



**HAL**  
open science

# On the product of generating functions for domino and bi-tableaux

Ekaterina A. Vassilieva

► **To cite this version:**

Ekaterina A. Vassilieva. On the product of generating functions for domino and bi-tableaux. 2019. hal-03092817

**HAL Id: hal-03092817**

**<https://hal.science/hal-03092817v1>**

Preprint submitted on 2 Jan 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the product of generating functions for domino and bi-tableaux

Ekaterina A. Vassilieva

November 23, 2019

**Abstract.** The connection between the generating functions of various sets of tableaux and the appropriate families of quasisymmetric functions is a significant tool to give a direct analytical proof of some advanced bijective results and provide new combinatorial formulas. In this paper we focus on two kinds of type B Littlewood-Richardson coefficients and derive new formulas using weak composition quasisymmetric functions and Chow's quasisymmetric functions. In the type A case these coefficients give the multiplication table for Schur functions, i.e. the generating functions for classical Young tableaux. We look at their generalisations involving a set of bi-tableaux and domino tableaux.

## 1 Type A quasisymmetric functions

### 1.1 Permutations, Young tableaux and descent sets

For any positive integer  $n$  write  $[n] = \{1, \dots, n\}$ ,  $S_n$  the symmetric group on  $[n]$ . A **partition**  $\lambda$  of  $n$ , denoted  $\lambda \vdash n$  is a sequence  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$  of  $\ell(\lambda) = p$  parts sorted in decreasing order such that  $|\lambda| = \sum_i \lambda_i = n$ . A partition  $\lambda$  is usually represented as a Young diagram of  $n = |\lambda|$  boxes arranged in  $\ell(\lambda)$  left justified rows so that the  $i$ -th row from the top contains  $\lambda_i$  boxes. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a **semistandard Young tableau**. If the entries of a semistandard Young tableau are restricted to the elements of  $[n]$  and strictly increasing along the rows, we call it a **standard Young tableau**. The partition  $\lambda$  is the **shape** of the tableau and we denote  $SYT(\lambda)$  (resp.  $SSYT(\lambda)$ ) the set of standard (resp. semistandard) Young tableaux of shape  $\lambda$ .

One important feature of a permutation  $\pi$  in  $S_n$  is its **descent set**, i.e. the subset of  $[n-1]$  defined as  $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ . Similarly, define the **descent set of a standard Young tableau**  $T$  as  $Des(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$ .

## 1.2 Quasisymmetric expansion of Schur polynomials

Let  $X = \{x_1, x_2, \dots\}$  and  $Y = \{y_1, y_2, \dots\}$  be two alphabets of commutative indeterminates. Denote also  $XY$  the product alphabet  $\{x_i y_j\}_{i,j}$  ordered by the lexicographical order. Let  $F_I(X)$  be the **Gessel's fundamental quasisymmetric function** on  $X$  indexed by the subset  $I \subset [n-1]$ :

$$F_I(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \quad (1)$$

The power series  $F_I(X)$  is not symmetric in  $X$  but verifies the property that for any strictly increasing sequence of indices  $i_1 < i_2 < \dots < i_p$  the coefficient of  $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$  is equal to the coefficient of  $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$ . Let  $s_\lambda(X)$  be the Schur polynomial indexed by partition  $\lambda$  on alphabet  $X$ . Schur polynomials are the generating functions for semistandard Young tableaux. As a result,

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{Des(T)}(X). \quad (2)$$

Using for any  $\pi \in S_n$  Gessel's formula for the coproduct of fundamental quasisymmetric functions in [4]

$$F_{Des(\pi)}(XY) = \sum_{\sigma, \rho \in S_n; \sigma\rho = \pi} F_{Des(\sigma)}(X) F_{Des(\rho)}(Y), \quad (3)$$

and the Cauchy formula for Schur polynomials

$$\sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) = s_{(n)}(XY), \quad (4)$$

one can derive

$$\sum_{\substack{\lambda \vdash n \\ T, U \in SYT(\lambda)}} F_{Des(T)}(X) F_{Des(U)}(Y) = F_\emptyset(XY) = \sum_{\pi \in S_n} F_{Des(\pi)}(X) F_{Des(\pi^{-1})}(Y). \quad (5)$$

This gives a direct analytical proof of one important property usually proved using the *Robinson-Schensted* correspondence, i.e. that permutations  $\pi$  in  $S_n$  are in bijection with pairs of standard Young tableaux  $T, U$  of the same shape such that  $Des(T) = Des(\pi)$  and  $Des(U) = Des(\pi^{-1})$ .

