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# COLORING GRAPHS WITH NO INDUCED SUBDIVISION OF $K_4^+$

LOUIS ESPERET AND NICOLAS TROTIGNON

**ABSTRACT.** Let  $K_4^+$  be the 5-vertex graph obtained from  $K_4$ , the complete graph on four vertices, by subdividing one edge precisely once (i.e. by replacing one edge by a path on three vertices). We prove that if the chromatic number of some graph  $G$  is much larger than its clique number, then  $G$  contains a subdivision of  $K_4^+$  as an induced subgraph.

Given a graph  $H$ , a *subdivision* of  $H$  is a graph obtained from  $H$  by replacing some edges of  $H$  (possibly none) by paths. We say that a graph  $G$  contains an *induced subdivision* of  $H$  if  $G$  contains a subdivision of  $H$  as an induced subgraph.

A class of graphs  $\mathcal{F}$  is said to be  $\chi$ -*bounded* if there is a function  $f$  such that for any graph  $G \in \mathcal{F}$ ,  $\chi(G) \leq f(\omega(G))$ , where  $\chi(G)$  and  $\omega(G)$  stand for the chromatic number and the clique number of  $G$ , respectively.

Scott [7] conjectured that for any graph  $H$ , the class of graphs without induced subdivisions of  $H$  is  $\chi$ -bounded, and proved it when  $H$  is a tree. But Scott's conjecture was disproved in [6]. Finding which graphs  $H$  satisfy the assumption of Scott's conjecture remains a fascinating question. It was proved in [1] that every graph  $H$  obtained from the complete graph  $K_4$  by subdividing at least 4 of the 6 edges once (in such a way that the non-subdivided edges, if any, are non-incident), is a counterexample to Scott's conjecture. On the other hand, Scott proved that the class of graphs with no induced subdivision of  $K_4$  has bounded chromatic number (see [5]). Le [4] proved that every graph in this class has chromatic number at most 24. If triangles are also excluded, Chudnovsky et al. [2] proved that the chromatic number is at most 3.

In this paper, we extend the list of graphs known to satisfy Scott's conjecture. Let  $K_4^+$  be the 5-vertex graph obtained from  $K_4$  by subdividing one edge precisely once.

**Theorem 1.** *The family of graphs with no induced subdivision of  $K_4^+$  is  $\chi$ -bounded.*

We will need the following result of Kühn and Osthus [3].

**Theorem 2** ([3]). *For any graph  $H$  and every integer  $s$  there is an integer  $d = d(H, s)$  such that every graph of average degree at least  $d$  contains the complete bipartite graph  $K_{s,s}$  as a subgraph, or an induced subdivision of  $H$ .*

*Proof of Theorem 1.* Let  $k$  be an integer, let  $d(\cdot, \cdot)$  be the function defined in Theorem 2, and let  $R(s, t)$  be the Ramsey number of  $(s, t)$ , i.e. the smallest  $n$  such that every graph on  $n$  vertices has a stable set of size  $s$  or a clique of size  $t$ .

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We will prove that every graph  $G$  with no induced subdivision of  $K_4^+$ , and with clique number at most  $k$ , is  $d$ -colorable, with  $d = \max(k, d(K_4^+, R(4, k)))$ . The proof proceeds by induction on the number of vertices of  $G$  (the result being trivial if  $G$  has at most  $k$  vertices). Observe that all induced subgraphs of  $G$  have clique number at most  $k$  and do not contain any induced subdivision of  $K_4^+$ . Therefore, by the induction, we can assume that all induced subgraphs of  $G$  are  $d$ -colorable. In particular, we can assume that  $G$  is connected.

Assume first that  $G$  does not contain  $K_{s,s}$  as a subgraph, where  $s = R(4, k)$ . Then by Theorem 2,  $G$  has average degree less than  $d$ , and hence contains a vertex of degree at most  $d - 1$ . By the induction,  $G - v$  has a  $d$ -coloring and this coloring can be extended to a  $d$ -coloring of  $G$ , as desired.

We can thus assume that  $G$  contains  $K_{s,s}$  as a subgraph. Since  $G$  has clique number at most  $k$ , it follows from the definition of  $R(4, k)$  that  $G$  contains  $K_{4,4}$  as an induced subgraph. Let  $M$  be a set of vertices of  $G$  inducing a complete multipartite graph with at least two partite sets containing at least 4 vertices. Assume that among all such sets of vertices of  $G$ ,  $M$  is chosen with maximum cardinality. Let  $V_1, V_2, \dots, V_t$  be the partite sets of  $M$ .

Let  $v$  be a vertex of  $G$ , and  $S$  be a set of vertices not containing  $v$ . The vertex  $v$  is *complete* to  $S$  if  $v$  is adjacent to all the vertices of  $S$ , *anticomplete* to  $S$  if  $v$  is not adjacent to any of the vertices of  $S$ , and *mixed* to  $S$  otherwise. Let  $R$  be the vertices of  $G$  not in  $M$ . We can assume that  $R$  is non-empty, since otherwise  $G$  is clearly  $k$ -colorable and  $k \leq d$ . We claim that:

If a vertex  $v$  of  $R$  has at least two neighbors in some set  $V_i$ , then it is not mixed to any set  $V_j$  with  $j \neq i$ . (1)

Assume for the sake of contradiction that  $v$  has two neighbors  $a, b$  in  $V_i$  and a neighbor  $c$  and a non-neighbor  $d$  in  $V_j$ , with  $j \neq i$ . Then  $v, a, b, c, d$  induce a copy of  $K_4^+$ , a contradiction. This proves (1).

