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ZETA-LIKE MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

HUY HUNG LE AND TUAN NGO DAC

Abstract. We study multiple zeta values (MZV's) over the rational function field over $\mathbb{F}_q$ which were introduced by Thakur as analogues of classical multiple zeta values of Euler. In this paper we affirmatively solve a conjecture of Lara Rodriguez and Thakur which gives a full list of zeta-like MZV's of weight at most q^2 and depth 2. Further, we completely determine all zeta-like MZV's of weight at most q^2 and arbitrary depth. Our method is based on a criterion which is derived from the Anderson-Thakur motivic interpretation of MZV's and the Anderson-Brownawell-Papanikolas criterion for linear independence in positive characteristic.

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1. Introduction

1.1. Classical multiple zeta values.

Multiple zeta values of Euler (MZV's for short) are real numbers of the form

$$\zeta(n_1,\dots,n_r)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\dots k_r^{n_r}},\quad \text{where } n_i\geq 1, n_r\geq 2.$$

Here r is called the depth and $w=n_1+\cdots+n_r$ is called the weight of the presentation $\zeta(n_1,\dots,n_r)$. For $r=1$ we recover the special values $\zeta(n)$ for $n\geq 2$ of the Riemann zeta function. These values have been studied in different contexts with deep connections to mathematical physics, knot theory, mixed Tate motives,

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and modular forms (see the survey of Zagier [34] and the book of Burgos Gil and Fresan [9] for more details and more complete references).

Relations among MZV's have been studied extensively for the last three decades. Of particular interest, we are interested in two special relations that were discovered by Euler.

- The first one states that
$$\frac{\zeta(n)}{(2i\pi)^n} = -\frac{1}{2}\frac{B_n}{n!} \quad \text{for all } n \geq 2,\ n \equiv 0 \pmod 2,$$
where B_n denotes the nth Bernoulli number. We say that $\zeta(n)$ for $n \geq 2$ and n even is Eulerian.
- The second one is the following identity
$$\zeta(1,2) = \zeta(3).$$
We say that $\zeta(1,2)$ is zeta-like.

More generally, we say that a MZV $\zeta(n_1,\ldots,n_r)$ is *Eulerian* (resp. *zeta-like*) if $\zeta(n_1,\ldots,n_r)/(2i\pi)^{n_1+\cdots+n_r}$ (resp. $\zeta(n_1,\ldots,n_r)/\zeta(n_1+\cdots+n_r)$) is rational.

Until now, we have extremely limited knowledge about zeta-like MZV's. We refer the reader to [19, Remark after Conjecture 4.3] and [30, §7.5] for a discussion about the known Eulerian and zeta-like MZV's. We also mention that a sufficient condition for Eulerian MZV's in terms of motivic MZV's was given by Brown (see [7, Theorem 3.3]), but this condition is not completely effective (see [12, §1] for a detailed discussion).

1.2. **Characteristic p multiple zeta values (MZV).**

By a well-known analogy between the arithmetic of number fields and that of global function fields conceived in the 1930s by Carlitz, we now switch to the function field setting.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞. Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$, and by $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.

In [10] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n \in \mathbb{N}$ given by
$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$
which are analogues of classical special zeta values in the function field setting. Here A_+ denotes the set of monic polynomials in A. For any tuple of positive integers $\mathfrak{s} = (s_1,\ldots,s_r) \in \mathbb{N}^r$, Thakur [27] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1,\ldots,s_r)$ by
$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1}\ldots a_r^{s_r}} \in K_\infty$$
where the sum runs through the set of tuples $(a_1,\ldots,a_r) \in A_+^r$ with $\deg a_1 > \ldots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s}) = s_1 + \cdots + s_r$ the weight of $\zeta_A(\mathfrak{s})$.

We note that Carlitz zeta values are exactly depth one MZV's. Thakur [28] showed that all the MZV's do not vanish.

Since their introduction many works have revealed the importance of these values for both their independent interest and for their applications to a wide variety of arithmetic applications, see for example [3, 4, 5, 6, 12, 22, 23, 24, 25, 29, 32, 33]. We refer the reader to the excellent surveys of Thakur [30, 31] for more details and more complete references.

As in the classical setting one can argue that the main goal of this theory is to determine all algebraic relations over K among MZV's. It is worth noting that analogues of the aforementioned identities of Euler were proved:

- In 1935 Carlitz [10] introduced analogues of the Bernoulli numbers BC_n and proved (see also [15, §9.2])
$$\frac{\zeta_A(n)}{\widetilde{\pi}^n} = \frac{BC_n}{\Gamma_n} \quad \text{for all } n \geq 1,\ n \equiv 0 \pmod{q-1}.$$
Here $\widetilde{\pi}$ is the Carlitz period which is the analogue of $2i\pi$ (see [15, 27]), and $\Gamma_n \in A$ is the nth Carlitz factorial (see §2.3 for more details).
- In [29] Thakur proved
$$(\theta^q - \theta)\zeta_A(1, q-1) = \zeta_A(q).$$

More precisely, we say that a MZV $\zeta_A(s_1, \ldots, s_r)$ is *Eulerian* (resp. *zeta-like*) if $\zeta_A(s_1, \ldots, s_r)/\widetilde{\pi}^{s_1+\cdots+s_r}$ (resp. $\zeta_A(s_1, \ldots, s_r)/\zeta_A(s_1 + \cdots + s_r)$) belongs to K.

In [19, 30] Lara Rodriguez and Thakur proved some families of zeta-like MZV's and made several conjectures on zeta-like MZV's based on the numerical evidence, which will be discussed below.

1.3. **A conjecture of Lara Rodriguez and Thakur.**

In [19] Lara Rodriguez and Thakur showed (see [19, Remark p. 796]):

Theorem 1.1 (Lara Rodriguez-Thakur [19])**.** *Let $1 \leq i \leq q$ and $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$. Then $\zeta_A(i, j(q-1))$ is zeta-like.*

They conjectured that the converse also holds. A weak form of this conjecture was stated in [19, Conjecture 4.4]. Later, Thakur [30] gave a slightly stronger form which is given below (see [30, the discussion after Conjecture 7.3, p. 1010]).

Conjecture 1.2 (Lara Rodriguez-Thakur [19, 30])**.** *All zeta-like tuples of weight at most q^2 and depth 2 are exactly $(i, j(q-1))$ such that $1 \leq i \leq q$ and $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$.*

The proof of Theorem 1.1 is of algebraic nature and based on explicit formulas of power sums (see [19, §5]). We mention that Lara Rodriguez and Thakur have extended their result for a more general setting (see [20, 26]). On the other hand, the statement that there are no other zeta-like MZV's is of a different nature, which may need some elaborated transcendental tools.

1.4. **Statement of main results.**

We are ready to state the main results of our paper. First we present an affirmative answer to Conjecture 1.2.

Theorem 1.3. *All zeta-like tuples of weight at most q^2 and depth 2 are exactly $(i, j(q-1))$ such that $1 \leq i \leq q$ and $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$.*

Next we extend our method and prove a similar result for zeta-like MZV's of weight at most q^2 and depth 3.

Theorem 1.4. *All zeta-like tuples of weight at most q^2 and depth 3 are exactly $(1, q-1, q(q-1))$.*

In particular, there are no Eulerian MZV's of weight at most q^2 and depth 3.

