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# On the product of generating functions for domino and bi-tableaux

Ekaterina A. Vassilieva

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**Abstract.** The connection between the generating functions of various sets of tableaux and the appropriate families of quasisymmetric functions is a significant tool to give a direct analytical proof of some advanced bijective results and provide new combinatorial formulas. In this paper we focus on two kinds of type B Littlewood-Richardson coefficients and derive new formulas using weak composition quasisymmetric functions and Chow's quasisymmetric functions. In the type A case these coefficients give the multiplication table for Schur functions, i.e. the generating functions for classical Young tableaux. We look at their generalisations involving a set of bi-tableaux and domino tableaux.

## 1 Type A quasisymmetric functions

### 1.1 Permutations, Young tableaux and descent sets

For any positive integer  $n$  write  $[n] = \{1, \dots, n\}$ ,  $S_n$  the symmetric group on  $[n]$ . A **partition**  $\lambda$  of  $n$ , denoted  $\lambda \vdash n$  is a sequence  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$  of  $\ell(\lambda) = p$  parts sorted in decreasing order such that  $|\lambda| = \sum_i \lambda_i = n$ . A partition  $\lambda$  is usually represented as a Young diagram of  $n = |\lambda|$  boxes arranged in  $\ell(\lambda)$  left justified rows so that the  $i$ -th row from the top contains  $\lambda_i$  boxes. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a **semistandard Young tableau**. If the entries of a semistandard Young tableau are restricted to the elements of  $[n]$  and strictly increasing along the rows, we call it a **standard Young tableau**. The partition  $\lambda$  is the **shape** of the tableau and we denote  $SYT(\lambda)$  (resp.  $SSYT(\lambda)$ ) the set of standard (resp. semistandard) Young tableaux of shape  $\lambda$ .

One important feature of a permutation  $\pi$  in  $S_n$  is its **descent set**, i.e. the subset of  $[n-1]$  defined as  $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ . Similarly, define the **descent set of a standard Young tableau**  $T$  as  $Des(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$ .

## 1.2 Quasisymmetric expansion of Schur polynomials

Let  $X = \{x_1, x_2, \dots\}$  and  $Y = \{y_1, y_2, \dots\}$  be two alphabets of commutative indeterminates. Denote also  $XY$  the product alphabet  $\{x_i y_j\}_{i,j}$  ordered by the lexicographical order. Let  $F_I(X)$  be the **Gessel's fundamental quasisymmetric function** on  $X$  indexed by the subset  $I \subset [n-1]$ :

$$F_I(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \quad (1)$$

The power series  $F_I(X)$  is not symmetric in  $X$  but verifies the property that for any strictly increasing sequence of indices  $i_1 < i_2 < \dots < i_p$  the coefficient of  $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$  is equal to the coefficient of  $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$ . Let  $s_\lambda(X)$  be the Schur polynomial indexed by partition  $\lambda$  on alphabet  $X$ . Schur polynomials are the generating functions for semistandard Young tableaux. As a result,

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{Des(T)}(X). \quad (2)$$

Using for any  $\pi \in S_n$  Gessel's formula for the coproduct of fundamental quasisymmetric functions in [4]

$$F_{Des(\pi)}(XY) = \sum_{\sigma, \rho \in S_n; \sigma\rho = \pi} F_{Des(\sigma)}(X) F_{Des(\rho)}(Y), \quad (3)$$

and the Cauchy formula for Schur polynomials

$$\sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) = s_{(n)}(XY), \quad (4)$$

one can derive

$$\sum_{\substack{\lambda \vdash n \\ T, U \in SYT(\lambda)}} F_{Des(T)}(X) F_{Des(U)}(Y) = F_\emptyset(XY) = \sum_{\pi \in S_n} F_{Des(\pi)}(X) F_{Des(\pi^{-1})}(Y). \quad (5)$$

This gives a direct analytical proof of one important property usually proved using the *Robinson-Schensted* correspondence, i.e. that permutations  $\pi$  in  $S_n$  are in bijection with pairs of standard Young tableaux  $T, U$  of the same shape such that  $Des(T) = Des(\pi)$  and  $Des(U) = Des(\pi^{-1})$ .

