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On the product of generating functions for domino and bi-tableaux

Ekaterina A. Vassilieva

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Abstract. The connection between the generating functions of various sets of tableaux and the appropriate families of quasisymmetric functions is a significant tool to give a direct analytical proof of some advanced bijective results and provide new combinatorial formulas. In this paper we focus on two kinds of type B Littlewood-Richardson coefficients and derive new formulas using weak composition quasisymmetric functions and Chow's quasisymmetric functions. In the type A case these coefficients give the multiplication table for Schur functions, i.e. the generating functions for classical Young tableaux. We look at their generalisations involving a set of bi-tableaux and domino tableaux.

1 Type A quasisymmetric functions

1.1 Permutations, Young tableaux and descent sets

For any positive integer n write $[n] = \{1, \dots, n\}$, S_n the symmetric group on $[n]$. A **partition** λ of n , denoted $\lambda \vdash n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ of $\ell(\lambda) = p$ parts sorted in decreasing order such that $|\lambda| = \sum_i \lambda_i = n$. A partition λ is usually represented as a Young diagram of $n = |\lambda|$ boxes arranged in $\ell(\lambda)$ left justified rows so that the i -th row from the top contains λ_i boxes. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a **semistandard Young tableau**. If the entries of a semistandard Young tableau are restricted to the elements of $[n]$ and strictly increasing along the rows, we call it a **standard Young tableau**. The partition λ is the **shape** of the tableau and we denote $SYT(\lambda)$ (resp. $SSYT(\lambda)$) the set of standard (resp. semistandard) Young tableaux of shape λ .

One important feature of a permutation π in S_n is its **descent set**, i.e. the subset of $[n-1]$ defined as $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$. Similarly, define the **descent set of a standard Young tableau** T as $Des(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$.

1.2 Quasisymmetric expansion of Schur polynomials

Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be two alphabets of commutative indeterminates. Denote also XY the product alphabet $\{x_i y_j\}_{i,j}$ ordered by the lexicographical order. Let $F_I(X)$ be the **Gessel's fundamental quasisymmetric function** on X indexed by the subset $I \subset [n-1]$:

$$F_I(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \dots x_{i_n}. \quad (1)$$

The power series $F_I(X)$ is not symmetric in X but verifies the property that for any strictly increasing sequence of indices $i_1 < i_2 < \dots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_p}^{k_p}$. Let $s_\lambda(X)$ be the Schur polynomial indexed by partition λ on alphabet X . Schur polynomials are the generating functions for semistandard Young tableaux. As a result,

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{Des(T)}(X). \quad (2)$$

Using for any $\pi \in S_n$ Gessel's formula for the coproduct of fundamental quasisymmetric functions in [4]

$$F_{Des(\pi)}(XY) = \sum_{\sigma, \rho \in S_n; \sigma\rho = \pi} F_{Des(\sigma)}(X) F_{Des(\rho)}(Y), \quad (3)$$

and the Cauchy formula for Schur polynomials

$$\sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) = s_{(n)}(XY), \quad (4)$$

one can derive

$$\sum_{\substack{\lambda \vdash n \\ T, U \in SYT(\lambda)}} F_{Des(T)}(X) F_{Des(U)}(Y) = F_\emptyset(XY) = \sum_{\pi \in S_n} F_{Des(\pi)}(X) F_{Des(\pi^{-1})}(Y). \quad (5)$$

This gives a direct analytical proof of one important property usually proved using the *Robinson-Schensted* correspondence, i.e. that permutations π in S_n are in bijection with pairs of standard Young tableaux T, U of the same shape such that $Des(T) = Des(\pi)$ and $Des(U) = Des(\pi^{-1})$.

1.3 Product of Schur polynomials

One may write a permutation $\alpha \in S_n$ as a word on the letters in $[n]$, $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ where $\alpha_i = \alpha(i)$. Given $\alpha \in S_n$ and $\beta \in S_m$, let $\alpha \sqcup \beta$ be the set of permutations $\gamma \in S_{m+n}$ obtained by shuffling the letters α and $\bar{\beta} = n + \beta_1 \ n + \beta_2 \dots n + \beta_m$ such that the initial order of the letters of α (resp. of $\bar{\beta}$) is preserved in γ . By extension if A and B are two sets of permutations, we

denote $A \sqcup B$ the set of all the shuffles of a permutation of A and a permutation of B . We have the product formula

$$F_{Des(\alpha)}(X)F_{Des(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}(X). \quad (6)$$

Given a standard Young tableau T of shape λ , Equation (5) proves the existence of classes of permutations C_T such that for any $\pi \in C_T$, $Des(\pi^{-1}) = Des(T)$ and there is a descent preserving bijection between C_T and $SYT(\lambda)$.

