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Convexity conditions for normal mean-variance mixture distribution in joint probabilistic constraints

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Abstract

In this paper, we study the linear programming with probabilistic constraints. We suppose that the distribution of the constraint rows is a normal mean-variance mixture distribution and the dependence of rows is represented by an Archimedean copula. We prove the convexity of the feasibility set in some additional conditions. Next, we propose a sequential approximation by linearization which provides a lower bound and a gradient descent method which provides an upper bound with numerical results.

Keywords:

Probabilistic constraints, Archimedean copulas, Normal mean-variance mixture distributions, Convex optimization

1. Introduction

We study the following linear programming with joint probabilistic constraints:

$$\begin{aligned}
& \min && c^T x \\
& \text{subject to} && \mathbb{P} \{Vx \leq D\} \geq 1 - \epsilon \\
& && x \in Q.
\end{aligned} \tag{1}$$

where Q is a closed convex subset of \mathbb{R}^n ; $c \in \mathbb{R}^n$, $D := (D_1, \dots, D_K) \in \mathbb{R}^K$ is a deterministic vector, $V := [v_1, \dots, v_K]^T$ is a random matrix with size $K \times n$, where v_k is a random vector in \mathbb{R}^n , $\forall k = \overline{1, K}$ and $\epsilon \in [0, 1]$.

1.1. Survey of literature

The probabilistic constraint optimisation has been widely studied since longtime ago. Prékopa studied the concavity and quasi-concavity properties for probability distribution functions in his article [16] in 1970. Sen introduced a relaxation method for probabilistic constraint programming with discrete random variable in [21]. Lobo studied some applications of second-order cone programming in [12] which gave a new approach for solving problems of probabilistic constraints. Henrion gave a general structural property for linear probabilistic constraints in [10]. In 2014, Cheng used the second-order cone programming approach for solving his joint probabilistic constraints problem in [4]. He supposed that the distribution of the constraint rows is elliptically distributed and the dependence of the rows follows an Archimedean copula.

In this paper, we study the same probabilistic constraint problem as in [4]. We suppose that the distribution of the row vectors is normal mean-variance mixture distributed and the dependence of the rows is an Archimedean copula. By assuming some conditions, we prove the convexity of the feasible set

of solutions. We propose two approximations method which gives a lower bound and an upper bound and present some numerical results.e

1.2. *Why is normal mean-variance mixture distribution?*

Definition 1.1. A random variable X in \mathbb{R}^n is a normal mean-variance mixture distribution if:

$$X = \mu + \gamma W + \sqrt{W}AZ,$$

where (1) Z is a n-dimension standard normal distribution $N_n(0, \mathbb{I}_n)$.

(2) W a positive random variable independent of Z .

(3) $A \in \mathbb{R}^{n \times k}$ is a matrix such that $AA^T = \Sigma$, where Σ a semidefinite positive matrix $\in \mathbb{R}^{n \times n}$.

(4) μ and γ are n-real vectors.

The relation between normal mean-variance distributions and elliptical distributions is represented by the following proposition:

Proposition 1. *[[13], theorem 3.25] Denote Ψ_∞ the set of characteristic generators which generate a spherical distribution in dimension n , for all $n \geq 1$. Hence, Y follows $S_n(\psi)$ with $\psi \in \Psi_\infty$ if and only if $Y = \sqrt{W}Z$, where Z is a n-dimension standard normal distribution $N_n(0, \mathbb{I}_n)$ independent of $W \geq 0$.*

Based on this proposition, we deduce that an elliptical distribution U can be represented in the form $U = \mu + \sqrt{W}AZ$ with $\mu \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times k}$ and Z is a n-dimension standard normal distribution $N_n(0, \mathbb{I}_n)$ if and only if $U = \mu + AY$, where Y follows $S_n(\psi)$ and $\psi \in \Psi_\infty$.

The family of distributions normal mean-variance mixture represents a comprehensive subset of the family of elliptical distributions. There exists some elliptical distributions which cannot be represented in the form of a normal mean-variance mixture distribution. However, it represents a big subset of the family of elliptical distributions and plays an important role in the elliptical world.

Next, we define an important subset of the family of normal mean-variance mixture distributions, the family of hyperbolic distributions.

Definition 1.2. A random variable X is a hyperbolic distribution if it is a normal mean-variance mixture where the random variable W in definition (1.1) is an inverse Gaussian distribution whose density function with respect to the measure of Lebesgue is:

$$g(w) = Cw^{\lambda-1} \exp\left(-\frac{1}{2}(\chi w^{-1} + \psi w)\right), \forall w \in [0, \infty).$$

where C is a constant and the domain of variation of the parameters is:

$$\chi > 0, \psi \geq 0 \quad \text{if } \lambda < 0.$$

$$\chi > 0, \psi > 0 \quad \text{if } \lambda = 0.$$

$$\chi \geq 0, \psi > 0 \quad \text{if } \lambda > 0.$$

The family of hyperbolic distributions is a generalization of many elliptical distributions. For example, the t-multivariate distribution with parameters (Σ, μ, ν) is a particular case of hyperbolic distributions when $\lambda = \frac{-\nu}{2}$; $\chi = \nu$

; $\psi = 0$; $\mu = \mu$; $\Sigma = \Sigma$; $\gamma = 0$. We summarize some important elliptical distributions (in dimension p) by the following table:

	Density	μ	Σ	γ	λ	χ	ψ
Normal	$C.exp(-\frac{1}{2}\ x\ ^2)$	μ	Σ	0	0	0	0
t-distribution	$C.(1 + \frac{\ x\ ^2}{\nu})^{-\frac{p+\nu}{2}}$	μ	Σ	0	$\frac{-\nu}{2}$	ν	0
Cauchy	$C.(1 + \ x\ ^2)^{-\frac{p+1}{2}}$	μ	Σ	0	$\frac{-1}{2}$	1	0
Laplace		μ	Σ	0	1	0	-2
Pearson VII	$C.(1 + \frac{\ x\ ^2}{m})^{-N}$	μ	Σ	0	$\frac{p}{2} - N$	m	0

A disadvantage of elliptical distributions is that they are symmetric. We should not use elliptical family for modelling in some cases. For example, we can find some applications of hyperbolic distributions in modelling financial in [7], [1], [20] and [19].

2. Normal mean-variance mixture constraints in linear programming

In this section, we study the linear programming (1). We suppose that v_i are random vectors. We find some sufficient conditions (necessary if possible) such as $\mathbb{P}\{Vx \leq D\}$ is a concave function with respect to x . A convex optimization can be defined formally as following:

$$\begin{aligned}
& \min && f(x) \\
\text{subject to} &&& g_k(x) \leq 0, \quad k = \overline{0, m-1} \\
&&& Gx \preceq h \\
&&& Ax = b \\
&&& x \in Q.
\end{aligned}$$

where Q is a closed convex subset of \mathbb{R}^n ; $f(x)$ and $g_i(x)$ are convex functions, $i = \overline{0, m-1}$; h and b are n -real vectors ; G and A are deterministic matrices.

2.1. Preliminaries

Proposition 2 ([11], lemme 3.1). *Given $F : \mathbb{R} \rightarrow [0, 1]$ a distribution function with $(r + 1)$ - decreasing density for some $r > 0$ and a threshold $t^*(r + 1) > 0$. Hence, the function $z \mapsto F(z^{-\frac{1}{r}})$ is a concave function on $(0, t^*(r + 1)^{-r})$. Moreover, $F(t) < 1, \forall t \in \mathbb{R}$.*

Definition 2.1. ([4]) A real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is K - monotone on an open interval $I \subseteq \mathbb{R}$ with $K \geq 2$ if it is differentiable up to $(K - 2)$ th - order and the derivatives are satisfied by:

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \forall 0 \leq k \leq K - 2 \text{ et } \forall t \in I.$$

and the function $(-1)^{K-2} \frac{d^{K-2}}{dt^{K-2}} f(t)$ is non-increasing and convex on I .

Proposition 3 ([14]). *Given $\psi : [0, 1] \rightarrow [0, +\infty)$ a strictly decreasing function such that $\psi(1) = 0$. Hence, it is the generator of an archimedean copula in dimension K if and only if ψ^{-1} is K -monotone on $(0, \psi(0))$.*

Definition 2.2. Given $f : Q \times \mathbb{R} \rightarrow \mathbb{R}$ a real function, where Q is a subset of \mathbb{R}^n . We say that f is differentiable at $x \in Q$ if there exists a function $g : Q \rightarrow \mathbb{R}^{\mathbb{R}^n}$ such that $\forall \theta \in \mathbb{R}$, there exists a neighbourhood $N(\theta)$ of θ such that if we note $f_{N(\theta)}$ the restriction of f on $N(\theta)$, we have:

$$\lim_{\epsilon \rightarrow 0, \epsilon \in \mathbb{R}^n} \left\| \frac{[f_{N(\theta)}(x + \epsilon, \cdot) - f_{N(\theta)}(x, \cdot) - \langle \epsilon, g(x) \rangle]}{\|\epsilon\|} \right\|_{max} = 0,$$

where $\mathbb{R}^{\mathbb{R}^n}$ denotes the set of functions from \mathbb{R} to \mathbb{R}^n and $\|\cdot\|_{max}$ is the maximum norm.

Moreover, we say that f is differentiable up to second-order at x if there exists a function $h : Q \rightarrow \mathbb{R}^{\mathbb{R}^{n \times n}}$ such that we have:

$$\lim_{\epsilon \rightarrow 0, \epsilon \in \mathbb{R}^n} \left\| \frac{[f_{N(\theta)}(x + \epsilon, \cdot) - f_{N(\theta)}(x, \cdot) - \langle \epsilon, g(x) \rangle - \frac{1}{2}\epsilon^T h(x)\epsilon]}{\|\epsilon\|^2} \right\|_{max} = 0$$

where $\mathbb{R}^{\mathbb{R}^{n \times n}}$ denotes the set of functions from \mathbb{R} to $\mathbb{R}^{n \times n}$.

Denote $\frac{df}{dx} := g$ the derivative at first-order of f according to x and $\frac{d^2f}{dx^2} := h$ the derivative at second-order of f according to x .

In the programming (1), we suppose that v_k is a normal mean-variance mixture distribution, for $1 \leq k \leq K$. Next, we will show that the feasible set of (1) is a convex set by adding some additional conditions.

2.2. Individual chance constraints

Suppose that $K = 1$ and V follows a normal mean-variance mixture distribution with parameters (μ, Σ, γ) (cf. definition (1.1)). Suppose that $0 \notin Q$.