## 1.3 Product of Schur polynomials

One may write a permutation  $\alpha \in S_n$  as a word on the letters in  $[n]$ ,  $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$  where  $\alpha_i = \alpha(i)$ . Given  $\alpha \in S_n$  and  $\beta \in S_m$ , let  $\alpha \sqcup \beta$  be the set of permutations  $\gamma \in S_{m+n}$  obtained by shuffling the letters  $\alpha$  and  $\bar{\beta} = n + \beta_1 \ n + \beta_2 \cdots n + \beta_m$  such that the initial order of the letters of  $\alpha$  (resp. of  $\bar{\beta}$ ) is preserved in  $\gamma$ . By extension if  $A$  and  $B$  are two sets of permutations, we

denote  $A \sqcup B$  the set of all the shuffles of a permutation of  $A$  and a permutation of  $B$ . We have the product formula

$$F_{Des(\alpha)}(X)F_{Des(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}(X). \quad (6)$$

Given a standard Young tableau  $T$  of shape  $\lambda$ , Equation (5) proves the existence of classes of permutations  $C_T$  such that for any  $\pi \in C_T$ ,  $Des(\pi^{-1}) = Des(T)$  and there is a descent preserving bijection between  $C_T$  and  $SYT(\lambda)$ .

**Remark 1.** *Classically the  $C_T$  are the Knuth classes of permutations but here we derive these results directly without using plactic relations and/or the Robinson-Schensted correspondence.*

Equation (6) directly implies for integer partitions  $\lambda$  and  $\mu$  and  $T, U \in SYT(\lambda) \times SYT(\mu)$ :

$$s_\lambda(X)s_\mu(X) = \sum_{\alpha \in C_T} F_{Des(\alpha)}(X) \sum_{\beta \in C_U} F_{Des(\beta)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (7)$$

For any  $\nu \vdash |\lambda| + |\mu|$  define formally the coefficients  $c_{\lambda\mu}^\nu$  as the coefficients of the expansion of  $s_\lambda(X)s_\mu(X)$  in the Schur basis  $(s_\nu)_\nu$ . Equation (7) shows for some  $V_\nu \in SYT(\nu)$  that

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu s_\nu(X) = \sum_{\substack{\nu \vdash |\lambda| + |\mu| \\ \gamma \in C_{V_\nu}}} c_{\lambda\mu}^\nu F_{Des(\gamma)}(X) = \sum_{\gamma \in C_T \sqcup C_U} F_{Des(\gamma)}(X). \quad (8)$$

As a result, we have the following proposition.

**Proposition 1.** *For  $I \subset [|\lambda| + |\mu| - 1]$  denote  $D_I$  the set of permutations  $\pi$  such that  $Des(\pi) = I$ . Using the notations above for integer partitions  $\lambda, \mu$  and for  $T, U$  in  $SYT(\lambda) \times SYT(\mu)$ , there is a set of tableaux  $V_\nu \in SYT(\nu)$  indexed by the set of integer partitions  $\nu \vdash |\lambda| + |\mu|$  such that*

$$\sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu |C_{V_\nu} \cap D_I| = |(C_T \sqcup C_U) \cap D_I|. \quad (9)$$

The following sections are dedicated to the extension of this method to weak composition quasisymmetric functions and Chow's type B quasisymmetric functions.

## 2 Type B extension to weak composition quasisymmetric functions

### 2.1 Weak composition quasisymmetric functions and coloured permutations

Weak composition quasisymmetric functions are a generalisation of Gessel's quasisymmetric functions introduced in [5] and [12] in order to provide a framework

for the study of a question a G.-C. Rota relating *Rota-Baxter algebras* and symmetric type functions.

We consider a finite alphabet  $X = \{x_1, x_2, \dots, x_M\}$ . For any non-negative integer  $n$ , let  $\mathcal{WC}(n)$  be the set of **weak compositions** of  $n$ , i.e. the set of ordered tuples of non-negative integers that sum up to  $n$ . A weak composition  $\alpha \in \mathcal{WC}(n)$  is of the form

$$\alpha = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

with  $i_j \geq 0$ ,  $1 \leq j \leq k+1$ ,  $s_i > 0$ ,  $1 \leq j \leq k$  and  $\sum s_i = n$ . We further define  $\ell(\alpha) = k + i_1 + \dots + i_{k+1}$  the **length** of  $\alpha$ ,  $\ell_0(\alpha) = i_1 + \dots + i_{k+1}$  as its **0-length**,  $|\alpha| = n$  the **weight** of  $\alpha$  and  $||\alpha|| = |\alpha| + \ell_0(\alpha)$  the **total weight** of  $\alpha$ . The descent set of  $\alpha$  is

$$D(\alpha) = \{a_1, \dots, a_k | a_j = i_1 + s_1 + \dots + i_j + s_j, j = 1, \dots, k\}.$$

