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LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF RANDOM WALKS IN RANDOM SCENERY

FRANÇOISE PÈNE

ABSTRACT. We study the asymptotic behaviour of additive functionals of random walks in random scenery. We establish bounds for the moments of the local time of the Kesten and Spitzer process. These bounds combined with a previous moment convergence result (and an ergodicity result) imply the convergence in distribution of additive observables (with a normalization in $n^{\frac{1}{4}}$). When the sum of the observable is null, the previous limit vanishes and we prove the convergence in the sense of moments (with a normalization in $n^{\frac{1}{8}}$).

1. INTRODUCTION

1.1. Description of the model and of some earlier results. We consider two independent sequences $(X_k)_{k \geq 1}$ (the increments of the random walk) and $(\xi_y)_{y \in \mathbb{Z}}$ (the random scenery) of independent identically distributed \mathbb{Z} -valued random variables. We assume in this paper that X_1 is centered and admits finite moments of all orders, and that its support generates the group \mathbb{Z} . We define the random walk $(S_n)_{n \geq 0}$ as follows

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{i=1}^n X_i \quad \text{for all } n \geq 1.$$

We assume that ξ_0 is centered, that its support generates the group \mathbb{Z} , and that it admits a finite second moment $\sigma_\xi^2 := \mathbb{E}[\xi_0^2] > 0$. The random walk in random scenery (RWRS) is the process defined as follows

$$Z_n := \sum_{k=0}^{n-1} \xi_{S_k} = \sum_{y \in \mathbb{Z}} \xi_y N_n(y), \quad (1)$$

where we set $N_n(y) = \#\{k = 0, \dots, n-1 : S_k = y\}$ for the local time of S at position y before time n . This process first studied by Borodin [7] and Kesten and Spitzer [32] describes the evolution of the total amount won until time n by a particle moving with respect to the random walk S , starting with a null amount at time 0 and winning the amount ξ_ℓ at each time the particle hits the position $\ell \in \mathbb{Z}$. This process is a natural example of (strongly) stationary process with long time dependence. Due to the first works by Borodin [7] and by Kesten and Spitzer [32], we know that $(n^{-\frac{3}{4}} Z_{\lfloor nt \rfloor})_t$ converges in distribution, as n goes to infinity, to the so-called Kesten and Spitzer process $(\sigma_\xi \Delta_t, t \geq 0)$, where Δ is defined by

$$\Delta_t := \int_{-\infty}^{+\infty} L_t(x) d\beta_x, \quad (2)$$

with $(\beta_x)_{x \in \mathbb{R}}$ a Brownian motion and $(L_t(x), t \geq 0, x \in \mathbb{R})$ a jointly continuous in t and x version of the local time process of a standard Brownian motion $(B_t)_{t \geq 0}$, where $((B_t)_t, (\beta_s)_s)$ is

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the limit in distribution of $n^{-\frac{1}{2}}((S_{\lfloor nt \rfloor})_t, (\sigma_\xi^{-1} \sum_{k=1}^{\lfloor ns \rfloor} \xi_k)_s)$ as $n \rightarrow +\infty$. Observe that Δ is the continuous time analog of the random walk in random scenery. To be convinced of this fact, one may compare the right hand side of (1) with (2). The process Δ is a classical and nice example of a (strongly) stationary process, self-similar with dependent (strongly) stationary increments and exhibiting long time dependence.

In [7], Borodin established the convergence in distribution of Z_n when X and ξ have second order moments. Kesten and Spitzer established in [32] a functional limit theorem when the distributions of X and ξ belong to the domain of attraction of stable distributions with respective parameters $\alpha \neq 1$ and $\beta \in (0, 2]$. Limit theorems have been extended by Bolthausen [6] (for the case $\alpha = \beta = 2$ for random walks of dimension $d = 2$), by Deligiannidis and Utev [19] ($\alpha = d \in \{1, 2\}$, $\beta = 2$, providing some correction to [6]) and by Castell, Guillin-Plantard and the author [12] (when $\alpha \leq d$ and $\beta < 2$), completing the picture for the convergence in the sense of distribution and for the functional limit theorem (except in the case $\alpha \leq 1$ and $\beta = 1$). Since the seminal works by Borodin and by Kesten and Spitzer, random walks in random scenery and the Kesten and Spitzer process Δ have been the object of various studies (let us mention for example [33, 50, 29, 3, 27, 25, 28, 2]).

Random walks in random scenery are related to other models, such as the Matheron and de Marsily Model [39] of transport in porous media, the transience of which has been established by Campanino and Petritis [11] and which has many generalizations (e.g. [26, 20, 23, 10, 9]), and such as the Lorentz-Lévy process (see [40] for a short presentation of some models linked to random walks in random scenery).

Random walks in random scenery constitute also a model of interest in the context of dynamical systems. They correspond indeed to Birkhoff sums of a transformation called the T, T^{-1} transformation appearing in [49, p. 682, Problem 2] where it was asked whether this Kolmogorov automorphism is Bernoulli or not. In [30], Kalikow answered negatively this question by proving that this transformation is not even loosely Bernoulli.

1.2. Main results. Before stating our main results, let us introduce some additional notations. Let $d \in \mathbb{N}$ be the greatest common divisor of the set $\{x \in \mathbb{Z}, \mathbb{P}(\xi_0 - \xi_1 = x) > 0\}$ and $\alpha \in \mathbb{Z}$ such that $\mathbb{P}(\xi_0 = \alpha) > 0$. This means that the random variables ξ_ℓ take almost surely their values in $\alpha + d\mathbb{Z}$ and that d is largest positive integer satisfying this property. Since the support of ξ generates the group \mathbb{Z} , necessarily α and d are coprime. Recall that the quantity d can be also simply characterized using the common characteristic function φ_ξ of the ξ_ℓ .¹

In the present paper we are interested in the asymptotic behaviour of additive functionals of the RWRS $(Z_n)_{n \geq 1}$ that is of quantities of the following form:

$$\mathcal{Z}_n := \sum_{k=1}^n f(Z_k)$$

where $f : \mathbb{Z} \rightarrow \mathbb{R}$ is absolutely summable. This quantity is strongly related to the local time \mathcal{N}_n of the RWRS Z , which is defined by

$$\mathcal{N}_n(a) = \#\{k = 1, \dots, n : Z_k = a\}.$$

Indeed if $f = \mathbf{1}_0$, then $\mathcal{Z}_n = \mathcal{N}_n(0)$ and if $f = \mathbf{1}_0 - \mathbf{1}_1$, then $\mathcal{Z}_n = \mathcal{N}_n(0) - \mathcal{N}_n(1)$. In the general case, \mathcal{Z}_n can be rewritten

$$\mathcal{Z}_n := \sum_{a \in \mathbb{Z}} f(a) \mathcal{N}_n(a).$$

¹Indeed $d \geq 1$ is such that $\{u : |\varphi_\xi(u)| = 1\} = (2\pi/d)\mathbb{Z}$ and a.s. $e^{\frac{2i\pi\xi}{d}} = e^{\frac{2i\pi\alpha}{d}}$ which is a primitive d -th root of the unity.

The asymptotic behaviour of $(\mathcal{N}_n(0))_n$ has been studied by Castell, Guillin-Plantard, Schapira and the author in [14, Corollary 6], in which it has been proved that the moments of $(n^{-\frac{1}{4}}\mathcal{N}_n(0))_{n \geq 1}$ converge to those of the local time $\mathcal{L}_1(0)$ at position 0 and until time 1 of the process Δ . The proof of this result was based on a multitime local limit theorem [14, Theorem 5] extending a local limit theorem contained in [13] and on the finiteness of the moments of $\mathcal{L}_1(0)$ (which was a delicate question). We complete this previous work by establishing in Section 2 the following bounds for the moments of $\mathcal{L}_1(0)$.

Theorem 1. *For any $\eta_0 > 0$, there exists $\mathfrak{a} > 0$ and $C > 0$ such that*

$$(Cm)^{\frac{3m}{4}} \leq \mathbb{E}[(\mathcal{L}_1(0))^m] = \mathcal{O}\left(\frac{\mathfrak{a}^m (m!)^{\frac{3}{2} + \eta_0}}{\Gamma(\frac{m}{4} + 1)}\right) \leq \mathcal{O}\left(m^{m(\frac{5}{4} + 2\eta_0)}\right).$$

Even if it uses some ideas that already existed in [14], the proof of Theorem 1 (given in Section 2) is different in many aspects. It requires indeed much more precise estimates which changes in the approach of the control of the moments. The proof of Theorem 1 relies on several auxiliary results. We summarize quickly its strategy. We will prove (see (9) coming from [14] and (10)) that

$$\mathbb{E}[(\mathcal{L}_1(0))^m] = \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \int_{0 < t_1 < \dots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} \mathbb{E}\left[\prod_{k=0}^{m-1} \left(d(L^{(k+1)}, W_k)\right)^{-1}\right] dt_1 \dots dt_m,$$

where we set $W_k := Vect(L^{(1)}, \dots, L^{(k)})$ and $L^{(k+1)} := (L_{t_{k+1}} - L_{t_k}) / (t_{k+1} - t_k)^{\frac{3}{4}}$ (normalized so that $|L^{(m)}|_{L^2(\mathbb{R})}$ has the same distribution as $|L_1|_{L^2(\mathbb{R})}$). We will prove, in Lemma 7, that

$$\exists c, C > 0, \quad m! \int_{0 < t_1 < \dots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} dt_1 \dots dt_m \sim c(Cm)^{\frac{3m}{4}},$$

as $m \rightarrow +\infty$ and, in Lemma 6, that

$$\left(\mathbb{E}\left[|L_1|_{L^2(\mathbb{R})}^{-1}\right]\right)^m \leq \mathbb{E}\left[\prod_{k=0}^{m-1} \left(d(L^{(k+1)}, W_k)\right)^{-1}\right] \leq \prod_{k=0}^{m-1} \left(\sup_{V \in \mathcal{V}_k} \mathbb{E}\left[(d(L_1, V))^{-1}\right]\right), \quad (3)$$

where $d(\cdot, \cdot)$ is the distance associated with the L^2 -norm on $L^2(\mathbb{R})$ and where \mathcal{V}_k is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most k . Theorem 1 will then follow from the next self-interesting estimate on the local time L_1 of the Brownian motion B up to time 1.

Theorem 2.

$$\sup_{V \in \mathcal{V}_k} \mathbb{E}\left[(d(L_1, V))^{-1}\right] = k^{\frac{1}{2} + o(1)}, \quad \text{as } k \rightarrow +\infty. \quad (4)$$

Now we use the following classical argument for positive random variables. The upper bound provided by Theorem 1 allows us to prove that the Carleman's criterion is satisfied for $\mathcal{E}\sqrt{\mathcal{L}_1(0)}$ where \mathcal{E} is a centered Rademacher distribution independent of $\mathcal{L}_1(0)$ and of Z , indeed:

$$\sum_{m \geq 1} \mathbb{E}[(\mathcal{L}_1(0))^m]^{-\frac{1}{2m}} \geq c_1 \sum_{m \geq 1} m^{-\frac{5}{8} - \eta_0} = \infty,$$

for every $\eta_0 \in (0, \frac{3}{8})$. This enables us to deduce from [14, Corollary 6] that $n^{-\frac{1}{8}}\mathcal{E}\sqrt{\mathcal{N}_n(0)}$ converges in distribution to $\mathcal{E}\sqrt{\sigma_\xi^{-1}\mathcal{L}_1(0)}$ and so that

$$n^{-\frac{1}{4}}\mathcal{N}_n(0) \xrightarrow{\mathcal{L}} \sigma_\xi^{-1}\mathcal{L}_1(0), \quad \text{as } n \rightarrow +\infty, \quad (5)$$

where $\xrightarrow{\mathcal{L}}$ means convergence in distribution. This convergence in distribution is extended to more general observables.

Theorem 3. *Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be such that $\sum_{a \in \mathbb{Z}} |f(a)| < \infty$. Then $n^{-\frac{1}{4}} \sum_{k=0}^{n-1} f(Z_k)$ converges in distribution and in the sense of moments to $\sum_{a \in \mathbb{Z}} f(a) \sigma_\xi^{-1} \mathcal{L}_1(0)$.*

The proof of the moments convergence in Theorem 3 is a straightforward adaptation of [14] and is given in Appendix B. Due to Theorem 1 and to the above argument that lead to (5), the convergence in distribution in Theorem 3 is a consequence of the moments convergence. Another strategy to prove the convergence in distribution in Theorem 3 consists in seeing this result as a direct consequence of (5) combined with Proposition 13 stating the ergodicity of the dynamical system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ corresponding to

$$\tilde{T}^k((X_{m+1})_{m \in \mathbb{Z}}, (\xi_m)_{m \in \mathbb{Z}}, Z_0) = ((X_{k+m+1})_{m \in \mathbb{Z}}, (\xi_{m+S_k})_{m \in \mathbb{Z}}, Z_k).$$

This dynamical system preserves the infinite measure $\tilde{\mu} := \mathbb{P}_{X_1}^{\otimes \mathbb{Z}} \otimes \mathbb{P}_{\xi_0}^{\otimes \mathbb{Z}} \otimes \lambda_{\mathbb{Z}}$, where $\lambda_{\mathbb{Z}}$ is the counting measure on \mathbb{Z} . Actually, thanks to (5) and to the recurrence ergodicity of $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$, we prove the following stronger version of the convergence in distribution of Theorem 3.

Theorem 4. *For any $\tilde{\mu}$ -integrable function $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{R}$,*

$$n^{-\frac{1}{4}} \sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k \xrightarrow{\mathcal{L}(\tilde{\mu})} \frac{\int_{\tilde{\Omega}} \tilde{f} d\tilde{\mu}}{\sigma_\xi} \mathcal{L}_1(0), \quad \text{as } n \rightarrow +\infty,$$

where $\xrightarrow{\mathcal{L}(\tilde{\mu})}$ means convergence in distribution with respect to any probability measure absolutely continuous with respect to $\tilde{\mu}$.

Theorem 3 can be seen as weak law of large numbers, with a non constant limit. When $\sum_{a \in \mathbb{Z}} f(a) = 0$, the limit given by Theorem 3 vanishes, but then the next result provides a limit theorem for $\mathcal{Z}_n = \sum_{k=0}^{n-1} f(Z_k)$ with another normalization. This second result corresponds to a central limit theorem for additive functionals of RWRS. Let us indicate that, contrarily to the moments convergence in Theorem 3, the next result is not an easy adaptation of [14], even if its proof (given in Section 4) uses the same initial idea (computation of moments using the local limit theorem) and, at the beginning, some estimates established in [13, 14]. Indeed, important technical difficulties arise from the cancellations coming from the fact that $\sum_{a \in \mathbb{Z}} f(a) = 0$.

Theorem 5. *Assume moreover that there exists some $\kappa \in (0, 1]$ such that ξ_0 admits a moment of order $2 + \kappa$. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be such that $\sum_{a \in \mathbb{Z}} (1 + |a|) |f(a)| < \infty$ and that $\sum_{a \in \mathbb{Z}} f(a) = 0$. Then*

$$\sum_{\ell \in \mathbb{Z}} \left| \sum_{\ell'=0}^{d-1} \sum_{a, b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{|\ell'+d\ell|} = a - b) \right| < \infty.$$

Moreover all the moments of $\left(n^{-\frac{1}{8}} \sum_{k=0}^{n-1} f(Z_k) \right)_n$ converges to those of $\sqrt{\frac{\sigma_f^2}{\sigma_\xi}} \mathcal{L}_1(0) \mathcal{N}$, where \mathcal{N} is a standard Gaussian random variable independent of $\mathcal{L}_1(0)$ and where

$$\sigma_f^2 := \sum_{k \in \mathbb{Z}} \sum_{a, b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{|k|} = a - b). \quad (6)$$

In particular, for any $a \in \mathbb{Z}$, $\left(n^{-\frac{1}{8}} (\mathcal{N}_n(a) - \mathcal{N}_n(0)) \right)_n$ converges in the sense of moments to $\sqrt{\frac{\sigma_{0,a}^2}{\sigma_\xi}} \mathcal{L}_1(0) \mathcal{N}$, with $\sigma_{0,a}^2 := \sum_{k \in \mathbb{Z}} [2\mathbb{P}(Z_{|k|} = 0) - \mathbb{P}(Z_{|k|} = a) - \mathbb{P}(Z_{|k|} = -a)]$.

Let us point out the similarity between these results and the classical Law of Large Numbers and Central Limit Theorem for sums of square integrable independent and identically distributed

random variables. Indeed Theorems 3 and 5 establish convergence results of the respective following forms

$$\frac{1}{a_n} \sum_{k=1}^n Y_k \rightarrow I(Y_1)\mathcal{Y} \quad \text{and} \quad \frac{1}{\sqrt{a_n}} \sum_{k=1}^n (Y_k - I(Y_1)Y_k^0) \rightarrow \sqrt{\sigma_Y^2} \mathcal{Y} \mathcal{Z}$$

as $n \rightarrow +\infty$, with $a_n \rightarrow +\infty$, I an integral (with respect to the counting measure on \mathbb{Z}) and Y_k^0 a reference random variable with integral 1 (e.g. $Y_k^0 = \mathbf{1}_0(Z_k)$, note that we cannot take $Y_k^0 = 1$ since it is not integrable with respect to the counting measure on \mathbb{Z}).

The summation order in the expression (6) of σ_f^2 is important. Indeed recall that $\mathbb{P}(Z_k = 0)$ has order $k^{-\frac{3}{4}}$ and so is not summable. The sum $\sum_{k \in \mathbb{Z}}$ appearing in (6) is a priori non absolutely convergent if $d \neq 1$. Indeed, considering for example that ξ_0 is a centered Rademacher random variable (i.e. $\mathbb{P}(\xi_0 = 1) = \mathbb{P}(\xi_0 = -1) = \frac{1}{2}$) and that $f = \mathbf{1}_0 - \mathbf{1}_1$, then, for any $k \geq 0$,

$$\sum_{a,b \in \mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|2k|} = a - b) = \mathbb{P}(Z_{|2k|} = 0 - 0) + \mathbb{P}(Z_{|2k|} = 1 - 1) = 2\mathbb{P}(Z_{|2k|} = 0)$$

and

$$\sum_{a,b \in \mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|2k+1|} = a - b) = -\mathbb{P}(Z_{|2k+1|} = 0 - 1) - \mathbb{P}(Z_{|2k+1|} = 1 - 0) = -\mathbb{P}(|Z_{|2k+1|}| = 1).$$

But, σ_f^2 corresponds to the following sum of an absolutely convergent series (in k):

$$\sigma_f^2 = \sum_{k \in \mathbb{Z}} \left(\sum_{\ell'=0}^{d-1} \sum_{a,b \in \mathbb{Z}^2} f(a)f(b)\mathbb{P}(Z_{|\ell'+dk|} = a - b) \right).$$

Finally, let us point out that σ_f^2 defined in (6) corresponds to the Green-Kubo formula, well-known to appear in central limit theorems for probability preserving dynamical systems (see Remark 14 at the end of Section 3).

Let us indicate that results similar to Theorem 5 exist for one-dimensional random walks, that is when the RWRS $(Z_n)_{n \geq 1}$ is replaced by the RW $(S_n)_{n \geq 1}$, with other normalizations and with an exponential random variable instead of $\mathcal{L}_1(0)$. Such results have been obtained by Dobrušin [21], Kesten in [31] and by Csáki and Földes in [17, 18]. The idea used therein was to construct a coupling using the fact that the times between successive return times of $(S_n)_{n \geq 1}$ to 0 are i.i.d., as well as the partial sum of the $f(S_k)$ between these return times to 0 and that these random variables have regularly varying tail distributions. This idea has been adapted to dynamical contexts by Thomine [47, 48]. Still in dynamical contexts, another approach based on moments has been developed in [41, 42] in parallel to the coupling method. This second method based on local limit theorem is well tailored to treat non-markovian situations, such as RWRS. Indeed, recall that the RWRS $(Z_n)_{n \geq 1}$ is (strongly) stationary but far to be not markovian (for example it has been proved in [14] that $Z_{n+m} - Z_n$ is more likely to be 0 if we know that $Z_n = 0$) and even more intricate conditionally to the scenery (it has been proved in [25] that the RWRS does not converge knowing the scenery). Luckily local limit theorem type estimates enables to prove moments convergence. But unfortunately Theorem 1 is not enough to conclude the convergence in distribution via Carleman's criterion.

The paper is organized as follows. In Section 2, we prove Theorem 1 (bounds on moments of the local time of the Kesten Spitzer process) and Theorem 2 (estimate on the distance in $L^2(\mathbb{R})$ between the local time of a Brownian motion and a k -dimensional vector space). In Section 3, we establish the recurrence ergodicity of the infinite measure preserving dynamical system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ and obtain the convergence in distribution of Theorem 3 (Law of Large Numbers) as a byproduct

of this recurrence ergodicity combined with (5). Section 3 is completed by Appendix B which contains the proof of the moments convergence of Theorem 3. In Section 4 (completed with Appendix A), we prove Theorem 5 (Central Limit Theorem).

2. UPPER BOUND FOR MOMENTS: PROOF OF THEOREM 1

This section is devoted to the study of the behaviour of $\mathbb{E}[(\mathcal{L}_1(0))^m]$ as $m \rightarrow +\infty$. It has been proved in [14] that this quantity is finite, but the estimate established therein was not enough to apply the Carleman criterion. The proof of Theorem 1 requires a much more delicate study, even if it uses some estimates used in [14]. We start by establishing bounds for $\mathbb{E}[(\mathcal{L}_1(0))^m]$.

Lemma 6.

$$\left(\mathbb{E} \left[|L_1|_{L^2(\mathbb{R})}^{-1} \right] \right)^m \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \int_{0 < t_1 < \dots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} dt_1 \dots dt_m \leq \mathbb{E}[(\mathcal{L}_1(0))^m] \quad (7)$$

and

$$\mathbb{E}[(\mathcal{L}_1(0))^m] \leq \prod_{j=0}^{m-1} \left(\sup_{V \in \mathcal{V}_k} \mathbb{E} \left[(d(L_1, V))^{-1} \right] \right) \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \int_{0 < t_1 < \dots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} dt_1 \dots dt_m, \quad (8)$$

where $d(f, g) = \|f - g\|_{L^2(\mathbb{R})}$ and where \mathcal{V}_k is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most k .

