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PADÉ APPROXIMATIONS TO THE LOGARITHM AND BERNOULLI POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. By using appropriate Padé approximations to the logarithm, we obtain some recurrence formulas for Bernoulli polynomials of the second kind. The proofs are based on suitable computations of some finite hypergeometric sums and the method of generating functions.

1. PADÉ APPROXIMATION TO THE LOGARITHM

The basic approximations to the logarithm, we consider here, are the so-called in [5], the *linear Padé approximations* to the logarithm

$$r_n(t) \log t + s_n(t) = O((t-1)^{2n+1}) \quad (t \rightarrow 1)$$

where n is a non negative integer,

$$r_n(t) = \sum_{k=0}^n \binom{n}{k}^2 t^k, \quad s_n(t) = 2 \sum_{k=0}^n \binom{n}{k}^2 (H_{n-k} - H_k) t^k$$

and H_k is the k -th harmonic number : $H_0 = 0$ and for $k \geq 1$, $H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$.

More precisely, we are concerned by the following equivalent approximations

$$(1.1) \quad r_n(1+t) \log(1+t) + s_n(1+t) = O(t^{2n+1}) \quad (t \rightarrow 0)$$

and then need expansions of $r_n(1+t)$ and $s_n(1+t)$ in powers of t . For instance, it is easy to see that $r_0(1+t) = 1$, $s_0(1+t) = 0$, and

$$\begin{aligned} r_1(1+t) &= 2+t, \quad r_2(1+t) = 6+6t+t^2, \quad r_3(1+t) = 20+30t+12t^2+t^3 \\ s_1(1+t) &= -2t, \quad s_2(1+t) = -6t-3t^2, \quad s_3(1+t) = -20t-20t^2-\frac{11}{3}t^3. \end{aligned}$$

For general values of n , we have the following expressions.

Lemma 1.1. *For any non negative integer n , one has*

$$(1.2) \quad r_n(1+t) = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} t^k \quad \text{and} \quad s_n(1+t) = 2 \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (H_n - H_{n-k}) t^k.$$

Proof. As done in [1] for instance, we transform sums containing binomial coefficients and/or harmonic numbers, into sums of hypergeometric type, and use that, for integers $m \geq k \geq 0$,

$$LD \binom{x+m}{x+k} = LD \binom{x+m}{m-k} = \binom{m}{k} (H_m - H_k)$$

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where LD is the operator of derivation and evaluation at $x = 0$, $LDf(x) := (D_x f)(0)$.
Let us now consider the two hypergeometric finite sums

$$p_n(x, t) := \sum_{k=0}^n \binom{n}{k} \binom{x+n}{n-k} t^k, \quad q_n(x, t) := \sum_{k=0}^n \binom{n}{k} \binom{x+n}{n} t^k$$

so that

$$r_n(t) = p_n(0, t) \quad \text{and} \quad s_n(t) = 2LD(p_n(x, t) - q_n(x, t)).$$

It remains now to investigate $p_n(x, 1+t)$ and $q_n(x, 1+t)$. By using the Gould's entry 3.17 in [2],

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k} z^k = \sum_{k=0}^n \binom{n}{k} \binom{x+n-k}{n} (z-1)^k$$

in which we transform x into $x+n$ and z into $1+t$, we get

$$q_n(x, 1+t) = \sum_{k=0}^n \binom{n}{k} \binom{x+n}{k} (1+t)^k = \sum_{k=0}^n \binom{n}{k} \binom{x+2n-k}{n} t^k.$$

For the sum $p_n(x, 1+t)$, we consider entry 3.18, of [2],

$$\sum_{k=0}^n \binom{n}{k} \binom{z+k}{k} (u-v)^{n-k} v^k = \sum_{k=0}^n \binom{n}{k} \binom{z}{k} u^{n-k} v^k = \sum_{k=0}^n \binom{n}{k} \binom{z}{n-k} u^k v^{n-k}$$

and make $z = x+n$, $u = 1+t$, $v = 1$, to obtain

$$p_n(x, 1+t) = \sum_{k=0}^n \binom{n}{k} \binom{x+n}{n-k} (1+t)^k = \sum_{k=0}^n \binom{n}{k} \binom{x+n+k}{k} t^{n-k} = \sum_{k=0}^n \binom{n}{k} \binom{x+2n-k}{n-k} t^k.$$

Finally

$$r_n(1+t) = p_n(0, 1+t) = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n-k} t^k = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} t^k$$

and

$$\begin{aligned} s_n(1+t) &= 2LD(p_n(x, 1+t) - q_n(x, 1+t)) \\ &= 2LD \left(\sum_{k=0}^n \binom{n}{k} \binom{x+2n-k}{n-k} t^k - \sum_{k=0}^n \binom{n}{k} \binom{x+2n-k}{n} t^k \right) \\ &= 2 \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (H_n - H_{n-k}) t^k \end{aligned}$$

and the proof of the lemma is done. □

2. APPLICATION TO BERNOULLI POLYNOMIALS OF THE SECOND KIND

The Bernoulli polynomials of the second kind $b_n(x)$ can be defined, see [3], by the way of the generating function,

$$(2.1) \quad \frac{t(1+t)^x}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(x) t^n \quad (|t| < 1).$$

Bernoulli numbers of the second kind are the numbers $b_n := b_n(0)$.