**Definition 1.** Given a weak composition  $\alpha = (0^{j_1}, s_1, 0^{j_2}, s_2, \dots, 0^{j_k}, s_k, 0^{j_{k+1}})$  with  $D(\alpha) = \{a_1, a_2, \dots, a_k\}$  define the **weak composition fundamental quasisymmetric function** indexed by  $\alpha$  the formal power series

$$\begin{aligned} \tilde{F}_\alpha(X) = & \\ & \sum_{\substack{1 \leq i_1 \leq \dots \leq i_{a_k+j_{k+1}} \\ j \in D(\alpha) \Rightarrow i_j < i_{j+1}}} x_{i_1}^0 \cdots x_{i_{j_1}}^0 x_{i_{j_1+1}} \cdots x_{i_{a_1}} \cdots x_{i_{a_{k-1}+j_k+1}} \cdots x_{i_{a_k}} x_{i_{a_k+1}}^0 \cdots x_{i_{a_k+j_{k+1}}}^0 \end{aligned} \quad (10)$$

Weak composition fundamental quasisymmetric functions may also be described in terms of  $P$ -partitions indexed by *coloured permutations*. Define the total order

$$\bar{1} < \bar{2} < \bar{3} < \dots < 1 < 2 < 3 \dots$$

and denote  $|\bar{i}| = i$  for  $i > 0$ . We look at chain posets  $P_\pi = \{\pi_1 <_{P_\pi} \pi_2 <_{P_\pi} \dots <_{P_\pi} \pi_l\}$  indexed by coloured permutations  $\pi$  for any non-negative integer  $l$  where the  $\pi_i$  are distinct letters in  $[l] \cup [\bar{l}]$  and such that  $|\pi_1| |\pi_2| \cdots |\pi_l|$  has no repeated letters. A  $P_\pi$ -**partition** is a map  $f : P_\pi \rightarrow [M]$  such that for  $x <_{P_\pi} y$ ,  $f(x) \leq f(y)$  with  $x > y \Rightarrow f(x) < f(y)$ . The **weight** of  $f$ ,  $w(f)$  is the monomial

$$w(f) = \prod_{\pi_i \in P_\pi} x_{f(\pi_i)}^{\delta_{\pi_i}}, \text{ where } \delta_{\pi_i} = \begin{cases} 0 & \text{if } \pi_i \text{ is an overlined letter,} \\ 1 & \text{otherwise.} \end{cases}$$

Denote  $\mathcal{A}(\pi)$  the set of  $P_\pi$ -partitions. We are interested in the generating functions  $\Gamma$  indexed by coloured permutations  $\pi$

$$\Gamma(\pi) = \sum_{f \in \mathcal{A}(\pi)} w(f). \quad (11)$$

Li shows in [6] that the  $\Gamma(\pi)$  can be expressed as a  $\mathbb{Z}$ -linear combination of the  $F_\alpha$  but not always positive. In particular if the indices  $j_1 < j_2 < \dots < j_p$  of the

overlined letters in a coloured permutation  $\pi$  are such that  $\pi_{j_1} < \pi_{j_2} < \dots < \pi_{j_p}$ , denote the weak composition

$$\text{comp}(\pi) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$$

where  $(s_1, s_2, \dots, s_k)$  is the tuple of lengths of the sequences of consecutive increasing non-overlined letters in  $\pi$  and  $i_r$  is the quantity of overlined letters between sequences  $r - 1$  and  $r$ . We have  $\ell_0(\text{comp}(\pi)) = p$  and we denote  $S_n^{(p)}$  the set of such coloured permutations with  $|\text{comp}(\pi)| = n$ . As an example for  $\pi = 597\bar{1}2\bar{6}3\bar{8}4$ , we have  $\text{comp}(\pi) = (2, 1, 0^2, 1, 0, 1, 0)$ . As a direct consequence of Equation (10), for  $n, p \geq 0$  and  $\pi \in S_n^{(p)}$ , one has ([6])