Proof. Recall that it has been proved in [14, Theorem 3] that

$$\mathbb{E}[(\mathcal{L}_1(0))^m] = \frac{m!}{(2\pi\sigma_\xi^2)^{\frac{m}{2}}} \int_{0 < t_1 < \dots < t_m < 1} \mathbb{E}[(\det \mathcal{D}_{t_1, \dots, t_m})^{-\frac{1}{2}}] dt_1 \dots dt_m, \quad (9)$$

with $\mathcal{D}_{t_1, \dots, t_m} := \left(\int_{\mathbb{R}} L_{t_i}(x) L_{t_j}(x) dx \right)_{i, j=1, \dots, m}$ where $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$ is the local time of the Brownian motion B . Since $\det \mathcal{D}_{t_1, \dots, t_m}$ is a Gram determinant, we have the iterative relation

$$\det \mathcal{D}_{t_1, \dots, t_{m+1}}^{\frac{1}{2}} = \det \mathcal{D}_{t_1, \dots, t_m}^{\frac{1}{2}} d(L_{t_{m+1}}, \text{Vect}(L_{t_1}, \dots, L_{t_m})),$$

where $d(f, g) = \|f - g\|_{L^2(\mathbb{R})}$ and where $\text{Vect}(L_{t_1}, \dots, L_{t_m})$ is the sublinear space of $L^2(\mathbb{R})$ generated by L_{t_1}, \dots, L_{t_m} . It follows that

$$\det \mathcal{D}_{t_1, \dots, t_m}^{-\frac{1}{2}} = \prod_{k=0}^{m-1} (d(L_{t_{k+1}}, \text{Vect}(L_{t_1}, \dots, L_{t_k})))^{-1}. \quad (10)$$

But, for any $m \geq 1$ and any $0 < t_1 < \dots < t_{m+1} < 1$ and any $k = 0, \dots, m-1$,

$$\begin{aligned} & \mathbb{E} \left[d(L_{t_{k+1}}, \text{Vect}(L_{t_1}, \dots, L_{t_k}))^{-1} \middle| (B_s)_{s \leq t_k} \right] \\ &= \mathbb{E} \left[d(L_{t_{k+1}} - L_{t_k}, \text{Vect}(L_{t_1}, \dots, L_{t_k}))^{-1} \middle| (B_s)_{s \leq t_k} \right] \\ &= \mathbb{E} \left[d((L_{t_{k+1}} - L_{t_k})(B_{t_k} + \cdot), \text{Vect}(L_{t_1}(B_{t_k} + \cdot), \dots, L_{t_k}(B_{t_k} + \cdot)))^{-1} \middle| (B_s)_{s \leq t_k} \right]. \end{aligned}$$

Therefore

$$\mathbb{E} \left[|L_{t_{k+1}} - L_{t_k}|_{L^2(\mathbb{R})}^{-1} \right] \leq \mathbb{E} \left[d(L_{t_{k+1}}, \text{Vect}(L_{t_1}, \dots, L_{t_k}))^{-1} \middle| (B_s)_{s \leq t_k} \right] \quad (11)$$

and

$$\mathbb{E} \left[d(L_{t_{k+1}}, \text{Vect}(L_{t_1}, \dots, L_{t_k}))^{-1} \middle| (B_s)_{s \leq t_k} \right] \leq \sup_{V \in \mathcal{V}_k} \mathbb{E} \left[d((L_{t_{k+1}} - L_{t_k})(B_{t_k} + \cdot), V)^{-1} \right], \quad (12)$$

where \mathcal{V}_k is the set of linear subspaces of dimension at most k of $L^2(\mathbb{R})$ and where we used the independence of $(L_{t_{k+1}} - L_{t_k})(B_{t_k} + \cdot)$ with respect to $(B_s)_{s \leq t_k}$ and the fact that $(L_{t_1}(B_{t_k} + \cdot), \dots, L_{t_k}(B_{t_k} + \cdot))$ is measurable with respect to $(B_s)_{s \leq t_k}$. Thus, by induction and using the fact that the increments of B are (strongly) stationary, it follows from (10) and (12) that

$$\begin{aligned} \prod_{k=0}^{m-1} \mathbb{E} \left[|L_{t_{k+1}} - L_{t_k}|_{L^2(\mathbb{R})}^{-1} \right] &\leq \mathbb{E} \left[\det \mathcal{D}_{t_1, \dots, t_m}^{-\frac{1}{2}} \right] \leq \prod_{k=0}^{m-1} \sup_{V \in \mathcal{V}_k} \mathbb{E} \left[(d((L_{t_{k+1}} - L_{t_k})(B_{t_k} + \cdot), V))^{-1} \right] \\ &= \prod_{k=0}^{m-1} \sup_{V \in \mathcal{V}_k} \mathbb{E} \left[(d(L_{t_{k+1}-t_k}, V))^{-1} \right], \end{aligned} \quad (13)$$

with the convention $t_0 = 0$. Recall that $(L_u(x))_{x \in \mathbb{R}}$ has the same distribution $(\sqrt{u}L_1(x/\sqrt{u}))_{x \in \mathbb{R}}$ and so $(d(L_u, Vect(g_1, \dots, g_k)))^2$ has the same distribution as

$$\begin{aligned} \min_{a_1, \dots, a_k} \int_{\mathbb{R}} \left(\sqrt{u}L_1 \left(\frac{x}{\sqrt{u}} \right) - \sum_{i=1}^k a_i g_i(x) \right)^2 dx &= u \min_{a'_1, \dots, a'_k} \int_{\mathbb{R}} \left(L_1 \left(\frac{x}{\sqrt{u}} \right) - \sum_{i=1}^k a'_i g_i(x) \right)^2 dx \\ &= u^{\frac{3}{2}} \min_{a'_1, \dots, a'_k} \int_{\mathbb{R}} \left(L_1(y) - \sum_{i=1}^k a'_i g_i(\sqrt{u}y) \right)^2 dy \\ &= u^{\frac{3}{2}} (d(L_1, Vect(h_1, \dots, h_k)))^2 \end{aligned}$$

setting $a'_i := a_i/\sqrt{u}$, and making the change of variable $y = x/\sqrt{u}$, with $h_i(x) = g_i(\sqrt{u}x)$ and so (13) becomes

$$\prod_{k=0}^{m-1} \left((t_{k+1} - t_k)^{-\frac{3}{4}} \mathbb{E} \left[|L_1|_{L^2(\mathbb{R})}^{-1} \right] \right) \leq \mathbb{E} \left[\det \mathcal{D}_{t_1, \dots, t_m}^{-\frac{1}{2}} \right] \leq \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} \sup_{V \in \mathcal{V}_k} \mathbb{E} \left[(d(L_1, V))^{-1} \right], \quad (14)$$

which ends the proof of the lemma. \square

We first study the behaviour, as $m \rightarrow +\infty$, of the integral appearing in Lemma 6.

Lemma 7.

$$m! \int_{0 < t_1 < \dots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{3}{4}} dt_1 \dots dt_m = \frac{m! \Gamma(\frac{1}{4})^m}{\Gamma(\frac{m}{4} + 1)} \sim c(Cm)^{\frac{3m}{4}},$$

as $m \rightarrow +\infty$.

Proof.

$$\begin{aligned}
 a_{m+1} &:= \int_{0 < t_1 < \dots < t_{m+1} < 1} \prod_{k=0}^m (t_{k+1} - t_k)^{-\frac{3}{4}} dt_1 \dots dt_{m+1} \\
 &= \int_{x_i > 0 : x_1 + \dots + x_{m+1} < 1} \prod_{k=1}^{m+1} x_k^{-\frac{3}{4}} dx_1 \dots dx_{m+1} \\
 &= \int_0^1 x_{m+1}^{-\frac{3}{4}} (1 - x_{m+1})^{-\frac{3m}{4}} \left(\int_{x_i > 0 : x_1 + \dots + x_m < 1 - x_{m+1}} \prod_{k=1}^m (x_k / (1 - x_{m+1}))^{-\frac{3}{4}} dx_1 \dots dx_m \right) dx_{m+1} \\
 &= \int_0^1 x_{m+1}^{-\frac{3}{4}} (1 - x_{m+1})^{\frac{m}{4}} \left(\int_{u_i > 0 : u_1 + \dots + u_m < 1} \prod_{k=1}^m u_k^{-\frac{3}{4}} du_1 \dots du_m \right) dx_{m+1} \\
 &= a_m \int_0^1 x_{m+1}^{-\frac{3}{4}} (1 - x_{m+1})^{\frac{m}{4}} dx_{m+1} = a_m B\left(\frac{1}{4}, \frac{m}{4} + 1\right) = a_m \frac{\Gamma(\frac{1}{4})\Gamma(\frac{m}{4} + 1)}{\Gamma(\frac{m+1}{4} + 1)},
 \end{aligned}$$

where $B(\cdot, \cdot)$ and Γ stand respectively for Euler's Beta and Gamma functions, and so, by induction, $a_m = \frac{\Gamma(\frac{1}{4})^m}{\Gamma(\frac{m}{4} + 1)}$ proving the first point of the lemma. Moreover

$$m! a_m \sim (\Gamma(1/4))^m m^{m+\frac{1}{2}} (m+4)^{-\frac{m}{4} - \frac{1}{2}} 4^{\frac{m}{4} + \frac{1}{2}} e^{-\frac{3m}{4} + 1},$$

where we used the Stirling formulas $m! = \Gamma(m+1)$ and $\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}$. This ends the proof of the lemma. \square

Observe that $\mathbb{E} \left[|L_1|_{L^2(\mathbb{R})}^{-1} \right] > 0$. Thus, the proof of Theorem 1 will be deduced from the two previous lemmas combined with Theorem 2, which can be rewritten as follows

$$\forall \eta_0 > 0, \exists C > 1, \forall k \in \mathbb{N}^*, \quad C^{-1} k^{\frac{1}{2} - \eta_0} \leq \sup_{V \in \mathcal{V}_k} \mathbb{E} \left[(d(L_1, V))^{-1} \right] \leq C k^{\frac{1}{2} + \eta_0}. \quad (15)$$

Due to [44, Cor. (1.8) of Chap. VI, Theorem (2.1) of Chap. I], L_1 is almost surely Hölder continuous of order $\frac{1}{2} - \eta_0$ and its Hölder constant admits moments of any order. The lower bound of theorem 2 follows directly from this fact.

Proof of the lower bound of Theorem 2. We prove the lower bound of (15). Let $\eta_0 \in (0, \frac{1}{2})$. Let \mathcal{C}_1 be the Hölder constant of order $\frac{1}{2} - \eta_0$ of L_1 . Let V_k be the linear subspace of $L^2(\mathbb{R})$ generated by the set

$$\left\{ \mathbf{1}_{[m/k, (m+1)/k]}, m = -\left\lfloor \frac{k}{2} \right\rfloor, \dots, \left\lceil \frac{k}{2} \right\rceil - 1 \right\},$$

and consider $\tilde{L}_k \in V_k$ given by

$$\tilde{L}_k := \sum_{m = -\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} L_1 \left(\frac{m}{k} \right) \mathbf{1}_{[\frac{m}{k}, \frac{m+1}{k}]}.$$

Let $K_0 > 0$. We will use the fact that

$$\mathbb{E} \left[(d(L_1, V_k))^{-1} \right] \geq \mathbb{E} \left[(d(L_1, V_k))^{-1} \mathbf{1}_{\{\mathcal{C}_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right].$$

Observe that, if $\sup_{[0,1]} |B| \geq \frac{k-1}{2k}$ and $\mathcal{C}_1 \leq K_0$, then

$$\begin{aligned} d(L_1, V_k)^2 &\leq d(L_1, \tilde{L}_k)^2 = \sum_{m=\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} \int_{\frac{m}{k}}^{\frac{m+1}{k}} (L_1(u) - L_1(m/k))^2 du \\ &\leq \sum_{m=\lfloor \frac{k}{2} \rfloor}^{\lceil \frac{k}{2} \rceil - 1} k^{-1} \left(K_0 k^{-\frac{1}{2} + \eta_0} \right)^2 \leq \left(K_0 k^{-\frac{1}{2} + \eta_0} \right)^2. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E} \left[(d(L_1, V_k))^{-1} \right] &\geq \mathbb{E} \left[(d(L_1, V_k))^{-1} \mathbf{1}_{\{\mathcal{C}_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right] \\ &\geq \mathbb{E} \left[\left(K_0 k^{-\frac{1}{2} + \eta_0} \right)^{-1} \mathbf{1}_{\{\mathcal{C}_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{k-1}{2k}\}} \right] \\ &\geq K_0^{-1} k^{\frac{1}{2} - \eta_0} \mathbb{P} \left(\mathcal{C}_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{1}{3} \right). \end{aligned}$$

□

The rest of this section is devoted to the proof of the upper bound of Theorem 2 (i.e. the upper bound of (15)), which is much more delicate to establish. To this end, we will prove a sequence of estimates. Let us first introduce the quantities used in this proof. We fix $\eta_0 > 0$ and $d = \frac{1}{2} + \eta_0 > 1/2$. Choose $\epsilon_0 \in (0, \frac{1}{10})$ such that

$$d > \frac{1 + \epsilon_0}{2(1 - \epsilon_0)}. \quad (16)$$

Fix $a, b, \eta, \gamma \in (0, \frac{1}{10})$ such that $0 < \frac{b}{8} < \frac{a}{2}$ and small enough so that

$$\frac{(1 + \gamma)(1 + \epsilon_0)}{2} + \frac{a}{2} + \frac{b}{8} < 1 \quad (17)$$

and

$$(2d(1 - \epsilon_0) - 1 - \epsilon_0)(1 - 2\eta) - 8\eta > 0. \quad (18)$$

Let $\theta > 0$ such that $(1 - 2\eta)\theta > 1$ and

$$1 - \frac{b}{4} - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} < \theta(1 - 2\eta) \left(1 - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8} \right) \quad (19)$$

and

$$(1 - \epsilon_0)(1 + 2d) < \theta[(2d(1 - \epsilon_0) - 1 - \epsilon_0)(1 - 2\eta) - 8\eta]. \quad (20)$$

The existence of such a θ is ensured by (17) and (18). Fix then K such that $\frac{1}{4a-b} < K$ and $v_0 = \lceil 16/b \rceil$. We will also consider the following quantities which will depend on $k \geq 1$. We set $M := \lceil \theta k \rceil$ and $M' := M^d$. For $x > M'$, we also set:

$$r_0 := (x/M')^{-(1+\gamma)(1+\epsilon_0)} M^{-\frac{1+\epsilon_0}{2}} M'^{-1-\epsilon_0}, \quad x_0 = (x/M')^a M, \quad x_1 = (x/M')^b. \quad (21)$$

Let V be a linear space generated by $g_1, \dots, g_k \in L^2(\mathbb{R})$. Observe that

$$\begin{aligned} \mathbb{E} \left[(d(L_1, V))^{-1} \right] &= \int_0^\infty \mathbb{P} \left((d(L_1, V))^{-1} > x \right) dx \\ &= \mathcal{O}(M') + \int_{M'}^\infty \mathbb{P} (d(L_1, V) < x^{-1}) dx. \end{aligned} \quad (22)$$

Lemma 8. *Uniformly on $x > M'$:*

$$\begin{aligned} & \mathbb{P}(d(L_1, V)) < x^{-1} \\ & \leq \mathcal{O}((x/M')^{-2}) + \mathbb{P}\left(\forall \ell = -v_0, \dots, v_0, D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_{n=1, \dots, M}, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1}r_0^{-\frac{1}{2}}\right). \end{aligned} \quad (23)$$

where $W_V^{(y_0)} := \text{Span}\left(\left(\int_{y_0+n/x_0}^{y_0+(n+1)/x_0} g_j(y') dy'\right)_{n=1, \dots, M}, 1, \dots, k\right) \subset \mathbb{R}^M$ and where D is the usual euclidean metric in \mathbb{R}^M .

Proof. We set

$$\mathcal{C}_1 := \sup_{y, z \in \mathbb{R}: y \neq z} \frac{|L_1(y) - L_1(z)|}{|y - z|^u}, \quad \text{with } u := \frac{1}{1 + \epsilon_0} - \frac{1}{2}.$$

Since \mathcal{C}_1 admits moments of every order, it follows that

$$\mathbb{P}(d(L_1, V) < 1/x) \leq \mathbb{P}(d(L_1, V) < 1/x, \mathcal{C}_1 \leq (x/M')^\gamma) + \mathcal{O}((x/M')^{-2}),$$

Note that, if $x > M'$, then

$$r_0 x_0 = (x/M')^{a-(1+\gamma)(1+\epsilon_0)} M^{\frac{1-\epsilon_0}{2}} M'^{-1-\epsilon_0} \leq 1,$$

as soon as $x > M'$, since $a < 1 < (1+\gamma)(1+\epsilon_0)$ and since $M' = M^d$ with $\frac{1}{2} \leq d$, and so $r_0 \leq x_0^{-1}$. Assume moreover that $d(L_1, V) < 1/x$ and $\mathcal{C}_1 \leq x^\gamma$. Let a_j such that $d\left(L_1, \sum_{j=1}^k a_j g_j\right) < x^{-1}$. Then, for every $\ell \in \mathbb{Z}$, the following estimate holds true

$$\begin{aligned} x^{-1} & > \left(\sum_{n=1}^M \int_{\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}}^{\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0} + r_0} \left(L_1(y) - \sum_{j=1}^k a_j g_j(y) \right)^2 dy \right)^{\frac{1}{2}} \\ & \geq \left(\sum_{n=1}^M \int_{\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}}^{\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0} + r_0} \left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right) - \sum_{j=1}^k a_j g_j(y) \right)^2 dy \right)^{\frac{1}{2}} - (Mr_0(x/M')^{2\gamma} r_0^{2u})^{\frac{1}{2}} \\ & \geq \sqrt{r_0} D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_{n=1, \dots, M}, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) - \sqrt{M}(x/M')^\gamma r_0^{\frac{1}{2}+u}. \end{aligned}$$

Since $\frac{1}{2}+u = \frac{1}{1+\epsilon_0}$ and $r_0 = (x/M')^{-(1+\gamma)(1+\epsilon_0)} M^{-\frac{1+\epsilon_0}{2}} M'^{-1-\epsilon_0}$, we conclude that $\sqrt{M}(x/M')^\gamma r_0^{\frac{1}{2}+u} = x^{-1}$ and so

$$\begin{aligned} & \mathbb{P}(d(L_1, V) < 1/x, \mathcal{C}_1 \leq x^\gamma) \\ & \leq \mathbb{P}\left(\forall \ell = -v_0, \dots, v_0, D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_n, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1}r_0^{-\frac{1}{2}}\right). \end{aligned}$$

□

Recall that $v_0 = \lceil 16/b \rceil$. Set

$$E_{0,W} := \left\{ D\left(\left(x_0 \int_{\frac{n}{x_0}}^{\frac{n+1}{x_0}} Y'(y) dy\right)_{n=1, \dots, M}, W\right) < 2x^{-1}\sqrt{x_0} \right\}.$$

Lemma 9. *The following estimate holds true uniformly on $x > M$:*

$$\mathbb{P}(d(L_1, V) < x^{-1}) \leq \mathcal{O}((x/M')^{-2}) + \sup_W \mathbb{P}(E_{0,W}), \quad (24)$$

where \sup_W means the supremum over the set of linear subspaces W of \mathbb{R}^M of dimension at most k and where Y' is a squared Bessel process of dimension 0 starting from $x_1^{-\frac{1}{4}}$.

Proof. We adapt the proof of [14, Lemma 9]. Setting $\epsilon' := x_1^{-\frac{1}{8}}$ and $T_u := \min\{s > 0 : |B_s| = u\}$ for the first hitting time of $\{\pm u\}$ by the Brownian motion B , we observe that

$$\mathbb{P}(T_{v_0\epsilon'} > 1) = \mathbb{P}\left(\sup_{s \in [0,1]} |B_s| \leq v_0\epsilon'\right) = \mathcal{O}(e^{-c_0(v_0\epsilon')^{-2}}) = \mathcal{O}\left((x/M')^{-bv_0/8}\right) = \mathcal{O}\left((x/M')^{-2}\right). \quad (25)$$

(using e.g. [43, Proposition 8.4, page 52]). Moreover, due to [44, Exercise 4.12, Chapter VI, p 265], for every $n = 0, \dots, v_0 - 1$,

$$\mathbb{P}\left(L_{T_{(n+1)\epsilon'}}(B_{T_{n\epsilon'}}) - L_{T_{n\epsilon'}}(B_{T_{n\epsilon'}}) \leq (\epsilon')^2 | (B_u)_{u \leq T_{n\epsilon'}}\right) \leq \mathbb{P}(L_{T_{\epsilon'}}(0) \leq (\epsilon')^2) \leq \epsilon'$$

and so, due to the strong Markov property,

$$\mathbb{P}\left(\forall n = 0, \dots, v_0 - 1, L_{T_{(n+1)\epsilon'}}(B_{T_{n\epsilon'}}) - L_{T_{n\epsilon'}}(B_{T_{n\epsilon'}}) \leq (\epsilon')^2\right) \leq (\epsilon')^{v_0},$$

and this, combined with (25), ensures that there exists $C_0 > 0$ such that $\mathbb{P}(\forall \ell = -v_0, \dots, v_0, L_1(\ell\epsilon') \leq (\epsilon')^2) \leq C_0(\epsilon')^{v_0}$ and so

$$\mathbb{P}(\forall \ell = -v_0, \dots, v_0, \mathfrak{t}_\ell(x_1) > 1) \leq C_0(x/M')^{-bv_0/8} \leq C_0(x/M')^{-2}, \quad (26)$$

setting $\mathfrak{t}_\ell(x_1) := \inf\{s > 0 : L_s(\ell x_1^{-\frac{1}{8}}) > x_1^{-\frac{1}{4}}\}$. Moreover, for any $\ell = 1, \dots, v_0$, we have

$$\begin{aligned} & \sup_V \mathbb{P}\left(\mathfrak{t}_\ell(x_1) < 1, D\left(\left(L_1\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_n, W_V^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1}\sqrt{x_0}\right) \\ & \leq \sup_V \mathbb{P}\left(\mathfrak{t}_\ell(x_1) < 1, D\left(\left(L_{\mathfrak{t}_\ell(x_1)}\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_n, W_{V,0}^{(\ell x_1^{-\frac{1}{8}})}\right) < 2x^{-1}\sqrt{x_0}\right) \\ & \leq \sup_W \mathbb{P}\left(D\left(\left(L_{\mathfrak{t}_\ell(x_1)}\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_n, W\right) < 2x^{-1}\sqrt{x_0}\right), \end{aligned} \quad (27)$$

setting $W_{V,0}^{(\ell x_1^{-\frac{1}{8}})} := W_V^{(\ell x_1^{-\frac{1}{8}})} + \left((L_{\mathfrak{t}_\ell(x_1)} - L_1)\left(\ell x_1^{-\frac{1}{8}} + \frac{n}{x_0}\right)\right)_n$. Due to the second Ray-Knight theorem (see [44, Theorem 2.3, page 456]), $(L_{\mathfrak{t}_\ell(x_1)}(\ell x_1^{-\frac{1}{8}} + y))_{y \geq 0}$ has the same distribution as Y' . The lemma follows from (26) and (27). \square

Recall that $4a - b > 0$. Set

$$E_1 := \left\{ \sup_{s \leq M/x_0} |Y'(s) - x_1^{-1/4}| < \frac{x_1^{-1/4}}{2} \right\}.$$

Lemma 10. *For every $K > (4a - b)^{-1}$, the following estimate holds true uniformly on $x > M$:*

$$\mathbb{P}(E_1) = 1 - \mathcal{O}\left((x/M')^{-K(4a-b)}\right). \quad (28)$$

Proof. Using the Burkholder-Davis-Gundy inequality, combined with the fact that Y' is dominated by the square of a Brownian motion starting from $x_1^{-1/4}$, we observe that

$$\begin{aligned} \mathfrak{p}_x &= \mathbb{P} \left[\sup_{s \leq M/x_0} |Y'(s) - x_1^{-1/4}| \geq \frac{x_1^{-1/4}}{2} \right] \\ &\leq C_K x_1^{2K} 2^{8K} \mathbb{E} \left[\left(\int_0^{M/x_0} Y'(u) du \right)^{4K} \right] \\ &\leq C'_K x_1^{2K} (M/x_0)^{4K-1} \int_0^{M/x_0} \mathbb{E} [Y'(u)^{4K}] du \end{aligned} \quad (29)$$

with $C'_K = 2^{8K} C_K$, and so

$$\begin{aligned} \mathfrak{p}_x &\leq C'_K x_1^{2K} (M/x_0)^{4K-1} \int_0^{M/x_0} \mathbb{E} \left[(x_1^{-1/8} + B_u)^{8K} \right] du \\ &\leq C'_K x_1^{2K} (M/x_0)^{4K} 2^{8K} \left(x_1^{-K} + (M/x_0)^{4K} \right) \\ &\leq C''_K \left(x_1^K (M/x_0)^{4K} + x_1^{2K} (M/x_0)^{8K} \right) \end{aligned}$$

with $C''_K = 2^{8K} C'_K$ and

$$x_1^K (M/x_0)^{4K} = (x/M')^{-K(4a-b)},$$

since $x_0 = (x/M')^a M$ and $x_1 = (x/M')^b$. \square

Lemma 11. *Uniformly on $x > M'$,*

$$\begin{aligned} \sup_W \mathbb{P}(E_{0,W} \cap E_1) &\leq C''^k (x/M') \left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2} \right]_{k-M(1-2\eta)} \left[1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8} \right] \\ &\quad \times M^{\frac{(1-\epsilon_0)k}{4} + \frac{(1+\epsilon_0)(1-2\eta)M}{4} + 2\eta M} M'^{\left(\frac{1-\epsilon_0}{2}\right)(k-(1-2\eta)M)}. \end{aligned}$$

Proof. Observe that

$$E_{0,W} \cap E_1 \subset \left\{ (Y'(n/x_0))_{n=1,\dots,M} \in \mathcal{B}_\infty \left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2} \right) \cap W_x \right\}, \quad (30)$$

where $\mathcal{B}_\infty \left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2} \right)$ is the ball (for the supremum norm) of radius $\frac{x_1^{-1/4}}{2}$ and centered on $(x_1^{-1/4}, \dots, x_1^{-1/4})$, and where W_x is the $\varepsilon = 2x^{-1}r_0^{-\frac{1}{2}}$ -neighbourhood of W for the metric D . Note that

$$\varepsilon = 2(x/M')^{\frac{(1+\gamma)(1+\epsilon_0)}{2}-1} M^{\frac{1+\epsilon_0}{4}} M'^{\frac{1+\epsilon_0}{2}-1} \quad (31)$$

and

$$\mathcal{R}_x := \frac{\sqrt{M} x_1^{-1/4}}{\varepsilon} = \frac{1}{2} (x/M')^{1-\frac{b}{4}-\frac{(1+\gamma)(1+\epsilon_0)}{2}} M^{\frac{1-\epsilon_0}{4}} M'^{1-\frac{1+\epsilon_0}{2}} \gg 1, \quad (32)$$

uniformly in $x > M'$, since $\frac{b}{4} + \frac{(1+\gamma)(1+\epsilon_0)}{2} < 1$. Observe that $\mathcal{B}_\infty \left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2} \right) \cap W_x$ is contained in $\mathcal{B}_2 \left(x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right) \cap W_x$ where $\mathcal{B}_2 \left(x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right)$ is the euclidean ball centered on $(x_1^{-1/4}, \dots, x_1^{-1/4})$ with radius $\sqrt{M} x_1^{-1/4}$.