Finally, we obtain a complete list of all zeta-like MZV's of weight at most q^2.

Theorem 1.5. *All zeta-like tuples of weight at most q^2 are exactly*

- *the tuples of depth 2: $(i, j(q-1))$ such that $1 \leq i \leq q$ and $i \leq j \leq \lfloor \frac{q^2-i}{q-1} \rfloor$,*
- *one tuple of depth 3: $(1, q-1, q(q-1))$.*

Let us briefly outline the main ideas of the proofs of Theorems 1.3, 1.4 and 1.5.

(1) First, by using a motivic interpretation of MZV's due to Anderson and Thakur in [4] and the Anderson-Brownawell-Papanikolas criterion for linear independence in positive characteristic in [2], Chang, Papanikolas and Yu [12] succeeded in devising a criterion called *the CPY criterion* deciding whether a MZV is zeta-like (resp. Eulerian). As a consequence we are led to find non-trivial solutions of a system of difference equations having Anderson-Thakur polynomials as parameters.
(2) Second, we apply the previous CPY criterion to determine all zeta-like MZV's of weight at most q^2 and depth 2. In order to do so we manage to completely solve the corresponding system of difference equations. We use explicit formulas for Anderson-Thakur polynomials of weight at most q^2 and carefully investigate both Eulerian and non-Eulerian cases. It settles Theorem 1.3.
(3) Third, we apply the CPY criterion to determine the full list of all zeta-like MZV's $\zeta_A(s_1, s_2, s_3)$ of weight at most q^2 and depth 3. We observe that the CPY criterion implies that $\zeta_A(s_2, s_3)$ is Eulerian. Thus by Theorem 1.3 above we obtain a very short list of (s_2, s_3). We then repeat the first two steps to determine s_1 and prove Theorem 1.4.
(4) Finally, we deduce from the CPY criterion and Theorem 1.4 that there are no zeta-like MZV's of weight at most q^2 and depth at least 4. Theorem 1.5 is shown, and we are all done.

1.5. **Organization of the paper.**

Our paper is organized as follows. In §2 we briefly review the CPY criterion deciding whether a MZV is zeta-like or Eulerian. We introduce the notion of dual t-motives and recall the work of Anderson and Thakur [4] connecting dual t-motives and MZV's. After recalling the Anderson-Brownawell-Papanikolas criterion in [2] we state the key CPY criterion deciding whether a MZV is zeta-like (resp. Eulerian). The rest of the paper is devoted to the proofs of the main results (see §3 for Theorem 1.3, §4 for Theorem 1.4, and §5 for Theorem 1.5, respectively). At the end we give some remarks in §6.

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2. A criterion for zeta-like and Eulerian MZV's

We continue with the notation given in the Introduction. Further, letting t be another independent variable, we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ equipped with the Gauss norm $\|.\|_\infty$, and by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

2.1. Dual t-motives.

We recall the notion of dual t-motives due to to Anderson (see [8, §4] and [17, §5] for more details). We refer the reader to [1] for the related notion of t-motives.

For $i \in \mathbb{N}$ we consider the i-fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t)) \\ f = \sum_j a_j t^j &\mapsto f^{(i)} := \sum_j a_j^{q^i} t^j. \end{aligned}$$

We extend i-fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise.

Definition 2.1. An effective dual t-motive is a $\overline{K}[t,\sigma]$-module $\mathcal{M}'$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t-\theta)^\ell(\mathcal{M}'/\sigma\mathcal{M}') = \{0\}.$$

We mention that effective dual t-motives are called Frobenius modules in [12, §2.2]. Note that Hartl and Juschka [17, §4] introduced a more general notion of dual t-motives. In particular, effective dual t-motives are always dual t-motives.

Throughout this paper we will always work with effective dual t-motives. Therefore, we will sometimes drop the word "effective" where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t-motives. Then a morphism of effective dual t-motives $\mathcal{M} \to \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t,\sigma]$-modules. We denote by $\mathcal{F}$ the category of effective dual t-motives equipped with the trivial object $\mathbf{1}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$-module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$-basis for $\mathcal{M}$. We recall that $\mathbb{L}$ denotes the fraction field of the Tate algebra $\mathbb{T}$. We say that an object $\mathcal{M}$ of $\mathcal{F}$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$. By [21, Proposition 3.3.9] there exists a rigid analytic trivialization Ψ_0 of $\mathcal{M}$ with $\Psi_0 \in \mathrm{GL}_r(\mathbb{T})$. Further, if Ψ is a rigid analytic trivialization of $\mathcal{M}$, then $\Psi = \Psi_0 M$ with $M \in \mathrm{Mat}_r(\mathbb{F}_q(t))$.

2.2. Ext^1-modules.

Let $\mathcal{M}'$ be an effective dual t-motive of rank r over $\overline{K}[t]$. We denote by $\Phi' \in \mathrm{Mat}_r(\overline{K}[t])$ the matrix defining the σ-action on $\mathcal{M}'$ with respect to some $\overline{K}[t]$-basis

of $\mathcal{M}'$. Let $\mathcal{M}$ be the dual t-motive given by the matrix

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix}, \quad \text{with } \mathbf{v} = (v_1, \dots, v_r) \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

We note that $\mathcal{M}$ fits into an exact sequence of the form

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathbf{1} \to 0,$$

and so is an extension of the trivial dual t-motive $\mathbf{1}$ by $\mathcal{M}'$, i.e. $\mathcal{M}$ represents a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$.

Note that $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ has a natural $\mathbb{F}_q[t]$-module structure defined as follows. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two objects of $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ defined by the matrices

$$\Phi_1 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_1 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_1 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]),$$

and

$$\Phi_2 = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]), \quad \mathbf{v}_2 \in \mathrm{Mat}_{1\times r}(\overline{K}[t]).$$

Then for any $a_1, a_2 \in \mathbb{F}_q[t]$, $a_1 * \mathcal{M}_1 + a_2 * \mathcal{M}_2$ is defined to be the class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}')$ represented by

$$\begin{pmatrix} \Phi' & 0 \\ a_1\mathbf{v}_1 + a_2\mathbf{v}_2 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

2.3. **Dual t-motives connected to MZV's.**

Following Anderson and Thakur [4] we introduce dual t-motives connected to MZV's. We briefly review Anderson-Thakur polynomials introduced in [3]. For $k \geq 0$, we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^{k} [\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$ we write $n - 1 = \sum_{j\geq 0} n_j q^j$ with $0 \leq n_j \leq q-1$ and define

$$\Gamma_n := \prod_{j\geq 0} D_j^{n_j}.$$

We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^{j} (\theta^{q^j} - t^{q^\ell})$ for $j \geq 1$. Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n\geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j\geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t

$$H_n(t) = \alpha_n(t)\big|_{t=\theta,\, \theta=t}. \tag{2.1}$$

By [3, Eq. (3.7.3)] we get $\|H_n\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$.

