Stability Analysis of Time-Varying Systems with Harmonic Oscillations Using IQC Frequency Domain Multipliers

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Abstract— **This paper considers stability analysis of uncertain and time-varying systems containing harmonic oscillations of the form cosine and sine simultaneously. The analysis is performed by investigating the stability of a feedback interconnection through Integral Quadratic Constraints (IQC) approach. A new class of IQC multipliers, derived in frequency domain, is proposed to deal with simultaneous cosine and sine harmonic oscillations. The proposed multiplier reduces the conservatism with respect to other multipliers by exploiting frequency and phase shift information. A multiplier parameterization admitting real state-space realization is proposed. Such a parameterization allows to perform stability analysis through a finite-dimensional LMI optimization problem with a reduced number of decision variables.**

I. INTRODUCTION

Systems operating with modulated signals are found in many engineering fields such as electrical engineering, telecommunications and microelectronics. Some of them operates in feedback loop as, for example, MEMS vibratory gyroscopes [1]. They use two mechanical resonators (modes) to measure the angular rate of an object. If one mode (drive mode) vibrates, Coriolis effect arises in the presence of angular movement, transferring the vibrations to the second mode (sense mode). Vibrations in the sense mode are then measured to obtain an image of the angular rate. The quality of the measure (precision, SNR) depends on the constancy of the frequency and amplitude of the oscillations. Hence, drive mode is operated in closed-loop to ensure that its displacement tracks a sinusoidal reference signal of frequency close to the natural frequency of the resonator.

MEMS vibratory gyroscopes are systems operating naturally with modulated signals at relatively high frequencies. From implementation reasons, sometimes it can be convenient to consider a baseband controller implementation [2]. In such case, instead of ensuring the output to follow the reference signal, the controller maintains the phase and amplitude of the output harmonic signals, demanding to track only constant references. This approach is possible thanks to synchronous demodulators in the feedback loop [3]. If we consider synchronous demodulation with ideal low-pass filters, the controller can be computed through a linear control design problem [4]. If non-ideal filters are taking into account, the system can be modeled as the interconnection of a Linear Time-Invariant (LTI) system with an operator containing cosine and sine functions of same frequency simultaneously. Then, we need to analyse the effect of

such harmonic oscillations on the feedback system stability. Robustness analysis framework offers appropriate tools to establish analysis tests with complete guarantees and reduced conservatism. In addition, MEMS sensors are particularly sensitive to different phenomena, resulting in systems which tend to be naturally uncertain. Therefore, the considered tool has to be modular, allowing the combination of the harmonic time-varying elements with other uncertainties.

Systems containing time-varying harmonic oscillations belong to the larger class of Linear Periodically Time-Varying (LPTV) systems. There exists several stability analysis results for LPTV systems based on Floquet theory [5], [6]. However, Floquet theory is generally used to test nominal stability, with no consideration of uncertainty. Analysis of uncertain LPTV systems dates from the 90's. Some approaches use the ν-gap metric [7] which considers only a particular type of unstructured uncertainty. Other approaches employs the lifting technique to obtain an equivalent discrete-time but time-invariant uncertain system [8], which can be analyzed using techniques for LTI uncertain systems, such as μ-analysis [9]. Unfortunately, the equivalent system demands to solve high-dimensional problems.

Integral Quadratic Constrains (IQC) approach has shown to be an attractive tool for the analysis of a large class of systems. This is due to its flexibility when dealing with several uncertainties, time-varying elements and non-linearities, as well as the possibility to obtain tests in the form of convex optimization problems, see [10] and [11]. The work of [12] proposed two approaches for the analysis of LPTV systems. The first approach uses a Fourier series expansion and represents the systems as a Linear Fractional Representation (LFR) whose "uncertain bloc" contains the harmonic functions and uncertainties; however, information about the harmonic frequency is not exploded. In fact, the authors argue that this approach offered more conservative results due to the lack of good IQCs for the harmonics. The second approach keeps the harmonics in the nominal system and derives periodic multipliers to establish an infinite-dimensional analysis test. In both cases, time-domain multipliers are derived. Then, [13] proposed a cutting-plane algorithm to solve the problem of the second approach of [12], and this work is extended to the study of forced periodic systems in [14]. The work of [10] includes a frequency domain multiplier for the analysis of the feedback interconnection of an LTI system and a repeated cosine gain. Nevertheless, no parameterization admitting a suitable state-space realization has been proposed for this type of multiplier, remaining unexploited in a tractable stability analysis test.

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The present work proposes a new class of multipliers that extends the the multiplier of [10] to the case of harmonic operators containing cosine and sine simultaneously. Such multiplier reduces conservatism of stability analysis with respect to other available multipliers. We also propose a parameterization for this multiplier which, by admitting a real state-space realization, allows to obtain tractable conditions in the form of Linear Matrix Inequality (LMI) constraints with real matrices and compact number of decision variables. The formalization of the system and problem under consideration are presented in Section II. The proposed multiplier is introduced in Section III. An appropriate parameterization for the multiplier is presented in Section IV. The main stability result for the considered system is given in Section V. Finally, an application case is introduced in Section VI.