Lemma 2.1 (Proposition 5, [5]). *The standard normal distribution has 3-decreasing density with a threshold $t^*(3) = \sqrt{3}$.*

The constraint in (1) can be rewritten as follows:

$$\begin{aligned}
& \mathbb{P}_{v_1} (v_1^T x \leq D) \geq 1 - \epsilon. \\
& \Leftrightarrow \mathbb{P}_{W,Z} \left((\mu^T + W\gamma^T + \sqrt{W}Z^T A^T)x \leq D \right) \geq 1 - \epsilon \\
& \Leftrightarrow \mathbb{P}_{W,Z} \left((W\gamma^T + \sqrt{W}Z^T A^T)x \leq D - x^T \mu \right) \geq 1 - \epsilon. \\
& \Leftrightarrow \mathbb{P}_{W,X} \left(\frac{W}{\|x^T \Sigma^{\frac{1}{2}}\|_2} x^T \gamma + \sqrt{W}X \leq \frac{D - x^T \mu}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \right) \geq 1 - \epsilon. \\
& \left(\text{By letting } X := \frac{Z^T A^T x}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \right). \\
& \Leftrightarrow \mathbb{E} \left[\mathbf{1} \left\{ X \leq \frac{-x^T \gamma}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x^T \mu}{\sqrt{W} \|x^T \Sigma^{\frac{1}{2}}\|_2} \right\} \right] \geq 1 - \epsilon. \\
& \Leftrightarrow \mathbb{E}_W \left[\mathbb{E} \left[\mathbf{1} \left\{ X \leq \frac{-x^T \gamma}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x^T \mu}{\sqrt{W} \|x^T \Sigma^{\frac{1}{2}}\|_2} \right\} \middle| W \right] \right] \geq 1 - \epsilon. \\
& \Leftrightarrow \mathbb{E}_W \left[\Phi_X \left(\frac{-x^T \gamma}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x^T \mu}{\sqrt{W} \|x^T \Sigma^{\frac{1}{2}}\|_2} \right) \right] \geq 1 - \epsilon. \tag{2}
\end{aligned}$$

where $\epsilon \in (0, 1)$, X follows a standard normal distribution $N(0, 1)$, Φ_X is the distribution function of X , W is a positive random variable independent of X .

We have the following theorem:

Theorem 2.2. *Consider the linear programming (1). Let*

$M := \{x \in Q \mid \mathbb{P}\{Vx \leq D\} \geq 1 - \epsilon\}$ is the feasible set of (1). Suppose that $K = 1$ and V follows a normal mean-variance mixture distribution with parameters (μ, Σ, γ) .

Suppose that:

(1) Σ is a definite positive matrix with $0 < \lambda_{\min} \leq \lambda_{\max}$, where λ_{\max} is the biggest eigenvalue of Σ and λ_{\min} is the smallest eigenvalue of Σ .

(2) W is a random variable in $[t_{\min}, t_{\max}]$ with $0 \leq t_{\min} \leq t_{\max} \leq t^+$.

(3) For all $x \in Q$, we have:

$$c > 0.$$

$$\text{and } \frac{1}{t^+} \geq \frac{\sqrt{b^2 - 4ac} - b}{2c} \text{ or } b^2 \leq 4ac.$$

where:

$$\begin{aligned}
a &= (-x^T \gamma)^2 \lambda_{\min} + \alpha_{x,\gamma,\Sigma}. \\
b &= 2(-x^T \gamma)(D - x^T \mu) \lambda_{\min} - 6(x^T \Sigma x) \|\gamma\| \|\mu\| + \beta_{x,\gamma,\mu,\Sigma}. \\
c &= (D - x^T \mu)^2 \lambda_{\min} + \theta_{x,\mu,\Sigma}. \\
\alpha_{x,\gamma,\Sigma} &:= u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}. \\
\beta_{x,\gamma,\mu,\Sigma} &:= v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}. \\
\theta_{x,\mu,\Sigma} &:= q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}. \\
u &= 4(-x^T \gamma) \gamma. \\
v &= 4(-x^T \gamma) \gamma + 4(D - x^T \mu) \mu. \\
q &= 4(D - x^T \mu) \mu. \\
z &= \Sigma x.
\end{aligned}$$

(4) For all $x \in Q$, we have:

$$\begin{aligned}
\frac{2\sqrt{(-x^T \gamma)(D - x^T \mu)}}{\sqrt{x^T \Sigma x}} &> \sqrt{3}. \\
(-x^T \gamma) &> 0. \\
D - x^T \mu &> 0.
\end{aligned}$$

Hence, M is a convex set.

Proof. The proof of this theorem follows this outline:

(i) We show that $f(x, W) := \frac{-x^T \gamma}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x^T \mu}{\sqrt{W} \|x^T \Sigma^{\frac{1}{2}}\|_2}$ is a (-2)-concave function according to x on Q , for all $W \geq 0$.

(ii) By using (i), we show that $\Phi_X(f(x, W))$ is a concave function according to x , for all $W \geq 0$.

(iii) By using (ii), we show that $\mathbb{E}_W [\Phi_X(f(x, W))]$ is a concave function according to x on Q .

(iv) By using (iii), we deduce that the feasible set of (1) is a convex set.

Proof of (i). Let $f(x, W) := \frac{-x^T \gamma}{\|x^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x^T \mu}{\sqrt{W} \|x^T \Sigma^{\frac{1}{2}}\|_2}$.

The (-2) -concavity of $f(x, W)$ is equivalent to the convexity of the following function:

$$\begin{aligned} h(x, W) &:= \frac{x^T \Sigma x}{\left(-x^T \gamma \sqrt{W} + \frac{1}{\sqrt{W}}(D - x^T \mu)\right)^2} \\ &= \frac{x^T \Sigma x}{W(x^T \gamma)^2 + \frac{1}{W}(x^T \mu - D)^2 + 2x^T \gamma(x^T \mu - D)}. \end{aligned}$$

Let $W(x^T \gamma)^2 + \frac{1}{W}(x^T \mu - D)^2 + 2x^T \gamma(x^T \mu - D) =: M$. Denote $H_x h(x, W)$ the gradient vector of h according to x and $H_x^2 h(x, W)$ the Hessian matrix of h according to x .

By a direct calculation, we deduce a formula of the gradient vector and the Hessian matrix of h according to x as follows:

$$\begin{aligned} (*) H_x h(x, W) &= 2.M^{-1}.\Sigma x - M^{-2}.(x^T \Sigma x). \\ &\cdot \left(2W x^T \gamma.\gamma + \frac{2}{W}(x^T \mu - D).\mu + 2(x^T \mu - D).\gamma + 2x^T \gamma.\mu \right). \end{aligned}$$

$$\begin{aligned}
(*) H_x^2 h(x, W) &= 2M^{-1} \cdot \Sigma - 8M^{-2} \cdot [(Wx^T \gamma + x^T \mu - D) \cdot (\Sigma x \gamma^T + \gamma x^T \Sigma)] \\
&\quad - 8M^2 \cdot \left[\left(\frac{1}{W} (x^T \mu - D) + x^T \gamma \right) \cdot (\Sigma x \mu^T + \mu x^T \Sigma) \right] \\
&\quad + 6 \cdot M^{-2} (x^T \Sigma x) \cdot \left(W \gamma \gamma^T + \frac{1}{W} \mu \mu^T + \gamma \mu^T + \mu \gamma^T \right). \\
&= 2 \cdot M^{-2} [W(-x^T \gamma) + D - x^T \mu]. \\
&\quad \cdot \left\{ [(-x^T \gamma) \Sigma + 4(\Sigma x \gamma^T + \gamma x^T \Sigma)] + \frac{1}{W} [(D - x^T \mu) \Sigma + 4(\Sigma x \mu^T + \mu x^T \Sigma)] \right\} \\
&\quad + 6 \cdot M^{-2} (x^T \Sigma x) \cdot \left(W \gamma \gamma^T + \frac{1}{W} \mu \mu^T + \gamma \mu^T + \mu \gamma^T \right). \\
&\Leftrightarrow \frac{H_x^2 h(x, W)}{2 \cdot M^{-2}} = \\
&= W [(-x^T \gamma)^2 \Sigma + 4(-x^T \gamma) \cdot (\Sigma x \gamma^T + \gamma x^T \Sigma) + 3(x^T \Sigma x) \gamma \gamma^T] + \\
&\quad + [2(-x^T \gamma)(D - x^T \mu) \Sigma + 4(-x^T \gamma)(\Sigma x \gamma^T + \gamma x^T \Sigma)] + \\
&\quad + [4(D - x^T \mu)(\Sigma x \mu^T + \mu x^T \Sigma) + 3(x^T \Sigma x)(\gamma \mu^T + \mu \gamma^T)] + \\
&\quad + \frac{1}{W} [(D - x^T \mu)^2 \Sigma + 4(D - x^T \mu)(\Sigma x \mu^T + \mu x^T \Sigma) + 3(x^T \Sigma x) \mu \mu^T]. \\
&= WA + B + \frac{1}{W} C.
\end{aligned}$$

where

$$\begin{aligned}
A &= (-x^T \gamma)^2 \Sigma + 4(-x^T \gamma) \cdot (\Sigma x \gamma^T + \gamma x^T \Sigma) + 3(x^T \Sigma x) \gamma \gamma^T. \\
B &= 2(-x^T \gamma)(D - x^T \mu) \Sigma + 4(-x^T \gamma)(\Sigma x \gamma^T + \gamma x^T \Sigma) + \\
&\quad + 4(D - x^T \mu)(\Sigma x \mu^T + \mu x^T \Sigma) + 3(x^T \Sigma x)(\gamma \mu^T + \mu \gamma^T). \\
C &= (D - x^T \mu)^2 \Sigma + 4(D - x^T \mu)(\Sigma x \mu^T + \mu x^T \Sigma) + 3(x^T \Sigma x) \mu \mu^T.
\end{aligned}$$

For that h is a convex function according to x for all $W \in [0, t^+]$, it is

necessary that the Hessian matrix $H_x^2 h(x, W)$ is semidefinite positive for all $(x, W) \in Q \times [0, t^+]$. That is equivalent to the semidefinite positivity of the matrix $WA + B + \frac{1}{W}C$ for all $(x, W) \in Q \times [0, t^+]$. We have:

$$(1) A = (-x^T \gamma)^2 \Sigma + 4(-x^T \gamma) \cdot (\Sigma x \gamma^T + \gamma x^T \Sigma) + 3(x^T \Sigma x) \gamma \gamma^T.$$

Given M, N two any symmetric matrix. Denote $M \succeq N$ if the matrix $M - N$ is a semidefinite positive matrix.

By using this notation, we deduce the following inequalities:

$$* \Sigma \succeq \lambda_{\min} \mathbb{I}d_n.$$

* $4(-x^T \gamma)(\Sigma x \gamma^T + \gamma x^T \Sigma)$ - is a symmetric matrix therefore diagonalizable. We can show that it has $(n - 2)$ - eigenvalues which is 0 and 2 - eigenvalues which are:

$$u^T z \pm \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}.$$

where $u = 4(-x^T \gamma) \gamma$ and $z = \Sigma x$.

$$* \gamma \gamma^T \succeq 0.$$

$$\text{Let } \alpha_{x, \gamma, \Sigma} := u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}.$$

Hence, we deduce the following inequality:

$$A \succeq [(-x^T \gamma)^2 \lambda_{\min} + \alpha_{x, \gamma, \Sigma}] \mathbb{I}d_n. \quad (3)$$

$$(2) B = 2(-x^T \gamma)(D - x^T \mu) \Sigma + 4(-x^T \gamma)(\Sigma x \gamma^T + \gamma x^T \Sigma) + 4(D - x^T \mu)(\Sigma x \mu^T + \mu x^T \Sigma) + 3(x^T \Sigma x)(\gamma \mu^T + \mu \gamma^T).$$

We have:

$$* \Sigma \succeq \lambda_{\min} \mathbb{I}d_n.$$

* $\gamma\mu^T + \mu\gamma^T \succeq -2\|\gamma\|\|\mu\|$.

* $4(-x^T\gamma)(\Sigma x\gamma^T + \gamma x^T\Sigma) + 4(D - x^T\mu)(\Sigma x\mu^T + \mu x^T\Sigma)$ - is a symmetric matrix, hence diagonalizable. We can show that it has $(n - 2)$ - eigenvalues which is 0 and 2 - eigenvalues which are:

$$z^T v \pm \sqrt{(z^T v)^2 + \sum_{1 \leq i < j \leq n} (z_i v_j - z_j v_i)^2}$$

where $z = \Sigma x$ and $v = 4(-x^T\gamma)\gamma + 4(D - x^T\mu)\mu$.