## 1.3 Product of Schur polynomials

One may write a permutation  $\alpha \in S_n$  as a word on the letters in  $[n]$ ,  $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$  where  $\alpha_i = \alpha(i)$ . Given  $\alpha \in S_n$  and  $\beta \in S_m$ , let  $\alpha \sqcup \beta$  be the set of permutations  $\gamma \in S_{m+n}$  obtained by shuffling the letters  $\alpha$  and  $\bar{\beta} = n + \beta_1 \ n + \beta_2 \cdots n + \beta_m$  such that the initial order of the letters of  $\alpha$  (resp. of  $\bar{\beta}$ ) is preserved in  $\gamma$ . By extension if  $A$  and  $B$  are two sets of permutations, we

denote  $A \sqcup B$  the set of all the shuffles of a permutation of  $A$  and a permutation of  $B$ . We have the product formula

$$F_{Des(\alpha)}(X)F_{Des(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}(X). \quad (6)$$

Given a standard Young tableau  $T$  of shape  $\lambda$ , Equation (5) proves the existence of classes of permutations  $C_T$  such that for any  $\pi \in C_T$ ,  $Des(\pi^{-1}) = Des(T)$  and there is a descent preserving bijection between  $C_T$  and  $SYT(\lambda)$ .

**Remark 1.** *Classically the  $C_T$  are the Knuth classes of permutations but here we derive these results directly without using plactic relations and/or the Robinson-Schensted correspondence.*

Equation (6) directly implies for integer partitions  $\lambda$  and  $\mu$  and  $T, U \in SYT(\lambda) \times SYT(\mu)$ :

$$s_\lambda(X)s_\mu(X) = \sum_{\alpha \in C_T} F_{Des(\alpha)}(X) \sum_{\beta \in C_U} F_{Des(\beta)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (7)$$

For any  $\nu \vdash |\lambda| + |\mu|$  define formally the coefficients  $c_{\lambda\mu}^\nu$  as the coefficients of the expansion of  $s_\lambda(X)s_\mu(X)$  in the Schur basis  $(s_\nu)_\nu$ . Equation (7) shows for some  $V_\nu \in SYT(\nu)$  that

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu s_\nu(X) = \sum_{\substack{\nu \vdash |\lambda| + |\mu| \\ \gamma \in C_{V_\nu}}} c_{\lambda\mu}^\nu F_{Des(\gamma)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (8)$$

As a result, we have the following proposition.

**Proposition 1.** *For  $I \subset [|\lambda| + |\mu| - 1]$  denote  $D_I$  the set of permutations  $\pi$  such that  $Des(\pi) = I$ . Using the notations above for integer partitions  $\lambda, \mu$  and for  $T, U$  in  $SYT(\lambda) \times SYT(\mu)$ , there is a set of tableaux  $V_\nu \in SYT(\nu)$  indexed by the set of integer partitions  $\nu \vdash |\lambda| + |\mu|$  such that*

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu |C_{V_\nu} \cap D_I| = |(C_T \sqcup C_U) \cap D_I|. \quad (9)$$

The following sections are dedicated to the extension of this method to weak composition quasisymmetric functions and Chow's type B quasisymmetric functions.

## 2 Type B extension to weak composition quasisymmetric functions

### 2.1 Weak composition quasisymmetric functions and coloured permutations

Weak composition quasisymmetric functions are a generalisation of Gessel's quasisymmetric functions introduced in [5] and [12] in order to provide a framework

for the study of a question a G.-C. Rota relating *Rota-Baxter algebras* and symmetric type functions.

We consider a finite alphabet  $X = \{x_1, x_2, \dots, x_M\}$ . For any non-negative integer  $n$ , let  $\mathcal{WC}(n)$  be the set of **weak compositions** of  $n$ , i.e. the set of ordered tuples of non-negative integers that sum up to  $n$ . A weak composition  $\alpha \in \mathcal{WC}(n)$  is of the form

$$\alpha = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

with  $i_j \geq 0$ ,  $1 \leq j \leq k+1$ ,  $s_i > 0$ ,  $1 \leq j \leq k$  and  $\sum s_i = n$ . We further define  $\ell(\alpha) = k + i_1 + \dots + i_{k+1}$  the **length** of  $\alpha$ ,  $\ell_0(\alpha) = i_1 + \dots + i_{k+1}$  as its **0-length**,  $|\alpha| = n$  the **weight** of  $\alpha$  and  $||\alpha|| = |\alpha| + \ell_0(\alpha)$  the **total weight** of  $\alpha$ . The descent set of  $\alpha$  is

$$D(\alpha) = \{a_1, \dots, a_k | a_j = i_1 + s_1 + \dots + i_j + s_j, j = 1, \dots, k\}.$$