Remark 1. *Classically the C_T are the Knuth classes of permutations but here we derive these results directly without using plactic relations and/or the Robinson-Schensted correspondence.*

Equation (6) directly implies for integer partitions λ and μ and $T, U \in SYT(\lambda) \times SYT(\mu)$:

$$s_\lambda(X)s_\mu(X) = \sum_{\alpha \in C_T} F_{Des(\alpha)}(X) \sum_{\beta \in C_U} F_{Des(\beta)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (7)$$

For any $\nu \vdash |\lambda| + |\mu|$ define formally the coefficients $c_{\lambda\mu}^\nu$ as the coefficients of the expansion of $s_\lambda(X)s_\mu(X)$ in the Schur basis $(s_\nu)_\nu$. Equation (7) shows for some $V_\nu \in SYT(\nu)$ that

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu s_\nu(X) = \sum_{\substack{\nu \vdash |\lambda| + |\mu| \\ \gamma \in C_{V_\nu}}} c_{\lambda\mu}^\nu F_{Des(\gamma)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (8)$$

As a result, we have the following proposition.

Proposition 1. *For $I \subset [|\lambda| + |\mu| - 1]$ denote D_I the set of permutations π such that $Des(\pi) = I$. Using the notations above for integer partitions λ, μ and for T, U in $SYT(\lambda) \times SYT(\mu)$, there is a set of tableaux $V_\nu \in SYT(\nu)$ indexed by the set of integer partitions $\nu \vdash |\lambda| + |\mu|$ such that*

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu |C_{V_\nu} \cap D_I| = |(C_T \sqcup C_U) \cap D_I|. \quad (9)$$

The following sections are dedicated to the extension of this method to weak composition quasisymmetric functions and Chow's type B quasisymmetric functions.

2 Type B extension to weak composition quasisymmetric functions

2.1 Weak composition quasisymmetric functions and coloured permutations

Weak composition quasisymmetric functions are a generalisation of Gessel's quasisymmetric functions introduced in [5] and [12] in order to provide a framework

for the study of a question a G.-C. Rota relating *Rota-Baxter algebras* and symmetric type functions.

We consider a finite alphabet $X = \{x_1, x_2, \dots, x_M\}$. For any non-negative integer n , let $\mathcal{WC}(n)$ be the set of **weak compositions** of n , i.e. the set of ordered tuples of non-negative integers that sum up to n . A weak composition $\alpha \in \mathcal{WC}(n)$ is of the form

$$\alpha = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

with $i_j \geq 0$, $1 \leq j \leq k+1$, $s_i > 0$, $1 \leq j \leq k$ and $\sum s_i = n$. We further define $\ell(\alpha) = k + i_1 + \dots + i_{k+1}$ the **length** of α , $\ell_0(\alpha) = i_1 + \dots + i_{k+1}$ as its **0-length**, $|\alpha| = n$ the **weight** of α and $||\alpha|| = |\alpha| + \ell_0(\alpha)$ the **total weight** of α . The descent set of α is

$$D(\alpha) = \{a_1, \dots, a_k | a_j = i_1 + s_1 + \dots + i_j + s_j, j = 1, \dots, k\}.$$

Definition 1. Given a weak composition $\alpha = (0^{j_1}, s_1, 0^{j_2}, s_2, \dots, 0^{j_k}, s_k, 0^{j_{k+1}})$ with $D(\alpha) = \{a_1, a_2, \dots, a_k\}$ define the **weak composition fundamental quasisymmetric function** indexed by α the formal power series

$$\begin{aligned} \tilde{F}_\alpha(X) = \\ \sum_{\substack{1 \leq i_1 \leq \dots \leq i_{a_k+j_{k+1}} \\ j \in D(\alpha) \Rightarrow i_j < i_{j+1}}} x_{i_1}^0 \dots x_{i_{j_1}}^0 x_{i_{j_1+1}} \dots x_{i_{a_1}} \dots x_{i_{a_{k-1}+j_k+1}} \dots x_{i_{a_k}} x_{i_{a_k+1}}^0 \dots x_{i_{a_k+j_{k+1}}}^0 \end{aligned} \quad (10)$$