Let $\beta_{x,\gamma,\mu,\Sigma} := v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}$.

Hence, we deduce the following inequality:

$$B \succeq [2(-x^T\gamma)(D - x^T\mu)\lambda_{\min} - 6(x^T\Sigma x)\|\gamma\|\|\mu\| + \beta_{x,\gamma,\mu,\Sigma}] \mathbb{I}d_n. \quad (4)$$

(3) $C = (D - x^T\mu)^2\Sigma + 4(D - x^T\mu)(\Sigma x\mu^T + \mu x^T\Sigma) + 3(x^T\Sigma x)\mu\mu^T$.

We have:

* $\Sigma \succeq \lambda_{\min}\mathbb{I}d_n$.

* $4(D - x^T\mu)(\Sigma x\mu^T + \mu x^T\Sigma)$ - is a symmetric matrix, therefore diagonalizable. We can show that it has $(n - 2)$ - eigenvalues which is 0 and 2 - eigenvalues which are:

$$z^T q \pm \sqrt{(z^T q)^2 + \sum_{1 \leq i < j \leq n} (z_i q_j - z_j q_i)^2}$$

where $q := 4(D - x^T\mu)\mu$ and $z = \Sigma x$.

* $\mu\mu^T \succeq 0$.

Let $\theta_{x,\mu,\Sigma} := q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}$.

Hence, we deduce the following inequality:

$$C \succeq (D - x^T \mu)^2 \lambda_{\min} + \theta_{x, \mu, \Sigma}. \quad (5)$$

By using (3), (4), (5), we deduce the following inequality:

$$\begin{aligned} WA + B + \frac{1}{W}C &\succeq W [(-x^T \gamma)^2 \lambda_{\min} + \alpha_{x, \gamma, \Sigma}] \mathbb{I}d_n + \\ &[2(-x^T \gamma)(D - x^T \mu) \lambda_{\min} - 6(x^T \Sigma x) \|\gamma\| \|\mu\| + \beta_{x, \gamma, \mu, \Sigma}] \mathbb{I}d_n + \\ &\frac{1}{W} [(D - x^T \mu)^2 \lambda_{\min} + \theta_{x, \mu, \Sigma}] \mathbb{I}d_n. \end{aligned} \quad (6)$$

where:

$$\begin{aligned} \alpha_{x, \gamma, \Sigma} &:= u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}. \\ \beta_{x, \gamma, \mu, \Sigma} &:= v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}. \\ \theta_{x, \mu, \Sigma} &:= q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}. \\ u &= 4(-x^T \gamma) \gamma. \\ v &= 4(-x^T \gamma) \gamma + 4(D - x^T \mu) \mu. \\ q &= 4(D - x^T \mu) \mu. \\ z &= \Sigma x. \end{aligned}$$

$$\begin{aligned} \text{Let } a &= (-x^T \gamma)^2 \lambda_{\min} + \alpha_{x, \gamma, \Sigma}, \\ b &= 2(-x^T \gamma)(D - x^T \mu) \lambda_{\min} - 6(x^T \Sigma x) \|\gamma\| \|\mu\| + \beta_{x, \gamma, \mu, \Sigma}, \\ c &= (D - x^T \mu)^2 \lambda_{\min} + \theta_{x, \mu, \Sigma}. \end{aligned}$$

Obviously, the semidefinite positivity of the Hessian matrix $H_x^2 h(x, W)$ is deduced by the positivity of $Wa + b + \frac{1}{W}c$, for all $W \in [0, t^+]$.

We deduce a sufficient condition as follows:

$$c > 0.$$

$$\text{and } \frac{1}{t^+} \geq \frac{\sqrt{b^2 - 4ac} - b}{2c} \text{ or } b^2 \leq 4ac.$$

where:

$$a = (-x^T \gamma)^2 \lambda_{\min} + \alpha_{x, \gamma, \Sigma}.$$

$$b = 2(-x^T \gamma)(D - x^T \mu) \lambda_{\min} - 6(x^T \Sigma x) \|\gamma\| \|\mu\| + \beta_{x, \gamma, \mu, \Sigma}.$$

$$c = (D - x^T \mu)^2 \lambda_{\min} + \theta_{x, \mu, \Sigma}.$$

$$\alpha_{x, \gamma, \Sigma} := u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}.$$

$$\beta_{x, \gamma, \mu, \Sigma} := v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}.$$

$$\theta_{x, \mu, \Sigma} := q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}.$$

$$u = 4(-x^T \gamma) \gamma.$$

$$v = 4(-x^T \gamma) \gamma + 4(D - x^T \mu) \mu.$$

$$q = 4(D - x^T \mu) \mu.$$

$$z = \Sigma x.$$

which is satisfied by the assumption (3) of the theorem.

We deduce that $f(x, W)$ is a (-2) - concave function according to x on Q , for all $W \in [0, t^+]$.

□

Proof of (ii). Given $x_1, x_2 \in Q$, $W \in [0, t^+]$, $\alpha \in [0, 1]$. We have:

$$f[\alpha x_1 + (1 - \alpha)x_2, W] \geq [\alpha f^{-2}(x_1, W) + (1 - \alpha)f^{-2}(x_2, W)]^{\frac{-1}{2}}.$$

(by the (-2) - concavity of f by (i))

$$\iff \Phi_X(f[\alpha x_1 + (1 - \alpha)x_2, W])$$

$$\geq \Phi_X\left([\alpha f^{-2}(x_1, W) + (1 - \alpha)f^{-2}(x_2, W)]^{\frac{-1}{2}}\right)$$

(by the strict increasing of Φ_X). (7)

By using the lemma (2.1), we deduce that the function $\Phi_X(t^{\frac{-1}{2}})$ is a concave function on $(0, (t^*)^{-2})$ where $t^* = \sqrt{3}$. We will show that $f^{-2}(x_1, W)$ is on $(0, (t^*)^{-2})$.

Proof. In fact, we have:

$$\begin{aligned} |f(x_1, W)| &= \left| \frac{-x_1^T \gamma}{\|x_1^T \Sigma^{\frac{1}{2}}\|_2} \sqrt{W} + \frac{D - x_1^T \mu}{\sqrt{W} \|x_1^T \Sigma^{\frac{1}{2}}\|_2} \right| \\ &\geq \frac{2\sqrt{(-x_1^T \gamma)(D - x_1^T \mu)}}{\sqrt{x_1^T \Sigma x_1}} \quad (\text{by Cauchy inequality}) \\ &> t^* \quad (\text{by the assumption (4) of the theorem}) \\ &\iff 0 < f(x_1, W)^{-2} < (t^*)^{-2}. \end{aligned}$$

□

Hence, by using the concavity of $\Phi_X(t^{\frac{-1}{2}})$ on $(0, (t^*)^{-2})$, we have:

$$\begin{aligned} & \Phi_X \left([\alpha f^{-2}(x_1, W) + (1 - \alpha) f^{-2}(x_2, W)]^{\frac{-1}{2}} \right) \\ & \geq \alpha \Phi_X (f(x_1, W)) + (1 - \alpha) \Phi_X (f(x_2, W)). \end{aligned}$$

Hence, by combining with (7), we deduce that $\Phi_X (f(x, W))$ is a concave function according to x on Q , for all $W \in [0, t^+]$.

□

Proof of (iii). Let $g(x, W) := \Phi_X(f(x, W))$. By (ii), we shown that $g(x, W)$ is a concave function according to x on Q , for all $W \in [0, t^+]$.

Noting that $\mathbb{E}_W : g \rightarrow \mathbb{E}_W(g)$ is a linear transformation and the concavity is conserved by the linear transformations, we deduce that:

$$\mathbb{E}_W [\Phi_X(f(x, W))] - \text{ is a concave function according to } x \text{ on } Q.$$

□

Proof of (iv). Given x, y two points which satisfy the constraint (2) and $\alpha \in [0, 1]$. We have:

$$\begin{aligned} & \mathbb{E}_W [\Phi_X(f(x, W))] \geq 1 - \epsilon. \\ & \mathbb{E}_W [\Phi_X(f(y, W))] \geq 1 - \epsilon. \\ & \mathbb{E}_W [\Phi_X(f(\alpha x + (1 - \alpha)y, W))] \geq \\ & \geq \alpha \mathbb{E}_W [\Phi_X(f(x, W))] + (1 - \alpha) \mathbb{E}_W [\Phi_X(f(y, W))]. \\ & \text{(by the concavity of } \mathbb{E}_W [\Phi_X(f(x, W))] \text{ by (iii)).} \end{aligned}$$

We deduce that $\mathbb{E}_W [\Phi_X(f(\alpha x + (1 - \alpha)y, W))] \geq 1 - \epsilon$, as well as $\alpha x + (1 - \alpha)y$ satisfies the constraint (2). Hence, the feasible set of (1) is a convex set.

□

□

2.3. Independent joint chance constraints

Suppose that v_i follows a normal mean-variance mixture with parameters $(\mu_i, \Sigma_i, \gamma_i)$, for $1 \leq i \leq K$ and the vectors v_i are independent. Suppose that $0 \notin Q$.

The constraint in (1) can be rewritten as follows:

$$\begin{aligned}
& \mathbb{P}\{Vx \leq D\} \geq 1 - \epsilon. \\
& \Leftrightarrow \prod_{i=1}^K \mathbb{P}\{v_i^T x \leq D_i\} \geq 1 - \epsilon \text{ (by the independence).} \\
& \Leftrightarrow \prod_{i=1}^K \mathbb{E}_{W_i} \left[\Phi_{X_i} \left(\frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2} \right) \right] \geq 1 - \epsilon \\
& \text{(by using the transformation of (2) in the section (2.2)).} \\
& \Leftrightarrow \sum_{i=1}^K \log \left(\mathbb{E}_{W_i} \left[\Phi_{X_i} \left(\frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2} \right) \right] \right) \geq \log(1 - \epsilon).
\end{aligned} \tag{8}$$

We have the following theorem:

Theorem 2.3. *Consider the linear programming (1). Let*

$M := \{x \in Q \mid \mathbb{P}\{Vx \leq D\} \geq 1 - \epsilon\}$ the feasible set of (1). Suppose that $\forall i = \overline{1, K}$, v_i follows a normal mean-variance mixture with parameters $(\mu_i, \Sigma_i, \gamma_i)$ and the vectors v_i are independent.

Suppose that $\forall i = \overline{1, K}$, we have:

(1) Σ_i is a definite positive matrix with $0 < \lambda_{i,\min} \leq \lambda_{i,\max}$, where $\lambda_{i,\max}$ is the biggest eigenvalue of Σ_i and $\lambda_{i,\min}$ is the smallest eigenvalue of Σ_i .

(2) W_i is a positive variable in $[t_{i,\min}, t_{i,\max}]$ with $0 \leq t_{i,\min} \leq t_{i,\max} \leq t_i^+$.