Notations: $j = \sqrt{-1}$. $\widehat{y}$ is the Fourier transform of y. $\bar{y}$ denotes the complex conjugate of y. A^T is the transpose of A. A^* is the Hermitian conjugate of A. $\bar{\sigma}(A)$ is the maximal singular value of A. I_n is the identity matrix in $\mathbb{R}^{n \times n}$. I and 0 in matrices denote the identity and zero matrices respectively of dimensions given by the context. $diag(A_1, ... A_n)$ is the diagonal concatenation of matrices A_1 until A_n. $A \succ 0$ ($A \succeq 0$) means that A is positive definite (semidefinite); and $A \prec 0$ ($A \preceq 0$) means that A is negative definite (semidefinite). $\otimes$ denotes the Kronecker product. $\mathscr{L}_2^n$ is the space of square integrable $\mathbb{R}^n$-valued functions. $\mathscr{L}_{2e}^n$ is the space of $\mathbb{R}^n$-valued functions whose truncation is square integrable. $G : \mathscr{L}_{2e}^{n_q} \to \mathscr{L}_{2e}^{n_p}$ is an operator taking any input $q \in \mathscr{L}_{2e}^{n_q}$ into the output $p = G(q) \in \mathscr{L}_{2e}^{n_p}$. In the case of linear operators, the output is $p = Gq$. $\mathscr{RL}_\infty^{m \times n}$ is the space of real-rational and proper transfer function matrices of dimension $m \times n$ without poles on the extended imaginary axis. $\mathscr{RH}_\infty^{m \times n} \subset \mathscr{RL}_\infty^{m \times n}$ consists of transfer function matrices without poles on the closed right-half complex plane. In the case of cumbersome quadratic forms, we use $\star$ to abbreviate the notation (i.e. for G containing several elements, $G^* M G$ is written $[\star]^* M G$). The state-space realizations of transfer functions matrices, i.e. $G(s)$, are denoted $G \sim (A, B, C, D)$ or $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$.

II. PROBLEM FORMULATION

Let us consider dynamic systems represented by an interconnection (G, Δ) of a so called "nominal system" G, and an operator Δ containing the "problematic elements" of the system (uncertainties, time-varying, non-linearities). The interconnection (G, Δ) is defined by the following equations:

$$q = Gp + q_{in}, \quad p = \Delta(q), \qquad (1)$$

where $q_{in} \in \mathscr{L}_{2e}^{n_q}$ represents an exogenous input, $G \in \mathscr{RH}_\infty^{n_p \times n_q}$ is a stable LTI system, $\Delta : \mathscr{L}_{2e}^{n_q} \to \mathscr{L}_{2e}^{n_p}$ is a bounded and causal operator belonging to a specific set $\mathbf{\Delta}$ that defines the nature and structure of the problematic elements. Synchronous demodulation operation can be modeled as a system represented by the interconnection (1) with $\Delta(q) = \theta(t, \omega_0) q(t)$ where θ is

$$\theta(t, \omega_0) = \left[\begin{array}{cc} \cos(\omega_0 t) I_{n_{q_1}} & 0 \\ 0 & \sin(\omega_0 t) I_{n_{q_2}} \end{array} \right] \qquad (2)$$

with $p = [p_1 \ p_2]^T$, $q = [q_1 \ q_2]^T$, $n_{q1} = n_{q2}$, $n_{p1} = n_{p2}$ and ω_0 being the frequency of the harmonic oscillations. The objective of this paper is then stated as follows:

Problem 1. *Given the system defined by the interconnection (1), with $\Delta(q) = \theta(t, \omega_0) q(t)$ defined by (2), test the stability of the feedback interconnection.*

As previously mentioned, robustness analysis tools seem to be appropriate to perform the stability analysis of such a system, more specifically, IQC approach is considered in this work. In robust stability analysis, the main objective is to evaluate whether or not the feedback interconnection (G, Δ) remains stable for all $\Delta \in \mathbf{\Delta}$. Let us first define some preliminary concepts. The interconnection (G, Δ) is well-posed if, for every input $q_{in} \in \mathscr{L}_{2e}^{n_q}$, the equations of (1) admit a unique solution q on $\mathscr{L}_{2e}$; this means that $(I - \Delta G)$ has a causal inverse. The interconnection (G, Δ) is stable if it is well-posed and if $(I - \Delta G)^{-1}$ is bounded. The interconnection (G, Δ) is robustly stable if it is stable for all $\Delta \in \mathbf{\Delta}$. The IQC approach proposed in [10] allows to verify stability of the interconnection (1) by studying G and Δ separately using Integral Quadratic Constrains (IQCs). The idea behind is to capture the main features of Δ by constraining input-output energetic relations.

Definition 1. *Considering $p = \Delta(q)$, Δ is said to satisfy the IQC defined by the multiplier Π ($\Delta \in IQC(\Pi)$) if, for all $q \in \mathscr{L}_2^{n_q}$,*

$$\int_{-\infty}^{\infty} \left[\begin{array}{c} \star \\ \star \end{array} \right]^* \left[\begin{array}{cc} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{array} \right] \left[\begin{array}{c} \widehat{q}(j\omega) \\ \widehat{p}(j\omega) \end{array} \right] d\omega \geq 0. \qquad (3)$$

In general Π can be any linear, bounded Hermitian-valued operator. To evaluate robustness property, we construct a whole set of multipliers $\mathbf{\Pi}$ such that the IQC (3) holds for all $\Pi \in \mathbf{\Pi}$ and for all $\Delta \in \mathbf{\Delta}$. We then state the following result concerning stability analysis.