Let $z_0, z'_0 \in \mathcal{B}_2 \left(x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right) \cap W_x$ and $z_1 \in W \cap \mathcal{B}_2(z_0, \varepsilon)$, $z'_1 \in W \cap \mathcal{B}_2(z'_0, \varepsilon)$. Then $z'_1 \in \mathcal{B}_2 \left(z_1, 3\sqrt{M} x_1^{-1/4} \right)$. Due to [45, Theorem 3, pages 157], there exists $c > 0$ such that

$W \cap \mathcal{B}_2\left(z_1, 3\sqrt{M}x_1^{-1/4}\right)$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius ε in W . Thus $W_x \cap \mathcal{B}_2\left(x_1^{-1/4}, \sqrt{M}x_1^{-1/4}\right)$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius 2ε . We conclude that $\mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right) \cap W_x$ is contained in the union of at most $(c\mathcal{R}_x)^k$ euclidean balls of radius 4ε centered at a point contained in $\mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right) \cap W_x$. It follows from this combined with (30) that

$$\mathbb{P}[E_{0,W} \cap E_1] \leq (c\mathcal{R}_x)^k \sup_{z \in \mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)} \mathbb{P}\left((Y'(n/x_0))_{n=1,\dots,M} \in \mathcal{B}_2(z, 4\varepsilon)\right). \quad (33)$$

Note that if $z \in \mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)$ and $(Y'(n/x_0))_{n=1,\dots,M} \in \mathcal{B}_2(z, 4\varepsilon)$, then $\max_{n=0,\dots,M-1} |z_{n+1} - x_1^{-1/4}| < \frac{x_1^{-1/4}}{2}$ and there exist at most ηM indices n' that $|Y'(n'/x_0) - z_{n'}| \geq 4\varepsilon/\sqrt{\eta M}$, and so at least $(1 - 2\eta)M$ indices $n = \{0, \dots, M-1\}$ such that

$$\left|Y'\left(\frac{n}{x_0}\right) - z_n\right|, \left|Y'\left(\frac{n+1}{x_0}\right) - z_{n+1}\right| < 4\varepsilon/\sqrt{\eta M},$$

with $z_0 = x_1^{-1/4}$. Due to [44, after Corollary 1.4, page 441], the distribution of $Y'((n+1)/x_0)$ knowing $Y'(n/x_0) = y$ is the sum of a Dirac mass at 0 and of a measure with density

$$z \mapsto q_{x_0}(y, z) := \frac{x_0}{2} \sqrt{\frac{y}{z}} \exp\left(-\frac{x_0(y+z)}{2}\right) I_1(x_0\sqrt{yz}),$$

where I_1 is the modified Bessel function of index 1 which satisfies $I_1(z) = \mathcal{O}(e^z/\sqrt{z})$, as $z \rightarrow \infty$, (see [35, (5.10.22) or (5.11.10)]). So

$$q_{x_0}(y, z) = \mathcal{O}\left(x_0^{\frac{1}{2}} x_1^{\frac{1}{8}} \exp\left(-\frac{x_0(\sqrt{y} - \sqrt{z})^2}{2}\right)\right) = \mathcal{O}\left(x_0^{\frac{1}{2}} x_1^{\frac{1}{8}}\right)$$

uniformly on $y, z \in \left[\frac{x_1^{-1/4}}{4}, 2x_1^{-1/4}\right]$. We will use the expression x_0, x_1 and ε given in (21) and (31).

Thus by using the Markov property (and $\frac{M!}{(M(1-2\eta))!(2\eta M)!} \leq M^{2\eta M}$), we get by induction, that, when $x > M'$,

$$\begin{aligned} & \sup_{z \in \mathcal{B}_\infty\left(x_1^{-1/4}, \frac{x_1^{-1/4}}{2}\right)} \mathbb{P}\left((Y'(n/x_0))_{n=1,\dots,M} \in \mathcal{B}_2(z, 4\varepsilon)\right) \\ & \leq M^{2\eta M} \left(C'(x/M')^{-\left(1 - \frac{(1+\gamma)(1+\varepsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right)} M^{\frac{1+\varepsilon_0}{4}} M'^{-1 + \frac{1+\varepsilon_0}{2}}\right)^{(1-2\eta)M}. \end{aligned}$$

Recalling that $M = \mathcal{O}(k)$, the previous estimate combined with (33) and (32) ensures that

$$\begin{aligned} \sup_W \mathbb{P}(E_{0,W} \cap E_1) & \leq C''^k (x/M')^{\left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\varepsilon_0)}{2}\right]k - M(1-2\eta)\left[1 - \frac{(1+\gamma)(1+\varepsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right]} \\ & M^{\frac{(1-\varepsilon_0)k}{4} + \frac{(1+\varepsilon_0)(1-2\eta)M}{4} + 2\eta M} M'^{\left(1 - \frac{1+\varepsilon_0}{2}\right)(k - (1-2\eta)M)}, \end{aligned} \quad (34)$$

which ends the proof of the lemma. \square

Proof of the upper bound of Theorem 2. Formula (15) follows from (22) and Lemmas 9, 10 and 11. We will use the fact that

$$\forall Q > 1, \quad \int_{M'}^{\infty} (x/M')^{-Q} dx = \mathcal{O}(M'). \quad (35)$$

Thanks to this, the error terms in Lemmas 9 and 10 gives directly a term in $\mathcal{O}(M') = \mathcal{O}(k^d)$. Let us detail the term coming from Lemma 11. We first observe that the exponent of (x/M') is strictly smaller than -1 for k large enough. Indeed this exponent is

$$\left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2}\right] k - M(1-2\eta) \left[1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right]$$

which is smaller than

$$k \left[1 - \frac{b}{4} - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \theta(1-2\eta) \left(1 - \frac{(1+\gamma)(1+\epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right)\right]$$

where we used the fact that $M = \lceil \theta k \rceil \geq \theta k$. The fact that this quantity is strictly smaller than -1 for any k large enough comes from our conditions (17) and (19). It follows from this combined with (35) and Lemma 11 that

$$\begin{aligned} & \int_{M'}^{+\infty} \sup_W \mathbb{P}(E_{0,W} \cap E_1 \cap E_2) dx \\ & \leq C'' k M^{\frac{(1-\epsilon_0)k}{4} + \frac{(1+\epsilon_0)(1-2\eta)M}{4} + 2\eta M} M'^{1 + \left(\frac{1-\epsilon_0}{2}\right)(k - (1-2\eta)M)} \\ & \leq C'' k M^{d + \frac{(1-\epsilon_0)(1+2d)M}{4\theta} + \frac{(1+\epsilon_0 - 2d(1-\epsilon_0))(1-2\eta)M}{4} + 2\eta M}, \end{aligned}$$

where we used the fact that $M' = M^d$ and that $k \leq \lceil \theta k \rceil / \theta = M/\theta$. Finally, we notice that $1 + \epsilon_0 - 2d(1 - \epsilon_0) < 0$ (due to (16)) and that (20) ensures that

$$\frac{(1-\epsilon_0)(1+2d)}{4\theta} + \frac{(1+\epsilon_0 - 2d(1-\epsilon_0))(1-2\eta)}{4} + 2\eta < 0$$

and conclude that

$$\int_{M'}^{+\infty} \sup_W \mathbb{P}(E_{0,W} \cap E_1 \cap E_2) dx = \mathcal{O}(1).$$

□

3. LAW OF LARGE NUMBERS: PROOF OF THEOREM 3

We complete the sequence $(X_n)_{n \geq 1}$ into a bi-infinite sequence $(X_n)_{n \in \mathbb{Z}}$ of i.i.d. random variables. Theorem 3 could be proved by an adaptation of the proof of [14, Corollary 6] (combined with Theorem 1). We use here another approach enabling the study of more general additive functionals. Recall that $(\xi_{m+S_k})_{m \in \mathbb{Z}}$ is the scenery seen from the particle at time k .

Proposition 12. *Let $\tilde{f} : \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z} \rightarrow \mathbb{R}$ be a measurable function such that*

$$\sum_{\ell \in \mathbb{Z}} |\mathbb{E}[\tilde{f}((X_{n+1})_{n \in \mathbb{Z}}, (\xi_n)_{n \in \mathbb{Z}}, \ell)]| < \infty.$$

Then

$$\left(\frac{\sum_{k=0}^{n-1} \tilde{f}((X_{m+k+1})_{m \in \mathbb{Z}}, (\xi_{m+S_k})_{m \in \mathbb{Z}}, Z_{k+m})}{\mathcal{N}_n(0)} \right)_{n \geq 0}$$

converges almost surely to $I(\tilde{f}) := \sum_{\ell \in \mathbb{Z}} \mathbb{E}[\tilde{f}((X_n)_{n \in \mathbb{Z}}, (\xi_n)_{n \in \mathbb{Z}}, \ell)]$.

In particular, this combined with (5) ensures that

$$\left(n^{-\frac{1}{4}} \sum_{k=0}^{n-1} \tilde{f}((X_{m+k+1})_{m \in \mathbb{Z}}, (\xi_{m+S_k})_{m \in \mathbb{Z}}, Z_k) \right)_{n \geq 0}$$

converges in distribution to $I(\tilde{f})\sigma_\xi^{-1}\mathcal{L}_1(0)$.

Our approach to prove Proposition 12 uses an ergodic point of view. Let us consider the probability preserving dynamical system (Ω, T, μ) given by

$$\Omega = \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}}, \quad T((x_k)_{k \in \mathbb{Z}}, (y_k)_{k \in \mathbb{Z}}) = ((x_{k+1})_{k \in \mathbb{Z}}, (y_{k+x_0})_{k \in \mathbb{Z}}), \quad \mu = \mathbb{P}_{X_1}^{\otimes \mathbb{Z}} \otimes \mathbb{P}_{\xi_0}^{\otimes \mathbb{Z}},$$

i.e. $T(\mathbf{x}, \mathbf{y}) = (\sigma \mathbf{x}, \sigma^{x_0} \mathbf{y})$, where we write $\sigma : \mathbb{Z}^{\mathbb{Z}} \rightarrow \mathbb{Z}^{\mathbb{Z}}$ for the usual shift transformation given by $\sigma((z_k)_{k \in \mathbb{Z}}) = (z_{k+1})_{k \in \mathbb{Z}}$.

This system is known to be ergodic (see [49, 30]). We set $\Phi(x, y) := y_0$. With these notations, Z_k corresponds to the Birkhoff sum $\sum_{k=0}^{n-1} \Phi \circ T^k$. Consider the \mathbb{Z} -extension $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ over (Ω, T, μ) with step function Φ . This system is given by

$$\tilde{\Omega} := \Omega \times \mathbb{Z}, \quad \tilde{\mu} = \mu \otimes \lambda_{\mathbb{Z}},$$

where $\lambda_{\mathbb{Z}} = \sum_{\ell \in \mathbb{Z}} \delta_\ell$ is the counting measure on \mathbb{Z} and with

$$\tilde{T}(x, y, \ell) = (T(x, y), \ell + y_0).$$

In particular

$$\tilde{T}^k((x_{m+1})_{m \in \mathbb{Z}}, (y_m)_{m \in \mathbb{Z}}, \ell) = \left((x_{m+k+1})_{m \in \mathbb{Z}}, (y_{m+x_0+\dots+x_{k-1}})_{m \in \mathbb{Z}}, \ell + \sum_{j=0}^{k-1} y_{x_0+\dots+x_j} \right).$$

Observe that $\mathcal{N}_n(0)$ corresponds to the Birkhoff sum $\sum_{k=0}^{n-1} h_0 \circ \tilde{T}^k(\mathbf{x}, \mathbf{y}, 0)$ with $h_0(\mathbf{x}, \mathbf{y}, \ell) = \mathbf{1}_0(\ell)$, and the sum studied in Proposition 12 corresponds to $\sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k(\mathbf{x}, \mathbf{y}, 0)$, while $I(\tilde{f}) = \int_{\tilde{\Omega}} \tilde{f} d\tilde{\mu}$.

Proposition 13. *The system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ is recurrent ergodic.*

Proof. Since (Ω, T, μ) is ergodic and since Φ is integrable and μ -centered, we know (by [46, Corollary 3.9] combined with the Birkhoff ergodic theorem) that $\mathbb{P}(Z_n = 0 \text{ i.o.}) = 1$, thus that $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ is recurrent (i.e. conservative). Now let us prove that this system is also ergodic. Let $g : \tilde{\Omega} \rightarrow (0, +\infty)$ be a positive $\tilde{\mu}$ -integrable function such that $g(\mathbf{x}, \mathbf{y}, \ell) = g_0(\ell)$ does not depend on $(\mathbf{x}, \mathbf{y}) \in \Omega$ and with unit integral (g is a probability density function with respect to $\tilde{\mu}$). By recurrence of $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$, we know that

$$\sum_{k \geq 1} g \circ \tilde{T}^k = \infty \tag{36}$$

$\tilde{\mu}$ -almost everywhere. Let $K \in \mathbb{N}$. Consider $f : \tilde{\Omega} \rightarrow \mathbb{R}$ a $\tilde{\mu}$ -integrable function constant on the K -cylinders of the first coordinate, i.e. such that $f(\mathbf{x}, \mathbf{y}, \ell) = f_0((x_m)_{|m| \leq k}, \mathbf{y}, \ell)$ does not depend on $(x_k)_{|k| > K}$.

Since $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ is recurrent, the Hopf-Hurewicz's theorem (see e.g. [1, p. 56]) ensures that

$$\lim_{|n| \rightarrow +\infty} \frac{\sum_{k=1}^n f \circ \tilde{T}^k}{\sum_{k=1}^n g \circ \tilde{T}^k} = H_{(f,g)} := \mathbb{E}_{g\tilde{\mu}} \left[\frac{f}{g} \middle| \tilde{\mathcal{I}} \right] \tag{37}$$

$\tilde{\mu}$ -almost everywhere, where $\tilde{\mathcal{I}}$ is the σ -algebra of \tilde{T} -invariant events. Thus the ergodicity of $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ will follow from the fact that $H_{(f,g)}$ is $\tilde{\mu}$ -almost everywhere constant for every f as above (g can be fixed). Observe that, for $k > K$,

$$\begin{aligned} f \circ \tilde{T}^k(\mathbf{x}, \mathbf{y}, \ell) &= f \left(\sigma^k \mathbf{x}, \sigma^{x_0 + \dots + x_{k-1}} \mathbf{y}, \ell + \sum_{m=0}^{k-1} y_{x_0 + \dots + x_m} \right) \\ &= f_0 \left(x_{k-K}, \dots, x_{K+k}, \sigma^{x_0 + \dots + x_{k-1}} \mathbf{y}, \ell + \sum_{m=0}^{k-1} y_{x_0 + \dots + x_m} \right) \end{aligned}$$

does not depend on $(x_k)_{k \leq -1}$. Analogously, for $k > K$,

$$\begin{aligned} f \circ \tilde{T}^{-k}(\mathbf{x}, \mathbf{y}, \ell) &= f \left(\sigma^{-k} \mathbf{x}, \sigma^{-x_{-1} - \dots - x_{-k}} \mathbf{y}, \ell - \sum_{m=1}^k y_{-x_{-1} - \dots - x_{-m}} \right) \\ &= f_0 \left(x_{-K-k}, \dots, x_{-(k-K)}, \sigma^{-x_{-1} - \dots - x_{-k}} \mathbf{y}, \ell - \sum_{m=1}^k y_{-x_{-1} - \dots - x_{-m}} \right) \end{aligned}$$

does not depend on $(x_k)_{k \geq 0}$. Of course $g \circ \tilde{T}^k$ satisfies the same property. Thus, due to (36) and (37), it follows that $H_{(f,g)}(\mathbf{x}, \mathbf{y}, \ell)$ does not depend on \mathbf{x} . Thus, $H_{(f,g)}(\mathbf{x}, \mathbf{y}, \ell) = H_{(f,g)}^{(0)}(\mathbf{y}, \ell)$ for $\tilde{\mu}$ -almost every $(\mathbf{x}, \mathbf{y}, \ell) \in \tilde{\Omega}$.

By \tilde{T} -invariance of $H_{(f,g)}$, given two distinct points $x_0, x'_0 \in \mathbb{Z}$ such that $\mathbb{P}(X_1 = x_0)\mathbb{P}(X_1 = x'_0) > 0$, the following equality holds true almost everywhere

$$H_{(f,g)}^{(0)}(\mathbf{y}, \ell) = H_{(f,g)}^{(0)}(\sigma^{x_0} \mathbf{y}, \ell + y_0) = H_{(f,g)}^{(0)}(\sigma^{x'_0} \mathbf{y}, \ell + y_0),$$

where we write σ for the usual shift on $\mathbb{Z}^{\mathbb{Z}}$ given by $\sigma((y_k)_{k \in \mathbb{Z}}) = (y_{k+1})_{k \in \mathbb{Z}}$. It follows that, for every $\ell \in \mathbb{Z}$, $H_{(f,g)}^{(0)}(\cdot, \ell)$ is $\sigma^{x_0 - x'_0}$ -invariant almost everywhere. By ergodicity of $\sigma^{x_0 - x'_0}$, we conclude that $H_{(f,g)}(\mathbf{x}, \mathbf{y}, \ell) = H_{f,g}^{(1)}(\ell)$ depends only on ℓ almost everywhere. Since it is \tilde{T} -invariant, for every $y_0 \in \mathbb{Z}$ such that $\mathbb{P}(\xi_0 = y_0) > 0$, $H_{f,g}^{(1)}(\ell) = H_{f,g}^{(1)}(\ell + y_0)$. Since the support of y_0 generates the group \mathbb{Z} , we conclude that $H_{(f,g)}$ is $\tilde{\mu}$ -almost everywhere equal to a constant. \square

Note that the system in infinite measure $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ describes the evolution in time m of $((X_{m+k+1})_{k \in \mathbb{Z}}, (\xi_{S_m+k})_k, Z_m)$. In comparison, the system corresponding to $((X_{m+k+1})_k, S_m)$ is also recurrent ergodic, but the analogous system corresponding to $((X_{m+k+1})_k, (\xi_{S_m+k})_k, S_m)$ is recurrent (since $\mathbb{P}(S_n = 0 \text{ i.o.}) = 1$) not ergodic (since the sets of the form $\{(x, y, \ell) : (y_{n-\ell})_n \in A_0\}$ are invariant).

Proof of Proposition 12. Since $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ is recurrent ergodic, the Hopf ergodic theorem ensures that, for any $\tilde{f} \in L^1(\tilde{\mu})$, the sequence $\left(\frac{\sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k}{\sum_{k=0}^{n-1} \tilde{h}_0 \circ \tilde{T}^k} \right)_{n \geq 0}$ converges $\tilde{\mu}$ -almost everywhere to

$$\frac{\int_{\tilde{\Omega}} \tilde{f} d\tilde{\mu}}{\int_{\tilde{\Omega}} \tilde{h}_0 d\tilde{\mu}} = I(\tilde{f}).$$
 Thus

$$\left(\frac{\sum_{k=0}^{n-1} \tilde{f}((X_{m+k+1})_{m \in \mathbb{Z}}, (\xi_{S_m+k})_{m \in \mathbb{Z}}, Z_{k+m})}{N_n(0)} = \frac{\sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k}{\sum_{k=0}^{n-1} \tilde{h}_0 \circ \tilde{T}^k}((X_m)_{m \in \mathbb{Z}}, (\xi_m)_{m \in \mathbb{Z}}, 0) \right)_{n \geq 0}$$

converges almost surely to $I(\tilde{f})$, and we have proved the first part of the proposition. The second part comes from the first part combined with (5) and the Slutsky theorem. \square

Proof of Theorem 4. Proposition 12 states that $n^{-\frac{1}{4}} \sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k$ converges in distribution, with respect to $\mu \otimes \delta_0 \ll \tilde{\mu}$, to $\int_{\tilde{\Omega}} \tilde{f} d\tilde{\mu} \sigma_{\xi}^{-1} \mathcal{L}_1(0)$. Thus, Theorem 4 follows from Proposition 12 combined with [51, Theorem 1]. \square

We end this section with an interpretation of σ_f^2 in terms of the famous Green-Kubo formula.

Remark 14. Assume the assumptions of Theorem 5. consider the function $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{Z}$ given by $\tilde{f}(\mathbf{x}, \mathbf{y}, \ell) := f(\ell)$. Then σ_f^2 can be rewritten

$$\sigma_f^2 = \sum_{k \in \mathbb{Z}} \int_{\tilde{\Omega}} \tilde{f} \cdot \tilde{f} \circ \tilde{T}^{|k|} d\tilde{\mu}.$$

4. PROOF OF THE CENTRAL LIMIT THEOREM: PROOF OF THEOREM 5

We start by stating key intermediate results. We recall that d and α have been introduced in the beginning of Section 1.2.

Proposition 15. Assume the assumptions of Theorem 5. Let $M \in \mathbb{N}^*$ and $\eta > 0$. There exists $L \in (0, 1)$ such that for every $\theta \in (0, 1)$ the following holds true with the notations $n_j := k_j - k_{j-1}$, with the convention $k_0 = 0$.

First,

$$\sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}} \mathbb{E} \left[\prod_{j=1}^m \left(f(Z_{k_j+k'_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}}) \right) \right] = \mathcal{O} \left(\left(\prod_{i=1}^m n_i^{-\frac{3}{4}} \right) \mathfrak{E}_{\mathbf{k}} \right), \quad (38)$$

uniformly over the $\mathbf{k} = (k_1, \dots, k_m)$ and $\ell = (\ell_{j,s})_{j=1, \dots, m; s=1, \dots, s_j}$ such that $n > k_j > k_{j-1} + n^\theta$ (with convention $k_0 := 0$) and $\ell_{j,s} \in \{0, \dots, \lfloor n^{L\theta} \rfloor\}$ with $M = \sum_{j=1}^m (s_j + 1)$, where we set $\mathcal{J} := \{j = 1, \dots, m : s_j = 0\}$ and $k'_j = 0$ if $j \notin \mathcal{J}$, and with

$$\mathfrak{E}_{\mathbf{k}} = \mathcal{O} \left(\sum_{\mathcal{J}' \subset \{1, \dots, m\} : \#\mathcal{J}' \geq \#\mathcal{J}/2} \left(\prod_{j \in \mathcal{J}'} n_j^{-\frac{1}{2} + \eta} \right) \right).$$

Second, if $s_j = 1$ for all j , then

$$\mathbb{E} \left[\prod_{j=1}^m (f(Z_{k_j}) f(Z_{k_j+\ell_j})) \right] = \frac{d^m E_{\mathbf{k}}}{(2\pi\sigma_{\xi}^2)^{\frac{m}{2}}} \prod_{j=1}^m \mathcal{A}_{k_j, \ell_j} + \mathcal{O} \left(n^{-L(M+1)\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}} \right),$$

uniformly on \mathbf{k}, ℓ as above, with $E_{\mathbf{k}}$ depending on \mathbf{k} but not on ℓ and such that $E_{\mathbf{k}} = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right)$

uniformly on \mathbf{k} as above, and $E_{\mathbf{k}} \sim n^{-\frac{3m}{4}} \mathbb{E} \left[\det \mathcal{D}_{t_1, \dots, t_m}^{-\frac{1}{2}} \right]$ as $k_j/n \rightarrow t_j$ and $n \rightarrow +\infty$, with $\mathcal{D}_{t_1, \dots, t_m} = (\int_{\mathbb{R}} L_{t_i}(x) L_{t_j}(x) dx)_{i,j=1, \dots, m}$ where L is the local time of the brownian motion B , limit of $(S_{\lfloor nt \rfloor} / \sqrt{n})_t$ as n goes to infinity, and where

$$\mathcal{A}_{k, \ell} := \sum_{a \in k\alpha + d\mathbb{Z}, b \in \mathbb{Z}} \left(f(a) \prod_{s=1}^m f(b) \right) \mathbb{P}(Z_{\ell} = b - a).$$

Third,

$$\sum_{k'_1, \dots, k'_m=0}^{d-1} \sum_{\ell_1, \dots, \ell_m=0}^{n \frac{\kappa \theta \eta}{10M}} 2^{\#\{j: \ell_j > 0\}} \prod_{j=1}^m \mathcal{A}_{k_j+k'_j, \ell_j} = \sigma_f^{2m} + o(1),$$

as $(k_1/n, \dots, k_m/n) \rightarrow (t_1, \dots, t_m)$ and $n \rightarrow +\infty$.