(3) For all $x \in Q$ and $i = \overline{1, K}$, we have:

$$c_i > 0.$$

$$\text{and } \frac{1}{t_i^+} \geq \frac{\sqrt{b_i^2 - 4a_i c_i} - b_i}{2c_i} \text{ or } b_i^2 \leq 4a_i c_i.$$

where:

$$a_i = (-x^T \gamma_i)^2 \lambda_{i,min} + \alpha_{x,\gamma_i,\Sigma_i}.$$

$$b_i = 2(-x^T \gamma_i)(D_i - x^T \mu_i) \lambda_{i,min} - 6(x^T \Sigma_i x) \|\gamma_i\| \|\mu_i\| + \beta_{x,\gamma_i,\mu_i,\Sigma_i}.$$

$$c_i = (D_i - x^T \mu_i)^2 \lambda_{i,min} + \theta_{x,\mu_i,\Sigma_i}.$$

$$\alpha_{x,\gamma_i,\Sigma_i} := u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}.$$

$$\beta_{x,\gamma_i,\mu_i,\Sigma_i} := v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}.$$

$$\theta_{x,\mu_i,\Sigma_i} := q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}.$$

$$u = 4(-x^T \gamma_i) \gamma_i.$$

$$v = 4(-x^T \gamma_i) \gamma_i + 4(D_i - x^T \mu_i) \mu_i.$$

$$q = 4(D_i - x^T \mu_i) \mu_i.$$

$$z = \Sigma_i x.$$

(4) For all $x \in Q$, we have:

$$\frac{2\sqrt{(-x^T \gamma_i)(D_i - x^T \mu_i)}}{\sqrt{x^T \Sigma_i x}} > \sqrt{3}, \quad \forall i = \overline{1, K}.$$

$$(-x^T \gamma) > 0.$$

$$D - x^T \mu > 0.$$

Hence, M is a convex set.

Proof. Based on the proof of theorem (2.2), we deduce that

$\mathbb{E}_{W_i} \left[\Phi_{X_i} \left(\frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2} \right) \right]$ is a concave function, i.e that is also a log-concave function.

Hence, $\sum_{i=1}^K \log \left(\mathbb{E}_{W_i} \left[\Phi_{X_i} \left(\frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2} \right) \right] \right)$ is a concave function. We deduce that the feasible set of (1) is convex. \square

2.3.1. Dependent joint chance constraints with independent copula

Suppose that v_i follows a normal mean-variance mixture distribution with parameters $(\mu_i, \Sigma_i, \gamma_i)$, for $1 \leq i \leq K$. Suppose that $0 \notin Q$.

The constraint in (1) can be rewritten as follows:

$$\begin{aligned} \mathbb{P}\{Vx \leq D\} &\geq 1 - \epsilon. \\ \Leftrightarrow \mathbb{P}\left\{ \bigcup_{i=1}^K \{v_i^T x \leq D_i\} \right\} &\geq 1 - \epsilon. \\ \Leftrightarrow \mathbb{E}_W \left[\mathbb{E} \left[\mathbf{1}_{\bigcup_{i=1}^K \left\{ X_i(x) \leq \frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2} \right\}} \mid \bigcup W_i \right] \right] &\geq 1 - \epsilon, \end{aligned}$$

following the same procedure as (2).

$$\Leftrightarrow \mathbb{E}_W [\Phi(g_1(x, W), \dots, g_K(x, W))] \geq 1 - \epsilon.$$

Here we suppose that W and $X(x)$ are independent, with

$$\begin{aligned} W &:= (W_1, \dots, W_K), \quad X(x) := (X_1(x), \dots, X_K(x)), \quad X_i(x) := \frac{Z_i^T A_i^T x}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2}, \\ g_i(x, W) &:= \frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2}. \end{aligned} \tag{9}$$

where Φ is the distribution function of X .

Remark. In general, unfortunately, we cannot prove the concavity of the function $\Phi(g_1(x, W), \dots, g_K(x, W))$, i.e. $\mathbb{E}_W [\Phi(g_1(x, W), \dots, g_K(x, W))]$.

Consequently, we suppose additionally that for $i = \overline{1, K}$, we have $t_{i,min} \approx t_{i,max}$. Hence, we deduce that $\mathbb{E}_W [\Phi (g_1(x, W), \dots, g_K(x, W))]$

$\approx \prod_{i=1}^K (t_{i,max} - t_{i,min}) \times \Phi (g_1(x, w_1), \dots, g_K(x, w_K))$, where w_i is an arbitrary point on $[t_{i,min}, t_{i,max}]$. Then the constraint in (1) can be rewritten as follows:

$$\Phi (g_1(x, w_1), \dots, g_K(x, w_K)) \geq \theta \quad (10)$$

where $X(x) := (X_1(x), \dots, X_K(x))$, $X_i(x) := \frac{Z_i^T A_i^T x}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2}$,
 $g_i(x, W) := \frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2}$
and Φ is the distribution function of X .

Suppose that there exists an archimedean copula C which does not depend on x such that

$\Phi (g_1(x, w_1), \dots, g_K(x, w_K)) = C[F_1(g_1(x, w_1)), \dots, F_K(g_K(x, w_K))]$, where F_i the distribution function of $X_i(x)$ which is a standard normal distribution and $C(u) = \psi^{(-1)} \left(\sum_{i=1}^K \psi(u_i) \right)$, where ψ is a generator of C . We reformulate the constraint (10) as follows:

$$\begin{aligned}
& \Phi(g_1(x, w_1), \dots, g_K(x, w_K)) \geq \theta \\
& \Leftrightarrow C[F_1(g_1(x, w_1)), \dots, F_K(g_K(x, w_K))] \geq \theta \\
& \Leftrightarrow \sum_{i=1}^K \psi[F_i(g_i(x, w_i))] \leq \psi(\theta) \\
& \text{(by the decreasing property of } \psi) \\
& \Leftrightarrow F_i(g_i(x, w_i)) \geq \psi^{(-1)}[\alpha_i \psi(\theta)], \quad \forall i = \overline{1, K} \\
& \sum_{i=1}^K \alpha_i = 1 \\
& \Leftrightarrow g_i(x, w_i)^{-2} \leq F_i^{(-1)}\{\psi^{(-1)}[\alpha_i \psi(\theta)]\}^{-2}, \quad \forall i = \overline{1, K} \\
& \sum_{i=1}^K \alpha_i = 1. \tag{11}
\end{aligned}$$

We have the following theorem:

Theorem 2.4. *Consider the linear programming (1). Let*

$M := \{x \in Q \mid \mathbb{P}\{Vx \leq D\} \geq 1 - \epsilon\}$ *the feasible set of (1). Suppose that* $\forall i = \overline{1, K}$, v_i *follows a normal mean-variance mixture with parameters* $(\mu_i, \Sigma_i, \gamma_i)$. *Suppose that* W_i *is a positive random variable in* $[t_{i,\min}, t_{i,\max}]$ *with* $t_{i,\min} \approx t_{i,\max}$. *Let* $\frac{Z_i^T A_i^T x}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} := X_i(x), \forall 1 \leq i \leq K$ *and* $x \in Q$.

Suppose that the multivariate copula of $X(x) := (X_1(x), \dots, X_K(x))$ *is independent of* x . *Moreover,* W *and* $X(x)$ *are independent. Denote* C *the multivariate copula of* $X(x)$. *Suppose that* C *is an archimedean copula where* ψ *is a generator. Suppose that* $\theta \geq \Phi_{\mathcal{N}(0,1)}(\sqrt{3})$.

Suppose that $\forall i = \overline{1, K}$ *we have:*

(1) Σ_i *is a definite positive matrix with* $0 < \lambda_{i,\min} \leq \lambda_{i,\max}$, *where* $\lambda_{i,\max}$ *is the biggest eigenvalue of* Σ_i *and* $\lambda_{i,\min}$ *is the smallest eigenvalue of* Σ_i .

(2) W_i is a random variable in $[t_{i,min}, t_{i,max}]$ with $0 \leq t_{i,min} \leq t_{i,max} \leq t_i^+$.

(3) For all $x \in Q$ and $i = \overline{1, K}$, we have:

$$c_i > 0.$$

$$\text{and } \frac{1}{t_i^+} \geq \frac{\sqrt{b_i^2 - 4a_i c_i} - b_i}{2c_i} \text{ or } b_i^2 \leq 4a_i c_i.$$

where:

$$a_i = (-x^T \gamma_i)^2 \lambda_{i,min} + \alpha_{x,\gamma_i,\Sigma_i}.$$

$$b_i = 2(-x^T \gamma_i)(D_i - x^T \mu_i) \lambda_{i,min} - 6(x^T \Sigma_i x) \|\gamma_i\| \|\mu_i\| + \beta_{x,\gamma_i,\mu_i,\Sigma_i}.$$

$$c_i = (D_i - x^T \mu_i)^2 \lambda_{i,min} + \theta_{x,\mu_i,\Sigma_i}.$$

$$\alpha_{x,\gamma_i,\Sigma_i} := u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}.$$

$$\beta_{x,\gamma_i,\mu_i,\Sigma_i} := v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}.$$

$$\theta_{x,\mu_i,\Sigma_i} := q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}.$$

$$u = 4(-x^T \gamma_i) \gamma_i.$$

$$v = 4(-x^T \gamma_i) \gamma_i + 4(D_i - x^T \mu_i) \mu_i.$$

$$q = 4(D_i - x^T \mu_i) \mu_i.$$

$$z = \Sigma_i x.$$

(4) For all $x \in Q$, we have:

$$\frac{2\sqrt{(-x^T\gamma_i)(D_i - x^T\mu_i)}}{\sqrt{x^T\Sigma_i x}} > \sqrt{3}, \quad \forall i = \overline{1, K}.$$

$$(-x^T\gamma) > 0.$$

$$D - x^T\mu > 0.$$

Hence, M is a convex set.

Proof. The proof of this theorem follows this outline:

(i) We show that $g_i(x, w_i)$ is a (-2)-concave function according to x on Q , $\forall w_i \in [t_{i,min}, t_{i,max}]$.

(ii) We show that $F_i^{(-1)} \{ \psi^{(-1)} [\alpha_i \psi(\theta)] \}^{-2}$ is a convex function according to α_i on $[0, 1]$.

(iii) By using (i) and (ii), we deduce that the feasible set of (1) is a convex set.

Proof of (i). The proof follows from theorem (2.2). □

Proof of (ii). Let $H = \psi^{(-1)} [\alpha_i \psi(\theta)]$. By a direct calculation, we deduce the following formulation:

$$\frac{d^2 H}{d\alpha_i^2} = -\frac{\psi(\theta)^2 \times \psi''(H)}{\psi'(H)^3}$$

By using the properties of a generator of an archimedean copula, we deduce that $\frac{d^2 H}{d\alpha_i^2} \geq 0$, i.e H is a convex function according to α_i (*).

We prove that $u(x) := F_i^{(-1)}(x)^{-2}$ is a concave function according to x if $x \geq F_i(\sqrt{3})$ (**).

Proof. In fact, let $F_i^{(-1)}(x) = v$, we have:

$$u(x)'' = \frac{6v^{-4} - 2v^{-2}}{\frac{1}{2\pi} \exp(-v^2)}.$$

Hence, we deduce that $u''(x) \leq 0$ if and only if $x \geq F_i(\sqrt{3})$. □

We prove that $\psi^{(-1)}[\alpha_i \psi(\theta)] \geq F_i(\sqrt{3})$ (***) .

Proof. In fact, by using the fact that $0 \leq \alpha_i \leq 1$ and $\theta \geq F_i(\sqrt{3})$, we deduce the proof. □

We need the following lemma:

Lemma 2.5. *[[15]] Let $M_1, M_2 \subset \mathbb{R}$ and M_1, M_2 are convex sets. Suppose $f_1 : M_1 \rightarrow M_2$ is a convex function on M_1 and $f_2 : M_2 \rightarrow \mathbb{R}$ is a decreasing and concave function on M_2 . Then, the composition function $f_2 \circ f_1$ is a convex function on M_1 .*

By applying (*), (**), (***) and the lemma (2.5), we deduce the convexity of $F_i^{(-1)} \{ \psi^{(-1)} [\alpha_i \psi(\theta)] \}^{-2}$.