Theorem 1. *Consider the interconnection (G, Δ) of (1) with $G \in \mathscr{RH}_\infty^{n_q \times n_p}$ and Δ a bounded and causal operator belonging to a "star-shaped" set $\mathbf{\Delta}$, i. e. if $\Delta \in \mathbf{\Delta} \Rightarrow \forall \ \tau \in [0, 1] \ \tau \Delta \in \mathbf{\Delta}$.*
Assume that
1) for all $\Delta \in \mathbf{\Delta}$ the interconnection (1) is well-posed;
2) for all $\Delta \in \mathbf{\Delta}$ and for all $\Pi \in \mathbf{\Pi}$ the IQC (3) is satisfied.
Then, the interconnection (1) is robustly stable if there exists a $\Pi \in \mathbf{\Pi}$ and $\varepsilon > 0$ satisfying the frequency domain inequality

$$\left[\begin{array}{c} G(j\omega) \\ I \end{array} \right]^* \Pi(j\omega) \left[\begin{array}{c} G(j\omega) \\ I \end{array} \right] \preceq -\varepsilon I \quad \forall \omega \in \mathbb{R}. \qquad (4)$$

Despite the importance of the previous result, as presented, this is an infinite-dimensional problem since there are an infinite number of multipliers satisfying (3); moreover, the inequality must be satisfied for all $\omega \in \mathbb{R}$. To deal with these aspects, we first constraint the multipliers to be rational functions $\Pi \in \mathscr{RL}_\infty$ of finite dimension that accept a suitable parameterization so that we obtain a frequency-dependent constraint on a limited set of parameters. Here, we consider sets

of multipliers parameterized as $\Pi = \{\Pi = \Phi^* M \Phi |\ M \in \boldsymbol{M}\}$, with $\boldsymbol{M}$ being a set of real symmetric matrices $M = M^T$ and $\Phi \in \mathcal{RH}_\infty^{\bullet \times n_p}$ a given transfer function matrix. Secondly, with $H = \Phi \begin{bmatrix} G \\ I \end{bmatrix}$ we exploit the Kalman-Yakubovich-Popov (KYP) lemma [15] in order to obtain an LMI constraint that satisfies (4) for all $\omega \in \mathbb{R}$.

Lemma 1. *Let $H \in \mathcal{RL}_\infty^{\bullet \times \bullet}$ admit the minimal realization (A, B, C, D) with $A \in \mathbb{R}^{n_h \times n_h}$, and let be $M = M^T \in \mathbb{R}^{\bullet \times \bullet}$. Then the following two statements are equivalent*

1) $H(j\omega)^* M H(j\omega) \prec 0 \quad \forall\ \omega \in \mathbb{R}$.
2) *There exists a matrix $P = P^T \in \mathbb{R}^{n_h \times n_h}$ such that*

$$\begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ C & D \end{bmatrix} \prec 0 \quad (5)$$

In this context, the chosen set of multipliers Π plays a central role on the conservatism of the stability result. It is necessary to find a suitable parameterization of Π in order to obtain tractable conditions in the form of LMI constrains. We aim to test stability of system (1) when the operator Δ is composed of the harmonic time-varying elements given in (2). To do so, the result of Theorem 1 is exploited. Therefore, we define the set

$$\boldsymbol{\Delta} = \{\Delta : \Delta(q) = \tau \theta(t, \omega_0) q(t), \tau \in [0, 1]\} \quad (6)$$

Theorem 1 provides stability conditions with respect to all $\Delta \in IQC(\Pi)$ for all $\Pi \in \boldsymbol{\Pi}$. In order to reduce conservatism of the analysis results for the considered Δ, a suitable set $\boldsymbol{\Pi}$ is proposed in following section.

III. COUPLED HARMONIC MULTIPLIER

Several IQC multipliers can be used to characterize the operator defined by (2). The next theorem proposes a new class of multipliers that are specially derived for harmonic time-varying elements.

Theorem 2. *For $\Delta(q) = \theta(t, \omega_0) q(t)$, with $\theta(t, \omega_0)$ defined in (2), the IQC (3) holds with Π belonging to the following set of multipliers*

$$\boldsymbol{\Pi} = \left\{ \Pi : \Pi(j\omega) = \begin{bmatrix} \tilde{X}_{D+}(j\omega) + \tilde{X}_{D-}(j\omega) & 0 \\ 0 & -2X_D(j\omega) \end{bmatrix} \right\} \quad (7)$$

with $X_D(j\omega)$ being Hermitian-valued and positive definite,

$$\tilde{X}_{D+}(j\omega) = \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix} X_D(j(\omega + \omega_0)) \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix},$$

$$\tilde{X}_{D-}(j\omega) = \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix} X_D(j(\omega - \omega_0)) \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix}.$$