Proof. The proof of Proposition 15 is based on several technical lemmas. For reader convenience, the most technical points are proved in Appendix A. Let $M \geq 1$, $\theta \in (0, 1)$ and $\eta \in (0, \frac{1}{100})$. Choose $L = \frac{\kappa \eta}{10M}$. Assume $n^\theta < n_j < n$ and let $\ell_{j,1}, \dots, \ell_{j,s_j} = 0, \dots, \lfloor n^{L\theta} \rfloor$ with $\sum_{j=1}^m (1+s_j) = M$. We set $N'_j(y) := \#\{s = 0, \dots, n_j - 1 : S_{k_{j-1}+s} = y\}$, $N_j^* := \sup_y N'_j$ and $R'_j := \#\{y \in \mathbb{Z} : N'_j(y) > 0\}$. Analogously, we set $N'_{j,s} = \#\{s = 0, \dots, \ell_{j,s} - 1 : S_{k_j+s} = y\}$. The left hand side of (38) can be written

$$B_{\mathbf{k}, \ell} = \sum_{\mathbf{a}, \mathbf{b}} \left(\prod_{j=1}^m \left(f(a_j) \prod_{s=1}^{s_j} f(b_{j,s}) \right) \right) p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}), \quad (39)$$

where $\sum_{\mathbf{a}, \mathbf{b}}$ means the sum over $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^M$ with $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_{j,s})_{j=1, \dots, m; s=1, \dots, s_j}$, with the convention $a_0 = 0$ and

$$p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}) = \mathbb{P}(\forall j = 1, \dots, m, Z_{k_j} = a_j, \forall s = 1, \dots, s_j, Z_{k_j+\ell_{j,s}} = b_{j,s}).$$

An classical computation (detailed in Appendix A) ensures the following.

Lemma 16.

$$p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}) = \mathbf{1}_{\{\forall i, a_i = k_i \alpha + d\mathbb{Z}\}} \frac{d^m}{(2\pi)^M} \int_{[-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}} e^{-i \sum_{j=1}^m [(a_j - a_{j-1})\theta_j + \sum_{s=1}^{s_j} (b_{j,s} - a_j)\theta'_{j,s}]} \varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}') d(\boldsymbol{\theta}, \boldsymbol{\theta}'). \quad (40)$$

with $\boldsymbol{\theta} = (\theta_j)_{j=1, \dots, m}$ and $\boldsymbol{\theta}' = (\theta'_{j,s})_{j=1, \dots, m; s=1, \dots, s_j}$ and

$$\varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}') = \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_\xi \left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right) \right]. \quad (41)$$

For any event E and any $I \subset [-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}$, we also set

$$\varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}', E) = \mathbb{E} \left[\mathbf{1}_E \prod_{y \in \mathbb{Z}} \varphi_\xi \left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right) \right], \quad (42)$$

$$p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}, I, E) = \mathbf{1}_{\{\forall i, a_i = k_i \alpha + d\mathbb{Z}\}} \frac{d^m}{(2\pi)^M} \int_I e^{-i \sum_{j=1}^m [(a_j - a_{j-1})\theta_j + \sum_{s=1}^{s_j} (b_{j,s} - a_j)\theta'_{j,s}]} \varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}', E) d(\boldsymbol{\theta}, \boldsymbol{\theta}'). \quad (43)$$

and

$$B_{\mathbf{k}, \ell, I, E} = \sum_{\mathbf{a}, \mathbf{b}} \left(\prod_{j=1}^m \left(f(a_j) \prod_{s=1}^{s_j} f(b_{j,s}) \right) \right) p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}, I, E). \quad (44)$$

Let $\gamma < \min(L\theta, \frac{\eta\theta}{2M})$. Let $\theta' \in (0, \frac{\theta\eta}{2})$ such that $\theta' \leq \frac{\theta}{2} - 2ML\theta$. We consider the set

$$\Omega_{\mathbf{k}} := \left\{ \det D_{\mathbf{k}} \geq n^{-\theta'} \prod_{i=1}^m n_i^{\frac{3}{2}} \right\} \cap \bigcap_{j=1}^m \Omega_{\mathbf{k}}^{(j)}, \quad (45)$$

with

$$\Omega_{\mathbf{k}}^{(j)} := \left\{ \sup_{r=0, \dots, n_j} |S_{r+k_{j-1}} - S_{k_{j-1}}| \leq \frac{n_j^{\frac{1}{2}+\gamma}}{3}, \quad \sup_{y \neq z} \frac{|N'_j(y) - N'_j(z)|}{|y - z|^{\frac{1}{2}}} \leq n_j^{\frac{1}{4}+\frac{\gamma}{2}} \right\},$$

and with $D_{\mathbf{k}} = \left(\sum_{y \in \mathbb{Z}} N'_i(y) N'_j(y) \right)_{i,j}$. The following lemma follows from [14] (see appendix A for details).

Lemma 17. *For any $p > 1$, $\mathbb{P}(\Omega_{\mathbf{k}}) = 1 - o(n^{-p})$, and so $B_{\mathbf{k}, \ell, [-\frac{\pi}{d}, \frac{\pi}{d}]^M, \Omega_{\mathbf{k}}^c} = o(n^{-p})$.*

Note that, on $\Omega_{\mathbf{k}}$,

$$R'_j \leq n_j^{\frac{1}{2}+\gamma}, \quad (46)$$

$$N_j^* = |N_j^* - 0| \leq n_j^{\frac{1}{4}+\frac{\gamma}{2}} ((n_j)^{\frac{1}{2}+\gamma})^{\frac{1}{2}} \ll n_j^{\frac{1}{2}+\frac{\eta}{2}}, \quad (47)$$

$$V_j := \sum_{z \in \mathbb{Z}} (N'_j(z))^2 \geq \frac{\left(\sum_{z \in \mathbb{Z}} N'_j(z) \right)^2}{R'_j} \geq \frac{n_j^2}{n_j^{\frac{1}{2}+\gamma}} \geq n_j^{\frac{3}{2}-\frac{\eta}{2}}, \quad (48)$$

$$V_j \leq R'_j (N_j^*)^2 \leq n_j^{\frac{3(1+\eta)}{2}}. \quad (49)$$

It will be useful to notice that

$$|\varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}', E)| \leq \mathbb{E} \left[\mathbf{1}_E \prod_{y \in \mathcal{F}} \left| \varphi_{\xi} \left(\sum_{j=1}^m \theta_j N'_j(y) \right) \right| \right] \quad (50)$$

with

$$\mathcal{F} := \{y \in \mathbb{Z} : \forall (j, s), N'_{j,s}(y) = 0\},$$

and that

$$\#(\mathbb{Z} \setminus \mathcal{F}) \leq \sum_{j=1}^m \sum_{s=1}^{s_j} \ell_{j,s} \leq Mn^{L\theta} = o(n^{\frac{1}{4}}). \quad (51)$$

Using a straightforward adaptation of the proof of [13, Proposition 10], we prove (see Appendix A) that

Lemma 18.

$$B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(1)}, \Omega_{\mathbf{k}}} = o(e^{-n^c}), \quad (52)$$

uniformly on \mathbf{k}, ℓ as in Proposition 15, where $I_{\mathbf{k}}^{(1)}$ is the set of $(\boldsymbol{\theta}, \boldsymbol{\theta}') \in [-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}$ such that there exists $j = 1, \dots, m$ so that $n_j^{-\frac{1}{2}+\eta} < |\theta_j|$.

Lemma 19.

$$B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(2)}, \Omega_{\mathbf{k}}} = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{5}{4}+\eta} \right), \quad (53)$$

uniformly on \mathbf{k}, ℓ as in Proposition 15, where $I_{\mathbf{k}}^{(2)}$ is the set of $(\boldsymbol{\theta}, \boldsymbol{\theta}') \in [-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}$ such that for all $j = 1, \dots, m$, $|\theta_j| < n_j^{-\frac{1}{2}+\eta}$ and there exists $j' = 1, \dots, M$ such that $n_{j'}^{-\frac{1}{2}-\eta} < |\theta_{j'}|$.

It remains to estimate the integral over $I_{\mathbf{k}}^{(3)}$, the set of $(\boldsymbol{\theta}, \boldsymbol{\theta}') \in [-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}$ such that for all $j = 1, \dots, m$, $|\theta_j| < n_j^{-\frac{1}{2}-\eta}$.

We set $\mathcal{J} := \{j = 1, \dots, m : s_j = 0\} = \{j(1), \dots, j(J)\}$.

Lemma 20. *Under the assumptions of Theorem 5 with $\sum_{a \in \mathbb{Z}} f(b + ad) = 0$ for all $b \in \mathbb{Z}$. Let $\mathcal{J}' \subset \mathcal{J}$, then*

$$\sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} B_{\mathbf{k}+\mathbf{k}', \boldsymbol{\ell}, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \sum_{\mathcal{J}'' \subset \mathcal{J}' \cup (\mathcal{J}' + 1) : \#\mathcal{J}'' \geq \mathcal{J}'/2} \left(\prod_{j \in \mathcal{J}''} n_j^{-\frac{1}{2} + \eta} \right) \right), \quad (54)$$

uniformly on $\mathbf{k}, \boldsymbol{\ell}$ as in Proposition 15, and where we set $\mathbf{k}' = (k'_1, \dots, k'_m)$ with $k'_j = 0$ if $j \notin \mathcal{J}'$.

Moreover, if $s_j = 1$ for all j (and $\mathcal{J}' = \emptyset$), then,

$$B_{\mathbf{k}+\mathbf{k}', \boldsymbol{\ell}, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \left(\frac{d}{\sqrt{2\pi}\sigma_{\xi}} \right)^m \sum_{a_1, \dots, a_m \in \mathbb{Z}} \mathbf{1}_{\{\forall i, a_i = k_i \alpha + d\mathbb{Z}\}} \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \\ \prod_{j=1}^m f(a_j) \mathbb{E} [f(a_j + Z_{\ell_j})] + \mathcal{O} \left(n^{-(M+1)L\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}} \right),$$

uniformly on $\mathbf{k}, \boldsymbol{\ell}$ as above, with

$$\mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right),$$

uniformly on \mathbf{k} as above, and

$$\mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \sim n^{-\frac{3m}{4}} \mathbb{E} \left[\det \mathcal{D}_{t_1, \dots, t_m}^{-\frac{1}{2}} \right].$$

as $k_j/n \rightarrow t_j$ and $n \rightarrow +\infty$.

We can now complete the proof of Proposition 15. The two first points of Proposition 15 comes from the upper bounds provided by Lemmas 16, 17, 18, 19 and 20, with $E_{\mathbf{k}} := \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right]$. It remains to prove the last point of Proposition 15. We assume that $s_j = 1$ for all j and that $k_j/n \rightarrow t_j$ and $n \rightarrow +\infty$. Recall that $d_0 = \min\{n \geq 1 : n\xi_0 \in d\mathbb{Z}\} = \min\{n \geq 1 : n\alpha \in d\mathbb{Z}\}$. Observe that, for every $a_j \in \mathbb{Z}$ there is a unique $k' \in \{0, \dots, d_0 - 1\}$ such that $a_j \in (k_j + k'_j)\alpha + d\mathbb{Z}$. Thus

$$\sum_{k'_1, \dots, k'_m=0}^{d-1} \sum_{\ell_1, \dots, \ell_m=0}^{\frac{\kappa\theta\eta}{10M}} 2^{\#\{j:\ell_j>0\}} \prod_{j=1}^m \mathcal{A}_{k_j+k'_j, \ell_j} = \sum_{\ell_1, \dots, \ell_m=0}^{\frac{\kappa\theta\eta}{10M}} 2^{\#\{j:\ell_j>0\}} \sum_{a_j, b_j \in \mathbb{Z}} \prod_{j=1}^m f(a_j) f(b_j) \mathbb{P}(Z_{\ell_j} = b_j - a_j).$$

Finally, due to the last point of Lemma 20 and to the next lemma, this quantity is equivalent to

$$\sum_{\ell_1, \dots, \ell_m \geq 0} 2^{\#\{j:\ell_j>0\}} \sum_{a_j, b_j \in \mathbb{Z}} \prod_{j=1}^m f(a_j) f(b_j) \mathbb{P}(Z_{\ell_j} = b_j - a_j),$$

as $k_j/n \rightarrow t_j$ and $n \rightarrow +\infty$. □

Lemma 21. *Under the assumptions² of Theorem 5,*

$$\sum_{\ell \geq 1} \left| \sum_{\ell'=0}^{d-1} \sum_{a,b \in \mathbb{Z}} f(a)f(b) \mathbb{P}(Z_{\ell'+\ell d} = b-a) \right| < \infty$$

Proof. The proof of this lemma only uses estimates established in [13]. Since $\sum_{a,b} |f(a)f(b)| < \infty$ and using Lemma 16, we observe that

$$\begin{aligned} & \left| \sum_{\ell'=0}^{d-1} \sum_{a,b \in \mathbb{Z}} f(a)f(b) \mathbb{P}(Z_{\ell'+\ell d} = b-a) \right| \\ &= \left| \sum_{\ell'=0}^{d-1} \sum_{a \in \mathbb{Z}} \sum_{b \in a + (\ell d + \ell')\alpha + d\mathbb{Z}} f(a)f(b) \frac{d}{2\pi} \int_{[-\frac{\pi}{d}, \frac{\pi}{d}]} e^{-it(b-a)} \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_{\xi}(tN_{\ell d + \ell'}(y)) \right] dt \right|. \end{aligned}$$

Moreover, due to [13, Propositions 8,9,10], $\mathbb{P}(\Omega_k) = 1 - o(k^{-1-\eta_0})$ (due to [14, Lemma 16]), and due the fact that $|\varphi_{\xi}(tN_k(y))| \leq e^{-\frac{\sigma_{\xi}^2(tN_k(y))^2}{4}}$ on Ω_k when $|t| \leq k^{-\frac{3}{4}+\eta}$, we have

$$\begin{aligned} & \int_{[-\frac{\pi}{d}, \frac{\pi}{d}]} e^{-it(b-a)} \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_{\xi}(tN_{\ell d + \ell'}(y)) \right] \\ &= \int_{|t| \leq \ell^{-\frac{3}{4}+\eta}} e^{-it(b-a)} \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_{\xi}(tN_{\ell d + \ell'}(y)) \mathbf{1}_{\Omega_{\ell d + \ell'}} \right] dt + o(\ell^{-1-\eta_0}) \\ &= \int_{|t| \leq \ell^{-\frac{3}{4}+\eta}} e^{-it(b-a)} \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_{\xi}(tN_{\ell d}(y)) \mathbf{1}_{\Omega_{\ell d}} \right] dt + o(\ell^{-1-\eta_0}), \end{aligned}$$

using also the fact that $\#\{y \in \mathbb{Z} : N_{\ell d}(y) \neq N_{\ell d + \ell'}(y)\} \leq d$. It follows that

$$\begin{aligned} & \left| \sum_{\ell'=0}^{d-1} \sum_{a,b \in \mathbb{Z}} f(a)f(b) \mathbb{P}(Z_{\ell'+\ell d} = b-a) \right| \\ &= \left| \frac{d}{2\pi} \int_{|t| \leq \ell^{-\frac{3}{4}+\eta}} \sum_{a,b \in \mathbb{Z}} f(a)f(b) \left(e^{-it(b-a)} - 1 \right) \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_{\xi}(tN_{\ell d}(y)) \mathbf{1}_{\Omega_{\ell d}} \right] dt \right| + o(\ell^{-1-\eta_0}) \\ &\leq \frac{d}{2\pi} \int_{|t| \leq \ell^{-\frac{3}{4}+\eta}} \sum_{a,b} |f(a)f(b)t(b-a)| \mathbb{E} \left[e^{-\frac{\sigma_{\xi}^2 t^2 V_{\ell d}}{4}} \mathbf{1}_{\Omega_{\ell d}} \right] dt + o(\ell^{-1-\eta_0}) \\ &\leq C \mathbb{E} [V_{\ell d}^{-1} \mathbf{1}_{\Omega_{\ell d}}] + o(\ell^{-1-\eta_0}), \end{aligned}$$

since $\sum_{a,b \in \mathbb{Z}} f(a)f(b) = 0$, $\sum_{a \in \mathbb{Z}} |af(a)| < \infty$ and using the change of variable $v = tV_{\ell d}^{\frac{1}{2}}$. Now, due to (48), $V_{\ell d}^{-1} \mathbf{1}_{\Omega_{\ell d}} \leq \ell^{-\frac{3}{2}-2\gamma} = \mathcal{O}(\ell^{-1-\eta_0})$ up to take η_0 small enough, which ends the proof of the lemma. \square

²Our proof is valid in a more general context. The assumptions on f and S can be relaxed in $\sum_{a \in \mathbb{Z}} |af(a)| < \infty$, $\sum_{a \in \mathbb{Z}} f(a) = 0$, and $\|S_n\|_{L^{\frac{8}{3}}} = \mathcal{O}(\sqrt{n})$.

Theorem 5 follows directly from the following corollary of Proposition 15 and Lemma 21, since $\mathbb{E}[\mathcal{N}^{2N}] = \frac{(2N)!}{N!2^N}$ and $\mathbb{E}[(\mathcal{L}_1(0))^N] = \int_{[0,1]^N} \frac{\mathbb{E}[\det \mathcal{D}_{t_1, \dots, t_N}^{-\frac{1}{2}}]}{(2\pi)^{\frac{N}{2}}} dt_1 \dots dt_N$ (due to [14]).

Corollary 22. *[A rewriting of Theorem 5] Under the assumptions of Theorem 5,*

$$\mathbb{E} \left[\left(\sum_{k=1}^n f(Z_k) \right)^{2N+1} \right] = o \left(n^{\frac{2N+1}{8}} \right),$$

and

$$\mathbb{E} \left[\left(\sum_{k=1}^n f(Z_k) \right)^{2N} \right] = \frac{(2N)!}{N!2^N} n^{\frac{2N}{8}} \frac{\sigma_f^{2N}}{(2\pi\sigma_\xi^2)^{\frac{N}{2}}} \int_{[0,1]^N} \mathbb{E} \left[\det \mathcal{D}_{t_1, \dots, t_N}^{-\frac{1}{2}} \right] dx_1 \dots dx_N.$$

Proof. Since f is bounded, it is enough to prove the result for $n = n'd$. We start by writing

$$\mathbb{E} \left[\left(\sum_{k=1}^n f(Z_k) \right)^M \right] = \sum_{1 \leq m_1 \leq \dots \leq m_M \leq n} c_{\mathbf{m}} \mathbb{E} \left[\prod_{j=1}^M f(Z_{m_j}) \right], \quad (55)$$

where $c_{\mathbf{m}}$ is the number of $(r_1, \dots, r_M) \in \{1, \dots, n\}^M$ such that r_1, \dots, r_M and m_1, \dots, m_M contain the same values with same multiplicities.

Let $\theta_0 \in \left(0, \frac{1}{M+1}\right)$. Given a sequence $1 \leq m_1 \leq \dots \leq m_M \leq n$ with convention $m_0 = 0$, we consider $p \in \{0, \dots, M\}$ such that no $m_j - m_{j-1}$ (for $j = 1, \dots, M$) is in $(n^{L^{p+1}\theta_0}, n^{L^p\theta_0}]$. Set $\theta = L^p\theta_0$. We write $k_1 = m_1$ and, inductively, if $k_j = m_{u(j)}$, we set $k_{j+1} = m_{u(j+1)}$ for the smallest integer m_r such that $m_r > k_j + n^\theta$, $s_j = u(j+1) - u(j) - 1$ and then $\ell_{j,s} = m_{u(j)+s}$. Thus each $\mathbf{m} = (m_1, \dots, m_M)$ with $1 \leq m_1 \leq \dots \leq m_M \leq n$ can be represented by at least one

$$(\mathbf{k}, \ell) \in \bigcup_{p=0}^M \bigcup_{m=1}^M \bigcup_{s_j \geq 0: M = \sum_{j=1}^m (1+s_j)} F_{n, L^p\theta_0, m, s_1, \dots, s_m}, \quad (56)$$

with $F_{n, \theta, m, s_1, \dots, s_m}$ the set of M -uple (\mathbf{k}, ℓ) of nonnegative integers with $\mathbf{k} = (k_j)_{j=1, \dots, m}$, $\ell = (\ell_{j,s})_{j=1, \dots, m; s=1, \dots, s_j}$ such that, for all $j = 1, \dots, m$, $k_j \geq k_{j-1} + n^\theta$ (with convention $k_0 = 0$) and, for all $j = 1, \dots, m$ and all $s = 1, \dots, s_j$, $0 \leq \ell_{j,s} \leq n^{L\theta}$ and, with this representation,

$$\mathbb{E} \left[\prod_{j=1}^M f(Z_{m_j}) \right] = \mathbb{E} \left[\prod_{j=1}^m \left(f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_{j,s}}) \right) \right]. \quad (57)$$

We first study separately the following sums

$$\sum_{(m, \mathbf{s}) \in G_M} \sum_{(\mathbf{k}, \ell) \in F_{n, \theta, m, s_1, \dots, s_m}} c(\mathbf{k}, \ell) \mathbb{E} \left[\prod_{j=1}^m \left(f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_{j,s}}) \right) \right],$$

with G_M the set of (m, \mathbf{s}) with $m \in \{1, \dots, M\}$ and $\mathbf{s} = (s_1, \dots, s_m)$ with $s_j \geq 0$ for all $j = 1, \dots, m$ and such that $M = \sum_{j=1}^m (s_j + 1)$.

Let us fix for the moment $(m, \mathbf{s}) \in G_M$. With the notation (39), we wish to study

$$\sum_{(\mathbf{k}, \ell) \in F_{n, \theta, m, s_1, \dots, s_m}} \mathbb{E} \left[\prod_{j=1}^m \left(f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_{j,s}}) \right) \right] = \sum_{(\mathbf{k}, \ell) \in F_{n, \theta, m, s_1, \dots, s_m}} B_{\mathbf{k}, \ell}. \quad (58)$$

We say that (\mathbf{k}, ℓ) and (\mathbf{k}', ℓ') belong to a same block if

$$\forall r \notin \mathcal{J}, k_r = k'_r, \quad \forall j \in \mathcal{J}, \quad \lfloor k_j/d \rfloor = \lfloor k'_j/d \rfloor, \quad \ell = \ell'.$$

A block is an equivalence class for this equivalence relation. We write $F'_{n,\theta,m,s_1,\dots,s_m}$ for the set of (\mathbf{k}, ℓ) such that their block is contained in $F_{n,\theta,m,s_1,\dots,s_m}$. We will see that the contribution of the sum over $F_{n,\theta,m,s_1,\dots,s_m} \setminus F'_{n,\theta,m,s_1,\dots,s_m}$ is neglectable in (58). Indeed, observe that if $(\mathbf{k}, \ell) \in F_{n,\theta,m,s_1,\dots,s_m} \setminus F'_{n,\theta,m,s_1,\dots,s_m}$, then at least one of the following condition holds true

- (a) $\lfloor k_j/d \rfloor d - k_{j-1} < n^\theta \leq (\lfloor k_j/d \rfloor + 1)d - 1 - k_{j-1}$ if $j-1 \notin \mathcal{J}$ (or $\lfloor k_j/d \rfloor d - (\lfloor k_{j-1}/d \rfloor + 1)d - d < n^\theta \leq (\lfloor k_j/d \rfloor + 1)d - 1 - \lfloor k_{j-1}/d \rfloor d$ if $j-1 \in \mathcal{J}$)
- (b) $m \in \mathcal{J}$ and $d \lfloor k_m/d \rfloor + \max_s \ell_{m,s} < n \leq d(\lfloor k_j/d \rfloor + 1) + \max_s \ell_{m,s}$

Let us fix $\mathcal{J}'' \subset \mathcal{J}$. Due to the first point of Lemma 20, the contribution to (58) of blocks having a type (a) or (b) problem at indices \mathcal{J}'' is in

$$\begin{aligned} & \sum_{(k_j)_{j \notin \mathcal{J}'', \ell} } \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \sum_{\mathcal{J}' \in \{1, \dots, m\} : \#\mathcal{J}' \geq \#(\mathcal{J} \setminus \mathcal{J}'')/2} \prod_{j \in \mathcal{J}'} n_j^{-\frac{1}{2} + \eta} \right) \\ &= \mathcal{O} \left(n^{LM\theta} \sum_{(n_j)_{j \notin \mathcal{J}'' = n^\theta} } \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \sum_{\mathcal{J}' \in \{1, \dots, m\} : \#\mathcal{J}' \geq \#(\mathcal{J} \setminus \mathcal{J}'')/2} \prod_{j \in \mathcal{J}'} n_j^{-\frac{1}{2} + \eta} \right). \end{aligned}$$

The study of this quantity corresponds to (59) up to replace m par $m - \#\mathcal{J}''$ and to delete indices \mathcal{J}'' , which thus will be in $o(n^{-\frac{M}{8}})$, as proved below.