□

Proof of (iii). By using (i) and (ii), we deduce that all constraints are convex.

□

□

2.4. Dependent joint chance constraints with general copula

We rewrite the constraint of (1) in the previous section as follows:

$$\begin{aligned}
& \Phi(g_1(x, w_1), \dots, g_K(x, w_K)) \geq \theta \\
& \Leftrightarrow C[F_1(g_1(x, w_1)), \dots, F_K(g_K(x, w_K))] \geq \theta \\
& \Leftrightarrow g_i(x, w_i)^{-2} \leq F_i^{(-1)} \left\{ \psi^{(-1)} [\alpha_i \psi(\theta)] \right\}^{-2}, \quad \forall i = \overline{1, K} \\
& \sum_{i=1}^K \alpha_i = 1.
\end{aligned}$$

In this section, we suppose that the copula C is a function $C(x)$ of x , and for all x , $C(x)$ is an Archimedean copula with generator ψ_x . The last constraint is rewritten as follows:

$$\begin{aligned}
& \Leftrightarrow g_i(x, w_i)^{-2} \leq F_i^{(-1)} \left\{ \psi_x^{(-1)} [\alpha_i \psi_x(\theta)] \right\}^{-2}, \quad \forall i = \overline{1, K} \\
& \sum_{i=1}^K \alpha_i = 1.
\end{aligned}$$

The only difference between this case and the independent copula case is that the element on the right-hand side $F_i^{(-1)} \left\{ \psi_x^{(-1)} [\alpha_i \psi_x(\theta)] \right\}^{-2}$ does not depend on x , then that is a function depending only on α_i . Based on the same proof of the previous section, we only need to find a family of generators ψ_x such that $\psi_x^{(-1)} [\alpha_i \psi_x(\theta)]$ is a convex function with respect to (x, α_i) . Unfortunately, we do not have the convexity in this case because of α_i . However, if there exists a lower bound $\epsilon > 0$ for the α_i (i.e $\alpha_i \geq \epsilon$), we can prove that if $\theta \approx 1$, we have the convexity. We have the following theorem:

Theorem 2.6. Consider the linear programming (1). Let

M the feasible set of (1). Suppose that $\psi_x(t) = g(x)(t^{-\frac{1}{g(x)}} - 1)$ (a Clayton copula family with $g(x) > 0$). Suppose that $g(x)$ is continuously differentiable up to second order (cf. definition (2.2)) where $g''(x) \leq 0$ and there exists a $\delta > 0$ such that $g(x) \geq \delta, \forall x \in Q$. Suppose que $\theta \geq \Phi_{\mathcal{N}(0,1)}(\sqrt{3})$.

Suppose that $\forall i = \overline{1, K}$ we have:

(1) Σ_i is a definite positive matrix with $0 < \lambda_{i,min} \leq \lambda_{i,max}$, where $\lambda_{i,max}$ is the biggest eigenvalue of Σ_i and $\lambda_{i,min}$ is the smallest eigenvalue of Σ_i .

(2) For all $x \in Q$ and $i = \overline{1, K}$, we have:

$$c_i > 0.$$

$$\text{and } \frac{1}{t_i^+} \geq \frac{\sqrt{b_i^2 - 4a_i c_i} - b_i}{2c_i} \text{ or } b_i^2 \leq 4a_i c_i.$$

where:

$$\begin{aligned}
a_i &= (-x^T \gamma_i)^2 \lambda_{i,min} + \alpha_{x,\gamma_i,\Sigma_i}. \\
b_i &= 2(-x^T \gamma_i)(D_i - x^T \mu_i) \lambda_{i,min} - 6(x^T \Sigma_i x) \|\gamma_i\| \|\mu_i\| + \beta_{x,\gamma_i,\mu_i,\Sigma_i}. \\
c_i &= (D_i - x^T \mu_i)^2 \lambda_{i,min} + \theta_{x,\mu_i,\Sigma_i}. \\
\alpha_{x,\gamma_i,\Sigma_i} &:= u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}. \\
\beta_{x,\gamma_i,\mu_i,\Sigma_i} &:= v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}. \\
\theta_{x,\mu_i,\Sigma_i} &:= q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}. \\
u &= 4(-x^T \gamma_i) \gamma_i. \\
v &= 4(-x^T \gamma_i) \gamma_i + 4(D_i - x^T \mu_i) \mu_i. \\
q &= 4(D_i - x^T \mu_i) \mu_i. \\
z &= \Sigma_i x.
\end{aligned}$$

(4) For all $x \in Q$, we have:

$$(-x^T \gamma) > 0.$$

$$D - x^T \mu > 0.$$

Hence, there exists $\omega(\epsilon, \delta)$ depending on ϵ and δ such that $\forall \theta \geq \omega(\epsilon, \delta)$, we have M is a convex set.

Proof. Based the same proof of theorem (2.4), we only need to prove that $U(\alpha_i, x) := \psi_x^{(-1)}[\alpha_i \psi_x(\theta)]$ is convex with respect to (x, α_i) , $\forall \theta \geq \omega(\epsilon, \delta)$, for some $\omega(\epsilon, \delta)$.

Let $u_x : \mathbb{R} \rightarrow \mathbb{R}$ a family of real functions depending on x such that there exists $q_x \in \mathcal{L}(\mathbb{R}, \mathbb{R}^n)$ and $v_x \in \mathbb{M}_{n \times n}(\mathbb{R})$ such that $q_x = \frac{d}{dx}u_x$ and $v_x = \frac{d^2}{dx^2}u_x$ (cf. definition (2.2)). For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we have the following equations:

$$\frac{d}{dx}[u_x(f(x))] = q_x(f(x)) + u'_x(f(x)) \cdot f'(x). \quad (12)$$

$$\frac{d}{dx}[q_x(f(x))] = v_x(f(x)) + q'_x(f(x)) \cdot f'(x)^T. \quad (13)$$

Let $J = \psi_x^{(-1)}(\theta)$; $K_x := \psi_x^{(-1)}$; $L_x = \frac{d}{dx}K_x$; $M_x = \frac{d^2}{dx^2}K_x$.

We deduce the following equations:

$$\begin{aligned} K_x(t) &= \left(\frac{t}{g(x) + 1} \right)^{-g(x)} \\ L_x(t) &= K_x(t) \left[\frac{t}{t + g(x)} - \log \left(\frac{t}{g(x)} + 1 \right) \right] g'(x) \\ M_x(t) &= \left[\left(\frac{t}{t + g(x)} - \log \left(\frac{t}{g(x)} + 1 \right) \right)^2 + \frac{t}{tg(x) + g(x)^2} - \frac{t}{(t + g(x))^2} \right] \\ &\quad \times K_x(t) g'(x) g'(x)^T + K_x(t) \left(\frac{t}{t + g(x)} - \log \left(\frac{t}{g(x)} + 1 \right) \right) g''(x) \\ K'_x(t) &= -K_x(t) \frac{g(x)}{t + g(x)} \\ K''_x(t) &= K_x(t) \frac{g(x)^2 + g(x)}{(t + g(x))^2} \\ L'_x(t) &= K_x(t) \left[\frac{g(x)}{t + g(x)} \log \left(\frac{t}{g(x)} + 1 \right) - \frac{t(g(x) + 1)}{(t + g(x))^2} \right] g'(x). \end{aligned} \quad (14)$$

We have: $\frac{d^2}{d\alpha_i^2}U(\alpha_i, x) = J^2 K_x''(\alpha_i J) > 0$. Hence, the necessary and sufficient condition for the convexity of $U(\alpha_i, x)$ is the semidefinite positivity of the following symmetric matrix:

$$\begin{aligned}
& \left[\frac{d^2}{dx^2} \frac{d^2}{d\alpha_i^2} - \left(\frac{d^2}{dx d\alpha_i} \right) \left(\frac{d^2}{dx d\alpha_i} \right)^T \right]_o [U(\alpha_i, x)] \\
&= Q_1 \times M_x(\alpha_i J) + Q_2 \times [L'_x(\alpha_i J)L_x(J)^T + L_x(J)L'_x(\alpha_i J)^T] \\
&+ Q_3 \times [L'_x(J)L_x(J)^T + L_x(J)L'_x(J)^T] + Q_4 \times [L_x(J)L_x(J)^T] \\
&+ Q_5 \times L'_x(\alpha_i J)L'_x(\alpha_i J)^T. \tag{15}
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &= J^2 K_x''(\alpha_i J) \left(1 - \alpha_i \frac{K'_x(\alpha_i J)}{K'_x(J)} \right) \\
Q_2 &= J \frac{K'_x(\alpha_i J)}{K'_x(J)} \\
Q_3 &= \alpha_i J^2 \frac{K'_x(\alpha_i J)K_x''(\alpha_i J)}{K'_x(J)^2} \\
Q_4 &= -\frac{\alpha_i J^2 K_x''(J)K'_x(\alpha_i J)K_x''(\alpha_i J)}{K'_x(J)^3} - \frac{2\alpha_i J K_x''(\alpha_i J)K'_x(\alpha_i J)}{K'_x(J)^2} - \frac{K'_x(\alpha_i J)^2}{K'_x(J)^2} \\
Q_5 &= -J^2. \tag{16}
\end{aligned}$$

By using the equations of (14), we deduce that the equation (15) is equivalent to the following equation:

$$\begin{aligned}
& \left[\frac{d^2}{dx^2} \frac{d^2}{d\alpha_i^2} - \left(\frac{d^2}{dx d\alpha_i} \right) \left(\frac{d^2}{dx d\alpha_i} \right)^T \right]_o [U(\alpha_i, x)] \\
& = A \times g''(x) + (B_1 + B_2 + B_3 + B_4 + B_5) \times g'(x)g'(x)^T
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
A &= J^2 K_x(\alpha_i J)^2 \frac{g(x) + g(x)^2}{(\alpha_i J + g(x))^2} \left[1 - \alpha_i \left(\frac{\alpha_i J + g(x)}{J + g(x)} \right)^{-g(x)-1} \right] \\
&\times \left[\frac{\alpha_i J}{\alpha_i J + g(x)} - \log \left(\frac{\alpha_i J}{g(x)} + 1 \right) \right]. \\
B_1 &= J^2 K_x(\alpha_i J)^2 \frac{g(x) + g(x)^2}{(\alpha_i J + g(x))^2} \left[1 - \alpha_i \left(\frac{\alpha_i J + g(x)}{J + g(x)} \right)^{-g(x)-1} \right] \times \\
&\left[\left(\frac{\alpha_i J}{\alpha_i J + g(x)} - \log \left(\frac{\alpha_i J}{g(x)} + 1 \right) \right)^2 + \frac{\alpha_i J}{\alpha_i J + g(x)} \left(\frac{1}{g(x)} - \frac{1}{\alpha_i J + g(x)} \right) \right]. \\
B_2 &= JK_x(\alpha_i J)^2 \left[\frac{g(x)}{\alpha_i J + g(x)} \log \left(\frac{\alpha_i J}{g(x)} + 1 \right) - \frac{\alpha_i J [g(x) + 1]}{[\alpha_i J + g(x)]^2} \right] \\
&\times \left[\frac{J}{J + g(x) - \log \left(\frac{J}{g(x)} + 1 \right)} \right] + JK_x(\alpha_i J)^2 \times \\
&\left[\frac{g(x)}{J + g(x)} \log \left(\frac{J}{g(x)} + 1 \right) - \frac{J [g(x) + 1]}{[J + g(x)]^2} \right] \left[\frac{\alpha_i J}{\alpha_i J + g(x)} - \log \left(\frac{\alpha_i J}{g(x)} + 1 \right) \right] \\
B_3 &= -2\alpha_i J^2 K_x(\alpha_i J)^2 \frac{[g(x) + 1][J + g(x)]^2}{[\alpha_i J + g(x)]^3} \left[\frac{J}{J + g(x)} - \log \left(\frac{J}{g(x)} + 1 \right) \right] \\
&\times \left[\frac{g(x)}{J + g(x)} \log \left(\frac{J}{g(x)} + 1 \right) - \frac{J [g(x) + 1]}{[J + g(x)]^2} \right]. \\
B_4 &= K_x(\alpha_i J)^2 \frac{[J + g(x)]}{[\alpha_i J + g(x)]^2} \times \left[\frac{J}{[J + g(x)]} - \log \left(\frac{J}{g(x)} + 1 \right) \right]^2 \times \\
&\times \left[2\alpha_i J \frac{[g(x) + 1][J + g(x)]}{\alpha_i J + g(x)} - \alpha_i J^2 \frac{[g(x) + 1]^2}{\alpha_i J + g(x)} - J - g(x) \right] \\
B_5 &= -J^2 K_x(\alpha_i J)^2 \left[\frac{g(x)}{\alpha_i J + g(x)} \log \left(\frac{\alpha_i J}{g(x)} + 1 \right) - \frac{\alpha_i J [g(x) + 1]^2}{[\alpha_i J + g(x)]^2} \right]^2 \quad (18)
\end{aligned}$$