$$\tilde{F}_{\text{comp}(\pi)} = \Gamma(\pi).$$

On the other hand, take  $\pi = 3\bar{2}\bar{1}$ , then it is easy to show that  $\Gamma(\pi) = \tilde{F}_{(1,0^2)} - \tilde{F}_{(1,0)}$ . Finally, as in Section 1.3, define for  $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$  the set  $\alpha \sqcup \beta$  of coloured permutations obtained by shuffling the letters of  $\tilde{\alpha}$  and  $\tilde{\beta}$  where the non-overlined letters of  $\alpha$  are shifted by  $q$  to get  $\tilde{\alpha}$  and the overlined (resp. non-overlined) letters of  $\beta$  are shifted by  $p$  (resp. by  $p + n$ ) to get  $\tilde{\beta}$ . For instance, let  $\alpha = 3\bar{2}\bar{1}$ ,  $\beta = \bar{1}2$ , then  $\tilde{\alpha} = 43\bar{1}$ ,  $\tilde{\beta} = \bar{2}5$  and

$$\alpha \sqcup \beta = 43\bar{1}25 + 43\bar{2}15 + 4\bar{2}315 + \bar{2}4315 + 43\bar{2}51 + 4\bar{2}351 + \bar{2}4351 + 4\bar{2}531 + \bar{2}4531 + \bar{2}5431.$$

One can see that despite the fact that  $\alpha$  and  $\beta$  belong to  $S_n^{(p)}$  and  $S_r^{(q)}$ , not all the permutations in  $\alpha \sqcup \beta$  fulfil the conditions of proper order of the overlined letters. We have for  $\alpha, \beta \in S_n^{(p)} \times S_r^{(q)}$ ,

$$\tilde{F}_{\text{comp}(\alpha)}(X)\tilde{F}_{\text{comp}(\beta)}(X) = \sum_{\gamma \in \alpha \sqcup \beta} \Gamma(\gamma). \quad (12)$$

In particular for  $n = r = 0$

$$\tilde{F}_{(0^p)}(X)\tilde{F}_{(0^q)}(X) = \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \tilde{F}_{(0^{p+q-j})}(X). \quad (13)$$

In some sense, weak composition fundamental quasisymmetric functions are a specialisation of *Poirier quasisymmetric functions* (see [9]), a classical type B generalisation of Gessel's quasisymmetric functions that we do not detail here. In [1], the authors provide a connexion between the generating functions for Young *bi-tableaux* and the fundamental quasisymmetric functions of Poirier. As a result, there should be a natural bi-tableaux interpretation of weak composition quasisymmetric functions. This is the focus of the next section.

## 2.2 Tableau-theoretic interpretation

Let  $\lambda$  be an integer partition and  $p$  a non-negative integer. We consider pairs of Young tableau  $(T^-, T^+)$  where  $T^-$  is of shape  $(p)$  and  $T^+$  has shape  $\lambda$ . We call such a bi-tableau a  $(p)$ -tableau. We proceed with the formal definition.

**Definition 2.** For any couple of non-negative integer  $p, n$  and an integer partition  $\lambda \vdash n$ , a **semistandard**  $(p)$ -**tableau**  $T = (T^-, T^+)$  is a pair of Young diagrams of bi-shape  $((p), \lambda)$  whose boxes are filled with the integers in  $[M]$  such that the entries are strictly increasing down the columns and non-decreasing along the rows. The **shape** of  $T$  is noted  $sh(T) = ((p), \lambda)$  and we denote  $SSBT^{(p)}(\lambda)$  the set of all  $(p)$ -tableaux of shape  $((p), \lambda)$ . If the boxes are filled with integers in  $[n+p]$  all used exactly once, we call  $T$  a **standard**  $(p)$ -**tableau** and we denote  $SBT^{(p)}(\lambda)$  the set of such tableaux.