Weak composition fundamental quasisymmetric functions may also be described in terms of P -partitions indexed by *coloured permutations*. Define the total order

$$\bar{1} < \bar{2} < \bar{3} < \dots < 1 < 2 < 3 \dots$$

and denote $|\bar{i}| = i$ for $i > 0$. We look at chain posets $P_\pi = \{\pi_1 <_{P_\pi} \pi_2 <_{P_\pi} \dots <_{P_\pi} \pi_l\}$ indexed by coloured permutations π for any non-negative integer l where the π_i are distinct letters in $[l] \cup [\bar{l}]$ and such that $|\pi_1| |\pi_2| \dots |\pi_l|$ has no repeated letters. A P_π -**partition** is a map $f : P_\pi \rightarrow [M]$ such that for $x <_{P_\pi} y$, $f(x) \leq f(y)$ with $x > y \Rightarrow f(x) < f(y)$. The **weight** of f , $w(f)$ is the monomial

$$w(f) = \prod_{\pi_i \in P_\pi} x_{f(\pi_i)}^{\delta_{\pi_i}}, \text{ where } \delta_{\pi_i} = \begin{cases} 0 & \text{if } \pi_i \text{ is an overlined letter,} \\ 1 & \text{otherwise.} \end{cases}$$

Denote $\mathcal{A}(\pi)$ the set of P_π -partitions. We are interested in the generating functions Γ indexed by coloured permutations π

$$\Gamma(\pi) = \sum_{f \in \mathcal{A}(\pi)} w(f). \quad (11)$$

Li shows in [6] that the $\Gamma(\pi)$ can be expressed as a \mathbb{Z} -linear combination of the F_α but not always positive. In particular if the indices $j_1 < j_2 < \dots < j_p$ of the

overlined letters in a coloured permutation π are such that $\pi_{j_1} < \pi_{j_2} < \dots < \pi_{j_p}$, denote the weak composition

$$\text{comp}(\pi) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

where (s_1, s_2, \dots, s_k) is the tuple of lengths of the sequences of consecutive increasing non-overlined letters in π and i_r is the quantity of overlined letters between sequences $r-1$ and r . We have $\ell_0(\text{comp}(\pi)) = p$ and we denote $S_n^{(p)}$ the set of such coloured permutations with $|\text{comp}(\pi)| = n$. As an example for $\pi = 597\overline{1}2\overline{6}3\overline{8}4$, we have $\text{comp}(\pi) = (2, 1, 0^2, 1, 0, 1, 0)$. As a direct consequence of Equation (10), for $n, p \geq 0$ and $\pi \in S_n^{(p)}$, one has ([6])

$$\tilde{F}_{\text{comp}(\pi)} = \Gamma(\pi).$$

On the other hand, take $\pi = 3\overline{2}1$, then it is easy to show that $\Gamma(\pi) = \tilde{F}_{(1,0^2)} - \tilde{F}_{(1,0)}$. Finally, as in Section 1.3, define for $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$ the set $\alpha \sqcup \beta$ of coloured permutations obtained by shuffling the letters of $\tilde{\alpha}$ and $\tilde{\beta}$ where the non-overlined letters of α are shifted by q to get $\tilde{\alpha}$ and the overlined (resp. non-overlined) letters of β are shifted by p (resp. by $p+n$) to get $\tilde{\beta}$. For instance, let $\alpha = 3\overline{2}1$, $\beta = \overline{1}2$, then $\tilde{\alpha} = 43\overline{1}$, $\tilde{\beta} = \overline{2}5$ and

$$\alpha \sqcup \beta = 43\overline{1}\overline{2}5 + 43\overline{2}\overline{1}5 + 4\overline{2}3\overline{1}5 + \overline{2}43\overline{1}5 + 43\overline{2}5\overline{1} + 4\overline{2}35\overline{1} + \overline{2}435\overline{1} + 4\overline{2}53\overline{1} + \overline{2}453\overline{1} + \overline{2}543\overline{1}.$$