Proof. The input-output relation of the operator θ of (2) can be rewritten in its exponential form

$$\begin{aligned} p_1(t) &= \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right) q_1(t) \\ p_2(t) &= \frac{1}{2} j \left(e^{-j\omega_0 t} - e^{j\omega_0 t} \right) q_2(t) \end{aligned} \quad (8)$$

Let us use a hierarchical approach by defining the sub components $p_{up} = \theta_{up} q_{up} = e^{j\omega_0 t} \begin{bmatrix} I_{n_{q_1}} & 0 \\ 0 & -jI_{n_{q_2}} \end{bmatrix} q_{up}$ and $p_{low} = \theta_{low} q_{low} = e^{-j\omega_0 t} \begin{bmatrix} I_{n_{q_1}} & 0 \\ 0 & jI_{n_{q_2}} \end{bmatrix} q_{low}$, so $p = \frac{1}{2}(p_{up} + p_{low})$. This configuration of θ is shown on Figure 1. Let us consider θ_{up} with q_{up} and p_{up} its input and
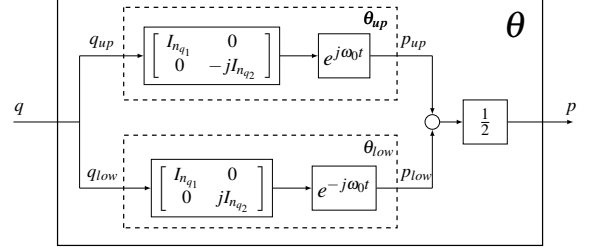

Fig. 1. Exponential form of θ

output signals respectively as shown in Figure 2 a). Here, we analyze the input-output relation of θ_{up} in the frequency domain. The operator θ_{up} is equivalent to $\tilde{\theta}_{up} = T^{-1} T \theta_{up}$,
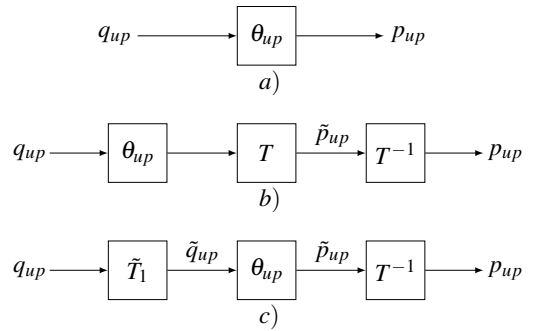

Fig. 2. Equivalent representation of θ_{up}

as presented in Figure 2 b), with $T(j\omega) \in \mathbb{C}^{n_q \times n_q}$ any invertible complex matrix. In turn, this is equivalent to $\bar{\theta} = T^{-1} \theta_{up} \tilde{T}_1$ as shown in 2 c), with $\tilde{T}_1(j\omega)$ given by

$$\tilde{T}_1 := \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix} T(j(\omega - \omega_0)) \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix} \quad (9)$$

In time domain we have $\bar{\sigma}(\theta_{up}) \leq 1$, which implies, for $\tilde{p}_{up} = \theta_{up} \tilde{q}_{up}$,

$$\int_0^\infty \tilde{p}_{up}(t)^T \tilde{p}_{up}(t) dt \leq \int_0^\infty \tilde{q}_{up}(t)^T \tilde{q}_{up}(t) dt \quad (10)$$

which, considering Parseval's equality, is equivalent to

$$\int_{-\infty}^\infty \widehat{\tilde{p}}_{up}(j\omega)^* \widehat{\tilde{p}}_{up}(j\omega) d\omega \leq \int_{-\infty}^\infty \widehat{\tilde{q}}_{up}(j\omega)^* \widehat{\tilde{q}}_{up}(j\omega) d\omega \quad (11)$$

We know that $\widehat{\tilde{q}}_{up}(j\omega) = \tilde{T}_1(j\omega) \widehat{q}_{up}(j\omega)$ and $\widehat{\tilde{p}}_{up} = \tilde{T}(j\omega) \widehat{p}_{up}(j\omega)$, which leads to the inequality

$$\begin{aligned} \int_{-\infty}^\infty &\widehat{p}_{up}(j\omega)^* T(j\omega)^* T(j\omega) \widehat{p}_{up}(j\omega) d\omega \leq \\ &\int_{-\infty}^\infty \widehat{q}_{up}(j\omega)^* \tilde{T}_1(j\omega)^* \tilde{T}_1(j\omega) \widehat{q}_{up}(j\omega) d\omega \end{aligned} \quad (12)$$

Let us define $X_D(j\omega) = T(j\omega)^*T(j\omega)$, so that $X_D(j\omega) = X_D(j\omega)^* \succ 0$. Similarly for $X_{D+} = T(j(\omega + \omega_0))^*T(j(\omega + \omega_0))$ and $X_{D-} = T(j(\omega - \omega_0))^*T(j(\omega - \omega_0))$. We have also that

$$\tilde{X}_{D-} = \tilde{T}_1(j\omega)^*\tilde{T}_1(j\omega) = \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix} X_{D-} \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix} \quad (13)$$

which yields to the inequality

$$\int_{-\infty}^{\infty} \widehat{p}_{up}(j\omega)^*X_D(j\omega)\widehat{p}_{up}(j\omega)d\omega \leq \\ \int_{-\infty}^{\infty} \widehat{q}_{up}(j\omega)^*\tilde{X}_{D-}(j\omega)\widehat{q}_{up}(j\omega)d\omega \quad (14)$$