We also need the following notions. The **standardisation**  $T^{st}$  of a semistandard  $(p)$ -tableau  $T$  is the standard  $(p)$ -tableau of shape  $((p), \lambda)$  obtained by relabelling the boxes of  $T$  with all the integers in  $[|\lambda| + p]$  such that the order of the labels is preserved. When a label in  $T$  is used both in  $T^-$  and  $T^+$ , the boxes of  $T^-$  are relabelled first. Finally, given a standard  $(p)$ -tableau  $T \in SBT^{(p)}(\lambda)$  with  $|\lambda| = n$  denote  $comp(T)$  the weak composition of  $n$ ,  $comp(T) = (0^{i_1}, s_1, 0^{i_2}, s_2, \dots, 0^{i_k}, s_k, 0^{i_{k+1}})$ , where  $i_1$  is the number of consecutive labels  $k = 1, 2, \dots, i_1$  in  $T^-$  ( $i_1 = 0$  if 1 is in  $T^+$ ). Then  $s_1$  is the number of consecutive labels  $k = i_1 + 1, \dots, i_1 + s_1$  in  $T^+$  such that no label  $k$  is in a higher row than  $k + 1$ . Then  $i_2$  is the number of consecutive labels  $k = i_1 + s_1 + 1, 2, \dots, i_1 + s_1 + i_2$  in  $T^-$ , etc...

**Example 1.** The following semistandard  $(3)$ -tableau belongs to  $SSBT^{(3)}((2, 2, 1))$ .

$$T = \left( T^- = \begin{array}{|c|c|c|} \hline 2 & 4 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 & 5 \\ \hline 9 & \\ \hline \end{array} \right)$$

Furthermore, its standardisation is:

$$T^{st} = \left( T^- = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline \end{array}, T^+ = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 6 & 7 \\ \hline 8 & \\ \hline \end{array} \right)$$

Finally the weak composition of  $T^{st}$  is  $comp(T^{st}) = (1, 0^3, 1, 2, 1)$ .

For  $\lambda \vdash n$  and  $p \geq 0$  denote  $s_\lambda^{(p)}$  the generating function for such tableaux. We have

$$s_\lambda^{(p)}(X) = \sum_{T \in SSBT^{(p)}(\lambda)} X^{T^+}. \quad (14)$$

Recall the notation of the Schur symmetric function  $s_\lambda$  indexed by  $\lambda$ . The following lemma is immediate.

**Lemma 1.** For any  $\lambda \vdash n$  and  $p \geq 0$ , one has

$$s_\lambda^{(p)}(X) = \binom{M+p-1}{p} s_\lambda(X). \quad (15)$$

We can now state the main result of this section.

**Proposition 2.** *For non-negative integers  $p$  and  $n$  and an integer partition  $\lambda \vdash n$  the generating function for  $(p)$ -tableau of shape  $((p), \lambda)$  is related to weak composition fundamental quasisymmetric functions through*

$$s_\lambda^{(p)}(X) = \sum_{T \in SBT^{(p)}(\lambda)} \tilde{F}_{comp(T)}(X). \quad (16)$$

*Proof.* (sketch) Starting from Equation (14), sort all the semi-standard  $(p)$ -tableaux according to their standardisation. Then, given a standard  $(p)$ -tableau  $T_0$  show that

$$\tilde{F}_{comp(T_0)}(X) = \sum_{T \in SSBT^{(p)}(\lambda); T^{st}=T_0} X^{T^+}.$$

Finally, sum on  $SBT^{(p)}(\lambda)$  to get the result.  $\square$

### 2.3 Product formulas

We use Proposition 2 to derive new results. We show the following formula.

**Theorem 1.** *Let  $p, q, n, m$  be non-negative integers and  $\lambda, \mu$  integer partitions of  $n$  and  $m$ . For  $\nu \vdash |\lambda| + |\mu|$  recall the coefficients  $c_{\lambda, \mu}^\nu$  introduced in Section 1.3. The following identity holds.*

$$\sum_{\substack{T \in SBT^{(p)}(\lambda) \\ U \in SBT^{(q)}(\mu)}} \tilde{F}_{comp(T)}(X) \tilde{F}_{comp(U)}(X) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda, \mu}^\nu \tilde{F}_{comp(V)}(X). \quad (17)$$

