Coloring graphs with no induced subdivision of $K_{4+}$
Louis Esperet, Nicolas Trotignon

To cite this version:
Louis Esperet, Nicolas Trotignon. Coloring graphs with no induced subdivision of $K_{4+}$. 2021. hal-03092640

HAL Id: hal-03092640
https://hal.science/hal-03092640
Preprint submitted on 2 Jan 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
COLORING GRAPHS WITH NO INDUCED SUBDIVISION OF $K_4^+$

LOUIS ESPERET AND NICOLAS TROTIGNON

Abstract. Let $K_4^+$ be the 5-vertex graph obtained from $K_4$, the complete graph on four vertices, by subdividing one edge precisely once (i.e. by replacing one edge by a path on three vertices). We prove that if the chromatic number of some graph $G$ is much larger than its clique number, then $G$ contains a subdivision of $K_4^+$ as an induced subgraph.

Given a graph $H$, a subdivision of $H$ is a graph obtained from $H$ by replacing some edges of $H$ (possibly none) by paths. We say that a graph $G$ contains an induced subdivision of $H$ if $G$ contains a subdivision of $H$ as an induced subgraph.

A class of graphs $\mathcal{F}$ is said to be $\chi$-bounded if there is a function $f$ such that for any graph $G \in \mathcal{F}$, $\chi(G) \leq f(\omega(G))$, where $\chi(G)$ and $\omega(G)$ stand for the chromatic number and the clique number of $G$, respectively.

Scott [7] conjectured that for any graph $H$, the class of graphs without induced subdivisions of $H$ is $\chi$-bounded, and proved it when $H$ is a tree. But Scott’s conjecture was disproved in [6]. Finding which graphs $H$ satisfy the assumption of Scott’s conjecture remains a fascinating question. It was proved in [1] that every graph $H$ obtained from the complete graph $K_4$ by subdividing at least 4 of the 6 edges once (in such a way that the non-subdivided edges, if any, are non-incident), is a counterexample to Scott’s conjecture. On the other hand, Scott proved that the class of graphs with no induced subdivision of $K_4$ has bounded chromatic number (see [5]). Le [4] proved that every graph in this class has chromatic number at most 24. If triangles are also excluded, Chudnovsky et al. [2] proved that the chromatic number is at most 3.

In this paper, we extend the list of graphs known to satisfy Scott’s conjecture. Let $K_4^+$ be the 5-vertex graph obtained from $K_4$ by subdividing one edge precisely once.

**Theorem 1.** The family of graphs with no induced subdivision of $K_4^+$ is $\chi$-bounded.

We will need the following result of Kühn and Osthus [3].

**Theorem 2 (3).** For any graph $H$ and every integer $s$ there is an integer $d = d(H, s)$ such that every graph of average degree at least $d$ contains the complete bipartite graph $K_{s,s}$ as a subgraph, or an induced subdivision of $H$.

**Proof of Theorem 7.** Let $k$ be an integer, let $d(\cdot, \cdot)$ be the function defined in Theorem 2 and let $R(s,t)$ be the Ramsey number of $(s,t)$, i.e. the smallest $n$ such that every graph on $n$ vertices has a stable set of size $s$ or a clique of size $t$. 

The authors are partially supported by ANR Project STINT (anr-13-bsts02-0007), and LabEx PERSYVAL-Lab (anr-11-labx-0025).
We will prove that every graph $G$ with no induced subdivision of $K_4^+$, and with clique number at most $k$, is $d$-colorable, with $d = \max(k, d(K_4^+, R(4, k)))$. The proof proceeds by induction on the number of vertices of $G$ (the result being trivial if $G$ has at most $k$ vertices). Observe that all induced subgraphs of $G$ have clique number at most $k$ and do not contain any induced subdivision of $K_4^+$. Therefore, by the induction, we can assume that all induced subgraphs of $G$ are $d$-colorable. In particular, we can assume that $G$ is connected.

Assume first that $G$ does not contain $K_{s,s}$ as a subgraph, where $s = R(4, k)$. Then by Theorem 2, $G$ has average degree less than $d$, and hence contains a vertex of degree at most $d - 1$. By the induction, $G - v$ has a $d$-coloring and this coloring can be extended to a $d$-coloring of $G$, as desired.

We can thus assume that $G$ contains $K_{s,s}$ as a subgraph. Since $G$ has clique number at most $k$, it follows from the definition of $R(4, k)$ that $G$ contains $K_{4,4}$ as an induced subgraph. Let $M$ be a set of vertices of $G$ inducing a complete multipartite graph with at least two partite sets containing at least 4 vertices. Assume that among all such sets of vertices of $G$, $M$ is chosen with maximum cardinality. Let $V_1, V_2, \ldots, V_t$ be the partite sets of $M$.

Let $v$ be a vertex of $G$, and $S$ be a set of vertices not containing $v$. The vertex $v$ is complete to $S$ if $v$ is adjacent to all the vertices of $S$, anticomplete to $S$ if $v$ is not adjacent to any of the vertices of $S$, and mixed to $S$ otherwise. Let $R$ be the vertices of $G$ not in $M$. We can assume that $R$ is non-empty, since otherwise $G$ is clearly $k$-colorable and $k \leq d$.