Using a similar procedure, we obtain the following inequality for the lower branch

$$\int_{-\infty}^{\infty} \widehat{p}_{low}(j\omega)^*X_D(j\omega)\widehat{p}_{low}(j\omega)d\omega \leq \\ \int_{-\infty}^{\infty} \widehat{q}_{low}(j\omega)^*\tilde{X}_{D+}(j\omega)\widehat{q}_{low}(j\omega)d\omega \quad (15)$$

We can add the inequalities (14) and (15) and exploit the following arguments

- $q_{up} = q_{low} = q$ and $2p = p_{up} + p_{low}$.
- $\int_{-\infty}^{\infty} \widehat{p}^*X_D\widehat{p}d\omega$ is a convex function of $\widehat{p}$, then we can easily show that $\frac{1}{2}\int_{-\infty}^{\infty}(\widehat{p}_{up} + \widehat{p}_{low})X_D(\widehat{p}_{up} + \widehat{p}_{low})d\omega \leq \int_{-\infty}^{\infty}(\widehat{p}_{up}X_D\widehat{p}_{up} + \widehat{p}_{low}X_D\widehat{p}_{low})d\omega$.

leading to an IQC defined by the multiplier Π of (7) ($\theta \in IQC(\Pi)$). $\qquad\square$

Alternative multipliers can be obtained by noting that $\theta(t, \omega_0)$ belongs to other larger sets, for instance

- $\boldsymbol{\Delta} = \{\Delta : \Delta(q)(t) = diag(\theta_1(t)I_{n_{q_1}}, \theta_2(t)I_{n_{q_2}})q(t)\}$, with $|\theta_1(t)| \leq 1$ and $|\theta_2(t)| \leq 1$ being bounded arbitrary time-varying parameters. In such case, $\boldsymbol{\Pi}$ of (7) is defined with $\tilde{X}_{D+} = \tilde{X}_{D-} = X_D = diag(X_{D1}, X_{D2}) \in \mathbb{R}^{n_q \times n_q}$ ($n_q = n_p$) with $X_D = X_D^T \succ 0$, leading to

$$\Pi = \begin{bmatrix} X_D & 0 \\ 0 & -X_D \end{bmatrix} \quad (16)$$

which is (without consideration of G-scaling [10]) the multiplier generally used for arbitrary time-varying parameters. In the sequel this multiplier is referred to as General Time-Varying Multiplier.

- $\boldsymbol{\Delta} = \{\Delta : \Delta(q)(t) = diag(\cos(\omega_0 t + \varphi_1)I_{n_{q_1}}, \cos(\omega_0 t + \varphi_2)I_{n_{q_2}})q(t)\}$ is the set containing two uncoupled harmonic oscillations, with φ_1 and φ_2 being arbitrary phase shifts. Hence, $\boldsymbol{\Pi}$ of (7) is defined with $\tilde{X}_{D+} = X_{D+} = diag(X_{D1+}, X_{D2+}) = diag(X_{D1}(j(\omega + \omega_0)), X_{D2}(j(\omega + \omega_0)))$, $\tilde{X}_{D-} = X_{D-} = diag(X_{D1-}, X_{D2-}) = diag(X_{D1}(j(\omega - \omega_0)), X_{D2}(j(\omega - \omega_0)))$ and $X_D = diag(X_{D1}(j\omega), X_{D2}(j\omega))$, leading to

$$\Pi(j\omega) = \begin{bmatrix} X_{D+}(j\omega) + X_{D-}(j\omega) & 0 \\ 0 & -2X_D(j\omega) \end{bmatrix} \quad (17)$$

In the sequel this multiplier is referred to as Separated Harmonic Multiplier.

The General Time-Varying Multiplier defined in (16) takes into account the bounded nature of $\theta(t, \omega_0)$, but not the fact

that $\theta(t, \omega_0)$ is an harmonic function at the frequency ω_0. In the other side, Separated Harmonic Multiplier in (17) considers the harmonic nature of $\theta(t, \omega_0)$, but it does not take into account the $\pi/2$ phase shift between cosine and sine functions. The Coupled Harmonic Multiplier presented in Theorem 2 exploits the bounds, frequency and phase shift characterizing $\theta(t, \omega_0)$. Section VI presents an example showing how important is to consider this information.

IV. PARAMETERIZATION

Multipliers need an appropriate parameterization to be exploitable in the form of finite-dimensional LMI optimization problems. For example, in the case of traditional D, G scaling, parameterization of an Hermitian positive definite matrix is well known. However, such parameterization is not always obvious to obtain, especially when particular multiplier structures arise as in the current case. The following lemma proposes a parameterization for the multiplier of (7), defining the subset $\underline{\boldsymbol{\Pi}} \subset \boldsymbol{\Pi}$.

