A compactness result for an equation with regular weight and boundary singularity on the annulus.

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A COMPACTNESS RESULT FOR AN EQUATION WITH REGULAR WEIGHT AND BOUNDARY SINGULARITY ON THE ANNULUS.

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ABSTRACT. We give a blow-up behavior for the solutions of an elliptic equation under some conditions. We also derive a compactness criterion for this equation.

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1. INTRODUCTION AND MAIN RESULTS

Let us consider the following operator:

$$L_\epsilon := \Delta + \epsilon \cdot \epsilon(x) < (x - x_0)|\nabla u_i >$$

We consider the following equation:

$$(P_\epsilon) \begin{cases} -\Delta u - \epsilon \cdot \epsilon(x) < x - x_0|\nabla u > = |x - x_0|^{2\beta} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

We denote by $C(1)$ and $C(1/2)$ the unit circle and the circle of radius $1/2$ respectively.

$$\Omega = A(0, 1/2, 1) \text{ is an annulus of center } 0 \text{ and radii } 1/2,\ 1,\ x_0 \in C(1/2)$$

and,

$$0 < \epsilon \to 0,\ \beta \in (0, +\infty),\ u \in W_0^{1,1}(\Omega),\ |x - x_0|^{2\beta} e^u \in L^1(\Omega),\ 0 \leq V \leq b.$$

We assume that:

Ω is an annulus of exterior circle the unit circle and interior circle the circle of radius $1/2$.

$\epsilon \equiv 1$ in a neighborhood of the unit circle $C(1)$ and $\epsilon \equiv -1$ in a neighborhood of the circle of radius $1/2$.

Important Remark: To give an example of a blow-up sequence of the previous type on the boundary. We use the counter-exemple of Brezis and Merle, this counterexample works if we replace the unit ball centered at $y_0 = (1, 0)$ by the annulus centered at $y_0 = (1, 0)$. After se use an inversion to have the blow-up point on the interior circle of the annulus. (do not forget to add the term $\epsilon \cdot \epsilon(x) < x - x_0|\nabla u_i >$ in the equation).

Here we present the result of Brezis-Merle in the regular case.

When $\epsilon = 0$ and $\beta = 0$, the previous equation was studied by many authors with or without the boundary condition, also for Riemann surfaces see [1-20] where one can find some existence and compactness results. Also we have a nice formulation in the sense of the distributions of this Problem in [7].

Among other results, we can see in [6] the following important Theorem,

Theorem A *(Brezis-Merle [6]) If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relative to the problem (P_0) with $\epsilon = 0$ and $\beta = 0$, and,*

$$0 < a \leq V_i \leq b < +\infty$$

then it holds, for all compact K of Ω:

$$\sup_K u_i \leq c,$$

with c depending on a, b, K and Ω.

We can find in [6] an interior estimate if we assume $a = 0$ but we need an assumption on the integral of e^{u_i}, namely:

Theorem B*(Brezis-Merle [6]).For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to the problem (P_0) ($\epsilon = \beta = 0$) with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,$$

then it holds, for all compact K of Ω:

$$\sup_K u_i \leq c,$$

with c depending on b, C, K and Ω.

The condition $\int_\Omega e^{u_i} dy \leq C$ is a necessary condition in the Problem (P_ϵ) as showed by the following counterexample for $\epsilon = 0$:

Theorem C *(Brezis-Merle [6]).There are two sequences $(u_i)_i$ and $(V_i)_i$ of the problem (P_0) ($\epsilon = \beta = 0$) with;*

$$0 \leq V_i \leq b < +\infty, \ \int_\Omega e^{u_i} dy \leq C,$$

such that,

$$\sup_\Omega u_i \to +\infty.$$

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument and they used Fatou's lemma and applied the maximum principle in $W_0^{1,1}(\Omega)$ which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

Note that for the problem (P_0) ($\epsilon = \beta = 0$), by using the Pohozaev identity, we can prove that $\int_\Omega e^{u_i}$ is uniformly bounded when $0 < a \leq V_i \leq b < +\infty$ and $||\nabla V_i||_{L^\infty} \leq A$ and Ω starshaped, when $a = 0$ and $\nabla \log V_i$ is uniformly bounded, we can bound uniformly $\int_\Omega V_i e^{u_i}$. In [17] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped in the case $a > 0$.