We consider the dual t-motives $\mathcal{M}_{\mathfrak{s}}$ and $\mathcal{M}'_{\mathfrak{s}}$ attached to $\mathfrak{s}$ given by the matrices

$$\Phi_{\mathfrak{s}} = \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ H_{s_1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ 0 & H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & H_{s_r}^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s}} \in \mathrm{Mat}_r(\overline{K}[t])$ is the upper left $r \times r$ sub-matrix of $\Phi_{\mathfrak{s}}$. Then $\mathcal{M}_{\mathfrak{s}}$ represents a class in $\mathrm{Ext}^1_{\mathcal{T}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.

Throughout this paper, we work with the Carlitz period $\widetilde{\pi}$ which is a fundamental period of the Carlitz module (see [15, 27]). We fix a choice of $(q-1)$st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t-\theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \widetilde{\pi}.$$

Given $\mathfrak{s}$ as above, Chang introduced the following series (see [11, Lemma 5.3.1] and also [12, Eq. (2.3.2)])

$$\mathfrak{L}(\mathfrak{s}) = \mathfrak{L}(s_1, \dots, s_r) := \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_r} H_{s_r})^{(i_r)} \dots (\Omega^{s_1} H_{s_1})^{(i_1)}. \tag{2.2}$$

Letting $\Gamma(\mathfrak{s}) = \Gamma_{s_1} \dots \Gamma_{s_r}$, by [11, Eq. (5.5.3)] we have

$$\mathfrak{L}(\mathfrak{s})(\theta) = \Gamma(\mathfrak{s})\zeta_A(\mathfrak{s})/\widetilde{\pi}^{w(\mathfrak{s})}. \tag{2.3}$$

In particular, $\mathfrak{L}(\mathfrak{s})(\theta)$ is non-zero since $\zeta_A(\mathfrak{s})$ is known to be non-zero by Thakur [28].

If we denote $\mathcal{E}$ the ring of series $\sum_{n \geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n \to +\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$, then any $f \in \mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}(\mathfrak{s}) \in \mathcal{E}$ (see [11, Lemma 5.3.1]).

Then the matrix given by

$$\Psi_{\mathfrak{s}} = \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1, \dots, s_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2, \dots, s_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1, \dots, s_r) & \mathfrak{L}(s_2, \dots, s_r) & \dots & \mathfrak{L}(s_r) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s}}^{(-1)} = \Phi_{\mathfrak{s}} \Psi_{\mathfrak{s}}.$$

Thus $\Psi_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}_{\mathfrak{s}}$.

We also denote by $\Psi'_{\mathfrak{s}}$ the upper $r \times r$ sub-matrix of $\Psi_{\mathfrak{s}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}'_{\mathfrak{s}}$.

To end this section, for $r \geq 2$ we let $\mathcal{N}_{\mathfrak{s}} \in \mathcal{F}$ be the dual t-motive defined by the matrix

$$\begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ H_{s_1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ 0 & H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ H_{w(\mathfrak{s})}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{r+1}(\overline{K}[t]).$$

Then $\mathcal{N}_{\mathfrak{s}}$ represents also a class in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.

2.4. **A criterion for zeta-like and Eulerian MZV's in positive characteristic.**

We recall the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

Theorem 2.2 (Anderson-Brownawell-Papanikolas)**.** *Let $\Phi \in \mathrm{Mat}_\ell(\overline{K}[t])$ be a matrix such that $\det \Phi = c(t-\theta)^s$ for some $c \in \overline{K}$ and $s \in \mathbb{Z}^{\geq 0}$. Let $\psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ be a vector satisfying $\psi^{(-1)} = \Phi\psi$ and $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{K})$ such that $\rho\psi(\theta) = 0$. Then there exists a vector $P \in \mathrm{Mat}_{1\times\ell}(\overline{K}[t])$ such that*

$$P\psi = 0 \quad \textit{and} \quad P(\theta) = \rho.$$

We now state the following result for zeta-like (resp. Eulerian) MZV's proved in [12].

Theorem 2.3 ([12], Theorems 4.4.2 and 4.2.2)**.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. Then $\zeta_A(\mathfrak{s})$ is zeta-like (resp. Eulerian) if and only if there exist $c, d \in \mathbb{F}_q[t]$ (resp. $c \in \mathbb{F}_q[t]$) with $c \neq 0$ such that $c * \mathcal{M}_{\mathfrak{s}} + d * \mathcal{N}_{\mathfrak{s}}$ (resp. $c * \mathcal{M}_{\mathfrak{s}}$) represents a trivial class in $Ext^1_{\mathcal{F}}(\mathbf{1}, \mathcal{M}'_{\mathfrak{s}})$.*

We stress that since $\zeta_A(\mathfrak{s})$ is non-zero, this result is an immediate consequence of [12, Theorem 2.5.2] whose key tool is the Anderson-Brownawell-Papanikolas (ABP) criterion as stated in Theorem 2.2. Roughly speaking, using rigid analytic trivializations $\Psi_{\mathfrak{s}}$ defined as above, one applies the ABP criterion to lift a K-linear relation among MZV's to a $\overline{K}[t]$-linear relation among corresponding series defined in (2.2), which gives enough information to conclude.

We also recall the following corollary of Theorem 2.3 which was also conjectured by Lara Rodriguez and Thakur (see [19, Conjecture 4.1, Part 2]).

Corollary 2.4 ([12], Corollary 4.4.1)**.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. Suppose that $\zeta_A(\mathfrak{s})$ is zeta-like. Then each of*

$$\zeta_A(s_2, \dots, s_r), \dots, \zeta_1(s_r)$$

is Eulerian.

In particular, each s_i is divisible by $q-1$ for all $2 \leq i \leq r$.

By [12, Remark 3.1] it implies the following criterion which will be used in the sequel.

Theorem 2.5 ([12]). *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ as above. Then $\zeta_A(\mathfrak{s})$ is zeta-like (resp. Eulerian) if and only if there exist $c, d \in \mathbb{F}_q[t]$ (resp. $c \in \mathbb{F}_q[t]$ and $d = 0$) with $c \neq 0$ and polynomials $\delta_1, \dots, \delta_r \in \overline{K}[t]$ such that*

$$\begin{aligned}
\delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w, \\
\delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} + \delta_3^{(-1)} H_{s_2}^{(-1)}(t-\theta)^{s_2+\cdots+s_r}, \\
&\cdots \\
\delta_{r-1} &= \delta_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} + \delta_r^{(-1)} H_{s_{r-1}}^{(-1)}(t-\theta)^{s_{r-1}+s_r}, \\
\delta_r &= \delta_r^{(-1)}(t-\theta)^{s_r} + cH_{s_r}^{(-1)}(t-\theta)^{s_r}.
\end{aligned} \tag{2.4}$$

Remark 2.6. 1) By [18, Theorem 2] we know that $\delta_1, \dots, \delta_r$ belong to $K[\theta] = \mathbb{F}_q[t, \theta]$ and

$$\deg_\theta \delta_i \leq \frac{q(s_i + \cdots + s_r)}{q-1}. \tag{2.5}$$

2) Note that if $(\delta_1, \dots, \delta_r, c, d) \in \mathbb{F}_q[\theta, t]^r \times \mathbb{F}_q[t]^2$ is a solution of the above system (2.4), then $(f\delta_1, \dots, f\delta_r, fc, fd) \in \mathbb{F}_q[\theta, t]^r \times \mathbb{F}_q[t]^2$ is also a solution of (2.4) for all $f \in \mathbb{F}_q[t]$.