We claim that:

If a vertex $v$ of $R$ has at least two neighbors in some set $V_i$, then it is not mixed to any set $V_j$ with $j \neq i$. \hfill (1)

Assume for the sake of contradiction that $v$ has two neighbors $a, b$ in $V_i$ and a neighbor $c$ and a non-neighbor $d$ in $V_j$, with $j \neq i$. Then $v, a, b, c, d$ induce a copy of $K_4^+$, a contradiction. This proves (1).

Each vertex $v$ of $R$ has at most one neighbor in each set $V_i$. \hfill (2)

Assume for the sake of contradiction that some vertex $v \in R$ has two neighbors $a, b$ in some set $V_i$. Then by (1), $v$ is complete or anticomplete to each set $V_j$ with $j \neq i$. Let $\mathcal{A}$ be the family of sets $V_j$ to which $v$ is anticomplete, and let $\mathcal{C}$ be the family of sets $V_j$ to which $v$ is complete. If $\mathcal{A}$ contains at least two elements, i.e. if $v$ is anticomplete to two sets $V_j$ and $V_j'$, then by taking $u \in V_j$ and $u' \in V_j'$, we observe that $v, a, b, u, u'$ induces a copy of $K_4^+$, a contradiction. It follows that $\mathcal{A}$ contains at most one element.

Next, we prove that $v$ is complete to $V_i$. Assume instead that $v$ is mixed to $V_i$. If $v$ is complete to some set $V_i$ containing at least two vertices, then we obtain a contradiction with (1). It follows that all the elements of $\mathcal{C}$ are singleton. By the definition of $M$, this implies that $\mathcal{A}$ contains exactly one set $V_j$, which has size at least 4. Let $c$ be a non-neighbor of $v$ in $V_i$, and let $d, d'$ be two vertices in $V_j$. Then $v, a, b, c, d, d'$ is an induced subdivision of $K_4^+$, a contradiction. We proved that $v$ is complete to $V_i$. Hence, every set
Each connected component of $G - M$ has at most one neighbor in each set $V_i$. (3)

Assume for the sake of contradiction that some connected component of $G - M$ has at least two neighbors in some set $V_i$. Then there is a path $P$ whose endpoints $u, v$ are in $V_i$, and whose internal vertices are in $R$. Choose $P, u, v, V_i$ such that $P$ contains the least number of edges. Note that by (2), $P$ contains at least 3 edges. Observe also that by the minimality of $P$, the only edges in $G$ between $V_i$ and the internal vertices of $P$ are the first and last edge of $P$. Let $V_j$ be a partite set of $M$ with at least 4 elements, with $j \neq i$ (this set exists, by the definition of $M$). By (2) and the minimality of $P$, at most two vertices of $V_j$ are adjacent to some internal vertex of $P$. Since $V_j$ contains at least four vertices, there exist $a, b \in V_j$ that are not adjacent to any internal vertex of $P$. If $V_i$ has at least three elements then it contains a vertex $w$ distinct from $u, v$. As $w$ is not adjacent to any vertex of $P$, the vertices $w, a, b$ together with $P$ induce a subdivision of $K^+_4$, a contradiction. If $V_i$ has at most two elements, then there must be an integer $\ell$ distinct from $i$ and $j$ such that $V_\ell$ has at least four elements. In particular, $V_\ell$ contains a vertex $c$ that is not adjacent to any internal vertex of $P$. As a consequence, the vertices $a, c$ together with $P$ induce a subdivision of $K^+_4$, which is again a contradiction. This proves (3).

Recall that we can assume that $R$ is non-empty. An immediate consequence of (3) is that the neighborhood of each connected component of $R$ is a clique. Since $G$ is connected, it follows that it contains a clique cutset $K$ (a clique whose deletion disconnects the graph). Let $C$ be a connected component of $G - K$, let $G_1 = G - C$, and let $G_2$ be the subgraph of $G$ induced by $C \cup K$. It follows from the induction that there exist $d$-colorings of $G_1$ and $G_2$. Furthermore, since $K$ is a clique, we can assume that the colorings coincide on $K$. This implies that $G$ is $d$-colorable and concludes the proof of Theorem 1. $\Box$

We remark that we could have used $K_{3,3}$ instead of $K_{4,4}$ in the proof, at the expense of a slightly more detailed analysis. The resulting bound on the chromatic number would have been $\max(k, d(K^+_4, R(3, k)))$ instead of $\max(k, d(K^+_4, R(4, k)))$.

Acknowledgement. The main result of this paper was proved in January 2016 during a meeting of the ANR project STINT at Saint Bonnet de Champsaur, France. We thank the organizers and participants for the friendly atmosphere. We also thank Alex Scott for spotting a couple of typos in a previous version of the draft.

References


Laboratoire G-SCOP (CNRS, Université Grenoble-Alpes), Grenoble, France
E-mail address: louis.esperet@ grenoble-inp.fr

LIP (CNRS, ENS de Lyon), Lyon, France
E-mail address: nicolas.trotignon@ens-lyon.fr