{\mu_{\pm} \rho_s \left[ (\Omega^2 - 2K^2 \beta^2)^2 - 4\beta^4 K^2 \chi_p \chi_s \right]} \quad (\text{S100})$$

370 we combine these two boundary conditions by subtracting  $r$  times the second equation to find a  
371 condition for the bottom velocities on the water side,

$$372 \quad \mu^+ (1 - r_+) e^{-\mu_+ h} W_+ + \mu^- (1 - r_-) \mu_- e^{-\mu_- h} W_- = 0. \quad (\text{S101})$$

373 We thus have the matrix equation

$$374 \quad \mathbf{M}(A_+, W_-, W_+)^T = (\Lambda_1, \Lambda_2, 0)^T \quad (\text{S102})$$

375 with

$$376 \quad \mathbf{M} = \begin{pmatrix} \nu_+ & -\mu_- & -\mu_+ \\ -m\Omega^2 - g(1-m)\nu_+ & \Omega^2 & \Omega^2 \\ 0 & (1-r_-)\mu_- e^{-\mu_- h} & (1-r_+)\mu_+ e^{-\mu_+ h} \end{pmatrix} \quad (\text{S103})$$

377 and we use the following simplification,

$$378 \quad \Lambda_1 = \left. \frac{\partial \Phi_{w,2,p}}{\partial z} \right|_0 - \left. \frac{\partial \Phi_{a,2,p}}{\partial z} \right|_0 - Q_{a,2} + Q_{w,2} \quad (\text{S104})$$

$$379 \quad \Lambda_2 = i\Omega F_{w,2}(0) + \frac{\partial R_2}{\partial t} \quad (\text{S105})$$

380 Assuming  $mu_+ \simeq -\mu_- \simeq \mu$  the matrix equation simplifies as:

$$381 \quad \mathbf{M} = \begin{pmatrix} \nu_+ & \mu & -\mu \\ -m\Omega^2 - g(1-m)\nu_+ & \Omega^2 & \Omega^2 \\ 0 & -(1+r)\mu e^{\mu h} & (1-r)\mu e^{-\mu h} \end{pmatrix} \quad (\text{S106})$$

382 **S6 FROM AMPLITUDE TO POWER**

383 **S6.1 Particular case of a pair of wave trains**

384 Here we first consider the pressure amplitude and variance in the water layer, which has been well  
385 studied and measured (Cox & Jacobs, 1989; Ardhuin et al., 2013).

386 In the case of only two wave trains of opposing direction with wave numbers  $k_1$  and  $k_2 \simeq -k_1$   
387 with surface elevation

$$388 \quad \zeta = a_1 \cos(k_1 x - \sigma_1 t) + a_2 \cos(k_2 x - \sigma_2 t) \quad (\text{S107})$$

389 and velocity field

$$390 \quad w(z=0) = a_1 \sigma_1 \sin(k_1 x - \sigma_1 t) + a_2 \sigma_2 \sin(k_2 x - \sigma_2 t) \quad (\text{S108})$$

$$392 \quad u(z=0) = a_1 \sigma_1 \cos(k_1 x - \sigma_1 t) - a_2 \sigma_2 \cos(k_2 x - \sigma_2 t) \quad (\text{S109})$$

393 the second order pressure is, keeping only the small wavenumber components,

$$394 \quad p_2 = \rho_w (u^2 + w^2) = -2\rho_w \sigma_1 \sigma_2 a_1 a_2 \cos [Kx + \Omega t] \quad (\text{S110})$$

395 Now we consider the variance of the pressure,

$$396 \quad \langle p_2^2 \rangle = 4\rho_w^2 \sigma_1^2 \sigma_2^2 a_1^2 a_2^2 / 2 \quad (\text{S111})$$

$$397 \quad = 2\rho_w^2 \sum_{k+k'=K} \sigma^2 \sigma'^2 a^2 a'^2 / 2 \quad (\text{S112})$$

$$398 \quad = 8\rho_w^2 \sigma_1^2 \sigma_2^2 \frac{a_1^2}{2} \frac{a_2^2}{2} \quad (\text{S113})$$

$$399 \quad \simeq \frac{1}{2} \rho_w^2 \Omega^4 E_1 E_2 \quad (\text{S114})$$

$$400 \quad = \frac{1}{4} \rho_w^2 \Omega^4 \sum_{k+k'=K} E E'. \quad (\text{S115})$$

401 **S6.2 Case of random waves**

$$402 \quad F_{p,2h}(\mathbf{K}, f_s) = 2 \lim_{|d\mathbf{K}| \rightarrow 0, df_s \rightarrow 0} \frac{\langle |P_{2h}^+|^2 \rangle}{dK_x dK_y df_s} \quad (\text{S116})$$