3. Proof of Theorem 1.3

This section aims to present a proof of Theorem 1.3.

3.1. Setup.

Let $\mathfrak{s} = (s_1, s_2) \in \mathbb{N}^2$ with $s_1 + s_2 \leq q^2$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. By Corollary 2.4 we can write $s_2 = \ell_2(q-1)$ for some $\ell_2 \in \mathbb{N}$. It suffices to show that we cannot have $\ell_2 < s_1$ and $s_1 + \ell_2(q-1) \leq q^2$.

Suppose that we do have $\ell_2 < s_1$ and $s_1 + \ell_2(q-1) \leq q^2$. In particular,

$$\ell_2 < q \tag{3.1}$$

since $\ell_2 q < s_1 + \ell_2(q-1) \leq q^2$. We will deduce a contradiction.

We start proving some preliminary results in §3.2 and then obtain a contradiction by distinguishing two cases for the zeta-like MZV $\zeta_A(\mathfrak{s})$: the non-Eulerian case in §3.3 and the Eulerian case in §3.4. To do so we use two key ingredients: the bound given in (2.5) and the explicit formulas for Anderson-Thakur polynomials of weight at most q^2. Roughly speaking, we consider δ_1 as a polynomial in θ with coefficients in $\mathbb{F}_q[t]$. The explicit formulas for Anderson-Thakur polynomials of weight at most q^2 forces that δ_1 is divisible by a product of certain factors. Then we manage to prove that the latter product has degree strictly bigger than the bound given in (2.5) so that $\delta_1 = \delta_2 = 0$, which is a trivial solution, and we are done.

From now on, we will use capital characters or Greek characters (e.g. F, G, δ) for polynomials in $\mathbb{F}_q[t, \theta]$ and usual characters (e.g. f, g) for polynomials in $\mathbb{F}_q[t]$.

3.2. Preliminaries.

We first recall Lucas' theorem and refer the reader to [16] for more details.

Lemma 3.1. *For $m, n \in \mathbb{N}$, we express m and n in base q*

$$m = m_k q^k + \cdots + m_1 q + m_0,$$
$$n = n_k q^k + \cdots + n_1 q + n_0,$$

with $0 \leq m_j, n_j \leq q-1$. Then we have the following equality in $\mathbb{F}_p$

$$\binom{m}{n} = \prod_{j=0}^{k} \binom{m_j}{n_j}.$$

We now prove some preliminary results which will be used later.

Lemma 3.2. *Let $F \in \mathbb{F}_q[t, \theta]$ be a polynomial. Suppose that $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$. Then $F = (t-\theta)^{q-1} G$ for some $G \in \mathbb{F}_q[t, \theta^q]$.*

Proof. We first suppose that $\deg_\theta F < q$. We write $F = a_0 + a_1\theta + \cdots + a_{q-1}\theta^{q-1}$ with $a_0, \ldots, a_{q-1} \in \mathbb{F}_q[t]$. Then

$$F(t-\theta) = a_0 t + \sum_{j=1}^{q-1} (a_j t - a_{j-1})\theta^j + a_{q-1}\theta^q.$$

Since $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$, it follows that for all $1 \leq j \leq q-1$ we have $a_j t - a_{j-1} = 0$, that means $a_j = a_{q-1} t^{q-1-j}$. Therefore,

$$F = a_{q-1}(t^{q-1} + t^{q-1}\theta + \cdots + \theta^{q-1}) = a_{q-1}(t-\theta)^{q-1}.$$

Here the last equality follows from the fact that for all $0 \leq j \leq q-1$, we have $(-1)^{q-1-j}\binom{q-1}{j} = 1$ in $\mathbb{F}_p$. We put $G = a_{q-1} \in \mathbb{F}_q[t]$ and we are done in this case.

We now move to the general case. We can always write $F = F_0 + \theta^q F_1 + \cdots + \theta^{kq} F_k$ for some $k \in \mathbb{N}$ and some polynomials $F_0, \ldots, F_k \in \mathbb{F}_q[t, \theta]$ with $\deg_\theta F_j < q$ for all $0 \leq j \leq k$. The hypothesis $F(t-\theta) \in \mathbb{F}_q[t, \theta^q]$ implies that $F_j(t-\theta) \in \mathbb{F}_q[t, \theta^q]$ for all $0 \leq j \leq k$. Thus by the previous discussion we deduce that there exist $G_0, \ldots, G_k \in \mathbb{F}_q[t]$ such that $F_j = G_j(t-\theta)^{q-1}$ for all $0 \leq j \leq k$. Therefore,

$$\begin{aligned} F &= F_0 + \theta^q F_1 + \cdots + \theta^{kq} F_k \\ &= G_0(t-\theta)^{q-1} + \theta^q G_1(t-\theta)^{q-1} + \cdots + \theta^{kq} G_k(t-\theta)^{q-1} \\ &= (G_0 + \theta^q G_1 + \cdots + \theta^{kq} G_k)(t-\theta)^{q-1}. \end{aligned}$$

We put $G = G_0 + \theta^q G_1 + \cdots + \theta^{kq} G_k \in \mathbb{F}_q[t, \theta^q]$, and we are also done. □

As an immediate consequence we obtain the following result.

Lemma 3.3. *Let $k \in \mathbb{N}$ and $F \in \mathbb{F}_q[t, \theta]$ be a polynomial. Suppose that $F(t-\theta)^k \in \mathbb{F}_q[t, \theta^q]$. We denote by ℓ the unique integer such that $0 \leq \ell \leq q-1$ and $k + \ell \equiv 0 \pmod{q}$. Then $F = (t-\theta)^\ell G$ for some $G \in \mathbb{F}_q[t, \theta^q]$.*

Proof. The hypothesis $F(t-\theta)^k \in \mathbb{F}_q[t, \theta^q]$ implies that $F(t-\theta)^{q-\ell} \in \mathbb{F}_q[t, \theta^q]$. Thus $F(t-\theta)^{q-\ell-1}(t-\theta) \in \mathbb{F}_q[t, \theta^q]$. By Lemma 3.2 there exists $G \in \mathbb{F}_q[t, \theta^q]$ such that $F(t-\theta)^{q-\ell-1} = (t-\theta)^{q-1} G$, i.e. $F = (t-\theta)^\ell G$ as required. □

We now give explicit formulas for Anderson-Thakur polynomials H_n with $n \leq q^2$. By direct calculations we deduce from (2.1)

- for $1 \leq n \leq q$, we have $H_n(t) = 1$,

- for $q+1 \le n \le q^2$, we put $k = \lfloor \frac{n-1}{q} \rfloor$ and get

$$H_n(t) = \sum_{j=0}^{k} \binom{(n-1)-jq+j}{j} (t^q - t)^{k-j} (t^q - \theta^q)^j. \tag{3.2}$$

For example, if $q+1 \le n \le 2q$, then

$$H_n(t) = (t^q - t) + n(t^q - \theta^q).$$

Furthermore, we prove the following results (see also [14, Proposition 4.10]).