**Definition 1.** Given a weak composition  $\alpha = (0^{j_1}, s_1, 0^{j_2}, s_2, \dots, 0^{j_k}, s_k, 0^{j_{k+1}})$  with  $D(\alpha) = \{a_1, a_2, \dots, a_k\}$  define the **weak composition fundamental quasisymmetric function** indexed by  $\alpha$  the formal power series

$$\begin{aligned} \tilde{F}_\alpha(X) = & \\ & \sum_{\substack{1 \leq i_1 \leq \dots \leq i_{a_k+j_{k+1}} \\ j \in D(\alpha) \Rightarrow i_j < i_{j+1}}} x_{i_1}^0 \cdots x_{i_{j_1}}^0 x_{i_{j_1+1}} \cdots x_{i_{a_1}} \cdots x_{i_{a_{k-1}+j_k+1}} \cdots x_{i_{a_k}} x_{i_{a_k+1}}^0 \cdots x_{i_{a_k+j_{k+1}}}^0 \end{aligned} \quad (10)$$

Weak composition fundamental quasisymmetric functions may also be described in terms of  $P$ -partitions indexed by *coloured permutations*. Define the total order

$$\bar{1} < \bar{2} < \bar{3} < \dots < 1 < 2 < 3 \dots$$

and denote  $|\bar{i}| = i$  for  $i > 0$ . We look at chain posets  $P_\pi = \{\pi_1 <_{P_\pi} \pi_2 <_{P_\pi} \dots <_{P_\pi} \pi_l\}$  indexed by coloured permutations  $\pi$  for any non-negative integer  $l$  where the  $\pi_i$  are distinct letters in  $[l] \cup [\bar{l}]$  and such that  $|\pi_1| |\pi_2| \cdots |\pi_l|$  has no repeated letters. A  $P_\pi$ -**partition** is a map  $f : P_\pi \rightarrow [M]$  such that for  $x <_{P_\pi} y$ ,  $f(x) \leq f(y)$  with  $x > y \Rightarrow f(x) < f(y)$ . The **weight** of  $f$ ,  $w(f)$  is the monomial

$$w(f) = \prod_{\pi_i \in P_\pi} x_{f(\pi_i)}^{\delta_{\pi_i}}, \text{ where } \delta_{\pi_i} = \begin{cases} 0 & \text{if } \pi_i \text{ is an overlined letter,} \\ 1 & \text{otherwise.} \end{cases}$$

Denote  $\mathcal{A}(\pi)$  the set of  $P_\pi$ -partitions. We are interested in the generating functions  $\Gamma$  indexed by coloured permutations  $\pi$

$$\Gamma(\pi) = \sum_{f \in \mathcal{A}(\pi)} w(f). \quad (11)$$

Li shows in [6] that the  $\Gamma(\pi)$  can be expressed as a  $\mathbb{Z}$ -linear combination of the  $F_\alpha$  but not always positive. In particular if the indices  $j_1 < j_2 < \dots < j_p$  of the

overlined letters in a coloured permutation  $\pi$  are such that  $\pi_{j_1} < \pi_{j_2} < \dots < \pi_{j_p}$ , denote the weak composition

$$\text{comp}(\pi) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

where  $(s_1, s_2, \dots, s_k)$  is the tuple of lengths of the sequences of consecutive increasing non-overlined letters in  $\pi$  and  $i_r$  is the quantity of overlined letters between sequences  $r - 1$  and  $r$ . We have  $\ell_0(\text{comp}(\pi)) = p$  and we denote  $S_n^{(p)}$  the set of such coloured permutations with  $|\text{comp}(\pi)| = n$ . As an example for  $\pi = 597\overline{1}2\overline{6}3\overline{8}4$ , we have  $\text{comp}(\pi) = (2, 1, 0^2, 1, 0, 1, 0)$ . As a direct consequence of Equation (10), for  $n, p \geq 0$  and  $\pi \in S_n^{(p)}$ , one has ([6])

$$\tilde{F}_{\text{comp}(\pi)} = \Gamma(\pi).$$

On the other hand, take  $\pi = 3\overline{2}1$ , then it is easy to show that  $\Gamma(\pi) = \tilde{F}_{(1,0^2)} - \tilde{F}_{(1,0)}$ . Finally, as in Section 1.3, define for  $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$  the set  $\alpha \sqcup \beta$  of coloured permutations obtained by shuffling the letters of  $\tilde{\alpha}$  and  $\tilde{\beta}$  where the non-overlined letters of  $\alpha$  are shifted by  $q$  to get  $\tilde{\alpha}$  and the overlined (resp. non-overlined) letters of  $\beta$  are shifted by  $p$  (resp. by  $p + n$ ) to get  $\tilde{\beta}$ . For instance, let  $\alpha = 3\overline{2}1$ ,  $\beta = \overline{1}2$ , then  $\tilde{\alpha} = 43\overline{1}$ ,  $\tilde{\beta} = \overline{2}5$  and

$$\alpha \sqcup \beta = 43\overline{1}25 + 43\overline{2}15 + 4\overline{2}3\overline{1}5 + \overline{2}43\overline{1}5 + 43\overline{2}5\overline{1} + 4\overline{2}35\overline{1} + \overline{2}435\overline{1} + 4\overline{2}53\overline{1} + \overline{2}453\overline{1} + \overline{2}543\overline{1}.$$