When $\theta \approx 1$, we deduce that $J \approx 0$. By using the Taylor development, we can show the following equalities:

$$\begin{aligned}
A &\leq 0, \quad \forall J \geq 0. \\
B_1 &\approx J^4 \times \frac{\alpha_i^2(1 - \alpha_i)[g(x) + 1]}{g(x)^4}. \\
B_2 &\approx J^3 \times \frac{\alpha_i + \alpha_i^2}{2g(x)^4}. \\
B_3 &\approx -J^5 \times \frac{\alpha_i[g(x) + 1]}{g(x)^5}. \\
B_4 &\approx -J^4 \times \frac{1}{4g(x)^4}. \\
B_5 &\approx -J^4 \times \frac{\alpha_i^2}{g(x)^4}.
\end{aligned}$$

Remark. Here, we denote $A \approx B$ if $\frac{A}{B} \rightarrow 1$, when $J \rightarrow 0$.

By using the assumptions that $g''(x) \leq 0$, $\alpha_i \geq \epsilon$ and $g(x) \geq \delta$, we deduce that there exists a $\omega(\epsilon, \delta)$ depending on (ϵ, δ) such that for all $\theta \geq \omega(\epsilon, \delta)$, we have:

$$\left[\frac{d^2}{dx^2} \frac{d^2}{d\alpha_i^2} - \left(\frac{d^2}{dx d\alpha_i} \right) \left(\frac{d^2}{dx d\alpha_i} \right)^T \right]_o [U(\alpha_i, x)] \geq 0.$$

We deduce that $U(\alpha_i, x)$ is a convex function with respect to (α_i, x) .

□

3. Approximation methods and numerical results

3.1. Individual chance constraints case

In theorem (2.2) where $K = 1$, n is the dimension, we choose , $D = 200 \times n$, $t^+ = 1$, $\mu = \gamma = (-1, -1, \dots, -1)$, $\Sigma = \mathbb{I}d_n$ (identity matrix) , i.e

$$V := \begin{pmatrix} -1 \\ \cdot \\ \cdot \\ -1 \end{pmatrix} + W \begin{pmatrix} -1 \\ \cdot \\ \cdot \\ -1 \end{pmatrix} + \sqrt{W}Z \text{ where } Z \text{ is a standard normal distribution}$$

in dimension n and W is a positive random variable taking values in $[0, 1]$, independent of Z . Here, we suppose that W follows an uniform random variable on $[t_{min}, t_{max}]$, where $0 \leq t_{min} \leq t_{max} \leq 1$. Moreover, we suppose that the domain $Q := \{x \mid 1 \leq x_i \leq 2, \forall i = \overline{1, n}\}$.

We show that the linear programming (1) is convex.

Proof. We show that these above assumptions satisfy the assumptions in theorem (2.2).

In fact, we can show the following inequalities:

$$0 > a > -30n^2. \text{ (condition on a)}$$

$$|b| \leq 12nD + 64n^2. \text{ (condition on b)}$$

$$c > (D + n)(D - 19n). \text{ (condition on c)}$$

where:

$$\begin{aligned}
a &= (-x^T \gamma)^2 \lambda_{\min} + \alpha_{x,\gamma,\Sigma}. \\
b &= 2(-x^T \gamma)(D - x^T \mu) \lambda_{\min} - 6(x^T \Sigma x) \|\gamma\| \|\mu\| + \beta_{x,\gamma,\mu,\Sigma}. \\
c &= (D - x^T \mu)^2 \lambda_{\min} + \theta_{x,\mu,\Sigma}. \\
\alpha_{x,\gamma,\Sigma} &:= u^T z - \sqrt{(u^T z)^2 + \sum_{1 \leq i < j \leq n} (u_i z_j - u_j z_i)^2}. \\
\beta_{x,\gamma,\mu,\Sigma} &:= v^T z - \sqrt{(v^T z)^2 + \sum_{1 \leq i < j \leq n} (v_i z_j - v_j z_i)^2}. \\
\theta_{x,\mu,\Sigma} &:= q^T z - \sqrt{(q^T z)^2 + \sum_{1 \leq i < j \leq n} (q_i z_j - q_j z_i)^2}. \\
u &= 4(-x^T \gamma) \gamma. \\
v &= 4(-x^T \gamma) \gamma + 4(D - x^T \mu) \mu. \\
q &= 4(D - x^T \mu) \mu. \\
z &= \Sigma x.
\end{aligned}$$

The assumptions of theorem (2.2) can be rewritten as follows:

$$1) \quad c > 0 \Leftrightarrow D + \sum_{i=1}^n x_i > 4 \left[\sum_{i=1}^n x_i + \sqrt{\left(\sum_{i=1}^n x_i \right)^2 + \sum_{1 \leq i < j \leq n} (x_i - x_j)^2} \right].$$

We have the following inequality where $D = 200 \times n$ and $1 \leq x_i \leq 2$, $\forall i = \overline{1, n}$.

$$\begin{aligned}
2) \quad \frac{1}{t^+} &\geq \frac{\sqrt{b^2 - 4ac} - b}{2c}. \\
&\Leftarrow \frac{1}{t^+} \geq \frac{\sqrt{b^2 - 4ac} + |b|}{2c}. \\
&\Leftarrow \frac{1}{t^+} \geq \frac{\sqrt{b^2 + 120n^2c} + |b|}{2c}. \quad (\text{by using the condition on a}) \\
&\Leftarrow \frac{1}{t^+} \geq \frac{\frac{|b| + 2\sqrt{30n}\sqrt{c}}{\sqrt{2}} + |b|}{2c}. \quad (\text{by using the Cauchy inequality})
\end{aligned}$$

By using the conditions on b and c , we have the following inequality.

$$\begin{aligned}
3) \quad &\frac{2\sqrt{(-x^T\gamma)(D - x^T\mu)}}{\sqrt{x^T\Sigma x}} > \sqrt{3}. \\
&\Leftarrow \frac{2\sqrt{(\sum_{i=1}^n x_i)(D + \sum_{i=1}^n x_i)}}{\sqrt{\sum_{i=1}^n x_i^2}} > \sqrt{3}.
\end{aligned}$$

By using the fact that $D = 200 \times n$ and $1 \leq x_i \leq 2$, $\forall i = \overline{1, n}$, we deduce the proof. □

Hence, by the theorem (2.2), if $x \neq 0$, the function $g(x) := \mathbb{P}(V^T x \leq 200 \times n) =$
 $= \mathbb{E}_W \left[\Phi_{\mathcal{N}(0,1)} \left(\frac{\sum_{i=1}^n x_i}{\|x\|} \sqrt{W} + \frac{200 \times n + \sum_{i=1}^n x_i}{\sqrt{W} \|x\|} \right) \right]$ is a concave function in the domain $Q = \{(x_1, \dots, x_n)^T \mid 1 \leq x_i \leq 2, 1 \leq i \leq n\}$ where $V :=$
 $\begin{pmatrix} -1 \\ \cdot \\ \cdot \\ -1 \end{pmatrix} + W \begin{pmatrix} -1 \\ \cdot \\ \cdot \\ -1 \end{pmatrix} + \sqrt{W} Z$ where Z is a standard normal distribution in

dimension n and W is an uniform random variable on $[0, 1]$, independent of Z .

As well as, the following linear programming:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.c} \quad & \mathbb{P} \{V^T x \leq 200 \times n\} \geq 1 - \epsilon \\
& 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n},
\end{aligned} \tag{19}$$

where $x := \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$, $V := \begin{pmatrix} -1 \\ \dots \\ -1 \end{pmatrix} + W \begin{pmatrix} -1 \\ \dots \\ -1 \end{pmatrix} + \sqrt{W}Z$ where Z is a standard normal distribution in dimension n and W is an uniform random variable on $[0, 1]$, independent of Z .

is a convex programming.

Let $g(x, W) := \frac{\sum_{i=1}^n x_i}{\|x\|} \sqrt{W} + \frac{200 \times n + \sum_{i=1}^n x_i}{\sqrt{W} \|x\|}$ and $\Psi(x, W) := \Phi_{\mathcal{N}(0,1)}(g(x, W))$.

Then, the linear programming (19) can be rewritten as following:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.c} \quad & \int_{t_{min}}^{t_{max}} \Psi(x, W) dW \geq 1 - \epsilon \\
& 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}.
\end{aligned} \tag{20}$$

When t_{min} and t_{max} are close, we can solve the following alternative problem:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.c} \quad & \Psi(x, w) \geq 1 - \epsilon \\
& \Leftrightarrow g^{-2}(x, w) \leq \left[\Phi_{\mathcal{N}(0,1)}^{-1}(1 - \epsilon) \right]^{-2} =: \theta \\
& \Leftrightarrow u(x, w) \leq \theta \\
& 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}.
\end{aligned} \tag{21}$$

with $w \in [0, 1]$.

By the proof of theorem (2.2), we have $u(x, W)$ is a convex function.

3.1.1. Lower bound approximation

Here, we use an outer-approximation algorithm (cf. [23]) to find a lower bound for the problem (21). The idea is to linearize the convex constraint $u(x, W) \leq \theta$ by its first order Taylor expansion $u(x_0, w) + (x - x_0) * u'(x_0, w) \leq \theta$, solve the problem with the linearized constraint, we obtain x_1 . Then, add the linearized constraint $u(x_1, w) + (x - x_1) * u'(x_1, w) \leq \theta$ in list of constraints, solve the new problem, we obtain x_2 . Repeat the process, we obtain a lower bound serie x_n of the original problem. More precisely, we have the following algorithm:

Algorithm 1 Outer-approximation algorithm

- Choose randomly a x_0 . Solve the following linear optimisation:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.c} \quad & u(x_0, w) + (x - x_0) * u'(x_0, w) \leq \theta \\ & 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}. \end{aligned} \tag{28-0}$$

Suppose that the solution of the problem (28-0) is x_1 .

- Solve the following linear optimisation:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.c} \quad & u(x_0, w) + (x - x_0) * u'(x_0, w) \leq \theta \\ & u(x_1, w) + (x - x_1) * u'(x_1, w) \leq \theta \\ & 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}. \end{aligned} \tag{28-1}$$

Suppose that the solution of the problem (28-1) is x_2 .