Now, using the d -block structure of $F'_{n,\theta,m,s_1,\dots,s_m}$, It follows from (38) that

$$\sum_{(\mathbf{k}, \ell) \in F'_{n,\theta,m,s_1,\dots,s_m}} B_{(\mathbf{k}, \ell)} = \mathcal{O} \left(n^{LM\theta} \sum_{n_1, \dots, n_m = n^\theta} \left(\prod_{i=1}^m n_i^{-\frac{3}{4}} \right) \left(\sum_{\mathcal{J}' \in \{1, \dots, m\} : \#\mathcal{J}' \geq \#(\mathcal{J})/2} \prod_{j \in \mathcal{J}'} n_j^{-\frac{1}{2} + \eta} \right) \right). \quad (59)$$

The above quantity is in

$$\begin{aligned} & \mathcal{O} \left(n^{LM\theta} \sum_{\mathcal{J}' : \#\mathcal{J}' \geq \#(\mathcal{J})/2} \sum_{n_1, \dots, n_m = n^\theta} \left(\prod_{i=1}^m n_i^{-\frac{3}{4}} \right) \prod_{r \in \mathcal{J}'} n_r^{-\frac{1}{2} - \eta} \right) \\ &= \mathcal{O} \left(\sum_{\mathcal{J}' : \#\mathcal{J}' \geq \#(\mathcal{J})/2} n^{LM\theta + \frac{1}{4}(m - \lceil \#(\mathcal{J})/2 \rceil) - (\frac{1}{4} - \eta)\theta \lceil \#(\mathcal{J})/2 \rceil} \right) \\ &= \mathcal{O} \left(n^{LM\theta + \frac{1}{4}(m - \lceil \#(\mathcal{J})/2 \rceil) - \frac{\theta}{4} \lceil \#(\mathcal{J})/2 \rceil + \theta J\gamma} \right), \end{aligned}$$

where we used the fact that $\sum_{r=1}^n r^{-\frac{3}{4}} = \mathcal{O} \left(n^{\frac{1}{4}} \right)$ and that $\sum_{r \geq n^\theta} r^{-\frac{5}{4}} = \mathcal{O} \left(n^{\frac{\theta}{4}} \right)$. Observe moreover that $M = \sum_{j=1}^m (s_j + 1) \geq 2(m - \#\mathcal{J}) + \#\mathcal{J} = 2m - \#\mathcal{J}$, with equality if and only if $s_j \in \{0, 1\}$ for all $j = 1, \dots, m$. It follows that

$$\sum_{(\mathbf{k}, \ell) \in F_{n,\theta,m,s_1,\dots,s_m}} \left| \mathbb{E} \left[\prod_{j=1}^m \left(f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_{j,s}}) \right) \right] \right| = \mathcal{O} \left(n^{LM\theta + \frac{M}{8} - \left[\frac{M - (2m - \#\mathcal{J})}{8} + \theta \left(\frac{\lceil \#(\mathcal{J})/2 \rceil}{4} - \#\mathcal{J}\eta \right) \right]} \right)$$

In particular this is in $o(n^{\frac{M}{8}})$ as soon as $M > 2m - \#\mathcal{J}$ or $\mathcal{J} \neq \emptyset$.

This ends the proof of the first point of Corollary 22 (since, when M is odd, we cannot have $M = 2m - \#\mathcal{J}$ and $\mathcal{J} = \emptyset$) and ensures that, for M even,

$$n^{-\frac{M}{8}} \mathbb{E} \left[\left(\sum_{k=1}^n f(Z_k) \right)^M \right] = n^{-\frac{M}{8}} \sum_{(\mathbf{k}, \boldsymbol{\ell}) \in \bigcup_{p=0}^M F_{n, L^p \theta_0, M/2, 1, \dots, 1}} c(\mathbf{k}, \boldsymbol{\ell}) \mathbb{E} \left[\prod_{j=1}^m (f(Z_{k_j}) f(Z_{k_j + \ell_{j,1}})) \right].$$

Assume from now on that $\theta = \theta_0$ and that M is even, $\mathcal{J} = \emptyset$ and $M = 2m$, which means that $s_j = 1$ for every $j = 1, \dots, m$ and let us estimate the following quantity

$$\mathcal{E}_{n, M, \theta} = \sum_{(\mathbf{k}, \boldsymbol{\ell}) \in F_{n, \theta, M/2, 1, \dots, 1}} c(\mathbf{k}, \boldsymbol{\ell}) \mathbb{E} \left[\prod_{j=1}^m (f(Z_{k_j}) f(Z_{k_j + \ell_{j,1}})) \right].$$

Note that, when $(\mathbf{k}, \boldsymbol{\ell}) \in F_{n, \theta, M/2, 1, \dots, 1}$, then $c(\mathbf{k}, \boldsymbol{\ell}) = \frac{(2m)!}{2^{\#\{j: \ell_j = 0\}}}$. Using this and applying Proposition 15 combined with the dominated convergence theorem, we obtain that

$$\begin{aligned} n^{-\frac{m}{4}} \mathcal{E}_{n, M, \theta} &= \frac{(2m)!}{2^m} n^{-\frac{m}{4}} \sum_{0 \leq k_1 < \dots < k_m \leq n: k_{i+1} - k_i > n^\theta} \sum_{\ell_1, \dots, \ell_m = 0}^{n^{L\theta}} 2^{\#\{j: \ell_j > 0\}} \mathbb{E} \left[\prod_{j=1}^m f(Z_{k_j}) f(Z_{k_j + \ell_j}) \right] \\ &= \frac{(2m)!}{2^m} n^{-m} \sum_{0 \leq k_1 < \dots < k_m \leq n/d: k_{i+1} - k_i > n^\theta} n^{\frac{3m}{4}} \sum_{k'_1, \dots, k'_m = 0}^{d-1} \sum_{\ell_1, \dots, \ell_m = 0}^{n^{L\theta}} 2^{\#\{j: \ell_j > 0\}} \mathbb{E} \left[\prod_{j=1}^m f(Z_{dk_j + k'_j}) f(Z_{dk_j + k'_j + \ell_j}) \right] + o(1) \\ &= \frac{(2m)!}{2^m} \int_{0 \leq t_1 < \dots < t_m \leq 1/d} \frac{d^m \sigma_f^{2m} \mathbb{E} \left[\det \mathcal{D}_{d_0 t_1, \dots, d_0 t_m}^{-\frac{1}{2}} \right]}{(2\pi \sigma_\xi^2)^{\frac{m}{2}}} dt_1 \dots dt_m + o(1). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{-\frac{m}{4}} \mathcal{E}_{n, M, \theta} &= \frac{(2m)!}{2^m} \int_{0 \leq s_1 < \dots < s_m \leq 1} \frac{\sigma_f^{2m} \mathbb{E} \left[\det \mathcal{D}_{s_1, \dots, s_m}^{-\frac{1}{2}} \right]}{(2\pi \sigma_\xi^2)^{\frac{m}{2}}} ds_1 \dots ds_m \\ &= \frac{(2m)! \sigma_f^{2m}}{m! 2^m (2\pi \sigma_\xi^2)^{\frac{m}{2}}} \int_{[0, 1]^m} \mathbb{E} \left[\det \mathcal{D}_{s_1, \dots, s_m}^{-\frac{1}{2}} \right] ds_1 \dots ds_m. \end{aligned}$$

It remains now to prove that we can neglect the contribution of the $(\mathbf{k}, \boldsymbol{\ell}) \in \bigcup_{p=1}^M F_{n, L^p \theta_0, M/2, 1, \dots, 1} \setminus F_{n, \theta_0, M/2, 1, \dots, 1}$. Fix some $p = 1, \dots, M$. It follows from (38) that

$$\begin{aligned} n^{-\frac{m}{4}} \sum_{(\mathbf{k}, \boldsymbol{\ell}) \in F_{n, L^p \theta_0, M/2, 1, \dots, 1} \setminus F_{n, \theta_0, M/2, 1, \dots, 1}} c(\mathbf{k}, \boldsymbol{\ell}) \mathbb{E} \left[\prod_{j=1}^m (f(Z_{k_j}) f(Z_{k_j + \ell_{j,1}})) \right] \\ = \mathcal{O} \left(n^{-\frac{m}{4}} \sum_{n_1, \dots, n_{m-1} = n^{L^p \theta_0}} \left(\prod_{i=1}^{m-1} n_i^{-\frac{3}{4}} \right) \sum_{n_m=1}^{n^{\theta_0}} n_m^{-\frac{3}{4}} n^{mL^{p+1}\theta_0} \right) = \mathcal{O} \left(n^{-\frac{1}{4} + \frac{\theta_0}{4} + mL^{p+1}\theta_0} \right) = o(1). \end{aligned}$$

□

The last part of Theorem 5 corresponds to the particular case $f = \delta_0 - \delta_a$. In this case

$$\sigma_f^2 = \sigma_{0,a}^2 = \sum_{k \in \mathbb{Z}} [2\mathbb{P}(Z|k| = 0) - \mathbb{P}(Z|k| = a) - \mathbb{P}(Z|k| = -a)].$$

APPENDIX A. PROOFS OF TECHNICAL LEMMAS FOR THEOREM 5

Recall the context. Let $M \geq 1$, $\theta \in (0, 1)$, $\eta \in (0, \frac{1}{100})$, $L = \frac{\kappa\eta}{10M}$. Recall that $n_j = k_j - k_{j-1}$ (with convention $k_0 = 0$). Assume $n^\theta < n_j < n$ and let $\ell_{j,1}, \dots, \ell_{j,s_j} = 0, \dots, \lfloor n^{L\theta} \rfloor$ with $\sum_{j=1}^m (1 + s_j) = M$.

Proof of Lemma 16. We start by writing

$$p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}) = \frac{1}{(2\pi)^M} \int_{[-\pi, \pi]^M} e^{-i \sum_{j=1}^m [(a_j - a_{j-1})\theta_j + \sum_{s=1}^{s_j} (b_{j,s} - a_j)\theta'_{j,s}]} \varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}') d(\boldsymbol{\theta}, \boldsymbol{\theta}'). \quad (60)$$

But, due to the definition of d , for any $u, v \in \mathbb{Z}$, $\varphi_\xi(u + \frac{2\pi v}{d}) = (\varphi_\xi(\frac{2\pi}{d}))^v \varphi_\xi(u)$ and so, for any $\mathbf{u} \in \mathbb{R}^M$ and $\mathbf{v} \in \mathbb{Z}^M$,

$$\begin{aligned} \varphi_{\mathbf{k}, \ell}(\mathbf{u} + \frac{2\pi}{d}\mathbf{v}) &= \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_\xi \left(\sum_{j=1}^m \left[\left(u_j + \frac{2\pi v_j}{d} \right) N'_j(y) + \sum_{s=1}^{s_j} \left(u_{j,s} + \frac{2\pi v_{j,s}}{d} \right) N'_{j,s}(y) \right] \right) \right] \\ &= \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \left(\varphi_\xi \left(\frac{2\pi}{d} \right) \right)^{\sum_{j=1}^m [v_j N'_j(y) + \sum_{s=1}^{s_j} v_{j,s} N'_{j,s}(y)]} \varphi_\xi \left(\sum_{j=1}^m \left[u_j N'_j(y) + \sum_{s=1}^{s_j} u_{j,s} N'_{j,s}(y) \right] \right) \right] \\ &= \left(\varphi_\xi \left(\frac{2\pi}{d} \right) \right)^{\sum_{j=1}^m [v_j n_j + \sum_{s=1}^{s_j} \ell_{j,s} v_{j,s}]} \varphi_{\mathbf{k}, \ell}(\mathbf{u}). \end{aligned}$$

and so

$$\begin{aligned} p_{\mathbf{k}, \ell}(\mathbf{a}, \mathbf{b}) &= \frac{1}{(2\pi)^M} \int_{[-\frac{\pi}{d}, \frac{\pi}{d}]^m \times [-\pi, \pi]^{M-m}} \sum_{r_j=0}^{d-1} e^{-i \sum_{j=1}^m [(a_j - a_{j-1})(\theta_j + \frac{2\pi r_j}{d}) + \sum_{s=1}^{s_j} (b_{j,s} - a_j)\theta'_{j,s}]} \\ &\quad \left(\varphi_\xi \left(\frac{2\pi}{d} \right) \right)^{\sum_{j=1}^m r_j n_j} \varphi_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}') d(\boldsymbol{\theta}, \boldsymbol{\theta}'). \end{aligned}$$

Moreover, for any $a \in \mathbb{Z}$, then $\sum_{r=0}^{d-1} e^{-\frac{2ia\pi r}{d}} (\varphi_\xi(\frac{2\pi}{d}))^{vr} = 0$ except if $e^{-\frac{2ia\pi}{d}} (\varphi_\xi(\frac{2\pi}{d}))^v = 1$ (i.e. if $va - a \in d\mathbb{Z}$) and then this sum is equal to d . This ends the proof of Lemma 16. \square

Proof of Lemma 17. Due to [14, Lemma 16], for any $\gamma > 0$, satisfies $\mathbb{P}(\Omega_{\mathbf{k}}^{(j)}) = 1 - o(n_j^{-\gamma})$ for any $\gamma > 1$ and so, since $n_j > n^\theta$, it follows that for all $p > 1$, $\mathbb{P}(\Omega_{\mathbf{k}}^{(j)}) = 1 - o(n^{-p})$. Moreover, since $\boldsymbol{\theta}' \in (0, \frac{\theta}{4})$, due to [14, Lemma 21],

$$\forall p > 1, \quad \mathbb{P} \left(\det D_{n_1, \dots, n_m} < n^{-\theta'} \prod_{i=1}^m n_i^{\frac{3}{2}} \right) = o(n^{-p}),$$

uniformly on \mathbf{k} as above. \square

Proof of Lemma 18. Recall that $\mathcal{F} = \{y \in \mathbb{Z} : \forall (j, s), N'_{j,s}(y) = 0\}$. Due to (50), Lemma 18 follows from the following estimate

$$\exists c > 0, \quad \int_{\{\exists j, n_j^{-\frac{1}{2} + \eta} < |\theta_j|\}} \mathbb{E} \left[\prod_{y \in \mathcal{F}} \left| \varphi_\xi \left(\sum_{j=1}^m \theta_j N'_j(y) \right) \right| \mathbf{1}_{\Omega_{\mathbf{k}}} \right] d\boldsymbol{\theta} = o(e^{-n^c}), \quad (61)$$

uniformly on \mathbf{k}, ℓ as in Proposition 15. To this end, we follow and slightly adapt the proof of [13, Proposition 10] as explained below. Observe that, up to conditioning with respect to

$(S_{k+1} - S_k)_{k \notin \{k_{j-1}, \dots, k_j-1\}}$, this will be a consequence of

$$\forall j = 1, \dots, m, \quad \forall u \in \mathbb{R}, \quad \int_{n_j^{-\frac{1}{2}+\eta} < |\theta| < \frac{\pi}{d}} \mathbb{E} \left[\prod_{y \in \mathcal{F}} |\varphi_\xi(u + \theta N'_j(y))| \mathbf{1}_{\Omega_k} \right] d\theta = o(e^{-n^c}), \quad (62)$$

uniformly on $k_j, \ell_{j,s}$ as above. Recall that $\#(\mathbb{Z} \setminus \mathcal{F}) \leq \sum_{j=1}^m \sum_{s=1}^{s_j} \ell_{j,s} \leq Mn^{L\theta}$. As in [13, after Lemma 16], we observe that, for n large enough,

$$\prod_{y \in \mathcal{F}} |\varphi_\xi(u + \theta N'_j(y))| \leq \exp \left(-\frac{\sigma_\xi^2}{4} n^{-\frac{1}{2}+4\gamma} \# \left\{ y : d \left(u + \theta N'_j(y), \frac{2\pi\mathbb{Z}}{d} \right) \geq n^{-\frac{1}{4}+2\gamma} \right\} \right), \quad (63)$$

and that

$$d \left(u + \theta N'_j(y), \frac{2\pi\mathbb{Z}}{d} \right) \geq n^{-\frac{1}{4}+2\gamma} \iff \frac{u}{\theta} + N'_j(y) \in \mathcal{I} := \bigcup_{k \in \mathbb{Z}} I_k, \quad (64)$$

where, for all $k \in \mathbb{Z}$,

$$I_k := \left[\frac{2k\pi}{d\theta} + \frac{n^{-\frac{1}{4}+2\gamma}}{\theta}, \frac{2(k+1)\pi}{d\theta} - \frac{n^{-\frac{1}{4}+2\gamma}}{\theta} \right].$$

In particular $\mathbb{R} \setminus \mathcal{I} = \bigcup_{k \in \mathbb{Z}} J_k$, where for all $k \in \mathbb{Z}$,

$$J_k := \left(\frac{2k\pi}{d\theta} - \frac{n^{-\frac{1}{4}+2\gamma}}{\theta}, \frac{2k\pi}{d\theta} + \frac{n^{-\frac{1}{4}+2\gamma}}{\theta} \right).$$

Let N_\pm be two positive integers such that $\mathbb{P}(X_1 = N_+) \mathbb{P}(X_1 = -N_-) > 0$. Let $\mathcal{C}^\pm = (\mathcal{C}_k^\pm)_{k=1, \dots, T} \in \mathbb{Z}^T$ with $T = N_+ + N_-$ and $\mathcal{C}_k^+ = N_+$ for $k \leq N_-$ and $\mathcal{C}_k^+ = -N_-$ otherwise, and symmetrically and $\mathcal{C}_k^- = -N_-$ for $k \leq N_+$ and $\mathcal{C}_k^- = N_+$ otherwise. It has been proved in [13] (see Lemma 15 therein combined with the estimate $\mathbb{P}(\mathcal{D}_n) = 1 - o(e^{-cn})$ in Section 2.8 therein) that, for n large enough,

$$\mathbb{P}(\Omega_k \setminus \mathcal{E}_j) = o(e^{-cn_j}), \quad (65)$$

with

$$\mathcal{E}_j = \left\{ \# \{ y \in \mathbb{Z} : C_j(y) \geq n_j^{\frac{1}{2}-2\gamma} \} \geq 3N_+N_- n_j^{\frac{1}{2}-2\gamma} \right\},$$

and where, for any $y \in \mathbb{Z}$,

$$C_j(y) := \# \left\{ k = 0, \dots, \left\lfloor \frac{n_j}{T} \right\rfloor - 1 : S_{k_{j-1}+kT} - S_{k_{j-1}} = y \text{ and } (X_{k_{j-1}+kT}, \dots, X_{k_{j-1}+(k+1)T-1}) = \mathcal{C}^\pm \right\}.$$

Now, on \mathcal{E}_j , we define Y_i for $i = 1, \dots, \left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor$, by

$$Y_1 := \min \left\{ y \in \mathbb{Z} : C_j(y) \geq n_j^{\frac{1}{2}-2\gamma} \right\},$$

and

$$Y_{i+1} := \min \left\{ y \geq Y_i + 3N_-N_+ : C_j(y) \geq n_j^{\frac{1}{2}-2\gamma} \right\} \quad \text{for } i \geq 1.$$

For every $i = 1, \dots, \left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor$, let $t_i^1, \dots, t_i^{\left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor}$ be the $\left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor$ first times (which are multiples of T) when a peak of the form \mathcal{C}^\pm is based on the site Y_i . We also define $N_j^0(Y_i + N_+N_-)$ as the number of visits of $(S_{k_{j-1}+k} - S_{k_{j-1}})_{k \geq 0}$ before time n_j to $Y_i + N_+N_-$, which do not occur

during the time intervals $[t_i^u, t_i^u + T]$, for $u \leq \left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor$. We proved in [13, Lemma 16] that, for any $H \geq 0$,

$$\mathbb{P}\left(\frac{u}{\theta} + N'_j(Y_i + N_+ N_-) \in \mathcal{I} \mid \mathcal{E}_n, N_j^0(Y_i + N_+ N_-) = H\right) = \mathbb{P}\left(H + \frac{u}{\theta} + b_j \in \mathcal{I}\right), \quad (66)$$

where b_j is a random variable with binomial distribution $\mathcal{B}\left(\left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor; \frac{1}{2}\right)$ and finally we proved in [13, Lemmas 17 and 18] (see in particular the last formula in the proof of Lemma 17) that

$$\forall H' \in \mathbb{R}, \quad \mathbb{P}(H' + b_n \in \mathcal{I}) \geq \frac{1}{3}.$$

Thus, conditionally to $(S_{k+1} - S_k)_{k \notin \{k_{j-1}, \dots, k_j-1\}}$, \mathcal{E}_j and $((N_j^0(Y_i + N_+ N_-), i \geq 1)$, the events $\{\frac{u}{\theta} + N_j(Y_i + N_+ N_-) \in \mathcal{I}\}$, $i \geq 1$, are independent of each other, and all happen with probability at least $1/3$. We conclude that

$$\mathbb{P}\left(\mathcal{E}_j \cap \left\{ \#\left\{i : \frac{u}{\theta} + N'_j(Y_i + N_+ N_-) \in \mathcal{I}\right\} \leq \frac{n_j^{\frac{1}{2}-2\gamma}}{4} \right\}\right) \leq \mathbb{P}\left(B_j \leq \frac{n_j^{\frac{1}{2}-2\gamma}}{4}\right) = o(e^{-c'n_j}), \quad (67)$$

where B_j has binomial distribution $\mathcal{B}\left(\left\lfloor n_j^{\frac{1}{2}-2\gamma} \right\rfloor; \frac{1}{3}\right)$.