Lemma 3.4. *Let $n \in \mathbb{N}$ such that $n = \ell(q-1)$ with $1 \le \ell \le q-1$. Then*

$$H_n = H_{\ell(q-1)} = (-1)^\ell \frac{(t-\theta^q)^\ell - (t^q - \theta^q)^\ell}{t^q - t}.$$

Proof. Since $1 \le \ell \le q-1$, we get $n = \ell(q-1) \le q^2$, and

$$\lfloor \frac{n-1}{q} \rfloor = \lfloor \frac{\ell(q-1)-1}{q} \rfloor = \lfloor \ell - \frac{\ell+1}{q} \rfloor = \ell - 1.$$

We use (3.2) to obtain

$$\begin{aligned}
H_n(t) &= \sum_{j=0}^{\ell-1} \binom{(n-1)-jq+j}{j} (t^q - t)^{k-j} (t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{(\ell(q-1)-1)-jq+j}{j} (t^q - t)^{\ell-j-1} (t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{(\ell-1-j)q+q-\ell-1+j}{j} (t^q - t)^{\ell-j-1} (t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} \binom{q-\ell-1+j}{j} (t^q - t)^{\ell-j-1} (t^q - \theta^q)^j \\
&= \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell}{j} (t^q - t)^{\ell-j-1} (t^q - \theta^q)^j \\
&= (-1)^\ell \frac{(t-\theta^q)^\ell - (t^q - \theta^q)^\ell}{t^q - t}.
\end{aligned}$$

Here the fourth equality holds by Lucas' theorem (see Lemma 3.1), and the last equality follows from the binomial expansion of $(t-\theta^q)^\ell = ((t^q - \theta^q) - (t^q - t))^\ell$. The proof is finished. □

Lemma 3.5. *We put $n = q(q-1)$. Then*

$$H_n = H_{q(q-1)} = (t^q - t)^{q-2}.$$

Proof. We have

$$\lfloor \frac{n-1}{q} \rfloor = \lfloor \frac{q(q-1)-1}{q} \rfloor = q-2.$$

We use (3.2) to obtain

$$\begin{aligned}
H_n(t) &= \sum_{j=0}^{q-2} \binom{(n-1)-jq+j}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\
&= \sum_{j=0}^{q-2} \binom{(q(q-1)-1)-jq+j}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\
&= \sum_{j=0}^{q-2} \binom{(q-1-j)q+j-1}{j} (t^q-t)^{q-2-j}(t^q-\theta^q)^j \\
&= (t^q-t)^{q-2}.
\end{aligned}$$

Here the last equality follows from the fact that for all $1 \le j \le q-2$, by Lucas' theorem (see Lemma 3.1),

$$\binom{(q-1-j)q+j-1}{j} = \binom{j-1}{j} = 0.$$

The proof is finished. □

3.3. **The non-Eulerian case:** $(q-1) \nmid w$.

The proof in this case is divided into three steps. By Theorem 2.5 and Remark 2.6 there exist $c, d \in \mathbb{F}_q[t]$ with $c \neq 0$ such that there exist $\delta_1, \delta_2 \in K[t]$ verifying

$$\begin{aligned}
\delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w, \\
\delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}.
\end{aligned}$$

Step 1. We first compute the Anderson-Thakur polynomials.

- Since $1 < s_1 < w \le q^2$, by (3.2) we get explicit formulas for $H_{s_1}, H_w \in \mathbb{F}_q[t,\theta]$. Further,

$$\begin{aligned}
\deg_\theta H_{s_1} &\le q\lfloor \frac{s_1-1}{q} \rfloor, \\
\deg_\theta H_w &\le q\lfloor \frac{w-1}{q} \rfloor.
\end{aligned} \tag{3.3}$$

- We know that $1 \le \ell_2 < q$ by (3.1). By Lemma 3.4 we have

$$H_{s_2} = H_{\ell_2(q-1)} = (-1)^{\ell_2} \frac{(t-\theta^q)^{\ell_2} - (t^q-\theta^q)^{\ell_2}}{t^q-t}.$$

Step 2. We solve the second equation

$$\delta_2 = \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}$$

for suitable $c \in \mathbb{F}_q[t]$. By the above explicit formula for H_{s_2} it is clear that we can take

$$\begin{aligned}
c &= t^q - t, \\
\delta_2 &= (-1)^{\ell_2}(t-\theta)^{\ell_2 q} = (\theta-t)^{\ell_2 q}.
\end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor$. We now solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + f\delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w \tag{3.4}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \leq n_1$ by (2.5),
- $f, d \in \mathbb{F}_q[t]$ with $f \neq 0$.

Note that by Remark 2.6 we need an extra polynomial $f \in \mathbb{F}_q[t]$.

Since the right-hand side of (3.4) is divisible by $(t-\theta)^w$, it implies that δ_1 is also divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we write

$$\delta_1 = F(t-\theta)^w, \quad F \in \mathbb{F}_q[t, \theta],\ \deg_\theta(F) \leq n_1 - w.$$

Replacing this expression in (3.4) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w + f(\theta - t)^{\ell_2 q} H_{s_1} + dH_w. \tag{3.5}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$.

We claim that

$$\deg_\theta F(t-\theta)^w \leq \deg_\theta(f(\theta-t)^{\ell_2 q} H_{s_1} + dH_w).$$

Otherwise $\deg_\theta F(t-\theta)^w > \deg_\theta(f(\theta-t)^{\ell_2 q} H_{s_1} + dH_w)$. It follows that $\deg_\theta F^{(1)} = \deg_\theta(F(t-\theta)^w)$. We deduce $\deg_\theta F = w/(q-1)$, which implies that $q-1$ divides w. We obtain then a contradiction.

We write

$$s_1 = \ell_1 q + k_1 + 1$$

with $0 \leq k_1 \leq q-1$. Then

$$w = s_1 + \ell_2(q-1) = (\ell_1 + \ell_2)q + k_1 + 1 - \ell_2.$$

By (3.3) $\deg_\theta H_{s_1} \leq \ell_1 q$ and $\deg_\theta H_w \leq (\ell_1 + \ell_2)q$. It follows that

$$\deg_\theta F(t-\theta)^w \leq \deg_\theta(f(\theta-t)^{\ell_2 q} H_{s_1} + dH_w) \leq (\ell_1 + \ell_2)q.$$

In particular, $w \leq (\ell_1 + \ell_2)q$. Thus

$$k_1 + 1 \leq \ell_2 \tag{3.6}$$

Therefore, $\ell_1 \geq 1$ since $s_1 = \ell_1 q + k_1 + 1 > \ell_2$.

On the other hand, $\deg_\theta F(t-\theta)^w \geq w = (\ell_1+\ell_2)q + k_1 + 1 - \ell_2 > (\ell_1+\ell_2)q + 1 - q$. Then the polynomial $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$ satisfies

$$(\ell_1 + \ell_2)q + 1 - q < \deg_\theta F(t-\theta)^w \leq (\ell_1 + \ell_2)q.$$

By Lemma 3.3 we conclude that

$$\delta_1 = F(t-\theta)^w = g(t-\theta)^{(\ell_1+\ell_2)q}$$

for some $g \in \mathbb{F}_q[t]$.