Here is our main result.

Proposition 2.1. *For positive integers k and n , with $k \leq 2n$, we have*

$$\sum_{m=0}^{\min\{n,k-1\}} \binom{n}{m} \binom{2n-m}{n} \binom{x}{k-m-1} + 2 \sum_{m=0}^{\min\{n,k\}} \binom{n}{m} \binom{2n-m}{n} (H_n - H_{n-m}) b_{k-m}(x) = 0.$$

Proof. For simplicity in computations, let us write

$$r_n(1+t) := \sum_{k=0}^n \rho_{n,k} t^k \quad \text{and} \quad s_n(1+t) := \sum_{k=0}^n \sigma_{n,m} t^k.$$

By the definition of the polynomials $b_k(x)$ 2.1, we have

$$\log(1+t) \sum_{k=0}^{\infty} b_k(x) t^k = t(1+t)^x$$

then

$$r_n(1+t) \log(1+t) \sum_{k=0}^{\infty} b_k(x) t^k = t(1+t)^x r_n(1+t)$$

and

$$(r_n(1+t) \log(1+t) + s_n(1+t)) \sum_{k=0}^{\infty} b_k(x) t^k = t(1+t)^x r_n(1+t) + s_n(1+t) \sum_{k=0}^{\infty} b_k(x) t^k.$$

Hence, according to the approximations 1.1, we have,

$$t(1+t)^x r_n(1+t) + s_n(1+t) \sum_{k=0}^{\infty} b_k(x) t^k = O(t^{2n+1}) \quad (t \rightarrow 0).$$

Now, we write the expansion, in powers of t , of the left hand side above. First, we have

$$\begin{aligned} t(1+t)^x r_n(1+t) &= \left(\sum_{m=0}^{\infty} \binom{x}{m} t^m \right) \left(\sum_{m=0}^n \rho_{n,m} t^{m+1} \right) = \sum_{m=0}^n \sum_{k=0}^{\infty} \rho_{n,m} \binom{x}{k} t^{m+k+1} \\ &= \sum_{m=0}^n \sum_{k=m+1}^{\infty} \rho_{n,m} \binom{x}{k-m-1} t^k = \sum_{k=1}^{\infty} \left(\sum_{m=0}^{\min\{n,k-1\}} \rho_{n,m} \binom{x}{k-m-1} \right) t^k. \end{aligned}$$

Secondly

$$\begin{aligned} s_n(1+t) \sum_{k=0}^{\infty} b_k(x) t^k &= \left(\sum_{m=0}^n \sigma_{n,m} t^m \right) \left(\sum_{k=0}^{\infty} b_k(x) t^k \right) = \sum_{m=0}^n \sum_{k=0}^{\infty} \sigma_{n,m} b_k(x) t^{m+k} \\ &= \sum_{m=0}^n \sum_{k=m}^{\infty} \sigma_{n,m} b_{k-m}(x) t^k = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\min\{n,k\}} \sigma_{n,m} b_{k-m}(x) \right) t^k. \end{aligned}$$

Finally, as $t \rightarrow 0$,

$$\sum_{k=1}^{\infty} \left(\sum_{m=0}^{\min\{n,k-1\}} \rho_{n,m} \binom{x}{k-m-1} \right) t^k + \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\min\{n,k\}} \sigma_{n,m} b_{k-m}(x) \right) t^k = O(t^{2n+1})$$

means that for $k = 1, 2, \dots, 2n$,

$$\sum_{m=0}^{\min\{n,k-1\}} \rho_{n,m} \binom{x}{k-m-1} + \sum_{m=0}^{\min\{n,k\}} \sigma_{n,m} b_{k-m}(x) = 0$$

and the proof is complete, when $\rho_{n,m}$ and $\sigma_{n,m}$ are replaced by their expressions. \square

2.1. Identities for Bernoulli numbers of the second kind. By specializing at $x = 0$, we obtain the following recurrence formulas for Bernoulli numbers of the second kind,

Corollary 2.1. *For positive integers k and n , with $k \leq 2n$, we have*

$$2 \sum_{m=0}^k \binom{n}{m} \binom{2n-m}{n} (H_n - H_{n-m}) b_{k-m} = - \binom{n}{k-1} \binom{2n-k+1}{n} \text{ for } k < n$$

$$\text{and } \sum_{m=0}^n \binom{n}{m} \binom{2n-m}{n} (H_n - H_{n-m}) b_{k-m} = 0 \text{ for } k \geq n .$$

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