But if $\#\{y \in \mathbb{Z} : N'_j(z) \in \mathcal{I}\} \geq n_j^{\frac{1}{2}-2\gamma}/4$, then, by (63) and (64) there exists a constant $c'' > 0$, such that, for any n large enough,

$$\prod_{y \in \mathcal{F}} |\varphi_\xi(u + \theta N'_j(y))| \leq \exp\left(-c'' n_j^{\frac{1}{2}-2\gamma} n_j^{-\frac{1}{2}+4\gamma}\right),$$

since $\#\mathbb{Z} \setminus \mathcal{F} \ll n_j^{\frac{1}{2}-2\gamma}/4$. This, combined with (65) and (67), ends the proof of (62) and so of Lemma 18. \square

Proof of Lemma 19. We have to estimate $B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(2)}, \Omega_{\mathbf{k}}}$ uniformly on \mathbf{k}, ℓ as in Proposition 15, where $I_{\mathbf{k}}^{(2)} = V_{\mathbf{k}} \times [-\pi, \pi]^{M-m}$ and where $V_{\mathbf{k}}$ is the set of $\boldsymbol{\theta} \in \mathbb{R}^m$ such that for all $j = 1, \dots, m$, $|\theta_j| < n_j^{-\frac{1}{2}+\eta}$ and such that there exists some $j_0 = 1, \dots, m$ satisfying $n_{j_0}^{-\frac{1}{2}-\eta} < |\theta_{j_0}|$. Let $\varepsilon_0 > 0$ be such that

$$\forall u \in [-\varepsilon_0, \varepsilon_0], \quad |\varphi_\xi(u)| \leq e^{-\frac{\sigma_\xi^2 u^2}{4}}. \quad (68)$$

We define the events $H_{\mathbf{k}} = \Omega_{\mathbf{k}} \cap \{\forall y \in \mathbb{Z}, |\sum_{j=1}^m \theta_j N'_j(y)| \leq \varepsilon_0/2\}$ and

$$H'_{\mathbf{k}} := \left\{ \#\left\{y \in \mathbb{Z} : \left| \sum_{j=1}^m \theta_j N'_j(y) \right| \in \left[\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{2} \right] \right\} > n^{\frac{1}{4}} \right\}.$$

Due to [14, Lemma 21 and last formula of p. 2446],

$$\exists c' > 0, \quad \mathbb{P}(\Omega_{\mathbf{k}} \setminus (H_{\mathbf{k}} \cup H'_{\mathbf{k}})) = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{3}{4}}\right),$$

uniformly on \mathbf{k} as above and uniformly on $\boldsymbol{\theta} \in V_{\mathbf{k}}$. Thus,

$$B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(2)}, \Omega_{\mathbf{k}} \setminus (H_{\mathbf{k}} \cup H'_{\mathbf{k}})} = \mathcal{O}\left(\prod_{j=1}^m n_j^{-\frac{5}{4}+\eta}\right), \quad (69)$$

where we used the fact that $\int_{V_{\mathbf{k}}} d\boldsymbol{\theta} \leq \prod_{j=1}^m n_j^{-\frac{1}{2}+\eta}$. Moreover, for n large enough, it follows from the definition of H'_k , from (51) and (68) that

$$B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(2)}, \Omega_{\mathbf{k}} \cap H'_k} = \mathcal{O} \left(\int_{V_{\mathbf{k}}} \mathbb{E} \left[\prod_{y \in \mathcal{F}} \left| \varphi_{\xi} \left(\sum_{j=1}^m \theta_j N'_j(y) \right) \right| \mathbf{1}_{\Omega_{\mathbf{k}} \cap H'_k} \right] d\boldsymbol{\theta} \right) \leq e^{-\frac{\sigma_{\xi}^2 \varepsilon_0^2 n^{\frac{1}{4}}}{64}}. \quad (70)$$

Finally, it remains to estimate $B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(2)}, \Omega_{\mathbf{k}} \cap H_k}$. To this end we write

$$\begin{aligned} & \int_{V_{\mathbf{k}}} \mathbb{E} \left[\prod_{y \in \mathcal{F}} \left| \varphi_{\xi} \left(\sum_{j=1}^m \theta_j N'_j(y) \right) \right| \mathbf{1}_{\Omega_{\mathbf{k}} \cap H_k} \right] d\boldsymbol{\theta} \\ & \leq \int_{V_{\mathbf{k}}} \mathbb{E} \left[e^{-\frac{\sigma_{\xi}^2}{4} \sum_{y \in \mathcal{F}} \left(\sum_{j=1}^m \theta_j N'_j(y) \right)^2} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] d\boldsymbol{\theta} \\ & \leq \int_{V_{\mathbf{k}}''} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathbb{E} \left[e^{-\frac{\sigma_{\xi}^2}{4} \sum_{y \in \mathcal{F}} \left(\sum_{j=1}^m \theta_j'' n_j^{-\frac{3}{4}} N'_j(y) \right)^2} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] d\boldsymbol{\theta}'' \\ & \leq \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathbb{E} \left[\int_{(\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} V_{\mathbf{k}}''} (\det \tilde{D}'_{\mathbf{k}})^{-\frac{1}{2}} e^{-\frac{\sigma_{\xi}^2 |\mathbf{v}|^2}{4}} \mathbf{1}_{\Omega_{\mathbf{k}}} d\mathbf{v} \right], \end{aligned} \quad (71)$$

with the successive changes of variable $\theta_j'' = n_j^{\frac{3}{4}} \theta_j$ and $\mathbf{v} = (\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} \boldsymbol{\theta}''$, with

$$\tilde{D}'_{\mathbf{k}} = \left((n_i n_j)^{-\frac{3}{4}} \sum_{y \in \mathcal{F}} N'_i(y) N'_j(y) \right)_{i,j} \quad \text{and} \quad V_{\mathbf{k}}'' = \text{Diag}(n_i^{\frac{3}{4}}) V_{\mathbf{k}}.$$

Note that $V_{\mathbf{k}}''$ is the set of $(\theta_1'', \dots, \theta_m'')$ such that $|\theta_j''| \leq n_j^{\frac{1}{4}+\eta}$ and such that there exists $j_0 = 1, \dots, m$ such that $|\theta_{j_0}''| \geq n_{j_0}^{\frac{1}{4}-\eta}$.

Let us prove that, in the above formula, we can approximate the determinant of $\tilde{D}'_{\mathbf{k}}$ by the one of $\tilde{D}_{\mathbf{k}} := \left((n_i n_j)^{-\frac{3}{4}} \sum_{y \in \mathbb{Z}} N'_i(y) N'_j(y) \right)_{i,j}$. To this end, writing Σ_m for the set of permutations of the set $\{1, \dots, m\}$ and $\varkappa(\sigma)$ for the signature of $\sigma \in \Sigma_m$, we observe that, on $\Omega_{\mathbf{k}}$,

$$\begin{aligned} & \left| \det \tilde{D}'_{\mathbf{k}} - \det \tilde{D}_{\mathbf{k}} \right| \\ & = \left(\prod_{j=1}^m n_j^{-\frac{3}{2}} \right) \left| \sum_{\sigma \in \Sigma_m} (-1)^{\varkappa(\sigma)} \prod_{j=1}^m \left(\sum_{y \in \mathcal{F}} N'_j(y) N'_{\sigma(j)}(y) \right) - \sum_{\sigma \in \Sigma_m} (-1)^{\varkappa(\sigma)} \prod_{j=1}^m \left(\sum_{y \in \mathbb{Z}} N'_j(y) N'_{\sigma(j)}(y) \right) \right| \\ & \leq \left(\prod_{j=1}^m n_j^{-\frac{3}{2}} \right) \sum_{\sigma \in \Sigma_m} \sum_{j=1}^m \sum_{z \in \mathbb{Z} \setminus \mathcal{F}} N'_j(z) N'_{\sigma(j)}(z) \prod_{j' \neq j} \left(\sum_{y \in \mathbb{Z}} N'_{j'}(y) N'_{\sigma(j')}(y) \right) \\ & \leq \left(\prod_{j=1}^m n_j^{-\frac{3}{2}} \right) \sum_{\sigma \in \Sigma_m} \sum_{j=1}^m \#(\mathbb{Z} \setminus \mathcal{F}) n_j^{\frac{1+2\gamma}{2}} n_{\sigma(j)}^{\frac{1+2\gamma}{2}} \prod_{j' \neq j} \sqrt{V_{j'} V_{\sigma(j')}} , \end{aligned}$$

where we used the Cauchy-Schwarz inequality together with the notations and estimates given after Lemma 17. Using (45) and (51), it follows that, on $\Omega_{\mathbf{k}}$,

$$\begin{aligned} \left| \det \tilde{D}'_{\mathbf{k}} - \det \tilde{D}_{\mathbf{k}} \right| &\ll \left(\prod_{j=1}^m n_j^{-\frac{3}{2}} \right) n^{L\theta} \sum_{j=1}^m n_j^{\frac{1+2\gamma}{2}} n_{\sigma(j)}^{\frac{1+2\gamma}{2}} \prod_{j' \neq j} n_{j'}^{\frac{3(1+2\gamma)}{4}} n_{\sigma(j')}^{\frac{3(1+2\gamma)}{4}} \\ &\ll \frac{1}{2} n^{m\gamma - \frac{\theta}{2} + L\theta} \ll n^{-\theta' - (M+1)L\theta} \leq \frac{n^{-(M+1)L\theta}}{2} \det \tilde{D}_{\mathbf{k}}, \end{aligned}$$

since $\theta' \leq \frac{\theta}{2} - 2ML\theta < \frac{\theta}{2} - m\gamma - ML\theta$ and where we used the fact that $\det \tilde{D}_{\mathbf{k}} = \det D_{\mathbf{k}} \prod_{j=1}^m n_j^{-\frac{3}{2}}$ together with the definition of $\Omega_{\mathbf{k}}$. Therefore, on $\Omega_{\mathbf{k}}$, $\det \tilde{D}'_{\mathbf{k}} \geq \frac{1}{2} \det \tilde{D}_{\mathbf{k}}$. Thus, due to (71),

$$\begin{aligned} &\int_{V_{\mathbf{k}}} \mathbb{E} \left[\prod_{y \in \mathcal{F}} \left| \varphi_{\xi} \left(\sum_{j=1}^m \theta_j N'_j(y) \right) \right| \mathbf{1}_{\Omega_{\mathbf{k}} \cap H_{\mathbf{k}}} \right] d\theta \\ &\leq \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathbb{E} \left[\int_{(\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} V''_{\mathbf{k}}} (\det \tilde{D}'_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} e^{-\frac{\sigma_{\xi}^2 |v|^2}{4}} dv \right] \right) \\ &= \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathbb{E} \left[(\det \tilde{D}_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \int_{(\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} V''_{\mathbf{k}}} e^{-\frac{\sigma_{\xi}^2 |v|^2}{4}} dv \right] \right), \end{aligned} \quad (72)$$

By definition of $V''_{\mathbf{k}}$, for any $v \in (\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} V''_{\mathbf{k}}$, $|v|_2 \geq (\tilde{\lambda}'_{\mathbf{k}})^{\frac{1}{2}} n^{\frac{1}{4} - \eta\theta}$, where $\tilde{\lambda}'_{\mathbf{k}}$ is the smallest eigenvalue of $\tilde{D}'_{\mathbf{k}}$. Since all the eigenvalues of $\tilde{D}'_{\mathbf{k}}$ are nonnegative ($\tilde{D}'_{\mathbf{k}}$ being symmetric and nonnegative), it follows that all the eigenvalues of $\tilde{D}'_{\mathbf{k}}$ are smaller than $\text{trace}(\tilde{D}'_{\mathbf{k}}) \leq \sum_{j=1}^m \frac{V_j}{n_j^{\frac{3}{2}}} \leq mn^{3\gamma}$ (on $\Omega_{\mathbf{k}}$). Thus, on $\Omega_{\mathbf{k}}$,

$$(\tilde{\lambda}'_{\mathbf{k}})^{\frac{1}{2}} n^{\frac{1}{4} - \eta\theta} \geq \frac{\det(\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}}}{(m^{\frac{1}{2}} n^{\frac{3\gamma}{2}})^{m-1}} n^{\frac{1}{4} - \eta\theta} \geq \frac{n^{(\frac{1}{4} - \eta)\theta - \frac{\theta'}{2} - \frac{3\gamma(m-1)}{2}}}{2m^{\frac{m-1}{2}}} \gg n^{\frac{\theta}{16}}, \quad (73)$$

since $\eta\theta$, $\frac{\theta'}{2}$, and $\frac{3\gamma(m-1)}{2}$ are all strictly smaller $\frac{\theta}{16}$. Hence

$$\begin{aligned} &\mathbb{E} \left[(\det \tilde{D}_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \int_{(\tilde{D}'_{\mathbf{k}})^{\frac{1}{2}} V''_{\mathbf{k}}} e^{-\frac{\sigma_{\xi}^2 |v|^2}{4}} dv \right] \\ &\leq \mathbb{E} \left[(\det \tilde{D}_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \int_{|v|_2 > n^{\frac{\theta}{16}}} e^{-\frac{\sigma_{\xi}^2 |v|^2}{4}} dv \right] = \mathcal{O}(n^{-p}), \end{aligned}$$

for any $p > 0$. This combined with (69), (70) and (72) ends the proof of the lemma. It will be worthwhile to note that the previous estimate also holds true when $\tilde{\lambda}'_{\mathbf{k}}$ is replaced by the smallest eigenvalue $\tilde{\lambda}_{\mathbf{k}}$ of $\tilde{D}_{\mathbf{k}}$. \square

Before proving Lemma 20, we state a useful coupling lemma allowing us to replace $\det D_{\mathbf{k}}$ by a copy independent of $(N'_{j,s})_{j,s}$.

Up to enlarging the probability space if necessary, we consider $X' = (X'_k)_{k \geq 1}$ an independent copy of the increments $X = (X_k)_{k \geq 0}$ of the random walk S . We then define the random walk S'' as follows: $S''_m = \sum_{k=1}^m X''_k$ with $X''_k = X_k$ if $k_{j-1} + \ell_{j-1} \leq k < k_j$ and $X''_k = X'_k$ if $k_j \leq k < k_j + \ell_j$, with $\ell_j := \max_{s=1, \dots, s_j} \ell_{j,s}$. We define $\Omega''_{\mathbf{k}}$, N''_j and $D''_{\mathbf{k}}$ for the space as we have defined $\Omega_{\mathbf{k}}$, N'_j , $D_{\mathbf{k}}$ (up to replace S by S'').

Lemma 23. *There exists $\Omega'_k \subset \Omega_k \cap \Omega''_k$ such that*

$$\forall p > 0, \quad \mathbb{P}((\Omega_k \cap \Omega''_k) \setminus \Omega'_k) = \mathcal{O}(n^{-p}) \quad (74)$$

and such that, on Ω'_k ,

$$\left| (\det D_k)^{-\frac{1}{2}} - (\det D''_k)^{-\frac{1}{2}} \right| \leq n^{-\frac{\theta}{8} - L\theta} (\det D_k)^{-\frac{3}{2}} + (\det D''_k)^{-\frac{3}{2}}.$$

Moreover

$$\mathbb{E} \left[\left((\det D_k)^{-\frac{1}{2}} - (\det D''_k)^{-\frac{1}{2}} \right) \mathbf{1}_{\Omega'_k} \right] \leq n^{-\frac{\theta}{8} - L\theta} \prod_{j=1}^m n_j^{-\frac{9}{4}}. \quad (75)$$

Proof of Lemma 23. Observe that

$$S''_{k_j} - S_{k_j} = h_j = \sum_{j' < j} \left(S'_{k_{j'} + \ell_{j'}} - S'_{k_{j'}} - (S_{k_{j'} + \ell_{j'}} - S_{k_{j'}}) \right)$$

and, on $\Omega_k \cap \Omega''_k$,

$$|N'_j(z) - N''_j(z)| = |N'_j(z) - N'_j(z + h_j)| \leq n_j^{\frac{1}{4} + \frac{\gamma}{2}} |h_j|^{\frac{1}{2}},$$

for all $z \in \mathbb{Z} \setminus \bigcup_{m=k_{j-1}}^{k_j-1+\ell_j} \{S_m, S''_m\}$.

We will prove that $\det D_k$ is close enough to $\det D''_k = \det(\sum_{y \in \mathbb{Z}} N'_i(y) N''_j(y))$. Due to the Markov inequality,

$$\forall p > 0, \quad \mathbb{P}(|S_{\ell_j}| > h) \leq \mathcal{O} \left(\frac{\ell_j^{\frac{p}{2}}}{h^p} \right) = \mathcal{O} \left(n^{-\gamma' p} \right),$$

where we set $h = n^{\gamma' + \frac{\kappa\theta\eta}{20M}} \geq n^{\gamma'} \ell_j^{\frac{1}{2}}$. Thus we set

$$\Omega'_k := \Omega_k \cap \Omega''_k \cap \{\forall j = 1, \dots, m, |h_j| \leq h\}$$

and we observe that $\mathbb{P}((\Omega_k \cap \Omega''_k) \setminus \Omega'_k) = \mathcal{O}(n^{-p})$ for all $p > 0$. Moreover, on Ω'_k ,

$$|N'_j(z) - N''_j(z)| \leq 2\ell_j + n_j^{\frac{1}{4} + \frac{\gamma}{2}} h^{\frac{1}{2}} \leq 3n_j^{\frac{1}{4} + \frac{\gamma}{2}} n^{\frac{\gamma'}{2} + \frac{\kappa\theta\eta}{40M}}.$$

Moreover

$$V_j'' := \sum_{y \in \mathbb{Z}} (N''_j(y))^2 \leq \sum_{y \in \mathbb{Z}} (N'_j(y))^2 + 2\ell_j^3 \leq n_j^{\frac{3}{2} + 3\gamma}.$$

This allows us to observe that, on Ω'_k ,

$$\begin{aligned} & \left| \det D_k - \det D''_k \right| \\ &= \left| \sum_{\sigma \in \Sigma_m} (-1)^{\varkappa(\sigma)} \prod_{j=1}^m \left(\sum_{y \in \mathbb{Z}} N'_j(y) (N'_{\sigma(j)}(y)) \right) - \sum_{\sigma \in \Sigma_m} (-1)^{\varkappa(\sigma)} \prod_{j=1}^m \left(\sum_{y \in \mathbb{Z}} N''_j(y) (N''_{\sigma(j)}(y)) \right) \right| \\ &\leq \sum_{\sigma \in \Sigma_m} \sum_{j=1}^m \sum_{z \in \mathbb{Z}} [N'_j(z) N'_{\sigma(j)}(z) - N''_j(z) N''_{\sigma(j)}(z)] \prod_{j' \neq j} \max \left(\sum_{y \in \mathbb{Z}} N'_{j'}(y) N'_{\sigma(j')}(y), \sum_{y \in \mathbb{Z}} N''_{j'}(y) N''_{\sigma(j')}(y) \right) \\ &\leq 3n^{\frac{\gamma'}{2} + \frac{\kappa\theta\eta}{40M}} \sum_{\sigma \in \Sigma_m} \sum_{j=1}^m \left[V_j^{\frac{1}{2}} n_{\sigma(j)}^{\frac{1}{2} + \gamma} + (V''_{\sigma(j)})^{\frac{1}{2}} n_j^{\frac{1}{2} + \gamma} \right] \prod_{j' \neq j} \max \left(V_{j'} V_{\sigma(j')}, V''_{j'} V''_{\sigma(j')} \right)^{\frac{1}{2}} \\ &\leq 3n^{\frac{\gamma'}{2} + \frac{\kappa\theta\eta}{40M}} m! \prod_{j'=1}^m n_{j'}^{\frac{3}{2} + 3\gamma} \sum_{j=1}^m n_j^{-\frac{1}{4} - \frac{\gamma}{2}} \ll \prod_{j'=1}^m n_{j'}^{\frac{3}{2}} \sum_{j=1}^m n_j^{-\frac{1}{8}} n^{-L\theta}, \end{aligned}$$

since $L\theta + 3m\gamma - \frac{\theta}{4} + \frac{\gamma'}{2} < -\frac{\theta}{8} - L\theta$, and so, on $\Omega'_{\mathbf{k}}$,

$$\left| (\det D_{\mathbf{k}})^{-\frac{1}{2}} - (\det D''_{\mathbf{k}})^{-\frac{1}{2}} \right| \leq n^{-\frac{\theta}{8} - L\theta} (\det D_{\mathbf{k}})^{-\frac{3}{2}} + (\det D''_{\mathbf{k}})^{-\frac{3}{2}}.$$

We conclude thanks to [14, Lemma 21] which ensures that $\mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{3}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{9}{4}} \right)$. \square

The proof of Lemma 20 will also use the following result. Recall that we set $\mathcal{J} = \{j = 1, \dots, m : s_j = 0\}$ and that $\mathcal{J}' \subset \mathcal{J}$.

Lemma 24. *Under the assumptions of Lemma 20,*

$$\sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} B_{\mathbf{k}+\mathbf{k}', \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \frac{d^m}{(2\pi)^M} \int_{I_{\mathbf{k}}^{(3)}} \mathbb{E} \left[\mathbf{1}_{\Omega_{\mathbf{k}}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') G(\boldsymbol{\theta}, \boldsymbol{\theta}') \right] d(\boldsymbol{\theta}, \boldsymbol{\theta}'),$$

with $\mathbf{k}' \in \mathbb{Z}^m$ such that $k'_j = 0$ for all $j \notin \mathcal{J}'$, and with

$$\begin{aligned} G(\boldsymbol{\theta}, \boldsymbol{\theta}') := & \prod_{j \notin \mathcal{J}'} \left(\sum_{a_j \in \alpha k_j + d\mathbb{Z}} \sum_{b_{j,s}, \dots, b_{j,s_j} \in \mathbb{Z}} f(a_j) \left(\prod_{v=1}^{s_j} f(b_{j,v}) \right) e^{ia_j(\theta_{j+1} - \theta_j) - i \sum_{s=1}^{s_j} (b_{j,s} - a_j) \theta'_{j,s}} \right) \\ & \times \prod_{y \in \mathbb{Z} \setminus \mathcal{S}'} \varphi_{\xi} \left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right), \end{aligned}$$

with $\mathcal{S}' = \bigcup_{j \in \mathcal{J}'} \mathcal{S}'_j$, $\mathcal{S}'_j := \{S_{k_j}, \dots, S_{k_j+d-1}\}$, so that $\{S_{k_j}, \dots, S_{k_j+d-1}\}$ and, uniformly on \mathbf{k}, ℓ and on $\Omega_{\mathbf{k}}$, $F(\boldsymbol{\theta}, \boldsymbol{\theta}') = \mathcal{O} \left(\sum_{\mathcal{J}'' \subset \mathcal{J}'} \prod_{j \in \mathcal{J}' \setminus \mathcal{J}''} (|\theta_j| + |\theta_{j+1}|) \mathbf{1}_{\cap_{j \in \mathcal{J}''} \mathcal{B}_j} \right)$ with $\mathcal{B}_j = \mathbf{1}_{\mathcal{S}'_j \cap \bigcup_{j' \in \mathcal{J}'' \setminus \{j\}} \mathcal{S}'_{j'} \neq \emptyset}$. If $\sum_{a \in \mathbb{Z}} f(b + d\mathbb{Z}) = 0$ for all $b \in \mathbb{Z}$ (true if $d = 1$), then $F(\boldsymbol{\theta}, \boldsymbol{\theta}') = \mathcal{O} \left(\prod_{j \in \mathcal{J}'} (|\theta_j| + |\theta_{j+1}|) \right)$ (with convention $\theta_{m+1} = 0$).

Proof. We start by writing

$$\sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} B_{\mathbf{k}+\mathbf{k}', \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \frac{d^m}{(2\pi)^M} \int_{I_{\mathbf{k}}^{(3)}} \mathbb{E} \left[\mathbf{1}_{\Omega_{\mathbf{k}}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') G(\boldsymbol{\theta}, \boldsymbol{\theta}') \right] d(\boldsymbol{\theta}, \boldsymbol{\theta}'), \quad (76)$$

where we set

$$\begin{aligned} F(\boldsymbol{\theta}, \boldsymbol{\theta}') := & \sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} \prod_{j \in \mathcal{J}'} \left(\sum_{a_j \in (k_j + k'_j)\alpha + d\mathbb{Z}} \left(f(a_j) e^{-ia_j(\theta_j - \theta_{j+1})} \right) \right) \\ & \times \prod_{y \in \mathcal{S}'} \varphi_{\xi} \left(\sum_{r=1}^m \left(\theta_r \tilde{N}'_{r, \mathbf{k}'}(y) + \sum_{s=1}^{s_r} \theta'_{r,s} \tilde{N}'_{r,s}(y) \right) \right), \end{aligned}$$

with

$$\tilde{N}'_{r, \mathbf{k}'}(y) = \#\{u = k_{r-1} + k'_{r-1}, \dots, k_r + k'_r - 1 : S_u = y\}.$$

If we had $\sum_{a \in u + d\mathbb{Z}} f(a) = 0$ for all $u \in \mathbb{Z}$, the proof of Lemma 24 will be ended by noticing that

$$\sum_{a_j \in (k_j + k'_j)\alpha + d\mathbb{Z}} \left(f(a_j) e^{-ia_j(\theta_j - \theta_{j+1})} \right) = \sum_{a_j \in (k_j + k'_j)\alpha + d\mathbb{Z}} \left(f(a_j) \left(e^{-ia_j(\theta_j - \theta_{j+1})} - 1 \right) \right),$$

which is in $\mathcal{O}(|\theta_j| + |\theta_{j+1}|)$ since $\sum_{a \in \mathbb{Z}} |af(a)| < \infty$. Since we just assume here that $\sum_{a \in \mathbb{Z}} f(a) = 0$, we need a more delicate approach. We rewrite F as follows

$$F(\boldsymbol{\theta}, \boldsymbol{\theta}') := \sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} \sum_{a_j \in (k_j + k'_j)\alpha + d\mathbb{Z} \forall j \in \mathcal{J}'} \left(\prod_{j \in \mathcal{J}'} H_{j, k'_j}(\theta_j - \theta_{j+1}) \right) \Psi(\mathbf{k}')$$

with

$$H_{j, k'_j}(\theta) := \sum_{a_j \in (k_j + k'_j)\alpha + d\mathbb{Z}} \left(f(a_j) e^{-ia_j \theta} \right),$$

$$\Psi(\mathbf{k}') = \prod_{y \in \mathcal{S}'} \varphi_\xi \left(\sum_{r=1}^m \left(\theta_r \tilde{N}'_{r, \mathbf{k}'}(y) + \sum_{s=1}^{s_r} \theta'_{r, s} \tilde{N}'_{r, s}(y) \right) \right),$$

and with $\tilde{N}'_{r, \mathbf{k}'}(y) = \#\{u = k_{r-1} + k'_{r-1}, \dots, k_r + k'_r - 1 : S_u = y\}$. Note that $\tilde{N}'_{r, \mathbf{k}'}(y) = N'_r(y)$ except maybe if $r \in \mathcal{J}'$ and $y \in \mathcal{S}'_r$ or if $r-1 \in \mathcal{J}'$ and $y \in \mathcal{S}'_{r-1}$. We order the elements of \mathcal{J}' as follows: $j'_1 < \dots < j'_{J'}$ and write

$$F(\boldsymbol{\theta}, \boldsymbol{\theta}') = F_0(\boldsymbol{\theta}, \boldsymbol{\theta}') + F_1(\boldsymbol{\theta}, \boldsymbol{\theta}')$$

with

$$F_1(\boldsymbol{\theta}, \boldsymbol{\theta}') = \sum_{k'_{j'_1}=0}^{d-1} H_{j'_1, k'_{j'_1}}(0) \sum_{k'_{j'_2}, \dots, k'_{j'_{J'}}=0}^{d-1} \left(\prod_{j \in \mathcal{J}' \setminus \{1\}} H_{j, k'_j}(\theta_j - \theta_{j+1}) \right) \Psi(\mathbf{k}')$$

and

$$F_0(\boldsymbol{\theta}, \boldsymbol{\theta}') = \sum_{k'_{j'_1}=0}^{d-1} \left(H_{j'_1, k'_{j'_1}}(\theta_{j'_1} - \theta_{j'_1+1}) - H_{j'_1, k'_{j'_1}}(0) \right) \sum_{k'_{j'_2}, \dots, k'_{j'_{J'}}=0}^{d-1} \left(\prod_{j \in \mathcal{J}' \setminus \{1\}} H_{j, k'_j}(\theta_j - \theta_{j+1}) \right) \Psi(\mathbf{k}').$$