One can see that despite the fact that  $\alpha$  and  $\beta$  belong to  $S_n^{(p)}$  and  $S_r^{(q)}$ , not all the permutations in  $\alpha \sqcup \beta$  fulfil the conditions of proper order of the overlined letters. We have for  $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$ ,

$$\tilde{F}_{\text{comp}(\alpha)}(X)\tilde{F}_{\text{comp}(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} \Gamma(\gamma). \quad (12)$$

In particular for  $n = r = 0$

$$\tilde{F}_{(0^p)}(X)\tilde{F}_{(0^q)}(X) = \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \tilde{F}_{(0^{p+q-j})}(X). \quad (13)$$

In some sense, weak composition fundamental quasisymmetric functions are a specialisation of *Poirier quasisymmetric functions* (see [9]), a classical type B generalisation of Gessel's quasisymmetric functions that we do not detail here. In [1], the authors provide a connexion between the generating functions for Young *bi-tableaux* and the fundamental quasisymmetric functions of Poirier. As a result, there should be a natural bi-tableaux interpretation of weak composition quasisymmetric functions. This is the focus of the next section.

## 2.2 Tableau-theoretic interpretation

Let  $\lambda$  be an integer partition and  $p$  a non-negative integer. We consider pairs of Young tableau  $(T^-, T^+)$  where  $T^-$  is of shape  $(p)$  and  $T^+$  has shape  $\lambda$ . We call such a bi-tableau a  $(p)$ -tableau. We proceed with the formal definition.

**Definition 2.** For any couple of non-negative integer  $p, n$  and an integer partition  $\lambda \vdash n$ , a **semistandard**  $(p)$ -**tableau**  $T = (T^-, T^+)$  is a pair of Young diagrams of bi-shape  $((p), \lambda)$  whose boxes are filled with the integers in  $[M]$  such that the entries are strictly increasing down the columns and non-decreasing along the rows. The **shape** of  $T$  is noted  $sh(T) = ((p), \lambda)$  and we denote  $SSBT^{(p)}(\lambda)$  the set of all  $(p)$ -tableaux of shape  $((p), \lambda)$ . If the boxes are filled with integers in  $[n+p]$  all used exactly once, we call  $T$  a **standard**  $(p)$ -**tableau** and we denote  $SBT^{(p)}(\lambda)$  the set of such tableaux.

We also need the following notions. The **standardisation**  $T^{st}$  of a semistandard  $(p)$ -tableau  $T$  is the standard  $(p)$ -tableau of shape  $((p), \lambda)$  obtained by relabelling the boxes of  $T$  with all the integers in  $[|\lambda| + p]$  such that the order of the labels is preserved. When a label in  $T$  is used both in  $T^-$  and  $T^+$ , the boxes of  $T^-$  are relabelled first. Finally, given a standard  $(p)$ -tableau  $T \in SBT^{(p)}(\lambda)$  with  $|\lambda| = n$  denote  $comp(T)$  the weak composition of  $n$ ,  $comp(T) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$ , where  $i_1$  is the number of consecutive labels  $k = 1, 2, \dots, i_1$  in  $T^-$  ( $i_1 = 0$  if 1 is in  $T^+$ ). Then  $s_1$  is the number of consecutive labels  $k = i_1 + 1, \dots, i_1 + s_1$  in  $T^+$  such that no label  $k$  is in a higher row than  $k + 1$ . Then  $i_2$  is the number of consecutive labels  $k = i_1 + s_1 + 1, 2, \dots, i_1 + s_1 + i_2$  in  $T^-$ , etc...

**Example 1.** The following semistandard  $(3)$ -tableau belongs to  $SSBT^{(3)}((2, 2, 1))$ .

$$T = \left( T^- = \begin{array}{|c|c|c|} \hline 2 & 4 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 & 5 \\ \hline 9 & \\ \hline \end{array} \right)$$

Furthermore, its standardisation is:

$$T^{st} = \left( T^- = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 6 & 7 \\ \hline 8 & \\ \hline \end{array} \right)$$

Finally the weak composition of  $T^{st}$  is  $comp(T^{st}) = (1, 0^3, 1, 2, 1)$ .