One can see that despite the fact that α and β belong to $S_n^{(p)}$ and $S_r^{(q)}$, not all the permutations in $\alpha \sqcup \beta$ fulfil the conditions of proper order of the overlined letters. We have for $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$,

$$\tilde{F}_{\text{comp}(\alpha)}(X) \tilde{F}_{\text{comp}(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} \Gamma(\gamma). \quad (12)$$

In particular for $n = r = 0$

$$\tilde{F}_{(0^p)}(X) \tilde{F}_{(0^q)}(X) = \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \tilde{F}_{(0^{p+q-j})}(X). \quad (13)$$

In some sense, weak composition fundamental quasisymmetric functions are a specialisation of *Poirier quasisymmetric functions* (see [9]), a classical type B generalisation of Gessel's quasisymmetric functions that we do not detail here. In [1], the authors provide a connexion between the generating functions for Young *bi-tableaux* and the fundamental quasisymmetric functions of Poirier. As a result, there should be a natural bi-tableaux interpretation of weak composition quasisymmetric functions. This is the focus of the next section.

2.2 Tableau-theoretic interpretation

Let λ be an integer partition and p a non-negative integer. We consider pairs of Young tableau (T^-, T^+) where T^- is of shape (p) and T^+ has shape λ . We call such a bi-tableau a (p) -tableau. We proceed with the formal definition.

Definition 2. For any couple of non-negative integer p, n and an integer partition $\lambda \vdash n$, a **semistandard** (p) -**tableau** $T = (T^-, T^+)$ is a pair of Young diagrams of bi-shape $((p), \lambda)$ whose boxes are filled with the integers in $[M]$ such that the entries are strictly increasing down the columns and non-decreasing along the rows. The **shape** of T is noted $sh(T) = ((p), \lambda)$ and we denote $SSBT^{(p)}(\lambda)$ the set of all (p) -tableaux of shape $((p), \lambda)$. If the boxes are filled with integers in $[n+p]$ all used exactly once, we call T a **standard** (p) -**tableau** and we denote $SBT^{(p)}(\lambda)$ the set of such tableaux.

We also need the following notions. The **standardisation** T^{st} of a semistandard (p) -tableau T is the standard (p) -tableau of shape $((p), \lambda)$ obtained by relabelling the boxes of T with all the integers in $[|\lambda| + p]$ such that the order of the labels is preserved. When a label in T is used both in T^- and T^+ , the boxes of T^- are relabelled first. Finally, given a standard (p) -tableau $T \in SBT^{(p)}(\lambda)$ with $|\lambda| = n$ denote $comp(T)$ the weak composition of n , $comp(T) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$, where i_1 is the number of consecutive labels $k = 1, 2, \dots, i_1$ in T^- ($i_1 = 0$ if 1 is in T^+). Then s_1 is the number of consecutive labels $k = i_1 + 1, \dots, i_1 + s_1$ in T^+ such that no label k is in a higher row than $k + 1$. Then i_2 is the number of consecutive labels $k = i_1 + s_1 + 1, 2, \dots, i_1 + s_1 + i_2$ in T^- , etc...

Example 1. The following semistandard (3)-tableau belongs to $SSBT^{(3)}((2, 2, 1))$.

$$T = \left(T^- = \begin{array}{|c|c|c|} \hline 2 & 4 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 & 5 \\ \hline 9 & \\ \hline \end{array} \right)$$

Furthermore, its standardisation is:

$$T^{st} = \left(T^- = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 6 & 7 \\ \hline 8 & \\ \hline \end{array} \right)$$

Finally the weak composition of T^{st} is $comp(T^{st}) = (1, 0^3, 1, 2, 1)$.

For $\lambda \vdash n$ and $p \geq 0$ denote $s_\lambda^{(p)}$ the generating function for such tableaux. We have

$$s_\lambda^{(p)}(X) = \sum_{T \in SSBT^{(p)}(\lambda)} X^{T^+}. \quad (14)$$

Recall the notation of the Schur symmetric function s_λ indexed by λ . The following lemma is immediate.