403 with

$$\begin{aligned}
 404 \quad P_{2h}^s &= \rho_a \mathcal{P}_{a,2,h} = -\rho_a \frac{\partial \phi_{a,2,h}}{\partial t} \\
 405 &= -\rho_a \frac{\partial}{\partial t} \left( \frac{R_a(\mathbf{K})}{\rho_w 2\sigma'} p_{\text{surf}}^{s,s'}(\mathbf{K}, \Omega) \right)
 \end{aligned}$$

406 remembering

$$407 \quad p_{\text{surf}}^{s,s'}(\mathbf{K}, \Omega) = \rho_w \sum_{\mathbf{k}, s, \mathbf{k}', s'} D_z(\mathbf{k}, s, \mathbf{k}', s') Z Z' e^{i\Theta} \quad (\text{S117})$$

408 one gets :

$$\begin{aligned}
 409 \quad P_{2h}^s &= \rho_a \sum_{\mathbf{k}, s, \mathbf{k}', s'} i R_a(\mathbf{K}) \frac{(s\sigma + s'\sigma')}{2\sigma'} D_z(\mathbf{k}, s, \mathbf{k}', s') Z Z' e^{i\Theta} \\
 410 &= \rho_a \sum_{\mathbf{k}, s, \mathbf{k}'} i s R_a(\mathbf{K}) \frac{(\sigma + \sigma')}{2\sigma'} D_z(\mathbf{k}, s, \mathbf{k}', s) Z Z' e^{i\Theta} \quad (\text{S118})
 \end{aligned}$$

411 Then,

$$\begin{aligned}
 412 \quad 2|P_{2h}^+|^2 &= 2\rho_a^2 \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} R_a(\mathbf{K}) \frac{(\sigma + \sigma')}{2\sigma'} D_z(\mathbf{k}, +, \mathbf{k}', +) Z Z' e^{i\Theta} \right|^2 \\
 413 &= 2\rho_a^2 \cdot 2 \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} |R_a(\mathbf{K})|^2 \frac{(\sigma + \sigma')^2}{4\sigma'^2} |D_z(\mathbf{k}, +, \mathbf{k}', +)|^2 |Z|^2 |Z'|^2 \quad (\text{S119})
 \end{aligned}$$

414 And the spectrum density of the source writes :

$$415 \quad F_{p,2h}(\mathbf{K}, f_s) = \lim_{|\mathbf{dK}| \rightarrow 0, df_s \rightarrow 0} \frac{1}{K_x dK_y df_s} \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \sigma+\sigma'=\Omega} \frac{(\sigma + \sigma')^2}{\sigma'^2} R_a(\mathbf{K})^2 \rho_a^2 |D_z(\mathbf{k}, +, \mathbf{k}', +)|^2 |Z|^2 |Z'|^2 \quad (\text{S120})$$

416 using the definition :

$$417 \quad E(k_x, k_y) = 2 \lim_{dk_x, dk_y \rightarrow 0} \frac{|Z|^2}{dk_x dk_y} \quad (\text{S121})$$

$$\begin{aligned}
 418 \quad F_{p,2h}(\mathbf{K}, f_s) &= \lim_{|\mathbf{dK}| \rightarrow 0, df_s \rightarrow 0} \frac{dk_x dk_x dk'_x dk'_y}{4dK_x dK_y df_s} \sum_{\mathbf{k}, s, \mathbf{k}'} \frac{(\sigma + \sigma')^2}{\sigma'^2} R_a(\mathbf{K})^2 \rho_a^2 |D_z(\mathbf{k}, +, \mathbf{k}', +)|^2 E(k_x, k_y) E(k'_x, k'_y) \\
 419 & \quad (\text{S122})
 \end{aligned}$$

420 Taking the limit to continuous sums and using a change of variable from  $(k_x, k_y, k'_x, k'_y)$  to  $(f_s, \varphi, K_x, K_y)$ ,

421 with  $K_x = k_x + k'_x$ ,  $K_y = k_y + k'_y$  and  $f_s = (\sqrt{gk} + \sqrt{gk'})/(2\pi)$  the Jacobian of the coordinate

422 transform is

$$\det \begin{pmatrix} \frac{\partial f_s \partial \varphi \partial K_x \partial K_y}{\partial k_x \partial k_y \partial k'_x \partial k'_y} \end{pmatrix} = \begin{vmatrix} g \cos \varphi / (4\pi\sigma) & -\sin \varphi / k & 1 & 0 \\ g \sin \varphi / (4\pi\sigma) & \cos \varphi / k & 0 & 1 \\ g \cos \varphi' / (4\pi\sigma') & 0 & 1 & 0 \\ g \sin \varphi' / (4\pi\sigma') & 0 & 0 & 1 \end{vmatrix} = \frac{g^2}{4\pi\sigma^3\sigma'} [\sigma' - \sigma \cos(\varphi - \varphi')],$$