- Repeat the process until we achieve some convergence tolerance $\epsilon > 0$.
-

3.1.2. Upper bound approximation

Consider a Lagrangien function as following:

$$L(x, \lambda) = c^T x - \lambda [u(x, w) - \theta].$$

By the conditions of Karush-Kuhn-Tucker, there exists $\lambda \geq 0$ such that if x^* is the solution of the original problem (21), (x^*, λ) is the solution of the following deterministic optimisation:

$$\begin{aligned} \min \quad & c^T x - \lambda [u(x, w) - \theta] \\ & 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}. \end{aligned}$$

To solve this problem, we will use a descent gradient method which is presented as following:

Algorithm 2 Descent gradient algorithm

- Choose randomly x_0 in the admissible set of the original problem (21), $\lambda_0 \geq 0$ as well as two positive sequences (ϵ^k) and (ρ^k) . Solve the following linear optimisation:
- Given x_k and λ_k , we update x_{k+1} and λ_{k+1} as following:

$$\begin{aligned} x_{k+1} &= x_k - \epsilon^k [c + \lambda_k u'(x, w)] \\ \lambda_{k+1} &= \lambda_k - \rho^k [\theta - u(x, w)] \end{aligned}$$

Here, we present some simulation results with different values of w and n (where the axe y present the objective value $c^T x$).

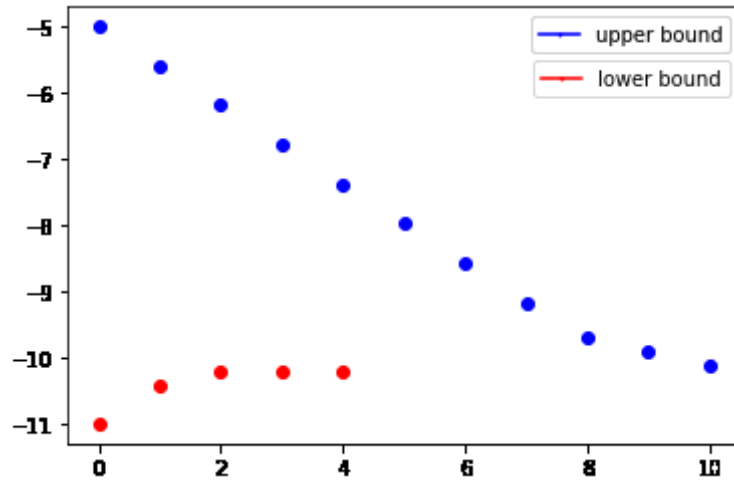


Figure 1: Series of lower bound and upper bound for $n=5, w=0.8$.

Gap between upper bound and lower bound: 0.079

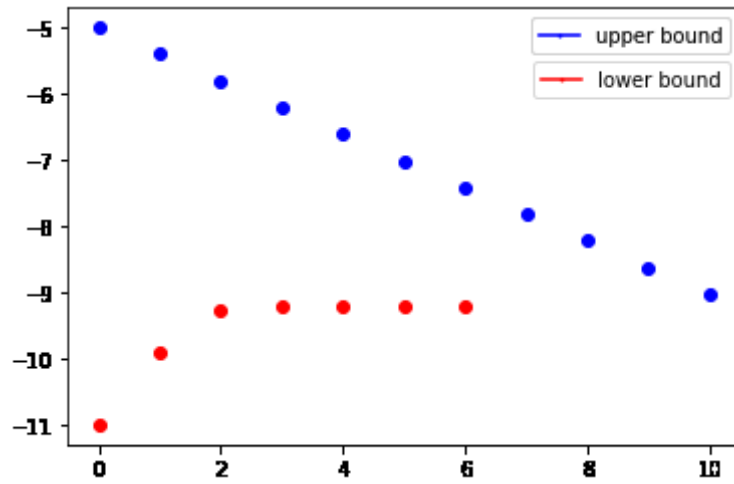


Figure 2: Series of lower bound and upper bound for $n=5, w=0.5$.

Gap between upper bound and lower bound: 0.2

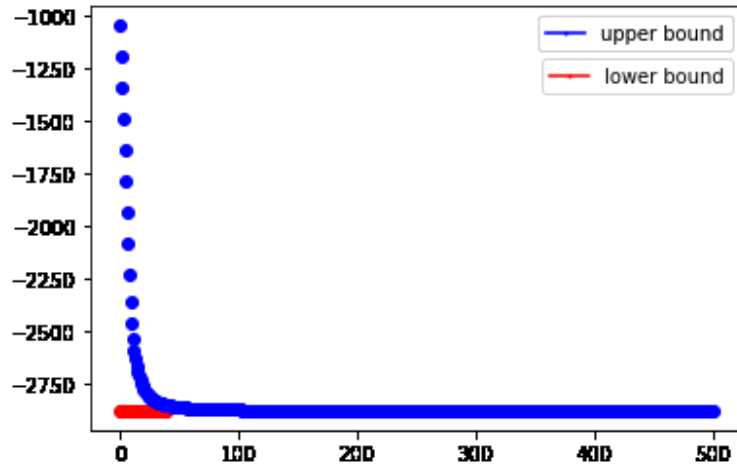


Figure 3: Series of lower bound and upper bound for $n=100, w=0.8$.

Gap between upper bound and lower bound: 0.0008

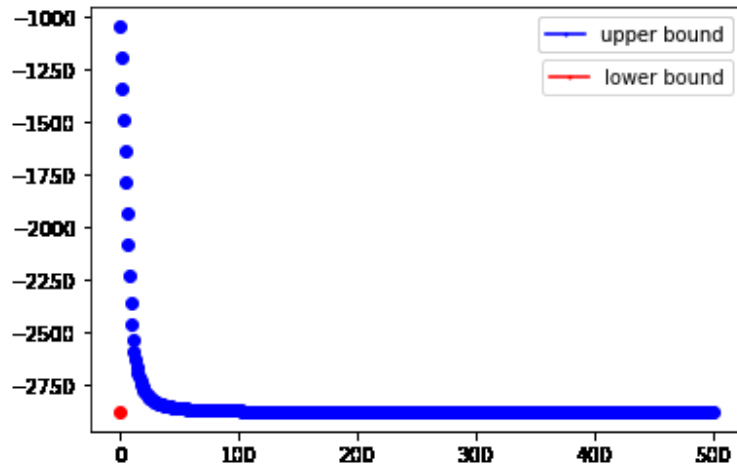


Figure 4: Series of lower bound and upper bound for $n=100, w=0.5$.

Gap between upper bound and lower bound: 0.0017

3.2. Independent joint chance constraints case

We can rewrite the problem (1) as follows:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.c} \quad & \mathbb{P} \{V^T x \leq D\} \geq 1 - \epsilon \\
& \Leftrightarrow \prod_{i=1}^K \int_{t_{i,\min}}^{t_{i,\max}} \Psi_i(x, W_i) dW_i \geq 1 - \epsilon \\
& 1 \leq x_j \leq 2, \quad \forall j = \overline{1, n},
\end{aligned} \tag{22}$$

where $g_i(x, W_i) := \frac{-x^T \gamma_i}{\|x^T \Sigma_i^{\frac{1}{2}}\|_2} \sqrt{W_i} + \frac{D_i - x^T \mu_i}{\sqrt{W_i} \|x^T \Sigma_i^{\frac{1}{2}}\|_2}$, for all $i = \overline{1, K}$.

When $t_{i,\min}$ and $t_{i,\max}$ are close, we can solve the following alternative problem:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.c} \quad & \prod_{i=1}^K \Psi_i(x, w_i) \geq \theta \\
& \Leftrightarrow g_i^{-2}(x, w_i) \leq \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\alpha_i}) \right]^{-2} \\
& \alpha_i \geq 0 \\
& \sum_{i=1}^K \alpha_i = 1 \\
& \Leftrightarrow u_i(x, \alpha, w_i) \leq 0, \forall i = \overline{1, K} \\
& \alpha_i \geq 0, \forall i = \overline{1, K} \\
& \sum_{i=1}^K \alpha_i = 1 \\
& 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}. \tag{23}
\end{aligned}$$

with $0 < \theta < 1$, $w_i \in [0, 1]$, $\alpha := (\alpha_1, \dots, \alpha_K)^T$,
 $u_i(x, \alpha, w_i) := g_i^{-2}(x, w_i) - \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\alpha_i}) \right]^{-2}$.

We show that under some additional conditions, u_i is a convex function with respect to (x, α) . That's enough to show that $\left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\alpha_i}) \right]^{-2}$ is a concave function with respect to α_i .

We need the following lemma:

Lemma 3.1. *For each $i = \overline{1, K}$, $\left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\alpha_i}) \right]^{-2}$ is a concave function with respect to α_i if $\theta \in [\Phi_{\mathcal{N}(0,1)}(\sqrt{3}), 1]$.*

Proof. Let $g_1 : [0, 1] \rightarrow [\theta, 1]$ such that $g_1(x) = \theta^x$, and $g_2 : [\theta, 1] \rightarrow [0, \infty]$ such that $g_2(x) = \Phi_{\mathcal{N}(0,1)}^{-1}(x)^{-2}$. We deduce that g_1 is a convex function and

g_2 is a decreasing function. Then, by applying lemma (2.5), we need to show that g_2 is a concave function.

In fact, let $\Phi_{\mathcal{N}(0,1)}^{-1}(x) = u$, we have:

$$g_2(x)'' = \frac{6u^{-4} - 2u^{-2}}{\frac{1}{2\pi} \exp(-u^2)}.$$

Hence, we deduce that $g_2(x) \leq 0$ if and only if $x \geq \Phi_{\mathcal{N}(0,1)}(\sqrt{3})$.

□

In theorem (2.3) where K is the number of constraints, n is the dimension, we choose , $D_i = 200 \times n$, $t_i^+ = 1$, $\mu_i = \gamma_i = (-1, -1, \dots, -1)$, $\Sigma_i = \mathbb{I}d_n$ (identity matrix), $w_i = 0.8$, for all $i = \overline{1, K-1}$ and $D_K = 400 \times n$, $t_K^+ = 1$, $\mu_K = \gamma_K = (-2, -2, \dots, -2)$, $\Sigma_K = \mathbb{I}d_n$ (identity matrix), $w_K = 0.8$.

The problem (1) can be rewritten as follows:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.c} \quad & g_1^{-2}(x, w_1) - \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^\alpha) \right]^{-2} \leq 0 \\ & g_1^{-2}(x, w_1) - 4 \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^\beta) \right]^{-2} \leq 0 \\ & \alpha, \beta \geq 0 \\ & (K-1)\alpha + \beta = 1 \\ & 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}. \end{aligned} \tag{24}$$

Let $M(\alpha) = \Phi_{\mathcal{N}(0,1)}^{-1}(\theta^\alpha)$. We approximate $M(\alpha)$ by $M(1)$.

For $k \gg 1$, we have $\Phi_{\mathcal{N}(0,1)}(k) \approx 1 - \frac{\phi(k)}{k}$, where $\phi(x)$ is the density function of a standard Gaussian distribution and for $k \approx 0$, α is bounded, we have $(1-k)^\alpha \approx 1 - \alpha k$. By applying that, we have:

$$\frac{\phi(M(\alpha))}{M(\alpha)} = \alpha \times \frac{\phi(M(1))}{M(1)},$$

where $M(1) = \Phi_{\mathcal{N}(0,1)}^{-1}(\theta)$

Here, we present some simulation results with $w = 0.8$, with different values of n (dimension) and K (number of constraints) (where the axe y present the objective value $c^T x$).