*Proof.* (sketch) First, note that for any integer  $p$

$$s_\emptyset^{(p)} = \tilde{F}_{(0^p)} = \binom{M+p-1}{p}.$$

Then use Equation (13) and the definition of the coefficients  $c_{\lambda, \mu}^\nu$  to write

$$\begin{aligned} s_\lambda^{(p)} s_\mu^{(q)} &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \binom{M+p-1}{p} \binom{M+q-1}{q} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} \binom{M+p+q-j-1}{p+q-j} s_\nu, \\ &= \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda, \mu}^\nu \sum_{j=0}^p (-1)^j \binom{p}{j} \binom{p+q-j}{p} s_\nu^{(p+q-j)}, \end{aligned}$$

and apply Proposition 2 to get the result.  $\square$



There is an interesting corollary of Theorem 1 in terms of generating functions for  $P_\pi$ -partitions. For a (classical) standard Young tableau  $T$ , recall the permutation class  $C_T$  of Section 1.3. Given a non-negative integer  $p$ , we denote  $\varepsilon_p$  the coloured permutation  $\varepsilon_p = \overline{12 \cdots p}$ . The result can be stated as follows.

**Corollary 1.** *For non-negative integers  $p, q$ , integers partitions  $\lambda$  and  $\mu$  and  $T \in SYT(\lambda)$ ,  $U \in SYT(\mu)$ , one has:*

$$\sum_{\gamma \in (\varepsilon_p \sqcup C_T) \sqcup (\varepsilon_q \sqcup C_U)} \Gamma(\gamma) = \sum_{\substack{0 \leq j \leq p \\ \nu \vdash |\lambda| + |\mu| \\ V \in SBT^{(p+q-j)}(\nu)}} (-1)^j \binom{p}{j} \binom{p+q-j}{p} c_{\lambda\mu}^\nu \tilde{F}_{comp(V)}(X). \quad (18)$$

*Proof.* We omit the proof in this extended abstract.  $\square$

### 3 Type B extension to Chow's quasisymmetric functions

#### 3.1 Signed permutations and domino tableaux

Let  $B_n$  be the **hyperoctahedral group** of order  $n$ .  $B_n$  is composed of all signed permutations  $\pi$  on  $\{-n, \dots, -2, -1, 0, 1, 2, \dots, n\}$  such that  $\pi(-i) = -\pi(i)$  for all  $i$ . The **total colour** of  $\pi \in B_n$  is the number  $tc(\pi) = |\{i \geq 1; \pi(i) < 0\}|$ . Its **descent set** is the subset of  $\{0\} \cup [n-1]$  equal to  $Des(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ . The main difference with respect to the case of the symmetric group is the possible descent in position 0 when  $\pi(1)$  is a negative integer.

For  $\lambda \vdash 2n$ , a **standard domino tableau**  $T$  of shape  $\lambda$  is a Young diagram of shape  $shape(T) = \lambda$  tiled by **dominoes**, i.e.  $2 \times 1$  or  $1 \times 2$  rectangles filled with the elements of  $[n]$  such that the entries are strictly increasing along the rows and down the columns. In the sequel we consider only the set  $\mathcal{P}^0(n)$  of *empty 2-core partitions*  $\lambda \vdash 2n$  that fit such a tiling. A standard domino tableau  $T$  has a descent in position  $i > 0$  if  $i+1$  lies strictly below  $i$  in  $T$  and has descent in position 0 if the domino filled with 1 is vertical. We denote  $Des(T)$  the set of all its descents. A **semistandard domino tableau**  $T$  of shape  $\lambda \in \mathcal{P}^0(n)$  and weight  $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \dots)$  with  $\mu_i \geq 0$  and  $\sum_i \mu_i = n$  is a tiling of the Young diagram of shape  $\lambda$  with horizontal and vertical dominoes labelled with integers in  $\{0, 1, 2, \dots\}$  such that labels are non decreasing along the rows, strictly increasing down the columns and exactly  $\mu_i$  dominoes are labelled with  $i$ . If the top leftmost domino is vertical, **it cannot be labelled 0**. Denote  $SDT(\lambda)$  (resp.  $SSDT(\lambda)$ ) the set of standard (resp. semistandard) domino tableaux of shape  $\lambda$ . Finally, we denote  $sp(T)$ , the **spin** of (semi-) standard domino tableau  $T$ , i.e. **half the number** of its vertical dominoes.