For  $\lambda \vdash n$  and  $p \geq 0$  denote  $s_\lambda^{(p)}$  the generating function for such tableaux. We have

$$s_\lambda^{(p)}(X) = \sum_{T \in SSBT^{(p)}(\lambda)} X^{T^+}. \quad (14)$$

Recall the notation of the Schur symmetric function  $s_\lambda$  indexed by  $\lambda$ . The following lemma is immediate.

**Lemma 1.** For any  $\lambda \vdash n$  and  $p \geq 0$ , one has

$$s_\lambda^{(p)}(X) = \binom{M+p-1}{p} s_\lambda(X). \quad (15)$$

We can now state the main result of this section.

**Proposition 2.** *For non-negative integers  $p$  and  $n$  and an integer partition  $\lambda \vdash n$  the generating function for  $(p)$ -tableau of shape  $((p), \lambda)$  is related to weak composition fundamental quasisymmetric functions through*

$$s_\lambda^{(p)}(X) = \sum_{T \in SBT^{(p)}(\lambda)} \tilde{F}_{comp(T)}(X). \quad (16)$$

*Proof.* (sketch) Starting from Equation (14), sort all the semi-standard  $(p)$ -tableaux according to their standardisation. Then, given a standard  $(p)$ -tableau  $T_0$  show that

$$\tilde{F}_{comp(T_0)}(X) = \sum_{T \in SSBT^{(p)}(\lambda); T^{st}=T_0} X^{T^+}.$$

Finally, sum on  $SBT^{(p)}(\lambda)$  to get the result.  $\square$

### 2.3 Product formulas

We use Proposition 2 to derive new results. We show the following formula.

**Theorem 1.** *Let  $p, q, n, m$  be non-negative integers and  $\lambda, \mu$  integer partitions of  $n$  and  $m$ . For  $\nu \vdash |\lambda| + |\mu|$  recall the coefficients  $c_{\lambda, \mu}^\nu$  introduced in Section 1.3. The following identity holds.*

$$\sum_{\substack{T \in SBT^{(p)}(\lambda) \\ U \in SBT^{(q)}(\mu)}} \tilde{F}_{comp(T)}(X) \tilde{F}_{comp(U)}(X) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda, \mu}^\nu \tilde{F}_{comp(V)}(X). \quad (17)$$

*Proof.* (sketch) First, note that for any integer  $p$

$$s_\emptyset^{(p)} = \tilde{F}_{(0^p)} = \binom{M+p-1}{p}.$$

Then use Equation (13) and the definition of the coefficients  $c_{\lambda, \mu}^\nu$  to write

$$\begin{aligned} s_\lambda^{(p)} s_\mu^{(q)} &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \binom{M+p-1}{p} \binom{M+q-1}{q} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \binom{M+p+q-j-1}{p+q-j} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} s_\nu^{(p+q-j)}, \end{aligned}$$

and apply Proposition 2 to get the result.  $\square$



There is an interesting corollary of Theorem 1 in terms of generating functions for  $P_\pi$ -partitions. For a (classical) standard Young tableau  $T$ , recall the permutation class  $C_T$  of Section 1.3. Given a non-negative integer  $p$ , we denote  $\varepsilon_p$  the coloured permutation  $\varepsilon_p = \overline{12 \cdots p}$ . The result can be stated as follows.

**Corollary 1.** *For non-negative integers  $p, q$ , integers partitions  $\lambda$  and  $\mu$  and  $T \in SYT(\lambda)$ ,  $U \in SYT(\mu)$ , one has:*

$$\sum_{\gamma \in (\varepsilon_p \sqcup C_T) \sqcup (\varepsilon_q \sqcup C_U)} \Gamma(\gamma) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda\mu}^\nu \tilde{F}_{comp(V)}(X). \quad (18)$$

*Proof.* We omit the proof in this extended abstract.  $\square$

### 3 Type B extension to Chow's quasisymmetric functions

#### 3.1 Signed permutations and domino tableaux

Let  $B_n$  be the **hyperoctahedral group** of order  $n$ .  $B_n$  is composed of all signed permutations  $\pi$  on  $\{-n, \dots, -2, -1, 0, 1, 2, \dots, n\}$  such that  $\pi(-i) = -\pi(i)$  for all  $i$ . The **total colour** of  $\pi \in B_n$  is the number  $tc(\pi) = |\{i \geq 1; \pi(i) < 0\}|$ . Its **descent set** is the subset of  $\{0\} \cup [n-1]$  equal to  $Des(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ . The main difference with respect to the case of the symmetric group is the possible descent in position 0 when  $\pi(1)$  is a negative integer.