In [8] Chen-Li have proved that if $a = 0$ and $\int_\Omega e^{u_i}$ is uniformly bounded and $\nabla \log V_i$ is uniformly bounded then $(u_i)_i$ is bounded near the boundary and we have directly the compactness result for the problem (P_0). Ma-Wei in [17] extend this result in the case where $a > 0$.

When $\epsilon = \beta = 0$ and if we assume V more regular we can have another type of estimates called $\sup + \inf$ type inequalities. It was proved by Shafrir see [19] that, if $(u_i)_i, (V_i)_i$ are two sequences of

functions solutions of the Problem (P_0) without assumption on the boundary and $0 < a \leq V_i \leq b < +\infty$ then it holds:

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

We can see in [9] an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature see [9].

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A its Lipschitz constant then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin see [9]. Also have in [15], an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [16] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here the notion of isolated blow-up point is used. Also one can see in [10] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we give the behavior of the blow-up points on the boundary and a proof of a compactness result with Lipschitz condition. Note that our problem is an extension of the Brezis-Merle Problem.

The Brezis-Merle Problem (see [6]) is:

Problem. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i$. Also, we consider a sequence of solutions (u_i) of (P_0) relative to (V_i) such that,

$$\int_\Omega e^{u_i} dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C = C(b, C, V, \Omega)?$$

Here we give blow-up analysis on the boundary when V (similar to the prescribed curvature when $\epsilon = 0$) are nonegative and bounded, and on the other hand, if we add the assumption that these functions (similar to the prescribed cruvature) are uniformly Lipschitzian, we have a compactness of the solutions of the problem (P_ϵ) for ϵ small enough. (In particular we can take a sequence of ϵ_i tending to 0):

For the behavior of the blow-up points on the boundary, the following condition is sufficient,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary. But for the compactness of the solutions we add the following condition:

$$||\nabla V_i||_{L^\infty} \leq A_i \to 0.$$

Our main results are:

Theorem 1.1. *Assume that* $\max_\Omega u_i \to +\infty$*, where* (u_i) *are solutions of the probleme* (P_{ϵ_i}) *with:*

$$\beta \in (0, +\infty),\ 0 \leq V_i \leq b,\ \text{and}\ \int_\Omega |x - x_0|^{2\beta} e^{u_i} dx \leq C,$$

then, after passing to a subsequence, there is a function u*, there is a number* $N \in \mathbb{N}$ *and* N *points* $x_1, \ldots, x_N \in \partial\Omega$*, such that,*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j},\ \ \alpha_1 \geq 4\pi \text{ and } \alpha_j \geq 4\pi, \quad \text{in the sense of measures on } \partial\Omega.$$

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}),$$

We have the following compactness result for the problem (P_ϵ):

Theorem 1.2. *Assume that (u_i) are solutions of (P_{ϵ_i}) relative to (V_i) with $\beta \in (0, +\infty)$ and the following conditions:*

$$0 \leq V_i \leq b, \ ||\nabla V_i||_{L^\infty} \leq A_i \to 0 \text{ and } \int_\Omega |x - x_0|^{2\beta} e^{u_i} \leq C.$$

Then we have:

$$||u_i||_{L^\infty} \leq c(b, \beta, ((A)_i), x_0, C, \Omega),$$

2. Proof of the theorems

Proof of theorem 1.1:

First remark that:

$$\begin{cases} -\Delta u_i = \epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i) + |x - x_0|^{2\beta} V_i e^{u_i} \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

and,

$$u_i \in W^{1,1}_0(\Omega).$$

By the corollary 1 of Brezis-Merle see [6] we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding see [1] imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Also remark that, we have for two positive constants $C_q = C(q, \Omega)$ and $C_1 = C_1(\Omega)$ (see [7]) :

$$||\nabla u_i||_{L^q} \leq C_q ||\Delta u_i||_{L^1} \leq (C'_q + \epsilon C_1 ||\nabla u_i||_{L^1}), \ \forall \ i \text{ and } 1 < q < 2.$$