Lemma 1. For any $\lambda \vdash n$ and $p \geq 0$, one has

$$s_\lambda^{(p)}(X) = \binom{M+p-1}{p} s_\lambda(X). \quad (15)$$

We can now state the main result of this section.

Proposition 2. *For non-negative integers p and n and an integer partition $\lambda \vdash n$ the generating function for (p) -tableau of shape $((p), \lambda)$ is related to weak composition fundamental quasisymmetric functions through*

$$s_\lambda^{(p)}(X) = \sum_{T \in SBT^{(p)}(\lambda)} \tilde{F}_{comp(T)}(X). \quad (16)$$

Proof. (sketch) Starting from Equation (14), sort all the semi-standard (p) -tableaux according to their standardisation. Then, given a standard (p) -tableau T_0 show that

$$\tilde{F}_{comp(T_0)}(X) = \sum_{T \in SSBT^{(p)}(\lambda); T^{st}=T_0} X^{T^+}.$$

Finally, sum on $SBT^{(p)}(\lambda)$ to get the result. \square

2.3 Product formulas

We use Proposition 2 to derive new results. We show the following formula.

Theorem 1. *Let p, q, n, m be non-negative integers and λ, μ integer partitions of n and m . For $\nu \vdash |\lambda| + |\mu|$ recall the coefficients $c_{\lambda, \mu}^\nu$ introduced in Section 1.3. The following identity holds.*

$$\sum_{\substack{T \in SBT^{(p)}(\lambda) \\ U \in SBT^{(q)}(\mu)}} \tilde{F}_{comp(T)}(X) \tilde{F}_{comp(U)}(X) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda, \mu}^\nu \tilde{F}_{comp(V)}(X). \quad (17)$$

Proof. (sketch) First, note that for any integer p

$$s_\emptyset^{(p)} = \tilde{F}_{(0^p)} = \binom{M+p-1}{p}.$$

Then use Equation (13) and the definition of the coefficients $c_{\lambda, \mu}^\nu$ to write

$$\begin{aligned} s_\lambda^{(p)} s_\mu^{(q)} &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \binom{M+p-1}{p} \binom{M+q-1}{q} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \binom{M+p+q-j-1}{p+q-j} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} s_\nu^{(p+q-j)}, \end{aligned}$$

and apply Proposition 2 to get the result. \square

There is an interesting corollary of Theorem 1 in terms of generating functions for P_π -partitions. For a (classical) standard Young tableau T , recall the permutation class C_T of Section 1.3. Given a non-negative integer p , we denote ε_p the coloured permutation $\varepsilon_p = \overline{1}2 \cdots p$. The result can be stated as follows.

Corollary 1. *For non-negative integers p, q , integers partitions λ and μ and $T \in SYT(\lambda)$, $U \in SYT(\mu)$, one has:*

$$\sum_{\gamma \in (\varepsilon_p \sqcup C_T) \sqcup (\varepsilon_q \sqcup C_U)} \Gamma(\gamma) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda\mu}^\nu \tilde{F}_{comp(V)}(X). \quad (18)$$

Proof. We omit the proof in this extended abstract. \square

3 Type B extension to Chow's quasisymmetric functions

3.1 Signed permutations and domino tableaux

Let B_n be the **hyperoctahedral group** of order n . B_n is composed of all signed permutations π on $\{-n, \dots, -2, -1, 0, 1, 2, \dots, n\}$ such that $\pi(-i) = -\pi(i)$ for all i . The **total colour** of $\pi \in B_n$ is the number $tc(\pi) = |\{i \geq 1; \pi(i) < 0\}|$. Its **descent set** is the subset of $\{0\} \cup [n-1]$ equal to $Des(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$. The main difference with respect to the case of the symmetric group is the possible descent in position 0 when $\pi(1)$ is a negative integer.