Lemma 2. *For $\Delta(q) = \theta(t, \omega_0)q(t)$, with $\theta(t, \omega_0)$ defined in (2), the IQC (3) holds with Π belonging to the subset $\underline{\boldsymbol{\Pi}}$ of (7)*

$$\underline{\boldsymbol{\Pi}} = \left\{ \Pi : \Pi(j\omega) = \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}^* \tilde{M}_D \begin{bmatrix} \tilde{\Psi}_+ + \tilde{\Psi}_- & 0 \\ -j(\tilde{\Psi}_+ - \tilde{\Psi}_-) & 0 \\ 0 & \Psi \end{bmatrix} \right\}, \quad (18)$$

$$\tilde{\Psi}_+ := \Psi(j(\omega + \omega_0)) \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix},$$

$$\tilde{\Psi}_- := \Psi(j(\omega - \omega_0)) \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix},$$

$\tilde{M}_D = diag(\frac{1}{2}M_D, \frac{1}{2}M_D, -2M_D)$, *if $M_D \in \boldsymbol{M_D}$, where $\boldsymbol{M_D} = \{M_D : M_D = M_D^T, \Psi^*M_D\Psi \succ 0 \ \forall \ \omega\}$ and $\Psi \in \mathscr{RH}_\infty^{(v+1) \times n_p}$ is a given fixed-basis function of order v.*

Proof. Considering the parameterization $X_D(j\omega) = \Psi(j\omega)^*M_D\Psi(j\omega)$, with $M_D = M_D^T$ and $\Psi \in \mathscr{RH}_\infty^{\bullet \times n_q}$, we obtain

$$X_D(j(\omega + \omega_0)) = \Psi(j(\omega + \omega_0))^*M_D \underbrace{\Psi(j(\omega + \omega_0))}_{\Psi_+}$$
$$X_D(j(\omega - \omega_0)) = \Psi(j(\omega - \omega_0))^*M_D \underbrace{\Psi(j(\omega - \omega_0))}_{\Psi_-} \quad (19)$$

Then, we denote $\tilde{\Psi}_+ = \Psi_+ \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix}$ and $\tilde{\Psi}_- = \Psi_- \begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix}$. Hence, the upper-left block of the multiplier Π of (7) is expressed as follows

$$\tilde{X}_{D+} + \tilde{X}_{D-} = \tilde{\Psi}_+^*M_D\tilde{\Psi}_+ + \tilde{\Psi}_-^*M_D\tilde{\Psi}_- \quad (20)$$

which can be rewritten as

$$\tilde{X}_{D+} + \tilde{X}_{D-} = \frac{1}{2}\left[\tilde{\Psi}_+ + \tilde{\Psi}_-\right]^* M_D \left[\tilde{\Psi}_+ + \tilde{\Psi}_-\right] \\ + \frac{1}{2}\left[-j(\tilde{\Psi}_+ + \tilde{\Psi}_-)\right]^* M_D \left[-j(\tilde{\Psi}_+ + \tilde{\Psi}_-)\right] \quad (21)$$

Then, combining this factorization with that of X_D, we can easily obtain the final factorization (18) of Π. $\qquad\square$

Here, we will consider the following basis Ψ: we set the denominator of Ψ as the scalar function $d(s) = s^v + d_{v-1}s^{v-1} + \ldots + d_0$ with roots in the open left-half complex plane $\mathbb{C}^-$. Then $\Psi(j\omega)$ is represented by $N(j\omega)/d(j\omega)$. Let us fix a basis for N defined by the vector function $B : \mathbb{C} \to \mathbb{C}^{(v+1)}$ as $B(s) = [1 \ s \ \cdots \ s^v]^T$. Then Ψ can be obtained as

$$\Psi(j\omega) = \left(\frac{B(j\omega)}{d(j\omega)} \otimes I_{n_q} \right). \qquad (22)$$

V. STABILITY ANALYSIS TEST

In order to obtain of an efficient finite-dimensional stability tests through Lemma 1, it is necessary to find a suitable state-space realization for the elements of (18), with Ψ parameterized as in (22). Let us then introduce the following auxiliary lemma.

Lemma 3. *Let be $\Psi \in \mathcal{RH}_\infty^{n_y \times n_u}$ admitting the minimal realization $(A_\Psi, B_\Psi, C_\Psi, D_\Psi)$ with $A_\Psi \in \mathbb{R}^{n \times n}$, $B_\Psi \in \mathbb{R}^{n \times n_u}$, $C_\Psi^{n_y \times n}$, $D_\Psi^{n_y \times n_u}$, then, with $B_\Psi = [B_{\Psi,1} \ B_{\Psi,2}]$ and $D_\Psi = [D_{\Psi,1} \ D_{\Psi,2}]$,*

$$\begin{bmatrix} \tilde{\Psi}_+ + \tilde{\Psi}_- \\ -j(\tilde{\Psi}_+ - \tilde{\Psi}_-) \end{bmatrix} \qquad (23)$$

admits the state-space realization

$$\left[\begin{array}{cc|cc} A_\Psi & \omega_0 I & B_{\Psi,1} & 0 \\ -\omega_0 I & A_\Psi & 0 & B_{\Psi,2} \\ \hline 2C_\Psi & 0 & 2D_{\Psi,1} & 0 \\ 0 & 2C_\Psi & 0 & 2D_{\Psi,2} \end{array} \right] \qquad (24)$$