For  $\lambda \vdash 2n$ , a **standard domino tableau**  $T$  of shape  $\lambda$  is a Young diagram of shape  $shape(T) = \lambda$  tiled by **dominoes**, i.e.  $2 \times 1$  or  $1 \times 2$  rectangles filled with the elements of  $[n]$  such that the entries are strictly increasing along the rows and down the columns. In the sequel we consider only the set  $\mathcal{P}^0(n)$  of *empty 2-core partitions*  $\lambda \vdash 2n$  that fit such a tiling. A standard domino tableau  $T$  has a descent in position  $i > 0$  if  $i+1$  lies strictly below  $i$  in  $T$  and has descent in position 0 if the domino filled with 1 is vertical. We denote  $Des(T)$  the set of all its descents. A **semistandard domino tableau**  $T$  of shape  $\lambda \in \mathcal{P}^0(n)$  and weight  $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \dots)$  with  $\mu_i \geq 0$  and  $\sum_i \mu_i = n$  is a tiling of the Young diagram of shape  $\lambda$  with horizontal and vertical dominoes labelled with integers in  $\{0, 1, 2, \dots\}$  such that labels are non decreasing along the rows, strictly increasing down the columns and exactly  $\mu_i$  dominoes are labelled with  $i$ . If the top leftmost domino is vertical, **it cannot be labelled 0**. Denote  $SDT(\lambda)$  (resp.  $SSDT(\lambda)$ ) the set of standard (resp. semistandard) domino tableaux of shape  $\lambda$ . Finally, we denote  $sp(T)$ , the **spin** of (semi-) standard domino tableau  $T$ , i.e. **half the number** of its vertical dominoes.

**Remark 2.** *The possible labelling of top leftmost horizontal dominoes in semistandard domino tableaux with 0 was first introduced by the authors in [8] and*

is essential to connect their generating functions with Chow's quasisymmetric functions.

$T_1 =$	<table style="border-collapse: collapse; text-align: center;"> <tr> <td style="border: 1px solid black; padding: 2px 5px;">1</td> <td style="border: 1px solid black; padding: 2px 5px;">2</td> <td style="border: 1px solid black; padding: 2px 5px;">3</td> <td style="border: 1px solid black; padding: 2px 5px;">5</td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">4</td> <td style="border: 1px solid black; padding: 2px 5px;">8</td> <td style="border: 1px solid black; padding: 2px 5px;">6</td> <td style="border: 1px solid black; padding: 2px 5px;"></td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">7</td> <td style="border: 1px solid black; padding: 2px 5px;"></td> <td style="border: 1px solid black; padding: 2px 5px;"></td> <td style="border: 1px solid black; padding: 2px 5px;"></td> </tr> </table>	1	2	3	5	4	8	6		7			
1	2	3	5										
4	8	6											
7													

$T_2 =$	<table style="border-collapse: collapse; text-align: center;"> <tr> <td style="border: 1px solid black; padding: 2px 5px;">1</td> <td style="border: 1px solid black; padding: 2px 5px;">2</td> <td style="border: 1px solid black; padding: 2px 5px;">3</td> <td style="border: 1px solid black; padding: 2px 5px;">5</td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">6</td> <td style="border: 1px solid black; padding: 2px 5px;">8</td> <td style="border: 1px solid black; padding: 2px 5px;">4</td> <td style="border: 1px solid black; padding: 2px 5px;"></td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">7</td> <td style="border: 1px solid black; padding: 2px 5px;"></td> <td style="border: 1px solid black; padding: 2px 5px;"></td> <td style="border: 1px solid black; padding: 2px 5px;"></td> </tr> </table>	1	2	3	5	6	8	4		7			
1	2	3	5										
6	8	4											
7													

$T_3 =$	<table style="border-collapse: collapse; text-align: center;"> <tr> <td style="border: 1px solid black; padding: 2px 5px;">0</td> <td style="border: 1px solid black; padding: 2px 5px;">0</td> <td style="border: 1px solid black; padding: 2px 5px;">5</td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">2</td> <td style="border: 1px solid black; padding: 2px 5px;">2</td> <td style="border: 1px solid black; padding: 2px 5px;">5</td> </tr> <tr> <td style="border: 1px solid black; padding: 2px 5px;">5</td> <td style="border: 1px solid black; padding: 2px 5px;">5</td> <td style="border: 1px solid black; padding: 2px 5px;">7</td> </tr> </table>	0	0	5	2	2	5	5	5	7
0	0	5								
2	2	5								
5	5	7								

Figure 1: Two standard domino tableaux  $T_1$  and  $T_2$  of shape  $(5, 5, 4, 1, 1)$ , descent set  $\{0, 3, 5, 6\}$ , a semistandard tableau  $T_3$  of shape  $(5, 5, 4, 3, 1)$  and weight  $\mu = (2, 0, 2, 0, 4, 0, 1)$ . All of the tableaux have a spin of 2.