For $\lambda \vdash 2n$, a **standard domino tableau** T of shape λ is a Young diagram of shape $shape(T) = \lambda$ tiled by **dominoes**, i.e. 2×1 or 1×2 rectangles filled with the elements of $[n]$ such that the entries are strictly increasing along the rows and down the columns. In the sequel we consider only the set $\mathcal{P}^0(n)$ of *empty 2-core partitions* $\lambda \vdash 2n$ that fit such a tiling. A standard domino tableau T has a descent in position $i > 0$ if $i+1$ lies strictly below i in T and has descent in position 0 if the domino filled with 1 is vertical. We denote $Des(T)$ the set of all its descents. A **semistandard domino tableau** T of shape $\lambda \in \mathcal{P}^0(n)$ and weight $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \dots)$ with $\mu_i \geq 0$ and $\sum_i \mu_i = n$ is a tiling of the Young diagram of shape λ with horizontal and vertical dominoes labelled with integers in $\{0, 1, 2, \dots\}$ such that labels are non decreasing along the rows, strictly increasing down the columns and exactly μ_i dominoes are labelled with i . If the top leftmost domino is vertical, **it cannot be labelled 0**. Denote $SDT(\lambda)$ (resp. $SSDT(\lambda)$) the set of standard (resp. semistandard) domino tableaux of shape λ . Finally, we denote $sp(T)$, the **spin** of (semi-) standard domino tableau T , i.e. **half the number** of its vertical dominoes.

Remark 2. *The possible labelling of top leftmost horizontal dominoes in semistandard domino tableaux with 0 was first introduced by the authors in [8] and*

is essential to connect their generating functions with Chow's quasisymmetric functions.

$T_1 =$

1	2	3	5
			6
4		8	
7			

$T_2 =$

1	2	3	
		4	5
6		8	
7			

$T_3 =$

	0	0	
2	2		5
	5		
5	7		

Figure 1: Two standard domino tableaux T_1 and T_2 of shape $(5, 5, 4, 1, 1)$, descent set $\{0, 3, 5, 6\}$, a semistandard tableau T_3 of shape $(5, 5, 4, 3, 1)$ and weight $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$. All of the tableaux have a spin of 2.

3.2 Chow's type B quasisymmetric functions

Chow defines in [3] another Type B extension of Gessel's algebra of quasisymmetric functions that is dual to Solomon's descent algebra of type B.

Let $X = \{\dots, x_{-i}, \dots, x_{-1}, x_0, x_1, \dots, x_i, \dots\}$ be an alphabet of totally ordered commutative indeterminates with the assumption that $x_{-i} = x_i$ and let I be a subset of $\{0\} \cup [n-1]$, he defines a type B analogue of the fundamental quasisymmetric functions

$$F_I^B(X) = \sum_{\substack{0=i_0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Note the particular role of the variable x_0 .

Given two signed permutations $\alpha \in B_n$ and $\beta \in B_m$, denote $\alpha \sqcup \beta$ the set of signed permutations in B_{m+n} obtained by shuffling the letters of α and $\bar{\beta}$ where we shift the absolute value of the β_i to get $\bar{\beta}$. Chow shows in [3, Prop. 2.2.6]

$$F_{Des(\alpha)}^B(X) F_{Des(\beta)}^B(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}^B(X). \quad (19)$$

Furthermore, we introduce in [8] modified generating functions for domino tableaux called domino functions. Given a semistandard domino tableau T of weight μ , denote X^T the monomial $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \dots$. For $\lambda \in \mathcal{P}^0(n)$ and an additional parameter q we define the **domino function** indexed by λ on alphabet X as

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SSDT(\lambda)} q^{sp(T)} X^T. \quad (20)$$

These functions are related to Chow's quasisymmetric functions through

Lemma 2 ([7]). *For $\lambda \in \mathcal{P}^0(n)$, the q -domino function indexed by λ can be expanded in the basis of Chow's quasisymmetric functions as*

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SDT(\lambda)} q^{sp(T)} F_{Des(T)}^B(X). \quad (21)$$

3.3 Type B Littlewood-Richardson coefficients

There is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([2]) built a bijection between signed permutations of B_n and pairs of standard domino tableaux of equal shape in $\mathcal{P}^0(n)$. An independent development on the subject is due to Garfinkle. Taşkin ([11, Prop. 26]) showed that the two standard domino tableaux T and U associated to a signed permutation π by the algorithm of Barbash and Vogan have respective descent sets $Des(T) = Des(\pi^{-1})$ and $Des(U) = Des(\pi)$ while Shimozono and White showed in [10] the **colour-to-spin** property i.e. $tc(\pi) = sp(T) + sp(U)$.