Note that $H_{j'_1, k'_{j'_1}}(\theta_{j'_1} - \theta_{j'_1+1}) - H_{j'_1, k'_{j'_1}}(0)$ is in $\mathcal{O}(|\theta_j| + |\theta_{j+1}|)$. Since $\sum_a f(a) = 0$, F_1 satisfies

$$F_1(\boldsymbol{\theta}, \boldsymbol{\theta}') = \sum_{k'_{j'_1}=0}^{d-1} H_{j'_1, k'_{j'_1}}(0) \sum_{k'_{j'_2}, \dots, k'_{j'_{J'}}=0}^{d-1} \left(\prod_{j \in \mathcal{J}' \setminus \{j'_1\}} H_{j, k'_j}(\theta_j - \theta_{j+1}) \right) \Delta_{j'_1} \Psi(\mathbf{k}')$$

with $\Delta_j \phi(\mathbf{k}') = \phi(\mathbf{k}') - \phi(\mathbf{k}'_j)$, where $\mathbf{k}'_j \in \mathbb{N}^m$ is such that $(\mathbf{k}'_j)_i = k'_i$ for $i \neq j$, and $(\mathbf{k}'_j)_j = 0$. Proceeding iteratively on \mathcal{J}' , we obtain

$$F(\boldsymbol{\theta}, \boldsymbol{\theta}') = \sum_{\epsilon_1, \dots, \epsilon_{J'} \in \{0, 1\}} F_{\epsilon_1, \dots, \epsilon_{J'}}(\boldsymbol{\theta}, \boldsymbol{\theta}'), \quad (77)$$

with

$$F_{\epsilon_1, \dots, \epsilon_{J'}}(\boldsymbol{\theta}, \boldsymbol{\theta}') = \left(\prod_{j': \epsilon_{j'}=0} \left(H_{j'_1, k'_{j'_1}}(\theta_{j'_1} - \theta_{j'_1+1}) - H_{j'_1, k'_{j'_1}}(0) \right) \right) \left(\prod_{j: \epsilon_j=1} H_{j, k'_j}(0) \right) \Delta_{j'_{J'}}^{\epsilon_{j'_{J'}}} \cdots \Delta_{j'_1}^{\epsilon_{j'_1}} \Psi(\mathbf{k}'),$$

with convention $\Delta_{j'}^0 = Id$. The first part will be easily dominated by $\mathcal{O}(\prod_{j': \epsilon_{j'}=0} (|\theta_{j'}| + |\theta_{j'+1}|))$. Let us study the second part of the formula exploiting the fact that $\sum_{a \in \mathbb{Z}} f(a) = 0$. The difficulty here is that \mathbf{k}' appears both in $(\prod_{j: \epsilon_j=1} H_{j, k'_j}(0))$ and in $\Delta_{\dots} \psi(\mathbf{k}')$. The value of $(\epsilon_1, \dots, \epsilon_{J'})$ being fixed, we consider the set \mathcal{J}'' of the $j' \in \mathcal{J}'$ such that $\epsilon_{j'} = 1$. Observe that, if $\mathcal{S}'_{j'} \cap \mathcal{S}'_j = \emptyset$, then

$$\Delta_{j'} \Delta_j \Psi(\mathbf{k}') = \left(\Delta_{j'} \Psi_{\mathcal{S}' \setminus \mathcal{S}'_j}(\mathbf{k}'_j) \right) \left(\Delta_j \Psi_{\mathcal{S}'_j}(\widehat{\mathbf{k}}'_j) \right)$$

with

$$\Psi_{S_0}(\mathbf{k}') = \prod_{y \in S_0} \varphi_\xi \left(\sum_{r=1}^m \left(\theta_r \tilde{N}'_{r,\mathbf{k}'}(y) + \sum_{s=1}^{s_r} \theta'_{r,s} \tilde{N}'_{r,s}(y) \right) \right),$$

and where we set $\hat{\mathbf{k}}'_j$ for the vector of \mathbb{Z}^m with j -th coordinate equal to k'_j , all the other coordinates being null. Let \mathcal{J}''_0 be the set of $j \in \mathcal{J}''$ such that $\mathcal{S}'_j \cap \bigcup_{j'' \in \mathcal{S}'' \setminus \{j\}} \mathcal{S}'_{j''} = \emptyset$. Then

$$\begin{aligned} & \sum_{k_j=0, \dots, d-1, \forall j \in \mathcal{J}''_0} \left(\prod_{j \in \mathcal{J}''_0} H_{j,k'_j}(0) \right) \Delta_{j'_{j'}}^{\epsilon_{j'}} \cdots \Delta_{j'_1}^{\epsilon_1} \Psi(\mathbf{k}') \\ &= \prod_{j \in \mathcal{J}''_0} \left(\sum_{k'_j=0}^{d-1} H_{j,k'_j}(0) \Delta_j \Psi_{\mathcal{S}'_j}(\hat{\mathbf{k}}'_j) \right) \Delta_{\mathcal{J}'' \setminus \mathcal{J}''_0} \Psi(\mathbf{k}'_{\mathcal{S}''_0}) \end{aligned}$$

with $\mathbf{k}'_{\mathcal{S}''_0} \in \mathbb{N}^m$ such that $(\mathbf{k}'_j)_i = k'_i$ for $i \notin \mathcal{S}''_0$, the other coordinates being null, the notation $\Delta_{\mathcal{J}'' \setminus \mathcal{J}''_0}$ standing for the composition of all the operators Δ_j for $j \in \mathcal{J}'' \setminus \mathcal{J}''_0$. We conclude by using (77) and by noticing that

$$\begin{aligned} & \left(\prod_{j' \in \mathcal{S} \setminus \mathcal{S}''} \left(H_{j'_1, k'_{j'_1}}(\theta_{j'_1} - \theta_{j'_1+1}) - H_{j'_1, k'_{j'_1}}(0) \right) \right) = \mathcal{O} \left(\prod_{j' \in \mathcal{S} \setminus \mathcal{S}''} (|\theta_{j'}| + |\theta_{j'+1}|) \right), \\ & \prod_{j \in \mathcal{J}''_0} \left(\sum_{k'_j=0}^{d-1} H_{j,k'_j}(0) \Delta_j \Psi_{\mathcal{S}'_j}(\hat{\mathbf{k}}'_j) \right) = \mathcal{O} \left(\prod_{j' \in \mathcal{S} \setminus \mathcal{S}''} (|\theta_{j'}| + |\theta_{j'+1}|) \right) \end{aligned}$$

and that

$$j \in \mathcal{J}'' \setminus \mathcal{J}''_0 \implies \mathcal{S}'_j \cap \bigcup_{j' \in \mathcal{S}''_0 \setminus \{j\}} \mathcal{S}'_{j'}.$$

□

The following lemma will be useful to estimate the term F appearing in Lemma 24. It is not needed when $\sum_{a \in \mathbb{Z}} f(b+ad) = 0$ for all $b \in \mathbb{Z}$.

Lemma 25. *For any $\mathcal{J}' \subset \mathcal{J}$,*

$$\mathbb{P} \left(\Omega_{\mathbf{k}} \cap \bigcap_{j \in \mathcal{J}'} \mathcal{B}_j \right) = \mathcal{O} \left(\sum_{\mathcal{J}'' \subset \mathcal{J}' \setminus \{\min \mathcal{J}'\}, \mathcal{J}'' \geq \#\mathcal{J}'/2} n^{J\gamma} \prod_{j \in \mathcal{J}''} (k_j - k_j^-)^{-\frac{1}{2}} \right),$$

where $k_j^- = \max\{k_s \leq k_j, s \in \mathcal{J}'\}$.

Proof. It is enough to study

$$\mathbb{P} \left(\Omega_{\mathbf{k}} \cap \bigcap_{j \in \mathcal{J}'} \left\{ S_{k_j+r_j} = S_{k_{m(j)}+s_j} \right\} \right)$$

for any $m(j) \in \mathcal{J}' \setminus \{j\}$, $r_j, s_j \in \{0, \dots, d-1\}$. This probability is dominated by

$$\mathbb{P} \left(\Omega_{\mathbf{k}} \cap \left\{ \forall j \in \mathcal{J}', |S_{k_j} - S_{k_{m(j)}}| \leq n^v \right\} \right) + o(n^{-p}),$$

for all $p, v > 0$. We partition the set \mathcal{J}' by the equivalence relation generated by the relation $j \sim m_j$. We write $\mathcal{R}(j)$ for the class of j and \mathcal{R} for the set of these equivalence classes. Observe

that the number of equivalent classes is at most $\lfloor \#\mathcal{J}'/2 \rfloor$. We order the set \mathcal{J}' in $j'_1 < \dots < j'_{J'}$. We wish to estimate

$$\sum_{A_r, r \in \mathcal{R}} \mathbb{P} \left(\Omega_{\mathbf{k}}, \forall i = 1, \dots, J' - 1, S_{k_{j'_{i+1}} - k_{j'_i}} = A_{\mathcal{R}(j'_{i+1})} - A_{\mathcal{R}(j'_i)} + \mathcal{O}(n^v) \right),$$

where the sum is over $(A_r)_{r \in \mathcal{R}} \in \mathbb{Z}^{\mathcal{R}}$ such that $A_{\mathcal{R}(1)} = 0$, $A_{\mathcal{R}(j'_{i+1})} - A_{\mathcal{R}(j'_i)} = \mathcal{O}((k_{j'_{i+1}} - k_{j'_i})^{\frac{1}{2} + \frac{\gamma}{2}})$. Due to the local limit theorem and the independence of the increments of S , the above probability is in

$$\sum_{A_r, r \in \mathcal{R}} \prod_{i=1}^{J'-1} n^v \left(\mathcal{O} \left((k_{j'_{i+1}} - k_{j'_i})^{-\frac{1}{2}} \right) \right).$$

Now let us control the cardinal of the admissible $(A_r, r \in \mathcal{R})$. To this end, consider the set $\overline{\mathcal{J}'}$ of the smallest representants of \mathcal{R} . Then the above quantity is smaller than

$$n^{J'(v+\frac{\gamma}{2})} \prod_{j \in \mathcal{J}' \setminus \overline{\mathcal{J}'}} (k_j - k_j^-)^{-\frac{1}{2}}.$$

□

Proof of Lemma 20. All the estimates below are uniformly in \mathbf{k} . For the first estimate, we have to estimate the following integral

$$\int_{\forall j, |\theta_j| < n_j^{-\frac{1}{2} - \eta}} \left(\prod_{j \notin \mathcal{J}'} \left(\sum_{a_j \in \alpha k_j + d\mathbb{Z}} f(a_j) e^{ia_j(\theta_{j+1} - \theta_j)} \prod_{s=1}^{s_j} \sum_{b_{j,s} \in \mathbb{Z}} \left(f(b_{j,s}) e^{-i(b_{j,s} - a_j)\theta'_{j,s}} \right) \right) \right) \times \mathbb{E} \left[\mathbf{1}_{\Omega_{\mathbf{k}}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') \prod_{y \in \mathbb{Z} \setminus \mathcal{S}'} \mathfrak{A}_y \right] d\boldsymbol{\theta}, \quad (78)$$

where we set

$$\mathfrak{A}_y := \varphi_{\xi} \left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right).$$

Let us study

$$E_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}') := \prod_{y \in \mathbb{Z} \setminus \mathcal{S}'} \mathfrak{A}_y - \prod_{y \in \mathbb{Z} \setminus \mathcal{S}'} \mathfrak{B}_y, \quad (79)$$

with

$$\mathfrak{B}_y := \exp \left(-\frac{\sigma_{\xi}^2}{2} \left(\sum_{j=1}^m \theta_j N'_j(y) \right)^2 \right) \varphi_{\xi} \left(\sum_{j=1}^m \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right).$$

But, on $\Omega_{\mathbf{k}}$, if $|\theta_j| \leq n_j^{-\frac{1}{2} - \eta}$ for all $j = 1, \dots, m$, and so

$$\forall y \in \mathbb{Z}, \quad \left| \sum_{j=1}^m \theta_j N'_j(y) \right| \leq \sum_{j=1}^m |\theta_j| N_j^* \leq \sum_{j=1}^m n_j^{-\frac{\eta}{2}} \leq mn^{-\frac{\theta\eta}{2}} < \varepsilon_0, \quad (80)$$

as soon as n is large enough (uniformly on $n_j \in [n^{\theta}, n]$). Thus $|E_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}')|$ is dominated by

$$\sum_{y \in \mathbb{Z}} |\mathfrak{A}_y - \mathfrak{B}_y| e^{-\frac{\sigma_{\xi}^2}{4} \sum_{z \in \mathcal{F} \setminus (\mathcal{S}' \cup \{y\})} (\sum_{j=1}^m \theta_j N'_j(z))^2}$$

for n large enough. Now, on $\Omega_{\mathbf{k}}$, according to (51),

$$\forall y \in \mathbb{Z}, \quad \sum_{z \in \mathcal{F} \setminus (S' \cup \{y\})} \left(\sum_{j=1}^m \theta_j N'_j(z) \right)^2 \geq \sum_{z' \in \mathbb{Z}} \left(\sum_{j=1}^m \theta_j N'_j(z') \right)^2 - M(d + n^{\frac{\theta}{10M}}) n^{-\theta\eta}. \quad (81)$$

It follows that

$$|E_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}')| \leq (A + B) \exp \left(-\frac{\sigma_\xi^2}{4} \sum_{z' \in \mathbb{Z}} \left(\sum_{j=1}^m \theta_j N'_j(z') \right)^2 - \mathcal{O} \left(n^{-\frac{9\theta}{10\eta}} \right) \right), \quad (82)$$

with

$$A := \sum_{y \in \mathcal{F} \setminus S'} \left| \varphi_\xi \left(\sum_{j=1}^m \theta_j N'_j(y) \right) - e^{-\frac{\sigma_\xi^2}{2} (\sum_{j=1}^m \theta_j N'_j(y))^2} \right| \leq \sum_{y \in \mathbb{Z}} \left| \sum_{j=1}^m \theta_j N'_j(y) \right|^2 C' n^{-\frac{\kappa\theta\eta}{2}} \quad (83)$$

where we used the fact that

$$\left| \varphi_\xi(u) - \exp \left(-\frac{\sigma_\xi^2 |u|^2}{2} \right) \right| \leq |u|^{2+\kappa} \quad \text{for all } u \in \mathbb{R},$$

since ξ admits a moment of order $2 + \kappa$ and there exists $C_0 > 0$ such that

$$\begin{aligned} B &:= \sum_{y \in \mathbb{Z} \setminus \mathcal{F}} \left| \varphi_\xi \left(\sum_{j=1}^m \left(\theta_j N'_j(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right) - e^{-\frac{\sigma_\xi^2}{2} (\sum_{j=1}^m \theta_j N'_j(y))^2} \varphi_\xi \left(\sum_{j=1}^m \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right) \right| \\ &\leq C_0 \sum_{y \in \mathbb{Z} \setminus \mathcal{F}} \left| \sum_{j=1}^m \theta_j N'_j(y) \right| \leq C_0 \sum_{j=1}^m \sum_{s=1}^{s_j} \ell_{j,s} n^{-\frac{\theta\eta}{2}} = \mathcal{O} \left(n^{\frac{\theta\eta}{10M} - \frac{\theta\eta}{2}} \right) = \mathcal{O} \left(n^{-\frac{\theta\eta}{4}} \right), \end{aligned} \quad (84)$$

since φ_ξ and $u \mapsto e^{-\frac{u^2}{2}}$ are Lipschitz continuous. Recall that it has been proved in [14, Lemma 21] that

$$\mathbb{E} \left[|\det D_{\mathbf{k}}|^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right), \quad (85)$$

uniformly on \mathbf{k} .

Combining Lemmas 24 and (25), (82), (83), (84), (85) and using the change of variable $\mathbf{v} = (D_{\mathbf{k}})^{\frac{1}{2}} \boldsymbol{\theta}$ with $D_{\mathbf{k}} = \left(\sum_{y \in \mathbb{Z}} N'_i(y) N'_j(y) \right)_{i,j}$, it follows that there exists $C_1 > 0$ such that

$$\begin{aligned} &\int_{\forall j, |\theta_j| \leq n_j^{-\frac{1}{2}-\eta}} \mathbb{E} \left[|F(\boldsymbol{\theta}, \boldsymbol{\theta}') E_{\mathbf{k}, \ell}(\boldsymbol{\theta}, \boldsymbol{\theta}')| \mathbf{1}_{\Omega_{\mathbf{k}}} \right] d(\boldsymbol{\theta}, \boldsymbol{\theta}') \\ &\leq C_1 \int_{\mathbb{R}^m} \left(n^{-\frac{\kappa\theta\eta}{2}} |\mathbf{v}|_2^2 + \mathcal{O}(n^{-\frac{\theta\eta}{4}}) \right) e^{-\frac{\sigma_\xi^2 |\mathbf{v}|^2}{4}} d\mathbf{v} \\ &\quad \sum_{\mathcal{J}_0 \subset \mathcal{J}'} \prod_{j \in \mathcal{J}' \setminus \mathcal{J}_0} \left(n_j^{-\frac{1}{2}-\eta} + n_{j+1}^{-\frac{1}{2}-\eta} \right) \mathbb{E} \left[|\det D_{\mathbf{k}}|^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}} \cap \bigcap_{j \in \mathcal{J}_0} \mathcal{B}_j} \right] \\ &= \mathcal{O} \left(n^{-\frac{\kappa\theta\eta}{4}} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathfrak{E}_{\mathbf{k}}(\mathcal{J}') \right), \end{aligned} \quad (86)$$

with

$$\begin{aligned} \mathfrak{E}_{\mathbf{k}}(\mathcal{J}') &= \sum_{\mathcal{J}'' \subset \mathcal{J}'} \prod_{j \in \mathcal{J}' \setminus \mathcal{J}''} \left(n_j^{-\frac{1}{2}-\eta} + n_{j+1}^{-\frac{1}{2}-\eta} \right) \sum_{\mathcal{J}_0 \subset \mathcal{J}'' \setminus \{\min \mathcal{J}''\}, \mathcal{J}_0 \geq \#\mathcal{J}''/2} n^{J\gamma + \frac{\theta'}{2}} \left(\prod_{j \in \mathcal{J}_0} (k_j - k_j^-)^{-\frac{1}{2}} \right) \\ &= \mathcal{O} \left(\sum_{\mathcal{J}'' \subset \mathcal{J}' \cup (\mathcal{J}'+1): \#\mathcal{J}'' \geq \#\mathcal{J}'/2} \left(\prod_{j \in \mathcal{J}''} n_j^{-\frac{1}{2}+\eta} \right) \right). \end{aligned}$$

where $k_j^- = \max\{k_s \leq k_j, s \in \mathcal{J}''\}$. Combining this last estimate with (78) and Lemmas 24 and 25,

$$\begin{aligned} \sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} B_{\mathbf{k}+\mathbf{k}', \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} &= \frac{d^m}{(2\pi)^M} \sum_{(a_j)_{j \notin \mathcal{J}'}, (b_{j,s})_{j,s}} \mathbf{1}_{\{\forall i \notin \mathcal{J}', a_i \in k_i \alpha + d\mathbb{Z}\}} \int_{[-\pi, \pi]^{M-m}} \mathbb{E} [I_1(\mathbf{a}) I_2(\mathbf{a}, \mathbf{b}) \mathbf{1}_{\Omega_{\mathbf{k}}}] d\boldsymbol{\theta}' \\ &+ \mathcal{O} \left(n^{-\frac{\kappa\theta\eta}{4}} \prod_{j=1}^m n_j^{-\frac{3}{4}} \mathfrak{E}_{\mathbf{k}}(\mathcal{J}') \right), \end{aligned} \quad (87)$$

with

$$\begin{aligned} I_1(\mathbf{a}) &:= \int_{\forall j, |\theta_j| \leq n_j^{-\frac{1}{2}-\eta}} \left(\prod_{j \notin \mathcal{J}'} e^{-i \sum_{j=1}^m (a_j - a_{j-1}) \theta_j} \right) F(\boldsymbol{\theta}, \boldsymbol{\theta}') e^{-\frac{\sigma_{\xi}^2}{2} \sum_{y \in \mathbb{Z} \setminus \mathcal{S}'} (\sum_{j=1}^m \theta_j N'_j(y))^2} d\boldsymbol{\theta} \\ &= \mathcal{O} \left(\int_{\forall j, |\theta_j| \leq n_j^{-\frac{1}{2}-\eta}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') e^{-\frac{\sigma_{\xi}^2}{2} (\sum_{y \in \mathbb{Z}} (\sum_{j=1}^m \theta_j N'_j(y))^2 - M d n_j^{-\eta})} d\boldsymbol{\theta} \right) \\ &= \mathcal{O} \left(\det D_{\mathbf{k}}^{-\frac{1}{2}} \sup_{\boldsymbol{\theta} \in V_{\mathbf{k}}} F(\boldsymbol{\theta}, \boldsymbol{\theta}') \int_{\mathbb{R}^m} e^{-\frac{\sigma_{\xi}^2 |v|_2^2}{2}} dv \right), \end{aligned} \quad (88)$$

with the change of variable $\mathbf{v} = D_{\mathbf{k}}^{\frac{1}{2}} \boldsymbol{\theta}$ and

$$\begin{aligned} I_2(\mathbf{a}, \mathbf{b}) &:= \left(\prod_{j \notin \mathcal{J}'} \left(f(a_j) \prod_{s=1}^{s_j} f(b_{j,s}) e^{-i \sum_{j,s} (b_{j,s} - a_j) \theta'_{j,s}} \right) \right) \prod_{y \in \mathbb{Z} \setminus \mathcal{S}'} \varphi_{\xi} \left(\sum_{j,s} (\theta'_{j,s} N'_{j,s}(y)) \right) \\ &= \mathcal{O} \left(\prod_{j \notin \mathcal{J}'} \left(f(a_j) \prod_{s=1}^{s_j} f(b_{j,s}) \right) \right). \end{aligned} \quad (89)$$

Since $\sum_{a \in \mathbb{Z}} |f(a)| < \infty$, it follows from (85), (87), (88) and (89) that

$$\sum_{k'_j=0, \dots, d-1, \forall j \in \mathcal{J}'} B_{\mathbf{k}+\mathbf{k}', \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \mathcal{O} \left(\mathfrak{E}_{\mathbf{k}}(\mathcal{J}') \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \right).$$

This ends the proof of the first point of Lemma 20.