Thus, if $\epsilon > 0$ is small enough and by the Holder inequality, we have the following estimate:

$$||\nabla u_i||_{L^q} \leq C''_q, \ \forall \ i \text{ and } 1 < q < 2.$$

Step 1: interior estimate

First remark that, if we consider the following equation:

$$\begin{cases} -\Delta w_i = \epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i) \in L^q, \ 1 < q < 2 & \text{in } \Omega \subset \mathbb{R}^2, \\ w_i = 0 & \text{in } \partial\Omega. \end{cases}$$

If we consider v_i the Newtonnian potential of $\epsilon_i(x_1\partial_1 u_i + x_2\partial_2 u_i)$, we have:

$$v_i \in C^0(\bar{\Omega}), \ \ \Delta(w_i - v_i) = 0.$$

By the maximum principle $w_i - v_i \in C^0(\bar{\Omega})$ and thus $w_i \in C^0(\bar{\Omega})$.

Also we have by the elliptic estimates that $w_i \in W^{2,1+\epsilon} \subset L^\infty$, and we can write the equation of the Problem as:

$$\begin{cases} -\Delta(u_i - w_i) = \tilde{V}_i e^{u_i - w_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i - w_i = 0 & \text{in } \partial\Omega. \end{cases}$$

with,

$$0 \leq \tilde{V}_i = V_i e^{w_i} \leq \tilde{b}, \int_\Omega |x - x_0|^{2\beta} e^{u_i - w_i} \leq \tilde{C}.$$

We apply the Brezis-Merle theorem to $u_i - w_i$ to have:

$$u_i - w_i \in L^\infty_{loc}(\Omega),$$

and, thus:

$$u_i \in L^\infty_{loc}(\Omega).$$

Step2: boundary estimate

Set $\partial_\nu u_i$ the inner derivative of u_i. By the maximum principle $\partial_\nu u_i \geq 0$.

We have:

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C.$$

We have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \to \mu(\varphi), \ \ \forall \ \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. Set $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$.

We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{ on } \ I_\epsilon, \ \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{ outside } \ I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} -\Delta\tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0,2)$ or $B(x_0,2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, \text{ on } f(I_{(1/k_2)\epsilon}), \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \text{ outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla\eta_\epsilon||_{L^\infty(I_{(2/k_1)\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, \text{ on a the connected set } J_\epsilon = f(I_\epsilon), \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \text{ outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla\eta_\epsilon||_{L^\infty(J'_\epsilon)} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, because f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} -\Delta\bar{\eta}_\epsilon = -\Delta\eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \bar{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finaly we set $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C(||\eta_\epsilon||_{L^\infty} + ||\nabla\eta_\epsilon||_{L^\infty} + ||\Delta\eta_\epsilon||_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω.

As we said in the beguening, see also [3, 7, 13, 20], we have:

$$||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega |x-x_0|^{2\beta} e^u < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} -\Delta u = |x-x_0|^{2\beta} V e^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [6], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have, $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$.

We denote by $f \cdot g$ the inner product of any two vectors f and g of $\mathbb{R}^2$.

We can write,

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = |x-x_0|^{2\beta}(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u)\cdot\nabla\tilde{\eta}_\epsilon + \epsilon_i(\nabla u_i \cdot (x-x_0))\tilde{\eta}_\epsilon. \quad (1)$$

We use the interior esimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u, we obtain,

$$\int_\Omega |x-x_0|^{2\beta} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C\epsilon = O(\epsilon) \quad (2)$$

We have,

$$\begin{cases} -\Delta u_i - \epsilon_i \nabla u_i \cdot (x-x_0) = |x-x_0|^{2\beta} V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\begin{aligned} \int_\Omega |x-x_0|^{2\beta} V_i e^{u_i} \tilde{\eta}_\epsilon dx &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma - \epsilon_i \int_\Omega (\nabla u_i \cdot (x-x_0))\tilde{\eta}_\epsilon = \\ &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma + o(1) \to \mu(\eta_\epsilon) \leq \mu(J'_\epsilon) \leq 4\pi - \epsilon_0, \ \ \epsilon_0 > 0 \end{aligned} \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is i_0 such that, for $i \geq i_0$,

$$\int_\Omega ||x-x_0|^{2\beta}(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2.1: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is an hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \leq meas(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$.