### 3.2 Chow's type B quasisymmetric functions

Chow defines in [3] another Type B extension of Gessel's algebra of quasisymmetric functions that is dual to Solomon's descent algebra of type B.

Let  $X = \{\dots, x_{-i}, \dots, x_{-1}, x_0, x_1, \dots, x_i, \dots\}$  be an alphabet of totally ordered commutative indeterminates with the assumption that  $x_{-i} = x_i$  and let  $I$  be a subset of  $\{0\} \cup [n - 1]$ , he defines a type B analogue of the fundamental quasisymmetric functions

$$F_I^B(X) = \sum_{\substack{0=i_0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Note the particular role of the variable  $x_0$ .

Given two signed permutations  $\alpha \in B_n$  and  $\beta \in B_m$ , denote  $\alpha \sqcup \beta$  the set of signed permutations in  $B_{m+n}$  obtained by shuffling the letters of  $\alpha$  and  $\bar{\beta}$  where we shift the absolute value of the  $\beta_i$  to get  $\bar{\beta}$ . Chow shows in [3, Prop. 2.2.6]

$$F_{Des(\alpha)}^B(X) F_{Des(\beta)}^B(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}^B(X). \quad (19)$$

Furthermore, we introduce in [8] modified generating functions for domino tableaux called domino functions. Given a semistandard domino tableau  $T$  of weight  $\mu$ , denote  $X^T$  the monomial  $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \dots$ . For  $\lambda \in \mathcal{P}^0(n)$  and an additional parameter  $q$  we define the **domino function** indexed by  $\lambda$  on alphabet  $X$  as

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SSDT(\lambda)} q^{sp(T)} X^T. \quad (20)$$

These functions are related to Chow's quasisymmetric functions through

**Lemma 2** ([7]). *For  $\lambda \in \mathcal{P}^0(n)$ , the  $q$ -domino function indexed by  $\lambda$  can be expanded in the basis of Chow's quasisymmetric functions as*

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SDT(\lambda)} q^{sp(T)} F_{Des(T)}^B(X). \quad (21)$$

### 3.3 Type B Littlewood-Richardson coefficients

There is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([2]) built a bijection between signed permutations of  $B_n$  and pairs of standard domino tableaux of equal shape in  $\mathcal{P}^0(n)$ . An independent development on the subject is due to Garfinkle. Taşkin ([11, Prop. 26]) showed that the two standard domino tableaux  $T$  and  $U$  associated to a signed permutation  $\pi$  by the algorithm of Barbash and Vogan have respective descent sets  $Des(T) = Des(\pi^{-1})$  and  $Des(U) = Des(\pi)$  while Shimozono and White showed in [10] the **colour-to-spin** property i.e.  $tc(\pi) = sp(T) + sp(U)$ .

Let  $X, Y$  be two alphabets of indeterminates. In [7], we show that Equation (21) implies the following proposition.

**Proposition 3** ([7]). *For any integer  $n$ , we have*

$$\sum_{\pi \in B_n} q^{tc(\pi)} F_{Des(\pi)}^B(X) F_{Des(\pi^{-1})}^B(Y) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in SDT(\lambda)}} q^{sp(T) + sp(U)} F_{Des(T)}^B(X) F_{Des(U)}^B(Y). \quad (22)$$

Proposition 3 provides a direct analytical proof of the properties showed in a bijective fashion by Barbash, Vogan, Taşkin, Shimozono and White. More formally it proves the existence for any standard domino tableau  $T$  of a class of signed permutations  $C_T^B$  such that for any  $\pi \in C_T^B$ ,  $Des(\pi^{-1}) = Des(T)$  and there is a descent preserving bijection  $C_T^B \rightarrow SDT(\lambda), \pi \mapsto U$  such that  $tc(\pi) = sp(T) + sp(U)$ . In a similar fashion as in Section 1.3, we use Equation (22) to show some properties for the *type B Littlewood-Richardson coefficients*.