Replacing this equality in (3.5) and using explicit formulas for H_{s_1} and H_w given in (3.2) yields

$$\begin{aligned}
&g(t-\theta^q)^{\ell_2-(k_1+1)}\\
&= g(t-\theta)^{(\ell_1+\ell_2)q} + f(\theta^q-t^q)^{\ell_2}\sum_{j=0}^{\ell_1}\binom{(\ell_1 q+k_1)-jq+j}{j}(t^q-t)^{\ell_1-j}(t^q-\theta^q)^j\\
&\quad + d\sum_{j=0}^{\ell_1+\ell_2-1}\binom{((\ell_1+\ell_2)q+k_1-\ell_2)-jq+j}{j}(t^q-t)^{\ell_1+\ell_2-1-j}(t^q-\theta^q)^j
\end{aligned}$$

We set

$$X := (t-\theta)^q = t^q-\theta^q$$

and rewrite the above equality as

$$\begin{aligned}
&g(X-(t^q-t))^{\ell_2-(k_1+1)}\\
&= gX^{\ell_1+\ell_2} + (-1)^{\ell_2} f X^{\ell_2}\sum_{j=0}^{\ell_1}\binom{\ell_1 q+k_1-jq+j}{j}(t^q-t)^{\ell_1-j}X^j\\
&\quad + d\sum_{j=0}^{\ell_1+\ell_2-1}\binom{(\ell_1+\ell_2)q+k_1-\ell_2-jq+j}{j}(t^q-t)^{\ell_1+\ell_2-1-j}X^j.
\end{aligned}$$

Comparing the coefficients of X^{ℓ_2} yields

$$\begin{aligned}
0 &= (-1)^{\ell_2} f(t^q-t)^{\ell_1} + d\binom{(\ell_1+\ell_2)q+k_1-\ell_2-\ell_2 q+\ell_2}{\ell_2}(t^q-t)^{\ell_1-1}\\
&= (-1)^{\ell_2} f(t^q-t)^{\ell_1} + d\binom{\ell_1 q+k_1}{\ell_2}(t^q-t)^{\ell_1-1}\\
&= (-1)^{\ell_2} f(t^q-t)^{\ell_1}.
\end{aligned}$$

Here the last equality holds by Lucas' theorem (see Lemma 3.1) since $\ell_2 < q$ by (3.1) and the fact that $k_1+1 \le \ell_2$ by (3.6). Thus $f=0$.

Next comparing the coefficients of $X^{\ell_1+\ell_2}$ yields

$$0 = g + (-1)^{\ell_2}\binom{\ell_1 q+k_1-\ell_1 q+\ell_1}{\ell_1} f.$$

Since $f=0$, it follows that $g=0$, which is a contradiction.

3.4. **The Eulerian case:** $(q-1) \mid w$.

Since $(q-1)\mid w$ and $s_2=\ell_2(q-1)$ for some $\ell_2\in\mathbb{N}$, it follows that $s_1=\ell_1(q-1)$ for some $\ell_1\in\mathbb{N}$. Since $s_1+s_2\le q^2$, we get $\ell_1+\ell_2\le q+1$.

As before, the proof in this case is also divided into three steps. By Theorem 2.5 and Remark 2.6 there exist $c\in\mathbb{F}_q[t]$ with $c\neq 0$ such that there exist $\delta_1,\delta_2\in K[t]$ verifying

$$\begin{aligned}
\delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)}H_{s_1}^{(-1)}(t-\theta)^w,\\
\delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}.
\end{aligned}$$

Step 1. We compute the Anderson-Thakur polynomials. By Lemma 3.4 and the fact that $\ell_2 < q$ by (3.1) we have

$$H_{s_2} = H_{\ell_2(q-1)} = (-1)^{\ell_2} \frac{(t-\theta^q)^{\ell_2} - (t^q-\theta^q)^{\ell_2 q}}{t^q - t}.$$

Step 2. We solve the equation

$$\delta_2 = \delta_2^{(-1)}(t-\theta)^{s_2} + cH_{s_2}^{(-1)}(t-\theta)^{s_2}$$

for suitable $c \in \mathbb{F}_q[t]$. As before, by the above explicit formula for H_{s_2} it is clear that we can take

$$\begin{aligned} c &= t^q - t, \\ \delta_2 &= (-1)^{\ell_2}(t-\theta)^{\ell_2 q} = (\theta - t)^{\ell_2 q}. \end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor = (\ell_1+\ell_2)q$. We have to solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} f H_{s_1}^{(-1)}(t-\theta)^w \tag{3.7}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \leq n_1$ by (2.5),
- $f \in \mathbb{F}_q[t]$ with $f \neq 0$.

We see that δ_1 is divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we can write

$$\delta_1 = F(t-\theta)^w, \quad F \in \mathbb{F}_q[t,\theta],\ \deg_\theta(F) \leq n_1 - w.$$

Replacing this expression in (3.7) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w + f(\theta-t)^{\ell_2 q} H_{s_1}(t^q - t). \tag{3.8}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$.

We distinguish four subcases.

3.4.1. **Subcase 1: $\ell_1 + \ell_2 < q$.**

Since $\ell_1 < \ell_1 + \ell_2 < q$, then by Lemma 3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q-\theta^q)^{\ell_1}}{t^q - t}.$$

We know $w = (q-1)(\ell_1+\ell_2) \leq \deg_\theta \delta_1 \leq n_1 = q(\ell_1+\ell_2)$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. By Lemma 3.3 we deduce that $\delta_1 = g(t-\theta)^{q(\ell_1+\ell_2)}$ with $g \in \mathbb{F}_q[t]$. Thus $F = g(t-\theta)^{\ell_1+\ell_2}$. Replacing it in (3.8) we get

$$g(t-\theta^q)^{\ell_1+\ell_2} = g(t-\theta)^{q(\ell_1+\ell_2)} + f(\theta-t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

If $f \neq 0$, then we obtain a contradiction since the right-hand side is divisible by $\theta - t$ but not the left-hand side.

3.4.2. **Subcase 2:** $\ell_1 + \ell_2 = q$ **with** $1 \leq \ell_2 < q$**.**

We have supposed that $\ell_2 < s_1 = \ell_1(q-1)$ (see §3.1). Thus $1 \leq \ell_2 \leq q-2$.

Since $\ell_1 q - \ell_2 < q$, then by Lemma 3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q - \theta^q)^{\ell_1}}{t^q - t}.$$

Since $w = (q-1)(\ell_1 + \ell_2) = (q-1)q$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$, it follows that $F \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta F \leq q$. By (3.8),

$$F^{(1)} = F(t-\theta)^{q(q-1)} + f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

The right-hand side is divisible by $(\theta - t)^{\ell_2 q}$. Thus F is divisible by $(t^q - \theta)^{\ell_2}$. Since $F \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta F \leq q$, we get $F = g(t^q - \theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q - \theta^q)^q = g(t^q - \theta)^q (t-\theta)^{q(q-1)} + f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Thus

$$g(t^q - t^{q^2})(t-\theta)^{q(q-1)} = f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Since $1 \leq \ell_2 \leq q-2$, we get $g = f = 0$, which is a contradiction.

3.4.3. **Subcase 3:** $\ell_1 + \ell_2 = q+1$ **with** $1 < \ell_2 < q$**.**

Since $\ell_1 = (q+1) - \ell_2 < q$, then by Lemma 3.4,

$$H_{s_1} = H_{\ell_1(q-1)} = (-1)^{\ell_1} \frac{(t-\theta^q)^{\ell_1} - (t^q - \theta^q)^{\ell_1}}{t^q - t}.$$

Since $\ell_1 + \ell_2 = q+1$, it follows that $w = (q-1)(\ell_1 + \ell_2) = q^2 - 1$. We know that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$. By Lemma 3.3 we get $\delta_1 = G(t-\theta)^{q^2}$ with $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$. Thus $F = G(t-\theta)$.