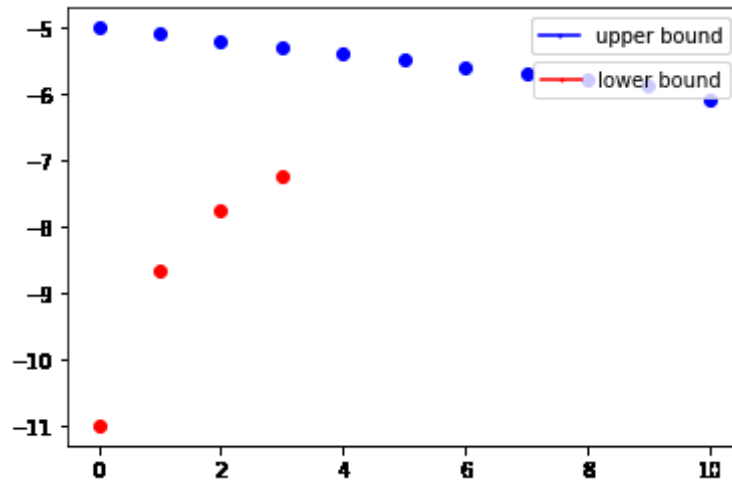


Figure 5: Series of lower bound and upper bound for $n=5$ (dimension), $K=5$ (number of constraints)

Gap between upper bound and lower bound: 1.167

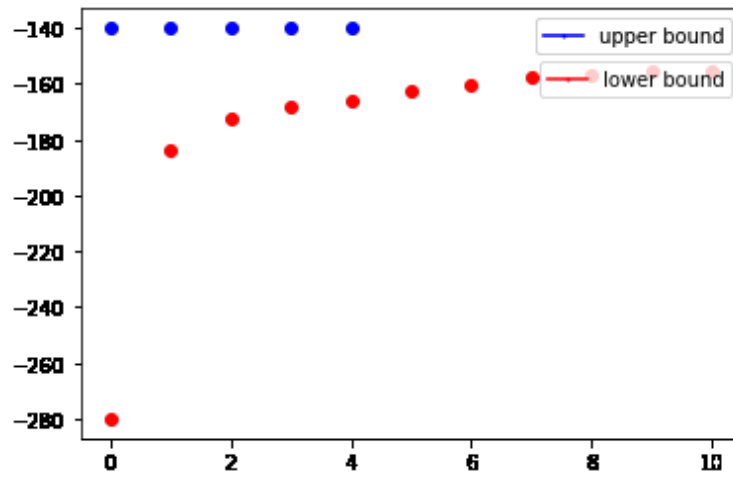


Figure 6: Series of lower bound and upper bound for $n=5$ (dimension), $K=100$ (number of constraints)

Gap between upper bound and lower bound: 15.621

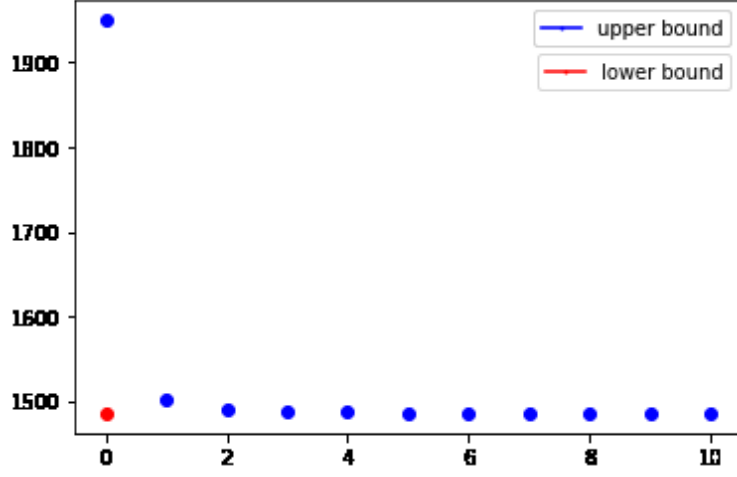


Figure 7: Series of lower bound and upper bound for $n=100$ (dimension), $K=100$ (number of constraints)

Gap between upper bound and lower bound: 0.099

3.3. Dependent joint chance constraints with independent copula case

We can see that the independent case is a particular case when we choose $\psi(x) = -\log(x)$. In this case, we choose $\psi(x) = \log(x)^2$ (a Gumbel-Hougaard generator type).

In theorem (2.4), where K is the number of constraints, n is the dimension, we choose , $D_i = 200 \times n$, $t_i^+ = 1$, $\mu_i = \gamma_i = (-1, -1, \dots, -1)$, $\Sigma_i = \mathbb{I}d_n$ (identity matrix), $w_i = 0.8$, for all $i = \overline{1, K-1}$ and $D_K = 400 \times n$, $t_K^+ = 1$, $\mu_K = \gamma_K = (-2, -2, \dots, -2)$, $\Sigma_K = \mathbb{I}d_n$ (identity matrix), $w_K = 0.8$.

The problem (1) can be rewritten as follows:

$$\begin{aligned}
& \min && c^T x \\
& \text{s.c} && g_1^{-2}(x, w_1) - \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\sqrt{\alpha}}) \right]^{-2} \leq 0 \\
& && g_1^{-2}(x, w_1) - 4 \left[\Phi_{\mathcal{N}(0,1)}^{-1}(\theta^{\sqrt{\beta}}) \right]^{-2} \leq 0 \\
& && \alpha, \beta \geq 0 \\
& && (K - 1)\alpha + \beta = 1 \\
& && 1 \leq x_i \leq 2, \quad \forall i = \overline{1, n}.
\end{aligned} \tag{25}$$

Here, we present some simulation results with $w = 0.8$, with different values of n (dimension) and K (number of constraints) (where the axe y present the objective value $c^T x$).

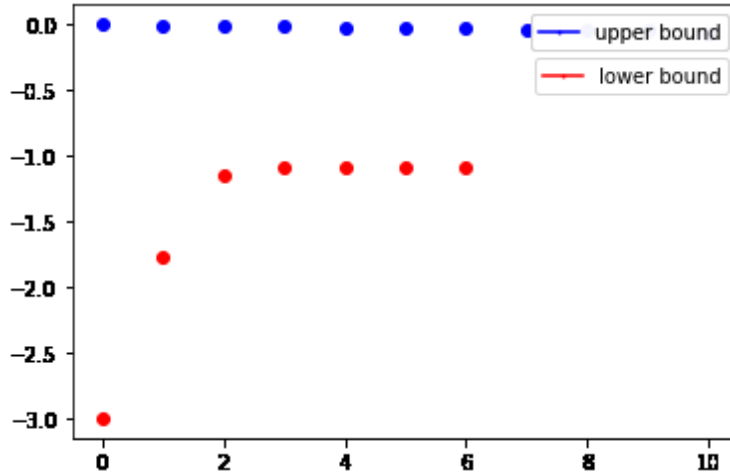


Figure 8: Series of lower bound and upper bound for $n=5$ (dimension), $K=5$ (number of constraints)

Gap between upper bound and lower bound: 1.0065

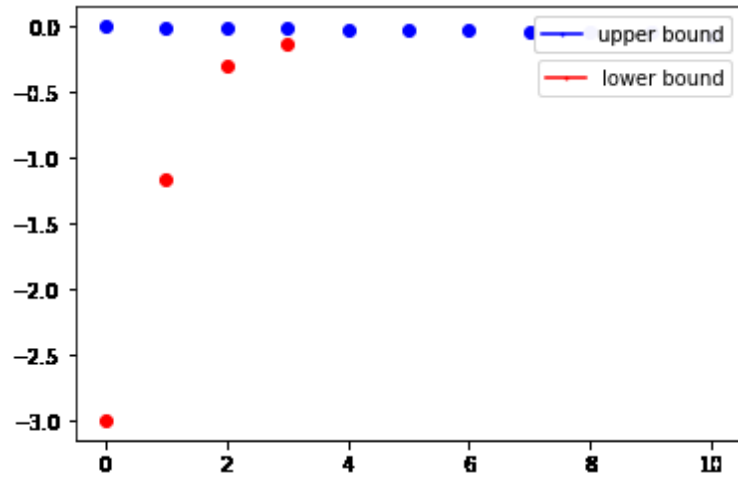


Figure 9: Series of lower bound and upper bound for $n=5$ (dimension), $K=100$ (number of constraints)

Gap between upper bound and lower bound: 0.0636

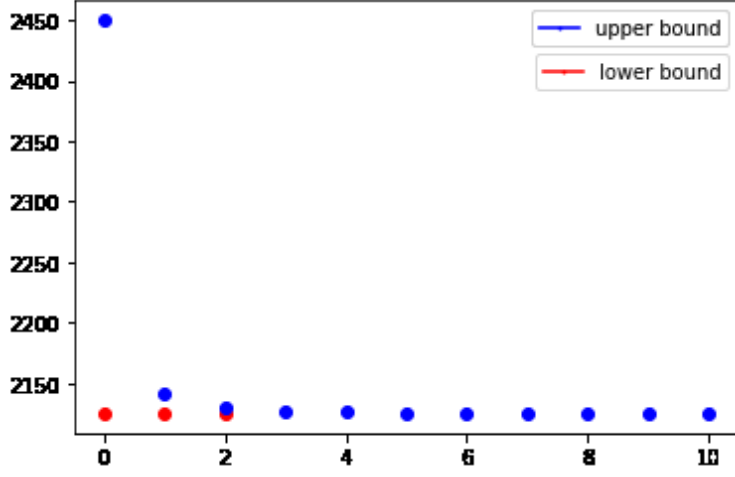


Figure 10: Series of lower bound and upper bound for $n=100$ (dimension), $K=100$ (number of constraints)

Gap between upper bound and lower bound: 0.0999

3.4. Dependent joint chance constraints with dependent copula case

This case is the same scenario of the independent copula case when $\theta \approx 1$.

We prove that as follows:

Let $M(\alpha_i, x) := F_i^{(-1)} \left\{ \psi_x^{(-1)} [\alpha_i \psi_x(\theta)] \right\}$. By applying $\psi_x(t) := g(x)[t^{\frac{-1}{g(x)}} - 1]$, we have:

$$M(\alpha_i, x) = F_i^{(-1)} \left\{ \left[\alpha_i \left(\theta^{\frac{-1}{g(x)}} - 1 \right) + 1 \right]^{-g(x)} \right\}$$

For $k \gg 1$, we have $F_i(k) \approx 1 - \frac{\phi(k)}{k}$, where $\phi(x)$ is the density function of a standard Gaussian distribution and for $k \approx 0$, α is bounded, we have $(1 - k)^\alpha \approx 1 - \alpha k$. By applying these approximations, we deduce that:

$$\frac{\phi(M(\alpha_i, x))}{M(\alpha_i, x)} = \alpha_i \times \frac{\phi(M(1, x))}{M(1, x)},$$

where $M(1, x) = F_i(\theta)$, for all x .

We can see that $M(\alpha_i, x)$ does not depend on x . Moreover, this value is exactly equal to $F_i^{(-1)}(\theta^{\alpha_i})$.

4. Conclusion

In this paper, we studied the problem of linear optimisation with joint probabilistic constraints. When the distribution of the constraint rows is a normal mean-variance mixture distribution and the dependence of random variables is represented by an archimedean copula, we shown the convexity of the feasibility set and found some new convexity results. There are some limitations in the simulation results because of the complexity of conditions for convexity. In this case, we cannot show the convexity for a wide subset of this family of distributions. We are going to relax the complexity in some next research results.

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