Proof. With $\Psi(s) = C_\Psi(sI_n - A_\Psi)^{-1}B_\Psi + D_\Psi$, a shift $+\omega_0$ of the frequency response gives $\Psi(s+j\omega_0) = C_\Psi(sI_n + j\omega_0 I_n - A_\Psi)^{-1}B_\Psi + D_\Psi$ and, similarly, $\Psi(s-j\omega_0) = C_\Psi(sI_n - j\omega_0 I_n - A_\Psi)^{-1}B_\Psi + D_\Psi$. Let us then introduce the state-space equations of $\tilde{\Psi}_+$ and $\tilde{\Psi}_-$

$$\dot{x}_+ = (A_\Psi - j\omega_0 I)x_+ + B_\Psi \underbrace{\begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix} u}_{\tilde{u}_+} \qquad (25)$$

$$y_+ = C_\Psi x_+ + D_\Psi \tilde{u}_+ \qquad (26)$$

$$\dot{x}_- = (A_\Psi + j\omega_0 I)x_- + B_\Psi \underbrace{\begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix} u}_{\tilde{u}_-} \qquad (27)$$

$$y_- = C_\Psi x_- + D_\Psi \tilde{u}_- \qquad (28)$$

We observe that Equation (25) is the complex conjugate of Equation (27). With $x_+(0) = 0$ and $x_-(0) = 0$, we have that $x_+ = \bar{x}_-$. We note that $x_+ = x_{Re} + jx_{Im}$ and $x_- = x_{Re} - jx_{Im}$, and we put the state-space vector $x = [x_{Re}^T \ x_{Im}^T]^T$. In this form, with $B_\Psi = [B_{\Psi,1} \ B_{\Psi,2}]$ and $D_\Psi = [D_{\Psi,1} \ D_{\Psi,2}]$, the output $y_+ + y_-$ is given by $2C_\Psi x_{Re} + [2D_{\Psi,1} \ 0]u$ and the output $-j(y_+ + y_-)$ by $2C_\Psi x_{Im} + [0 \ 2D_{\Psi,2}]u$. Thus, the complete system (23) has the state-space representation (24). $\qquad\square$

Given the previous parameterization, we can state the main result of this paper, solving Problem 1.

Theorem 3. *Consider the interconnection (1) with $G \in \mathcal{RH}_\infty^{n_p \times n_q}$ admitting the state-space realization $(A, B, C = [C_1^T \ C_2^T]^T, D = [D_1^T \ D_2^T]^T)$ and $\mathbf{\Delta} = \{\Delta : \Delta(q) = \theta(t, \omega_0)q(t)\}$ with $\theta(t, \omega_0)$ defined in (2). Consider the set of multipliers (18) with $\mathbf{M_D} = \{M_D : M_D = M_D^T, \ \Psi^* M_D \Psi \succ 0 \ \forall \ \omega\}$ and Ψ with minimal state-space realization $(A_\Psi, B_\Psi = [B_{\Psi,1} \ B_{\Psi,2}], C_\Psi, D_\Psi = [D_{\Psi,1} \ D_{\Psi,2}])$.*

Then, the interconnection (1) is stable if there exists $M_D \in \mathbf{M_D}$, $P = P^T$ and $\varepsilon > 0$ such that

$$\begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & \tilde{M}_D \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \preceq -\varepsilon I \qquad (29)$$

with $\tilde{M}_D = diag(\frac{1}{2}M_D, \frac{1}{2}M_D, -2M_D)$, and

$$\tilde{A} = \begin{bmatrix} A_\Psi & \omega_0 I & 0 & B_{\Psi,1}C_1 \\ -\omega_0 I & A_\Psi & 0 & B_{\Psi,2}C_2 \\ 0 & 0 & A_\Psi & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_{\Psi,1}D_1 \\ B_{\Psi,2}D_2 \\ B_\Psi \\ B \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} 2C_\Psi & 0 & 0 & 2D_{\Psi,1}C_1 \\ 0 & 2C_\Psi & 0 & 2D_{\Psi,2}C_2 \\ 0 & 0 & C_\Psi & 0 \end{bmatrix}, \ \tilde{D} = \begin{bmatrix} 2D_{\Psi,1}D_1 \\ 2D_{\Psi,2}D_2 \\ D_\Psi \end{bmatrix}.$$

Proof. Stability is tested using Theorem 1. First, please observe that if (29) is satisfied, then $\tilde{D}^T M_D \tilde{D} \prec -\varepsilon I$. It in turn implies $D^T M_{Dm} D \prec M_{Dm}$, with $M_{Dm} \in \mathbb{R}^{n_q \times n_q}$ being the lower-right minor of M_D (also symmetric). The last condition ensures $(I - D\Delta)$ is invertible and, since Δ is defined by continuous varying gain θ in (2), $(I - G\Delta)$ has a causal inverse. As a consequence, the interconnection (G, Δ) is well-posed for any $\Delta \in \mathbf{\Delta}$ defined by (6).