Remark: for the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0,1-\epsilon^3)$.

(Proof of this fact; let's consider $d(x,\partial\Omega) = d(x,z_0), z_0 \in \partial\Omega$, this imply that $(d(x,z_0))^2 \leq (d(x,z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z-z_0)\cdot(2x-z-z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t)-\gamma(t_0)\cdot(2x-\gamma(t)-\gamma(t_0)) \leq 0$ if we divide by $(t-t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0)\cdot(x-\gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0))

With this fact, we can say that $S = \{x, d(x,\partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega,\ -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z,4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z,\epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k,\epsilon_k)), k = 1,...,m$, by the area formula the measure of $S \cap B(z_k,\epsilon_k)$ is less than a $k\epsilon$ (a ϵ-rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary).

We write,

$$\int_\Omega |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx + \int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx. \tag{5}$$

Step 2.1.1: Estimate of $\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by the Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_\Omega |\nabla u_i| f dx \to \int_\Omega |\nabla u| f dx$$

If we take $f = 1_{\Omega-\Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0\ \exists\ i_1 = i_1(\epsilon) \in \mathbb{N},\ \ i \geq i_1,\ \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq meas(\Omega-\Omega_{\epsilon^3})||\nabla u||_{L^\infty} + \epsilon^3 = \epsilon^3(k_1||\nabla u||_{L^\infty}+1) = O(\epsilon^3).$$

Thus, we obtain,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty}+1) = O(\epsilon) \tag{6}$$

The constant C_1 does not depend on ϵ but on Ω.

Step 2.1.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon|dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3.$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq ||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} ||\nabla\tilde{\eta}_\epsilon||_{L^\infty} = C_1\epsilon = O(\epsilon) \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq i'$,

$$\int_\Omega |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2) = O(\epsilon) \tag{7}$$

From (4) and (7), we have, for $\epsilon > 0$, there is i'' such that, $i \geq i''$,

$$\int_\Omega |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^\infty} + 2 + C) = 4\pi - \epsilon_0 + O(\epsilon) \tag{8}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use Theorem 1 of [6] to conclude that there are $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_\Omega e^{q|u_i - u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighberhooh of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_0 > 0, q_0 > 1$ such that:

$$\int_{B(x_0,\epsilon_0)} e^{q_0 u_i} dx \leq C, \ \forall \ i.$$

By the elliptic estimate see [14] we have:

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \ \forall \ i.$$

We have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded in $C^{1,\theta}$, $(\theta > 0)$ on $\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

Proof of theorem 1.2:

We have:

$$u_i \in C^{2,\epsilon}(\bar{\Omega}),$$

Thus, we can use integration by parts. The Pohozaev identity gives locally around each blow-up :

a) When x is in the neighborhood of the unit circle, the Pohozaev identity gives

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \tag{9}$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx \to \int_{\partial\Omega} \partial_\nu u \varphi dx, \ \ \forall \ \varphi \in L^2(\partial\Omega),$$

Thus, $\alpha_j = 0$, $j = 1, \ldots, N$ and (u_i) is uniformly bounded.

b) When x is in the neighborhood of x_0 or $y_0 \in C(1/2)$, and the circle of radius $1/2$, we use again the Pohozaev identity, by multplying by $< x - x_0|\nabla u_i >$. Here we use $\nabla V_i \to 0$ (becasue we do not multiply by $< x|\nabla u_i >$ but by $< x - x_0|\nabla u_i >$) and the radius ϵ of the neighborhood, $\epsilon \to 0$ in the Pohozaev identity.