By (3.8),

$$G^{(1)}(t-\theta^q) = G(t-\theta)^{q^2} + f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

The right-hand side is divisible by $(t-\theta)^{\ell_2 q}$. It implies that G is divisible by $(t^q - \theta)^{\ell_2}$. Since $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$, then $G = g(t^q - \theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q - \theta^q)^q (t - \theta^q) = g(t^q - \theta)^q (t-\theta)^{q^2} + f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

We get

$$g(t^q - \theta^q)^q (t - t^{q^2}) = f(\theta - t)^{\ell_2 q}(-1)^{\ell_1}((t-\theta^q)^{\ell_1} - (t-\theta)^{\ell_1 q}).$$

Since $1 < \ell_2 < q$, comparing the power of $(t-\theta)$ yields $g = f = 0$, which is a contradiction.

3.4.4. **Subcase 4: $\ell_1 = q$ and $\ell_2 = 1$.**

The arguments are similar to those of Case 3 except the explicit formula for H_{s_1}. By Lemma 3.5 we have

$$H_{s_1} = H_{q(q-1)} = (t^q - t)^{q-2}.$$

Since $w = (q-1)(\ell_1 + \ell_2) = q^2 - 1$ and $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$, it follows that $\delta_1 = G(t-\theta)^{q^2}$ with $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$. Thus $F = G(t-\theta)$. By (3.8),

$$G^{(1)}(t - \theta^q) = G(t-\theta)^{q^2} + f(\theta - t)^q (t^q - t)^{q-1}.$$

The right-hand side is divisible by $(t-\theta)^q$. It follows that G is divisible by $(t^q - \theta)$. Since $G \in \mathbb{F}_q[t, \theta^q]$ and $\deg_\theta G \leq q$, $G = g(t^q - \theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q - \theta^q)^q (t - \theta^q) = g(t^q - \theta)^q (t-\theta)^{q^2} + f(\theta - t)^q (t^q - t)^{q-1}.$$

We get

$$g(t^q - \theta^q)^q (t - t^{q^2}) = f(\theta - t)^q (t^q - t)^{q-1}.$$

Comparing the power of $(t - \theta)$ yields $g = f = 0$, which is a contradiction.

To summarize, in all cases we obtain a contradiction. Then the proof of Theorem 1.3 is finished.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4.

Let $\mathfrak{s} = (s_1, s_2, s_3) \in \mathbb{N}^3$ with $s_1 + s_2 + s_3 \leq q^2$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. Corollary 2.4 implies that $\zeta_A(s_2, s_3)$ is Eulerian. By Theorem 1.3 either $(s_2, s_3) = (q-1, (q-1)^2)$ or $(s_2, s_3) = (q-1, q(q-1))$.

If $(s_2, s_3) = (q-1, q(q-1))$, then $s_1 \leq q^2 - s_2 - s_3 = 1$. Thus $s_1 = 1$. It turns out that $\zeta_A(1, q-1, q(q-1)$ is zeta-like (see [19, Theorem 3.2]), and we are done.

To conclude, we have to show that for all $1 \leq s_1 \leq q$, $\zeta_A(s_1, s_2, s_3)$ where $s_2 = q-1$ and $s_3 = (q-1)^2$ is not zeta-like. Suppose that it is not the case, i.e. $\zeta_A(s_1, s_2, s_3)$ is zeta-like where $1 \leq s_1 \leq q$, $s_2 = q-1$ and $s_3 = (q-1)^2$. Thus

$$w = s_1 + s_2 + s_3 = s_1 + q(q-1).$$

Lemma 4.1. *With the above notation, we have $H_{s_1} = 1$ and*

$$H_w(t) = \sum_{j=0}^{q-s_1} \binom{s_1 - 1 + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j. \tag{4.1}$$

Proof. Since $1 \leq s_1 \leq q$, $H_{s_1} = 1$, and it is clear that $\lfloor \frac{w-1}{q} \rfloor = \lfloor \frac{s_1 + q(q-1) - 1}{q} \rfloor = q - 1$. Thus by (3.2) we get

$$\begin{aligned} H_w(t) &= \sum_{j=0}^{q-1} \binom{(w-1) - jq + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j \\ &= \sum_{j=0}^{q-1} \binom{(s_1 + q(q-1) - 1) - jq + j}{j} (t^q - t)^{q-1-j} (t^q - \theta^q)^j. \end{aligned}$$

By Lucas' theorem (see Lemma 3.1), for all $0 \le j \le q-1$,

$$\binom{(s_1+q(q-1)-1)-jq+j}{j} = \binom{s_1-1+j}{j}.$$

Since $1 \le s_1 \le q$, this term is equal to 0 if $q-s_1+1 \le j \le q-1$. Thus

$$\begin{aligned} H_w(t) &= \sum_{j=0}^{q-1} \binom{(s_1+q(q-1)-1)-jq+j}{j} (t^q-t)^{q-1-j}(t^q-\theta^q)^j \\ &= \sum_{j=0}^{q-s_1} \binom{s_1-1+j}{j} (t^q-t)^{q-1-j}(t^q-\theta^q)^j. \end{aligned}$$

The proof is finished. □

By Theorem 2.5 and Remark 2.6 there exist $c, d \in \mathbb{F}_q[t]$ with $c \neq 0$ such that there exist $\delta_1, \delta_2, \delta_3 \in K[t]$ verifying

$$\begin{aligned} \delta_1 &= \delta_1^{(-1)}(t-\theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w, \\ \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+s_3} + \delta_3^{(-1)} H_{s_2}^{(-1)}(t-\theta)^{s_2+s_3}, \\ \delta_3 &= \delta_3^{(-1)}(t-\theta)^{s_3} + cH_{s_3}^{(-1)}(t-\theta)^{s_3}. \end{aligned}$$

As before, if $q-1 \mid w$, then we can suppose that $d=0$ (see Theorem 2.5) and divide the proof into three steps.

Step 1. We first compute the Anderson-Thakur polynomials. By Lemma 3.4,

$$\begin{aligned} H_{s_2} &= H_{q-1} = 1, \\ H_{s_3} &= H_{(q-1)^2} = \frac{(t-\theta^q)^{q-1} - (t^q-\theta^q)^{q-1}}{t^q-t}. \end{aligned}$$

Step 2. We solve the equations

$$\begin{aligned} \delta_2 &= \delta_2^{(-1)}(t-\theta)^{s_2+s_3} + \delta_3^{(-1)} H_{s_2}^{(-1)}(t-\theta)^{s_2+s_3}, \\ \delta_3 &= \delta_3^{(-1)}(t-\theta)^{s_3} + cH_{s_3}^{(-1)}(t-\theta)^{s_3}. \end{aligned}$$

for suitable $c \in \mathbb{F}_q[t]$. By the above explicit formula for H_{s_2} and H_{s_3} it is clear that we can take

$$\begin{aligned} c &= (t^q-t)^{q+1}, \\ \delta_3 &= (t^q-t)^q(t-\theta)^{(q-1)q}, \\ \delta_2 &= -(t^q-\theta)^q(t-\theta)^{(q-1)q}. \end{aligned}$$

Step 3. We put $n_1 = \lfloor \frac{qw}{q-1} \rfloor = \lfloor \frac{qs_1}{q-1} \rfloor + q^2$ and recall that $1 \le s_1 \le q$. We have to solve the equation

$$\delta_1 = \delta_1^{(-1)}(t-\theta)^w + f\delta_2^{(-1)} H_{s_1}^{(-1)}(t-\theta)^w + dH_w^{(-1)}(t-\theta)^w. \tag{4.2}$$

where

- $\delta_1 \in \mathbb{F}_q[\theta, t]$, $\delta_1 \neq 0$ and $\deg_\theta \delta_1 \le n_1$ by (2.5),
- $f, d \in \mathbb{F}_q[t]$ with $f \neq 0$.