First we focus on the case  $q = 1$ . There is a well known (not descent preserving) bijection between semistandard domino tableaux of weight  $\mu$  and semistandard bi-tableaux of respective weights  $\mu^-$  and  $\mu^+$  such that  $\mu_i^+ + \mu_i^- = \mu_i$  for all  $i$ . The respective shapes of the two Young tableaux depend only on the shape of the initial semistandard domino tableau. Denote  $(T^-, T^+)$  the bi-tableau associated to a semistandard domino tableau  $T$  of shape  $\lambda$  and  $(\lambda^-, \lambda^+)$  their respective shapes.  $(T^-, T^+)$  (resp.  $(\lambda^-, \lambda^+)$ ) is the **2-quotient** of  $T$  (resp.  $\lambda$ ). Denote  $X^* = X \setminus x_0$ . For  $\lambda \in \mathcal{P}^0(n)$  one has

$$\mathcal{G}_\lambda(X; 1) = s_{\lambda^-}(X^*) s_{\lambda^+}(X)$$

and for  $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$ ,

$$\mathcal{G}_\lambda(X; 1)\mathcal{G}_\mu(X; 1) = \sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} \mathcal{G}_\nu(X; 1). \quad (23)$$

The following result extends Proposition 1 to signed permutations.

**Theorem 2.** *Let  $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$ . For  $I \subset [0] \cup [n+m-1]$  denote  $D_I^B$  the set of signed permutations  $\pi$  such that  $\text{Des}(\pi) = I$  and  $\Delta_I^B$  the set of standard domino tableaux of descent set  $I$ . Using the notations above for integer partitions  $\lambda, \mu$  and  $T, U$  in  $\text{SDT}(\lambda) \times \text{SDT}(\mu)$*

$$\sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} |\text{SDT}(\nu) \cap \Delta_I^B| = |(C_T^B \sqcup C_U^B) \cap D_I^B|. \quad (24)$$

*Proof.* (sketch) Theorem 2 is a consequence of Equations (23) and (13).  $\square$

On the other hand, the general case for  $q$  remains open. Indeed, using Proposition 3, one can show for  $\lambda \in \mathcal{P}^0(n)$  and  $T \in \text{SDT}(\lambda)$

$$\mathcal{G}_\lambda(X; q) = \sum_{\pi \in C_T^B} q^{tc(\pi) - sp(T)} F_{\text{Des}(\pi)}^B.$$

Then, if  $\mu \in \mathcal{P}^0(m)$  and  $U \in \text{SDT}(\mu)$

$$\mathcal{G}_\lambda(X; q)\mathcal{G}_\mu(X; q) = q^{-(sp(T)+sp(U))} \sum_{\gamma \in C_T^B \sqcup C_U^B} q^{tc(\gamma)} F_{\text{Des}(\gamma)}^B.$$

However, even if we showed that the set of signed permutations  $C_T^B \sqcup C_U^B$  is in descent preserving bijection with a set of cardinality  $c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} |\text{SDT}(\nu)|$ , we cannot say *a priori* how this bijection preserves or transforms the statistic  $tc$ .

## References

- [1] R.M. Adin et al. “Character formulas and descents for the hyperoctahedral group”. In: *Advances in Applied Mathematics* 87 (2017), pp. 128–169.
- [2] D. Barbash and D. Vogan. “Primitive ideals and orbital integrals in complex classical groups”. In: *Math. Ann.* 259 (1982), pp. 153–199.
- [3] C.O. Chow. “Noncommutative symmetric functions of type B”. PhD thesis. MIT, 2001.
- [4] I. Gessel. “Multipartite P-partitions and inner products of skew Schur functions”. In: *Contemporary Mathematics* 34 (1984), pp. 289–317.
- [5] L. Guo, H. Yu, and J. Zhao. “Rota–Baxter algebras and left weak composition quasi-symmetric functions”. In: *The Ramanujan Journal* 44.3 (2017), pp. 567–596.

- [6] Y. Li. “On weak peak quasisymmetric functions”. In: *Journal of Comb. Theory, Series A* 158 (2018), pp. 449–491.
- [7] A.R. Mayorova and E.A. Vassilieva. “A  $q$ -deformed type B Cauchy identity and Chow’s quasisymmetric functions”. In: *Séminaire Lotharingien de Combinatoire* 82B.87 (2019).
- [8] A.R. Mayorova and E.A. Vassilieva. “On the structure constants of the descent algebra of the hyperoctahedral group”. In: *Electronic Notes in Discrete Mathematics* 61 (2017), pp. 847–853.
- [9] S. Poirier. “Cycle type and descent set in wreath products”. In: *Discrete Math.* 180 (1998), pp. 315–343.
- [10] M. Shimozono and D. White. “A color-to-spin domino Schensted algorithm”. In: *The Electronic Journal of Combinatorics* 8 (2001), R2.
- [11] M. Taşkin. “Plactic relations for  $r$ -domino tableaux”. In: *The Electronic Journal of Combinatorics* 19 (2012), P38.
- [12] H. Yu, L. Guo, and J.Y. Thibon. “Weak quasi-symmetric functions, Rota–Baxter algebras and Hopf algebras”. In: *Advances in mathematics* 344 (2019), pp. 1–34.