Let X, Y be two alphabets of indeterminates. In [7], we show that Equation (21) implies the following proposition.

Proposition 3 ([7]). *For any integer n , we have*

$$\begin{aligned} \sum_{\pi \in B_n} q^{tc(\pi)} F_{Des(\pi)}^B(X) F_{Des(\pi^{-1})}^B(Y) = \\ \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in SDT(\lambda)}} q^{sp(T) + sp(U)} F_{Des(T)}^B(X) F_{Des(U)}^B(Y). \end{aligned} \quad (22)$$

Proposition 3 provides a direct analytical proof of the properties showed in a bijective fashion by Barbash, Vogan, Taşkin, Shimozono and White. More formally it proves the existence for any standard domino tableau T of a class of signed permutations C_T^B such that for any $\pi \in C_T^B$, $Des(\pi^{-1}) = Des(T)$ and there is a descent preserving bijection $C_T^B \rightarrow SDT(\lambda), \pi \mapsto U$ such that $tc(\pi) = sp(T) + sp(U)$. In a similar fashion as in Section 1.3, we use Equation (22) to show some properties for the *type B Littlewood-Richardson coefficients*.

First we focus on the case $q = 1$. There is a well known (not descent preserving) bijection between semistandard domino tableaux of weight μ and semistandard bi-tableaux of respective weights μ^- and μ^+ such that $\mu_i^+ + \mu_i^- = \mu_i$ for all i . The respective shapes of the two Young tableaux depend only on the shape of the initial semistandard domino tableau. Denote (T^-, T^+) the bi-tableau associated to a semistandard domino tableau T of shape λ and (λ^-, λ^+) their respective shapes. (T^-, T^+) (resp. (λ^-, λ^+)) is the **2-quotient** of T (resp. λ). Denote $X^* = X \setminus x_0$. For $\lambda \in \mathcal{P}^0(n)$ one has

$$\mathcal{G}_\lambda(X; 1) = s_{\lambda^-}(X^*) s_{\lambda^+}(X)$$

and for $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$,

$$\mathcal{G}_\lambda(X; 1) \mathcal{G}_\mu(X; 1) = \sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda^- \mu^-}^{\nu^-} c_{\lambda^+ \mu^+}^{\nu^+} \mathcal{G}_\nu(X; 1). \quad (23)$$

The following result extends Proposition 1 to signed permutations.

Theorem 2. *Let $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$. For $I \subset [0] \cup [n + m - 1]$ denote D_I^B the set of signed permutations π such that $\text{Des}(\pi) = I$ and Δ_I^B the set of standard domino tableaux of descent set I . Using the notations above for integer partitions λ, μ and T, U in $\text{SDT}(\lambda) \times \text{SDT}(\mu)$*

$$\sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda^- \mu^-}^{\nu^-} c_{\lambda^+ \mu^+}^{\nu^+} |\text{SDT}(\nu) \cap \Delta_I^B| = |(C_T^B \sqcup C_U^B) \cap D_I^B|. \quad (24)$$

Proof. (sketch) Theorem 2 is a consequence of Equations (23) and (13). \square

On the other hand, the general case for q remains open. Indeed, using Proposition 3, one can show for $\lambda \in \mathcal{P}^0(n)$ and $T \in \text{SDT}(\lambda)$

$$\mathcal{G}_\lambda(X; q) = \sum_{\pi \in C_T^B} q^{tc(\pi) - sp(T)} F_{\text{Des}(\pi)}^B.$$

Then, if $\mu \in \mathcal{P}^0(m)$ and $U \in \text{SDT}(\mu)$

$$\mathcal{G}_\lambda(X; q) \mathcal{G}_\mu(X; q) = q^{-(sp(T) + sp(U))} \sum_{\gamma \in C_T^B \sqcup C_U^B} q^{tc(\gamma)} F_{\text{Des}(\gamma)}^B.$$

However, even if we showed that the set of signed permutations $C_T^B \sqcup C_U^B$ is in descent preserving bijection with a set of cardinality $c_{\lambda^- \mu^-}^{\nu^-} c_{\lambda^+ \mu^+}^{\nu^+} |\text{SDT}(\nu)|$, we cannot say *a priori* how this bijection preserves or transforms the statistic tc .

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