We see that δ_1 is divisible by $(t-\theta)^w$. Since $\deg_\theta \delta_1 \leq n_1$, we write

$$\delta_1 = F(t-\theta)^w$$

for some $F \in \mathbb{F}_q[t,\theta]$ with $\deg_\theta(F) \leq n_1 - w = \lfloor \frac{s_1}{q-1} \rfloor + q$.

Replacing this expression in (4.2) and twisting one time yields

$$F^{(1)} = F(t-\theta)^w - f(t^q-\theta)^q(t-\theta)^{(q-1)q}H_{s_1} + dH_w. \tag{4.3}$$

It follows that $\delta_1 = F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. By Lemma 3.3 we get $F = G(t-\theta)^{q-s_1}$ with $G \in \mathbb{F}_q[t,\theta^q]$. In particular,

$$\deg_\theta G \leq \deg_\theta F - (q-s_1) \leq \lfloor \frac{s_1}{q-1} \rfloor + q - (q-s_1) = s_1 + \lfloor \frac{s_1}{q-1} \rfloor.$$

We distinguish three subcases.

Subcase 1: $1 \leq s_1 < q-1$**.**

In this case, since $\deg_\theta G \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor < q$ and $G \in \mathbb{F}_q[t,\theta^q]$, it follows that $G = g \in \mathbb{F}_q[t]$. Further, H_w is given as in (4.1). Putting all together into (4.3) we obtain

$$\begin{aligned} g(t-\theta^q)^{q-s_1} &= g(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q} \\ &\quad + d\sum_{j=0}^{q-s_1}\binom{s_1-1+j}{j}(t^q-t)^{q-1-j}(t^q-\theta^q)^j. \end{aligned}$$

We set

$$X := (t-\theta)^q = t^q - \theta^q$$

and rewrite the above equality as

$$\begin{aligned} &g(X-(t^q-t))^{q-s_1} \\ &= gX^q - f(t^{q^2}-t^q+X)X^{q-1} + d\sum_{j=0}^{q-s_1}\binom{s_1-1+j}{j}(t^q-t)^{q-1-j}X^j. \end{aligned}$$

We compare the coefficients of X^q yields $g = f$.

- If $1 < s_1 \leq q-1$, then comparing the coefficients of X^{q-1} yields $f = 0$, which is a contradiction.
- Otherwise $s_1 = 1$, then by replacing $g = f$ in the above equation we obtain

  $$f(X-(t^q-t))^{q-1} = -f(t^{q^2}-t^q)X^{q-1} + d\sum_{j=0}^{q-1}(t^q-t)^{q-1-j}X^j.$$

  We compare the constant coefficients and get $d = f$. Then using $d = f$ and looking at the coefficients of X^{q-1} yields $f = -f(t^{q^2}-t^q) + d = -f(t^{q^2}-t^q) + f$. Thus $f = 0$, and we also get a contradiction.

Subcase 2: $s_1 = q-1$**.**

In this case, $F = G(t-\theta)$ with $\deg_\theta G \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor = q$ and $G \in \mathbb{F}_q[t,\theta^q]$. Further, we know that $q-1 \mid w$, then we can suppose that $d = 0$ in Eq. (4.3) (see Theorem 2.5). Putting all together into (4.3) yields

$$G^{(1)}(t-\theta^q) = G(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

The right-hand side is divisible by $(t-\theta)^{(q-1)q}$. It implies that G is divisible by $(t^q-\theta)^{q-1}$. Since $G \in \mathbb{F}_q[t,\theta^q]$ and $\deg_\theta G \leq q$, then $G = g(t^q-\theta)^q$ with $g \in \mathbb{F}_q[t]$. Hence

$$g(t^q-\theta^q)^q(t-\theta^q) = g(t^q-\theta)^q(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

We get

$$g(t^q-\theta^q)^q(t-t^{q^2}) = -f(t^q-\theta)^q(t-\theta)^{(q-1)q}.$$

Comparing the power of $(t-\theta)$ yields $f=g=0$, which is a contradiction.

Subcase 3: $s_1 = q$**.**

In this case, we know that $F = G \in \mathbb{F}_q[t,\theta^q]$ with $\deg_\theta F \leq s_1 + \lfloor \frac{s_1}{q-1} \rfloor = q+1$. Thus we can write $F = a + \theta^q b$ with $a, b \in \mathbb{F}_q[t]$. Further, by (4.1) we get

$$H_w = H_{q^2} = (t^q-t)^{q-1}.$$

Putting all together into (4.3) yields

$$a + \theta^{q^2} b = (a+\theta^q b)(t-\theta)^{q^2} - f(t^q-\theta)^q(t-\theta)^{(q-1)q} + d(t^q-t)^{q-1}.$$

Comparing the coefficients of θ^{q^2+q} yields $b=0$. Then we use $b=0$ and compare the coefficients of $\theta^{(q-1)q}$ to get $0 = -f(t^{q^2}-t^q)$. Thus $f=0$, which is a contradiction.

To summarize, in all cases we obtain a contradiction. Then the proof of Theorem 1.4 is finished.

5. Proof of Theorem 1.5

In this short section we present a proof of Theorem 1.5.

It suffices to prove that there is no zeta-like MZV's of weight at most q^2 and depth at least 4. Suppose that it is not the case. Then there exists $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $s_1 + \cdots + s_r \leq q^2$ and $r \geq 4$ such that $\zeta_A(\mathfrak{s})$ is zeta-like. Corollary 2.4 implies that $\zeta_A(s_{r-2}, s_{r-1}, s_r)$ is Eulerian. By Theorem 1.4 this is impossible. Thus the proof of Theorem 1.5 is complete.

6. Final remarks

We end this paper with some remarks.

Remark 6.1. We refer the reader to [12, 18, 19] for numerous numerical data concerning zeta-like and Eulerian MZV's in positive characteristic.

Remark 6.2. In this paper we have succeeded in determining completely all zeta-like MZV's of weight at most q^2. Thus it is tempting to ask whether we could go further.

- Eulerian MZV's are at least conjecturally understood (see for example [12, §6.2]).
- However, one should be aware that there are plenty of zeta-like MZV's of weight greater than q^2 (see for example [13, 14, 18]). At the moment, it seems very difficult to formulate a conjecture in a reasonable way to include all these examples. We hope to work on this question in a future work.

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