**Remark 2.** *The possible labelling of top leftmost horizontal dominoes in semistandard domino tableaux with 0 was first introduced by the authors in [8] and*

is essential to connect their generating functions with Chow's quasisymmetric functions.

|         |   |   |   |   |
|---------|---|---|---|---|
| $T_1 =$ | 1 | 2 | 3 | 5 |
|         |   |   |   | 6 |
|         |   | 4 | 8 |   |
|         |   | 7 |   |   |

|         |   |   |   |   |
|---------|---|---|---|---|
| $T_2 =$ | 1 | 2 | 3 | 5 |
|         |   |   |   |   |
|         |   | 6 | 8 |   |
|         |   | 7 |   |   |

|         |   |   |   |   |
|---------|---|---|---|---|
| $T_3 =$ | 0 | 0 | 5 | 5 |
|         |   |   |   |   |
|         |   | 2 | 5 |   |
|         |   | 5 | 7 |   |

Figure 1: Two standard domino tableaux  $T_1$  and  $T_2$  of shape  $(5, 5, 4, 1, 1)$ , descent set  $\{0, 3, 5, 6\}$ , a semistandard tableau  $T_3$  of shape  $(5, 5, 4, 3, 1)$  and weight  $\mu = (2, 0, 2, 0, 4, 0, 1)$ . All of the tableaux have a spin of 2.

### 3.2 Chow's type B quasisymmetric functions

Chow defines in [3] another Type B extension of Gessel's algebra of quasisymmetric functions that is dual to Solomon's descent algebra of type B.

Let  $X = \{\dots, x_{-i}, \dots, x_{-1}, x_0, x_1, \dots, x_i, \dots\}$  be an alphabet of totally ordered commutative indeterminates with the assumption that  $x_{-i} = x_i$  and let  $I$  be a subset of  $\{0\} \cup [n-1]$ , he defines a type B analogue of the fundamental quasisymmetric functions

$$F_I^B(X) = \sum_{\substack{0=i_0 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Note the particular role of the variable  $x_0$ .

Given two signed permutations  $\alpha \in B_n$  and  $\beta \in B_m$ , denote  $\alpha \sqcup \beta$  the set of signed permutations in  $B_{m+n}$  obtained by shuffling the letters of  $\alpha$  and  $\bar{\beta}$  where we shift the absolute value of the  $\beta_i$  to get  $\bar{\beta}$ . Chow shows in [3, Prop. 2.2.6]

$$F_{Des(\alpha)}^B(X) F_{Des(\beta)}^B(X) = \sum_{\gamma \in \alpha \sqcup \beta} F_{Des(\gamma)}^B(X). \quad (19)$$

Furthermore, we introduce in [8] modified generating functions for domino tableaux called domino functions. Given a semistandard domino tableau  $T$  of weight  $\mu$ , denote  $X^T$  the monomial  $x_0^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} \dots$ . For  $\lambda \in \mathcal{P}^0(n)$  and an additional parameter  $q$  we define the **domino function** indexed by  $\lambda$  on alphabet  $X$  as

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SSDT(\lambda)} q^{sp(T)} X^T. \quad (20)$$

These functions are related to Chow's quasisymmetric functions through

**Lemma 2** ([7]). *For  $\lambda \in \mathcal{P}^0(n)$ , the  $q$ -domino function indexed by  $\lambda$  can be expanded in the basis of Chow's quasisymmetric functions as*

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SDT(\lambda)} q^{sp(T)} F_{Des(T)}^B(X). \quad (21)$$

### 3.3 Type B Littlewood-Richardson coefficients

There is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([2]) built a bijection between signed permutations of  $B_n$  and pairs of standard domino tableaux of equal shape in  $\mathcal{P}^0(n)$ . An independent development on the subject is due to Garfinkle. Taşkin ([11, Prop. 26]) showed that the two standard domino tableaux  $T$  and  $U$  associated to a signed permutation  $\pi$  by the algorithm of Barbash and Vogan have respective descent sets  $Des(T) = Des(\pi^{-1})$  and  $Des(U) = Des(\pi)$  while Shimozono and White showed in [10] the **colour-to-spin** property i.e.  $tc(\pi) = sp(T) + sp(U)$ .

Let  $X, Y$  be two alphabets of indeterminates. In [7], we show that Equation (21) implies the following proposition.