428 (S123)

$$\begin{aligned}
 425 \int F_{p,2h}(\mathbf{K}, f_s) dK_x dK_y df_s &= \rho_a^2 \int \frac{(\sigma + \sigma')^2}{4\sigma'^2} |R_a|^2 |D_z|^2 E(k_x, k_y) E(k_x, k_y) dk_x dk_y dk'_x dk'_y \\
 426 &= \rho_a^2 \int \frac{(\sigma + \sigma')^2}{4\sigma'^2} |R_a|^2 |D_z|^2 \frac{E(k_x, k_y) E(k'_x, k'_y) 4\pi\sigma^3\sigma'}{g^2 [\sigma' - \sigma \cos(\varphi - \varphi')]} df_s d\varphi dK_x dK_y.
 \end{aligned}$$

427 To transform the spectra to frequency-direction spectra we use the Jacobian :

$$428 E(f, \varphi) = \frac{4\pi\sigma^3}{g^2} E(k_x, k_y) \quad (S124)$$

429 And then obtain :

$$430 \int F_{p,2h}(\mathbf{K}, f_s) dK_x dK_y df_s = \frac{1}{2} g^2 \rho_a^2 \int f_s \frac{(\sigma + \sigma')}{4\sigma'^4} |R_a|^2 |D_z|^2 \frac{E(f, \varphi) E(f', \varphi')}{[\sigma' - \sigma \cos(\varphi - \varphi')]} df_s d\varphi dK_x dK_y.$$

431 Now we use the unicity of the Fourier transform to identify the spectral density in the left and right  
 432 hand sides and considering  $|D_z(\mathbf{k}, +, \mathbf{k}', +)| \simeq 2\sigma\sigma'$  :

$$433 F_{p,2h}(\mathbf{K}, f_s) = \frac{1}{2} g^2 \rho_a^2 f_s \int_0^{2\pi} \frac{\sigma^2(\sigma + \sigma')}{\sigma'^2} |R_a|^2 \frac{E(f, \varphi) E(f', \varphi')}{\sigma' - \sigma \cos(\varphi - \varphi')} d\varphi. \quad (S125)$$

### 434 S6.3 Acoustic energy in the water column

435 We take the acoustic energy per unit of horizontal surface to be twice the kinetic energy. Consid-  
 436 ering only  $K < \Omega/\alpha_w$ , we have

$$437 E_w = \rho_w \int_{-h}^0 u^2 + w^2 dz \quad (S126)$$

438 Now expressing the velocities as a function of the velocity potential in the water given by eq.  
 439 (S80)

$$440 E_w = \rho_w \int_{-h}^0 \sum (K^2 + \mu^2) W_-^2 \left( \frac{1+r}{1-r} \right)^2 \cos^2(|\mu|z) dz \quad (S127)$$

$$\begin{aligned}
 441 &= \rho_w \int_{\theta_{a,1}}^{\theta_{a,2}} (K^2 + |\mu|^2) F_{p,2h}(\theta_a, \varphi_2, f_s) \left| \frac{A}{P_{2,h}^+} \frac{W_-}{A} \frac{1+r}{1-r} \right|^2 \left( \frac{h}{2} + \frac{\sin 2|\mu|h}{4|\mu|} \right)^2 d\theta_a d\varphi_2 \\
 442 & \quad (S128)
 \end{aligned}$$

443 with

$$444 \quad \left| \frac{W_-}{A} \frac{1+r}{1-r} \right| = \left| \frac{2\nu(1+r)}{\mu [i \sin(|\mu|h) + r \cos(|\mu|h)]} \right| \quad (\text{S129})$$

445 and

$$446 \quad \left| \frac{A}{P_{2,h}^+} \right| = \frac{1}{(\sigma + \sigma')\rho_a}. \quad (\text{S130})$$

447 Now, looking at the ratio of the acoustic energy and radiated power for any  $\theta_a$  and  $\varphi_2$  we have,

$$448 \quad Q_{max} = \frac{\Omega E_w}{F_{p,2h}(\theta_a, \varphi_2, f_s)/(\rho_a \alpha_a)} \quad (\text{S131})$$

$$449 \quad = \Omega \rho_w \rho_a \alpha_a (K^2 + |\mu|^2) \left| \frac{A}{P_{2,h}^+} \frac{W_-}{A} \frac{1+r}{1-r} \right|^2 \left( \frac{h}{2} + \frac{\sin 2|\mu|h}{4|\mu|} \right) \quad (\text{S132})$$

$$450 \quad = \frac{\rho_w \alpha_a}{\rho_a \Omega} (K^2 + |\mu|^2) \left| \frac{W_-}{A} \frac{1+r}{1-r} \right|^2 \left( \frac{h}{2} + \frac{\sin 2|\mu|h}{4|\mu|} \right) \quad (\text{S133})$$

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