From Lemma 2, we know that $\theta(t, \omega_0) \in IQC(\Pi)$ with $\Pi \in \underline{\mathbf{\Pi}}$ defined by (18). By construction of Π in Theorem 2 (see proof, Equation (10)), it is also the case for $\tau\theta, \forall \ \tau \in [0, 1]$. By applying Theorem 1 with parameterization (18) of Lemma 2 and the state-space realization (24) of Lemma 3, the stability is ensured if

$$H(j\omega)^* \tilde{M}_D H(j\omega) \preceq -\varepsilon I \ \ \forall \omega \in \mathbb{R} \qquad (30)$$

with

$$H = \left[\begin{array}{c|c} \tilde{\Psi}_+ + \tilde{\Psi}_- & 0 \\ -j(\tilde{\Psi}_+ - \tilde{\Psi}_-) & 0 \\ \hline 0 & \Psi \end{array} \right] \begin{bmatrix} G \\ I \end{bmatrix} \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right] \qquad (31)$$

Condition (30) is equivalent to condition (29) by Lemma 1. $\qquad\square$

VI. APPLICATION CASE

We consider a closed-loop controlled MEMS resonator with synchronous demodulation in the loop. This system can be modeled as the interconnection (1) with G containing the resonator, the controller and demodulation non-ideal filters, and

$$\Delta(q) = \theta(t, 2\omega_{exc})q = \begin{bmatrix} \cos(2\omega_{exc}t)I_2 & 0 \\ 0 & \sin(2\omega_{exc}t)I_2 \end{bmatrix} q$$

TABLE I
COMPARISON OF MULTIPLIERS WITH RESPECT TO $\omega_c^\star$

Multiplier	Critical frequency (rad/s)	Computation time (s)
General TV	Unfeasible	—
Separated Harmonic	Unfeasible	—
Coupled Harmonic	68.6	50.173

TABLE II
COMPARISON OF MULTIPLIERS WITH RESPECT TO MAXIMAL SIZE

Multiplier	Maximal size α	Computation time (s)
General TV	0.0412	1.32
Separated Harmonic	0.4305	16.63
Coupled Harmonic	19.67	21.57

with ω_{exc} being the excitation frequency or frequency of modulation/demodulation, which is close to the natural frequency of the resonator (see the details in [4]). The controller was designed to ensure the stability and tracking performance in the nominal case, that is, when ideal synchronous demodulation is considered. The purpose here is to analyse the stability in the non-ideal case represented by the interconnection (G, θ) . Typically, the instability arises when the cut-off frequency ω_c of low-pass filters in synchronous demodulation block is close to the bandwidth of the feedback controlled system. Hence, we will test the proposed stability analysis approach by searching for the smallest cut-off frequency $\omega_c^\star$ for which the system is guaranteed to be stable. Such analysis tool can be helpful to properly choose the low-pass filter cut-off frequency that ensures the closed-loop stability. Three classes of the multipliers were used to analyze the stability of the system (state-space matrices of G can be found in [4]): General Time-Varying Multipliers (16), Separated Harmonic Multipliers (17), and the Coupled Harmonic Multiplier of (7). Then we search an upper bound over the uncertainty of ω_c, which leads us to the the smallest $\omega_c^\star$ for which the system is ensured to be stable.

The results are presented in Table I. They allow to compare the conservatism of each multiplier. Using General Time-Varying multipliers and Separated Harmonic Multipliers is not possible to asses stability of the system for any cut-off frequency ω_c. In the case of the Coupled Harmonic Multiplier, we obtain a minimal cut-off frequency of 68.6 rad/s with a computation time of the LMI optimization problem of 50.173 seconds. This critical cut-off frequency is close to the obtained simulation values (around 65 rad/s), which confirms the relevance of the results. In order to reinforce the previous comparison, we set a cut-off frequency ω_c at some value (500 rad/s) at which the system is guaranteed to be stable. Now, we consider $\Delta(q) = \alpha\theta(t, 2\omega_{exc})q(t)$ and we search the maximal α for which the considered system is ensured to be stable by testing it with the different multipliers. In this case, it must exist a solution α for each multiplier since an $\alpha = 0$ implies to test the nominal stability of G, which is already guaranteed. The obtained results are summarized in Table II, showing the significant difference in the size of Δ allowed by each multiplier.

VII. CONCLUSIONS AND FUTURE WORKS

We present a new class of multipliers for the analysis of uncertain systems containing cosine and sine harmonic oscillations simultaneously. Stability analysis is performed using IQC approach. The proposed multiplier exploits not only the information related to the bounds of time-varying elements, but also the oscillation frequency and the $\pi/2$ phase between sine and cosine. A parameterization is proposed to obtain a finite-dimensional set of multipliers. We have shown that the proposed parameterization admits a real state-space parametrization, making possible the use of KYP Lemma to obtain a finite dimensional LMI optimization problem. The proposed class of multipliers is compared with other available multipliers through an application case. Such example illustrates the reduction of conservatism.

Some perspectives are projected for the presented multipliers. In one side, we will further investigate the reduction of conservatism by introducing the non-diagonal elements of the multiplier. In the other side, it is possible to derive similar classes of multipliers that can be extended to the analysis of systems containing rotation matrices.

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