Each vertex  $v$  of  $R$  has at most one neighbor in each set  $V_i$ . (2)

Assume for the sake of contradiction that some vertex  $v \in R$  has two neighbors  $a, b$  in some set  $V_i$ . Then by (1),  $v$  is complete or anticomplete to each set  $V_j$  with  $j \neq i$ . Let  $\mathcal{A}$  be the family of sets  $V_j$  to which  $v$  is anticomplete, and let  $\mathcal{C}$  be the family of sets  $V_j$  to which  $v$  is complete. If  $\mathcal{A}$  contains at least two elements, i.e. if  $v$  is anticomplete to two sets  $V_j$  and  $V_{j'}$  then by taking  $u \in V_j$  and  $u' \in V_{j'}$ , we observe that  $v, a, b, u, u'$  induces a copy of  $K_4^+$ , a contradiction. It follows that  $\mathcal{A}$  contains at most one element.

Next, we prove that  $v$  is complete to  $V_i$ . Assume instead that  $v$  is mixed to  $V_i$ . If  $v$  is complete to some set  $V_\ell$  containing at least two vertices, then we obtain a contradiction with (1). It follows that all the elements of  $\mathcal{C}$  are singleton. By the definition of  $M$ , this implies that  $\mathcal{A}$  contains exactly one set  $V_j$ , which has size at least 4. Let  $c$  be a non-neighbor of  $v$  in  $V_i$ , and let  $d, d'$  be two vertices in  $V_j$ . Then  $v, a, b, c, d, d'$  is an induced subdivision of  $K_4^+$ , a contradiction. We proved that  $v$  is complete to  $V_i$ . Hence, every set

$V_j$  is either in  $\mathcal{A}$  or in  $\mathcal{C}$ . Since  $\mathcal{A}$  contains at most one element, the graph induced by  $M \cup \{v\}$  is a complete multipartite graph, with at least two partite sets containing at least 4 elements. This contradicts the maximality of  $M$ , and concludes the proof of (2).

Each connected component of  $G - M$  has at most one neighbor in each set  $V_i$ . (3)

Assume for the sake of contradiction that some connected component of  $G - M$  has at least two neighbors in some set  $V_i$ . Then there is a path  $P$  whose endpoints  $u, v$  are in  $V_i$ , and whose internal vertices are in  $R$ . Choose  $P, u, v, V_i$  such that  $P$  contains the least number of edges. Note that by (2),  $P$  contains at least 3 edges. Observe also that by the minimality of  $P$ , the only edges in  $G$  between  $V_i$  and the internal vertices of  $P$  are the first and last edge of  $P$ . Let  $V_j$  be a partite set of  $M$  with at least 4 elements, with  $j \neq i$  (this set exists, by the definition of  $M$ ). By (2) and the minimality of  $P$ , at most two vertices of  $V_j$  are adjacent to some internal vertex of  $P$ . Since  $V_j$  contains at least four vertices, there exist  $a, b \in V_j$  that are not adjacent to any internal vertex of  $P$ . If  $V_i$  has at least three elements then it contains a vertex  $w$  distinct from  $u, v$ . As  $w$  is not adjacent to any vertex of  $P$ , the vertices  $w, a, b$  together with  $P$  induce a subdivision of  $K_4^+$ , a contradiction. If  $V_i$  has at most two elements, then there must be an integer  $\ell$  distinct from  $i$  and  $j$  such that  $V_\ell$  has at least four elements. In particular,  $V_\ell$  contains a vertex  $c$  that is not adjacent to any internal vertex of  $P$ . As a consequence, the vertices  $a, c$  together with  $P$  induce a subdivision of  $K_4^+$ , which is again a contradiction. This proves (3).

Recall that we can assume that  $R$  is non-empty. An immediate consequence of (3) is that the neighborhood of each connected component of  $R$  is a clique. Since  $G$  is connected, it follows that it contains a *clique cutset*  $K$  (a clique whose deletion disconnects the graph). Let  $C$  be a connected component of  $G - K$ , let  $G_1 = G - C$ , and let  $G_2$  be the subgraph of  $G$  induced by  $C \cup K$ . It follows from the induction that there exist  $d$ -colorings of  $G_1$  and  $G_2$ . Furthermore, since  $K$  is a clique, we can assume that the colorings coincide on  $K$ . This implies that  $G$  is  $d$ -colorable and concludes the proof of Theorem 1. □

We remark that we could have used  $K_{3,3}$  instead of  $K_{4,4}$  in the proof, at the expense of a slightly more detailed analysis. The resulting bound on the chromatic number would have been  $\max(k, d(K_4^+, R(3, k)))$  instead of  $\max(k, d(K_4^+, R(4, k)))$ .

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