**Proposition 3** ([7]). *For any integer  $n$ , we have*

$$\sum_{\pi \in B_n} q^{tc(\pi)} F_{Des(\pi)}^B(X) F_{Des(\pi^{-1})}^B(Y) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in SDT(\lambda)}} q^{sp(T) + sp(U)} F_{Des(T)}^B(X) F_{Des(U)}^B(Y). \quad (22)$$

Proposition 3 provides a direct analytical proof of the properties showed in a bijective fashion by Barbash, Vogan, Taşkin, Shimozono and White. More formally it proves the existence for any standard domino tableau  $T$  of a class of signed permutations  $C_T^B$  such that for any  $\pi \in C_T^B$ ,  $Des(\pi^{-1}) = Des(T)$  and there is a descent preserving bijection  $C_T^B \rightarrow SDT(\lambda), \pi \mapsto U$  such that  $tc(\pi) = sp(T) + sp(U)$ . In a similar fashion as in Section 1.3, we use Equation (22) to show some properties for the *type B Littlewood-Richardson coefficients*.

First we focus on the case  $q = 1$ . There is a well known (not descent preserving) bijection between semistandard domino tableaux of weight  $\mu$  and semistandard bi-tableaux of respective weights  $\mu^-$  and  $\mu^+$  such that  $\mu_i^+ + \mu_i^- = \mu_i$  for all  $i$ . The respective shapes of the two Young tableaux depend only on the shape of the initial semistandard domino tableau. Denote  $(T^-, T^+)$  the bi-tableau associated to a semistandard domino tableau  $T$  of shape  $\lambda$  and  $(\lambda^-, \lambda^+)$  their respective shapes.  $(T^-, T^+)$  (resp.  $(\lambda^-, \lambda^+)$ ) is the **2-quotient** of  $T$  (resp.  $\lambda$ ). Denote  $X^* = X \setminus x_0$ . For  $\lambda \in \mathcal{P}^0(n)$  one has

$$\mathcal{G}_\lambda(X; 1) = s_{\lambda^-}(X^*) s_{\lambda^+}(X)$$

and for  $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$ ,

$$\mathcal{G}_\lambda(X; 1)\mathcal{G}_\mu(X; 1) = \sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} \mathcal{G}_\nu(X; 1). \quad (23)$$

The following result extends Proposition 1 to signed permutations.

**Theorem 2.** *Let  $\lambda, \mu \in \mathcal{P}^0(n) \times \mathcal{P}^0(m)$ . For  $I \subset [0] \cup [n+m-1]$  denote  $D_I^B$  the set of signed permutations  $\pi$  such that  $\text{Des}(\pi) = I$  and  $\Delta_I^B$  the set of standard domino tableaux of descent set  $I$ . Using the notations above for integer partitions  $\lambda, \mu$  and  $T, U$  in  $\text{SDT}(\lambda) \times \text{SDT}(\mu)$*

$$\sum_{\nu \in \mathcal{P}^0(n+m)} c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} |\text{SDT}(\nu) \cap \Delta_I^B| = |(C_T^B \sqcup C_U^B) \cap D_I^B|. \quad (24)$$

*Proof.* (sketch) Theorem 2 is a consequence of Equations (23) and (13).  $\square$

On the other hand, the general case for  $q$  remains open. Indeed, using Proposition 3, one can show for  $\lambda \in \mathcal{P}^0(n)$  and  $T \in \text{SDT}(\lambda)$

$$\mathcal{G}_\lambda(X; q) = \sum_{\pi \in C_T^B} q^{tc(\pi) - sp(T)} F_{\text{Des}(\pi)}^B.$$

Then, if  $\mu \in \mathcal{P}^0(m)$  and  $U \in \text{SDT}(\mu)$

$$\mathcal{G}_\lambda(X; q)\mathcal{G}_\mu(X; q) = q^{-(sp(T)+sp(U))} \sum_{\gamma \in C_T^B \sqcup C_U^B} q^{tc(\gamma)} F_{\text{Des}(\gamma)}^B.$$

However, even if we showed that the set of signed permutations  $C_T^B \sqcup C_U^B$  is in descent preserving bijection with a set of cardinality  $c_{\lambda-\mu}^{\nu-} c_{\lambda+\mu}^{\nu+} |\text{SDT}(\nu)|$ , we cannot say *a priori* how this bijection preserves or transforms the statistic  $tc$ .

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