Indeed, we have,

$$-\Delta u_i - \epsilon \cdot \epsilon(x) < x - x_0|\nabla u_i >= |x - x_0|^{2\beta} V e^{u_i} \tag{10}$$

Let Ω^1_ϵ be a neighborhood of $C(1)$ or $C(1/2)$; first, we have $\epsilon \equiv 1$ in a neighborhood of $C(1)$, we multiply by $< x - x_0|\nabla u_i >$ the previous equation and we integrate by parts, we obtain around the blow-up y_0, and,

also, we consider blow-up from $C(1/2)$, in the neighborhood of $C(1/2)$, $\epsilon \equiv -1$, we multiply the equation of u_i, (10), by $< x - x_0|\nabla u_i >$ and we integrate by parts,

Thus,

$$\int -\Delta u_i(< x - x_0|\nabla u_i >) - \int \epsilon \cdot \epsilon(x)(< x - x_0|\nabla u_i >)^2 = \int |x - x_0|^{2\beta} V_i(x) < x - x_0|\nabla(e^{u_i}) >$$

We have:

$$\int_{\Omega^1_\epsilon} \Delta u_i(< x - x_0|\nabla u_i >) = \int_{\partial^+\Omega^1_\epsilon} < x - x_0|\nabla u_i >< \nu|\nabla u_i > - \frac{< x - x_0|\nu >}{2}|\nabla u_i|^2 +$$

$$+ \int_{\partial\Omega} \frac{1}{2} < x - x_0|\nu > (\partial_\nu u_i)^2 d\sigma,$$

and,

$$-\int |x-x_0|^{2\beta}V_i(x) < x-x_0|\nabla(e^{u_i}) >= 2(1+\beta)\int |x-x_0|^{2\beta}V_i(x)e^{u_i}+$$

$$+\int |x-x_0|^{2\beta} < x-x_0|\nabla V_i > e^{u_i} - \int_{\partial\Omega^1_\epsilon} |x-x_0|^{2\beta}V_i(x) < x-x_0|\nu > e^{u_i}.$$

We have:

$$\int_{C(1)\cap B_\epsilon(y_0)} \frac{1}{2}(||x||^2 - < x_0|x >)(\partial_\nu u_i)^2 d\sigma + \int_{C(1)\cap B_\epsilon(y_0)} (||x||^2 - < x_0|x >)|x-x_0|^{2\beta}V_i +$$

$$+\int_{\Omega^1_\epsilon} (< x-x_0|\nabla u_i >)^2 dx = O(\epsilon) + O(1),$$

but, on $C(1)$, $\nu = x$, and, $||x_0|| = 1/2$ and $||x|| = 1$, thus $||x||^2 - < x_0|x > \geq 1/2$, and thus:

$$\int_{C(1)} (\partial_\nu u_i)^2 \leq C$$

Now, we consider blow-up from $C(1/2)$, in the neighborhood of $C(1/2)$, $\epsilon \equiv -1$, we multiply the equation of u_i, (10), by $< x-x_0|\nabla u_i >$ and we integrate by parts, we obtain (here $\nu = -2x$ on $C(1/2)$):

$$2(1+\beta)\int_{\Omega^1_\epsilon} |x-x_0|^{2\beta}V_i e^{u_i} + \int_{C(1/2)\cap B_\epsilon(y_0)} -(-||x||^2 + < x_0|x >)(\partial_\nu u_i)^2 d\sigma +$$

$$+\int_{C(1/2)\cap B_\epsilon(y_0)} -2(-||x||^2 + < x_0|x >)|x-x_0|^{2\beta}V_i + \int_{\Omega^1_\epsilon} (< x-x_0|\nabla u_i >)^2 dx =$$

$$= \left| \int_{\Omega^1_\epsilon} < x-x_0|\nabla V_i > |x-x_0|^{2\beta} e^{u_i} \right| + O(\epsilon).$$

The previous left hand side is non-negative because $||x_0|| = 1/2$ and $||x|| = 1/2$ and $| < x_0|x > | \leq ||x_0|| \times ||x|| = ||x||^2$.

We tend $i \to +\infty$ and then $\epsilon \to 0$, $\nabla V_i \to 0$, to obtain:

$$\lim_{\epsilon\to 0}\lim_{i\to+\infty} \int_{\Omega^1_\epsilon} |x-x_0|^{2\beta}V_i e^{u_i} = 0,$$

however:

$$\int_{\Omega^1_\epsilon} |x-x_0|^{2\beta}V_i e^{u_i} dx = \int_{\partial\Omega^1_\epsilon} \partial_\nu u_i d\sigma + O(\epsilon) + o(1) \to \alpha_1 > 0,$$

it is a contradiction.

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