Assume now that $s_j = 1$ for all $j = 1, \dots, m$ (in particular $\mathcal{J} = \emptyset$). Then

$$\begin{aligned} I_1(\mathbf{a}) &= \int_{\forall j, |\theta_j| \leq n_j^{-\frac{1}{2}-\eta}} e^{-i \sum_{j=1}^m (a_j - a_{j-1}) \theta_j} e^{-\frac{\sigma_\xi^2}{2} \sum_{y \in \mathbb{Z} \setminus \mathcal{S}'} (\sum_{j=1}^m \theta_j N'_j(y))^2} d\boldsymbol{\theta} \\ &= \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \int_{\forall j, |\theta''_j| \leq n_j^{\frac{1}{4}-\eta}} e^{-i \sum_{j=1}^m n_j^{-\frac{3}{4}} (a_j - a_{j-1}) \theta''_j} e^{-\frac{\sigma_\xi^2}{2} \sum_{y \in \mathbb{Z}} (\sum_{j=1}^m \theta''_j n_j^{-\frac{3}{4}} N'_j(y))^2} d\boldsymbol{\theta}'' \\ &= \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \int_{\tilde{D}_{\mathbf{k}}^{\frac{1}{2}} U_{\mathbf{k}}} (\det \tilde{D}_{\mathbf{k}})^{-\frac{1}{2}} e^{-i \langle \tilde{D}_{\mathbf{k}}^{-\frac{1}{2}} (n_j^{-\frac{3}{4}} (a_j - a_{j-1}))_j, \mathbf{v} \rangle} e^{-\frac{\sigma_\xi^2 \|\mathbf{v}\|_2^2}{2}} d\mathbf{v}, \end{aligned}$$

where $U_{\mathbf{k}}$ is the set of $\boldsymbol{\theta}'' = (\theta''_1, \dots, \theta''_m)$ such that $|\theta''_j| \leq n_j^{\frac{1}{4}-\eta}$ for all $j = 1, \dots, m$ and with $\tilde{D}_{\mathbf{k}} = \left((n_i n_j)^{-\frac{3}{4}} \sum_{y \in \mathbb{Z}} N'_i(y) N'_j(y) \right)_{i,j}$. Moreover

$$\begin{aligned} I_2(\mathbf{a}, \mathbf{b}) &= (2\pi)^{\sum_{j=1}^m s_j} \left(\prod_{j=1}^m (f(a_j) f(b_{j,1})) \right) \mathbb{P} \left(\forall j, \sum_{y \in \mathbb{Z} \setminus \mathcal{S}'} N'_{j,1}(y) \xi_y = b_{j,1} - a_j \mid (N'_{j,1})_j \right) \\ &= (2\pi)^{M-m} \left(\prod_{j=1}^m f(a_j) \right) \mathbb{E} \left[f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,1}(y) \xi_y \right) \mathbf{1}_{\{a_j + \sum_{y \in \mathbb{Z}} N'_{j,1}(y) \xi_y = b_{j,1}\}} \mid (N'_{j,1})_j \right]. \end{aligned}$$

Thus, it follows that, uniformly in \mathbf{k} and on $\Omega_{\mathbf{k}}$,

$$\begin{aligned} \frac{d^m}{(2\pi)^M} \sum_{b_{1,1}, \dots, b_{m,1} \in \mathbb{Z}} I_1(\mathbf{a}) I_2(\mathbf{a}, \mathbf{b}) &= \left(\frac{d}{2\pi} \right)^m \left(\prod_{j=1}^m f(a_j) \right) \\ &\quad (\det D_{\mathbf{k}})^{-\frac{1}{2}} \left(\int_{\mathbb{R}^m} e^{-i \langle \tilde{D}_{\mathbf{k}}^{-\frac{1}{2}} (n_j^{-\frac{3}{4}} (a_j - a_{j-1}))_j, \mathbf{v} \rangle} e^{-\frac{\sigma_\xi^2 \|\mathbf{v}\|_2^2}{2}} d\mathbf{v} + \mathcal{O}(n^{-p}) \right) \\ &\quad \mathbb{E} \left[f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,1}(y) \xi_y \right) \mid (N'_{j,1})_j \right] \end{aligned}$$

for all $p > 0$, as seen at the end of the proof of Lemma 19 (applied with $\tilde{D}_{\mathbf{k}}$) and so

$$\begin{aligned} \frac{d^m}{(2\pi)^M} \sum_{b_{1,1}, \dots, b_{m,1} \in \mathbb{Z}} I_1(\mathbf{a}) I_2(\mathbf{a}, \mathbf{b}) &= \left(\frac{d}{2\pi} \right)^m \left(\prod_{j=1}^m f(a_j) \right) (\det D_{\mathbf{k}})^{-\frac{1}{2}} \\ &\quad \times \left(\int_{\mathbb{R}^m} \left(1 + \mathcal{O} \left(\left(\langle \tilde{D}_{\mathbf{k}}^{-\frac{1}{2}} (n_j^{-\frac{3}{4}} (a_j - a_{j-1}))_j, \mathbf{v} \rangle \right)^2 \right) \right) e^{-\frac{\sigma_\xi^2 \|\mathbf{v}\|_2^2}{2}} d\mathbf{v} + \mathcal{O}(n^{-p}) \right) \\ &\quad \times \mathbb{E} \left[f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \mid (N'_{j,1})_j \right], \end{aligned}$$

for all p . Due to (87), we obtain that

$$B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} = \left(\frac{d}{2\pi} \right)^m \sum_{a_1, \dots, a_m \in \mathbb{Z}} \mathbf{1}_{\{\forall i, a_i = k_i \alpha + d\mathbb{Z}\}} \quad (90)$$

$$\times \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \prod_{j=1}^m f(a_j) f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \right] \left(\frac{\sqrt{2\pi}}{\sigma_{\xi}} \right)^m \quad (91)$$

$$+ \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) n^{-\frac{\kappa\theta\eta}{4}} + \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} (\min_j n_j)^{-\frac{3}{2}} \tilde{\lambda}_{\mathbf{k}}^{-1} \mathbf{1}_{\Omega_{\mathbf{k}}} \right], \quad (92)$$

where $\tilde{\lambda}_{\mathbf{k}}$ is the smallest eigenvalue of $\tilde{D}_{\mathbf{k}}$. For the last term, we use (73) (applied for $\tilde{D}_{\mathbf{k}}$), which ensures that on $\Omega_{\mathbf{k}}$,

$$\tilde{\lambda}_{\mathbf{k}} \geq \frac{\det \tilde{D}_{\mathbf{k}}}{(mn^{3\gamma})^{m-1}}$$

and so

$$\begin{aligned} (\min_j n_j)^{-\frac{3}{2}} \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \lambda_{\mathbf{k}}^{-1} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] &\leq (mn^{3\gamma})^{m-1} (\min_j n_j)^{-\frac{3}{2}} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \mathbb{E} \left[(\det \tilde{D}_{\mathbf{k}})^{-\frac{3}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \\ &= \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) n^{-\frac{3\theta}{2} + 3(m-1)\gamma} \right), \end{aligned} \quad (93)$$

where we used [14, Lemma 21] which ensures that $\mathbb{E} \left[(\det \tilde{D}_{\mathbf{k}})^{-\frac{3}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] = \mathcal{O}(1)$ uniformly in \mathbf{k} . This combined with (92) implies that

$$\begin{aligned} B_{\mathbf{k}, \ell, I_{\mathbf{k}}^{(3)}, \Omega_{\mathbf{k}}} &= \mathcal{O} \left(\left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right) n^{-(M+1)L\theta} \right) \\ &+ \left(\frac{d}{\sqrt{2\pi}\sigma_{\xi}n^{\frac{3}{4}}} \right)^m \sum_{a_1, \dots, a_m \in \mathbb{Z}} \mathbf{1}_{\{\forall i, a_i = k_i \alpha + d\mathbb{Z}\}} \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \prod_{j=1}^m f(a_j) f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \right], \end{aligned} \quad (94)$$

$$(95)$$

since $L < \min \left(\frac{3m}{4M}, \frac{\kappa\eta}{4} \right)$ and since $L(M+1)\theta < \frac{3\theta}{2} - 3(m-1)\gamma$.

The last step of the proof of the lemma consists in studying the following quantity

$$G_{\mathbf{k}} := \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \prod_{j=1}^m f(a_j) f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \right]. \quad (96)$$

Due to Lemma 23,

$$\begin{aligned} G_{\mathbf{k}} &= \mathbb{E} \left[(\det D'_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega'_{\mathbf{k}}} \prod_{j=1}^m f(a_j) f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \right] + \mathcal{O} \left(n^{-\frac{\theta}{8} - L\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}} \right) \\ &= \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \prod_{j=1}^m f(a_j) \mathbb{E} \left[f \left(a_j + \sum_{y \in \mathbb{Z}} N'_{j,s}(y) \xi_y \right) \right] + \mathcal{O} \left(n^{-\frac{\theta}{8} - L\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}} \right), \end{aligned}$$

where we used the fact that $D_{\mathbf{k}}''$ has the same distribution as $D_{\mathbf{k}}$ and is independent of $N_{j,s}'$. This combined with (95), (93), (74) and (75) ensures that

$$B_{\mathbf{k},\ell,I_{\mathbf{k}}^{(3)},\Omega_{\mathbf{k}}} = \left(\frac{d}{\sqrt{2\pi}\sigma_{\xi}} \right)^m \sum_{a_1,\dots,a_m \in \mathbb{Z}} \mathbf{1}_{\{\forall i, a_i = k_i\alpha + d\mathbb{Z}\}} \mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \\ \prod_{j=1}^m f(a_j) \mathbb{E} [f(a_j + Z_{\ell_j})] + \mathcal{O} \left(n^{-L(M+1)\theta} \prod_{j=1}^m n_j^{-\frac{3}{4}} \right)$$

Moreover [14, Lemmas 21 and 23] ensure that

$$\mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] = \mathcal{O} \left(\prod_{j=1}^m n_j^{-\frac{3}{4}} \right),$$

and that

$$\mathbb{E} \left[(\det D_{\mathbf{k}})^{-\frac{1}{2}} \mathbf{1}_{\Omega_{\mathbf{k}}} \right] \sim n^{-\frac{3m}{4}} \mathbb{E} \left[\det \mathcal{D}_{t_1,\dots,t_m}^{-\frac{1}{2}} \right]$$

as $k_j/n \rightarrow t_j$ and $n \rightarrow +\infty$. This ends the proof of the lemma. \square

APPENDIX B. MOMENT CONVERGENCE IN THEOREM 3

Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be such that $\sum_{a \in \mathbb{Z}} |f(a)| < \infty$. In this appendix we prove that all the moments of $n^{-\frac{1}{4}} \sum_{k=0}^{n-1} f(Z_k)$ converge to those of $\sum_{a \in \mathbb{Z}} f(a) \sigma_{\xi}^{-1} \mathcal{L}_1(0)$, as $n \rightarrow +\infty$.

Due to Theorem 1, it is enough to prove the convergence of every moment. The key result is the following proposition.

Proposition 26. *For all $a_1, \dots, a_k \in \mathbb{Z}$,*

$$\mathbb{P}(Z_{n_1} = a_1, \dots, Z_{n_k} = a_k) \sim \mathbf{1}_{\{\forall i, a_i \in n_i\alpha + d\mathbb{Z}\}} \left(\frac{d}{\sqrt{2\pi}\sigma_{\xi}} \right)^k \mathbb{E}[\det \mathcal{D}_{T_1,\dots,T_k}^{-\frac{1}{2}}] n^{-3k/4},$$

as $n \rightarrow +\infty$ and $n_i/n \rightarrow T_i$, where $\mathcal{D}_{t_1,\dots,t_k} = (\int_{\mathbb{R}} L_{t_i}(x)L_{t_j}(x) dx)_{i,j=1,\dots,k}$ where L is the local time of the brownian motion B , limit of $(S_{\lfloor nt \rfloor} / \sqrt{n})_t$ as n goes to infinity.

Moreover, for every $k \geq 1$ and every $\vartheta \in (0, 1)$, there exists $C = C(k, \theta) > 0$, such that

$$\mathbb{P}[Z_{n_1} = a_1, \dots, Z_{n_1+\dots+n_k} = a_k] \leq C \prod_{j=1}^k n_j^{-3/4},$$

for all $n \geq 1$, all $a_1, \dots, a_k \in \mathbb{Z}$ and all $n_1, \dots, n_k \in [n^{\vartheta}, n]$.

Proof. The lemma has been proved for $a_i \equiv 0$ in [14, Theorem 5]. The proof in the general case is the straightforward adaptation of [14, Section 5]. For completeness, we explain the required adaptations. The proof of the present result follows line by line the same proof with the adjonction of a term $e^{-i \sum_{j=1}^k (a_j - a_{j-1})\theta_j}$ (with convention $a_0 = 0$) in the integrals appearing in [14, Lemma 15] (see Lemma 16 with $M = m = k$ and $s_j \equiv 0$). Lemma 16 (definition of the good set) and Propositions 18 and 19 (estimates of the integral of the absolute values) of [14] are unchanged. The only difference in the proof concern [14, Proposition 17] and more specifically [14, Lemma 23] for which there is a multiplication by $e^{-i \sum_{j=1}^k (a_j - a_{j-1})\theta_j}$ in the integral. The only difference in the proof of [14, Lemma 23] is that the quantity I_{n_1,\dots,n_k} considered therein (n_i corresponding to $\lfloor nT_i \rfloor - \lfloor nT_{i-1} \rfloor$) is slightly modified with the multiplication in the integral by a quantity

converging in probability to 1 (with the notations of the proof of [14, Lemma 23]). Indeed, considering the real part of the integral, this quantity is $\cos(\sum_{j=0}^k (a_j - a_{j-1})(A_{n_1, \dots, n_k}^{-\frac{1}{2}} r)_j)$ (with the notations of [14, Lemma 23]) which is equal to 1 up to an error in $\mathcal{O}(\min(1, \mu_{n_1, \dots, n_k}^{-1} |r|^2))$ where μ_{n_1, \dots, n_k} is the smallest eigenvalue of A_{n_1, \dots, n_k} , which is proved to converges to 0 in [14, Lemma 23], and so the asymptotic behaviour of I_{n_1, \dots, n_k} is the same as when $a_j \equiv 0$. \square

Proof of the convergence of moments in Theorem 3. Take $\vartheta < \frac{1}{4}$. Note that the last point of the lemma ensures that

$$\mathbb{P}[Z_{n_1} = a_1, \dots, Z_{n_1 + \dots + n_k} = a_k] \leq C \left(\prod_{i: n_i > n^\vartheta} n_i \right)^{-3/4}.$$

Let α_0 be such that $\alpha\alpha_0 \in 1 + d\mathbb{Z}$. Then $a_i = q_i\alpha + d\mathbb{Z}$ is equivalent to $q_i \in a_i\alpha_0 + d\mathbb{Z}$. Thus

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{q=0}^{n-1} f(Z_q) \right)^k \right] &= \sum_{q_1, \dots, q_k=0}^{n-1} \mathbb{E}[f(Z_{q_1}) \dots f(Z_{q_k})] = \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) \sum_{q_1, \dots, q_k=0}^{n-1} \mathbb{P}(Z_{q_1} = a_1, \dots, Z_{q_k} = a_k) \\ &= O(n^{\frac{k-1}{4}}) + \sum_{r_1, \dots, r_k=0}^{d-1} \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) \sum_{q_1, \dots, q_k=0}^{\lfloor \frac{n}{d} \rfloor - 1} \mathbb{P}(Z_{r_1 + q_1 d} = a_1, \dots, Z_{r_k + q_k d} = a_k) \\ &= O(n^{\frac{k-1}{4}}) + \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) \sum_{q_1, \dots, q_k=0}^{\lfloor \frac{n}{d} \rfloor - 1} \mathbb{P}(Z_{\overline{a_1 \alpha_0} + q_1 d} = a_1, \dots, Z_{\overline{a_k \alpha_0} + q_k d} = a_k), \end{aligned}$$

with \bar{x} the representant of $x + d\mathbb{Z}$ belonging to $\{0, \dots, d-1\}$. It follows that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{q=0}^{n-1} f(Z_q) \right)^k \right] &= o(n^{\frac{k}{4}}) + \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) n^k H_k \\ &= o(n^{\frac{k}{4}}) + \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) n^k H'_k, \end{aligned}$$

with

$$\begin{aligned} H_k &:= \int_{[0, 1/d]^k} \mathbb{P}(Z_{\overline{a_1 \alpha_0} + \lfloor t_1 n \rfloor d} = a_1, \dots, Z_{\overline{a_k \alpha_0} + \lfloor t_k n \rfloor d} = a_k) dt_1 \dots dt_k \\ H'_k &= \int_{[0, 1/d]^k} n^{\frac{3k}{4}} \mathbb{P}(Z_{\overline{a_1 \alpha_0} + \lfloor t_1 n \rfloor d} = a_1, \dots, Z_{\overline{a_k \alpha_0} + \lfloor t_k n \rfloor d} = a_k) \mathbf{1}_{\min_{i,j} |\lfloor t_i n \rfloor - \lfloor t_j n \rfloor| > 2n^\vartheta} dt_1 \dots dt_k. \end{aligned}$$

Due to the dominated convergence theorem, we conclude that

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{q=0}^{n-1} f(Z_q) \right)^k \right] \\ &= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \sum_{a_1, \dots, a_k \in \mathbb{Z}} f(a_1) \dots f(a_k) \int_{[0,1/d]^k} \left(\frac{d}{\sqrt{2\pi}\sigma_\xi} \right)^k \mathbb{E}[\det \mathcal{D}_{t_1 d, \dots, t_k d}^{-\frac{1}{2}}] dt_1 \dots dt_k \\ &= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \left(\sum_{a \in \mathbb{Z}} f(a) \right)^k \int_{[0,1]^k} (\sqrt{2\pi}\sigma_\xi)^{-k} \mathbb{E}[\det \mathcal{D}_{t_1 d, \dots, t_k d}^{-\frac{1}{2}}] dt_1 \dots dt_k \\ &= o(n^{\frac{k}{4}}) + n^{\frac{k}{4}} \left(\sum_{a \in \mathbb{Z}} f(a) \sigma_\xi^{-1} \right)^k \mathbb{E}[(\mathcal{L}_1(0))^k], \end{aligned}$$

due to [14, Theorem 3]. □

REFERENCES

- [1] Aaronson, J. *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997. xii+284 pp.
- [2] Aurzada, F.; Guillin-Plantard, F.; Pène, F. *Persistence probabilities for stationary increment processes*, Stochastic Process. Appl. **128** (2018), no. 5, 1750–1771.
- [3] Berger, N.; Peres, Y. *Detecting the trail of a random walker in a random scenery*. Electron. J. Probab. **18** (2013), no. 87, 18 pp.
- [4] Billingsley, P. *Probability and measure*, third edition. Wiley Series in Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1995), xiv+593 pp.
- [5] Blachère, S.; den Hollander, F.; Steif, J. E. *A crossover for the bad configurations of random walk in random scenery*. Ann. Probab. **39** (2011), no. 5, 2018–2041.
- [6] Bolthausen, E. *A central limit theorem for two-dimensional random walks in random sceneries*. Ann. Probab. **17** (1989), no. 1, 108–115.
- [7] Borodin, A. N. *A limit theorem for sums of independent random variables defined on a recurrent random walk*. (Russian) Dokl. Akad. Nauk SSSR **246** (1979), no. 4, 786–787.
- [8] Borodin, A. N. *On the character of convergence to Brownian local time. II*, Probab. Theory Relat. Fields **72** (1986), 251–277.
- [9] Brémont J. *Planar random walk in a stratified quasi-periodic environment*, preprint, Hal-02887510, 21 pp.
- [10] Brémont J. *On planar random walks in environments invariant by horizontal translations*, Markov Processes and Related Fields (2016), vol. 22, issue 2, 267–310.
- [11] Campanino M., Petritis D. *Random walks on randomly oriented lattices*, Mark. Proc. Relat. Fields **9** (2003), 391–412.
- [12] Castell, F.; Guillin-Plantard, N.; Pène, F. *Limit theorems for one and two-dimensional random walks in random scenery*. Ann. I.H.P. (B) Probabilités et Statistiques (2013), Vol. 49, No 2, 506–528
- [13] Castell, F.; Guillin-Plantard, N.; Pène, F.; Schapira, Br. *A local limit theorem for random walks in random scenery and on randomly oriented lattices*. Ann. Probab. **39**, (2011), 2079–2118.
- [14] Castell, F.; Guillin-Plantard, N.; Pène, F.; Schapira, Br. *On the local time of random processes in random scenery*. Ann. Probab. **42**, No 6, (2014), 2417–2453.
- [15] Chen, X. *Random Walk Intersections: Large Deviations and Related Topics*. Mathematical Surveys and Monographs, AMS, Vol. **157**, Providence, RI (2009).
- [16] Chen, X.; Li, W. V.; Rosiński, J.; Shao, Q.-M. *Large deviations for local times and intersection local times of fractional Brownian motions and Riemann-Liouville processes*. Ann. Probab. **39**, (2011), 729–778.
- [17] Csáki, E.; Földes, A. *On asymptotic independence and partial sums*, Asymptotic methods in probability and statistics, A volume in honour of Miklós Csörgö, Elsevier (2000), 373–381.
- [18] Csáki, E.; Földes, A. *Asymptotic independence and additive functionals*, Journal of Theoretical Probability, **13** (2000), 1123–1144.
- [19] Deligiannidis, G.; Utev, S. *Asymptotic variance of the self-intersections of stable random walks using Darboux-Wiener theory*, Sib Math J. **52**, 639 (2011), 14 p.

- [20] Devulder, A.; Pène, F. *Random walk in random environment in a two-dimensional stratified medium with orientations*, Electron. J. Probab. **18** (2013), no. 88, 1–23
- [21] Dobrušin, R. L. *Two limit theorems for the simplest random walk on a line.* (Russian) Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 139–146.
- [22] Dombry, C.; Guillin-Plantard, N. *Discrete approximation of a stable self-similar stationary increments process.* Bernoulli **15** (2009), no. 1, 195–222.
- [23] Gantert, N.; Kochler, M.; Pène, F. *On the recurrence of some random walks in random environment*, ALEA **11** (2014), 483–502.
- [24] Geman, D.; Horowitz, J. *Occupation densities.* Ann. Probab. **8**, (1980), 1–67.
- [25] Guillin-Plantard, N.; Hu, Y.; Schapira, Br. *The quenched limiting distributions of a one-dimensional random walk in random scenery.* Electron. Commun. Probab. **18** (2013), no. 85, 7 pp.
- [26] Guillin-Plantard, N.; Le Ny, A. *Transient random walks on 2d-oriented lattices*, Theory of Probability and Its Applications (TVP) **52** (2007), No 4, 815–826.
- [27] Guillin-Plantard, N., Pène, F. *Renewal theorems for random walks in random scenery.* Electron. J. Probab. **17** (2012), no. 78, 1–22.
- [28] Guillin-Plantard, N.; Pène, F.; Wendler, M. *Empirical processes for recurrent and transient random walks in random scenery*, ESAIM Probab. Stat. **24** (2020), 127–137.
- [29] Guillin-Plantard, N.; Poisat, J. *Quenched central limit theorems for random walks in random scenery.* Stochastic Process. Appl. **123** (2013), no. 4, 1348–1367.
- [30] Kalikow S. A. *T, T-1 Transformation is Not Loosely Bernoulli*, Ann. Math. **115** (1982), No 2 , pp. 393-409
- [31] Kesten, H. *Occupation times for Markov and semi-Markov chains.* Trans. Amer. Math. Soc. **103** (1962), 82–112.
- [32] Kesten, H.; Spitzer, F. *A limit theorem related to a new class of self-similar processes.* Z. Wahrsch. Verw. Gebiete **50** (1979), 5–25.
- [33] Khoshnevisan, D. *The codimension of the zeros of a stable process in random scenery.* Séminaire de Probabilités XXXVII, 236–245, Lecture Notes in Math. 1832, Springer, Berlin, (2003).
- [34] Khoshnevisan, D.; Lewis, T.M. *Iterated Brownian motion and its intrinsic skeletal structure.* Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996), 201–210, In: Progr. Probab. **45**, Birkhäuser, Basel, (1999).
- [35] Lebedev, N. N. *Special functions and their applications*, Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication. Dover Publications, Inc., New York, (1972), xii+308 pp,
- [36] Le Doussal, P. *Diffusion in layered random flows, polymers, electrons in random potentials, and spin depolarization in random fields.* J. Statist. Phys. **69** (1992), no. 5-6, 917–954.
- [37] Le Gall, J.-F. *Mouvement brownien, processus de branchement et superprocessus*, Master course, available on <http://www.math.u-psud.fr/~jfgall>.
- [38] Marcus, M. B.; Rosen, J. *Markov processes, Gaussian processes, and local times.* Cambridge Studies in Advanced Mathematics **100**, Cambridge University Press, Cambridge, (2006), x+620 pp.
- [39] Matheron G., de Marsily G. *Is transport in porous media always diffusive? A counterexample*, Water Resources Res. **16** (1980) 901–907.
- [40] Pène F. *Random walks in random sceneries and related models*, ESAIM: Proceedings and surveys **68** (2020), 35–51
- [41] Thomine D., Pène F. *Potential kernel, hitting probabilities and distributional asymptotics.* Ergodic Theory and Dynamical Systems
- [42] Thomine, D.; Pène, F. *Central limit theorems for the $Z^{\hat{2}}$ -periodic Lorentz gas*, to appear in Israel Journal of Mathematics
- [43] Port, S. C.; Stone, C. J. *Brownian motion and classical potential theory*, Probability and Mathematical Statistics. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, (1978), xii+236 pp.
- [44] Revuz, D.; Yor, M. *Continuous martingales and Brownian motion*, Third edition. Grundlehren der Mathematischen Wissenschaften, 293. Springer-Verlag, Berlin, (1999). xiv+602 pp.
- [45] Rogers, C. A. *Covering a sphere with spheres*, Mathematika, **10**, (1963), 157-164.
- [46] Schmidt, K. *On recurrence.* Z. Wahrsch. Verw. Gebiete **68** (1984) 75–95.
- [47] Thomine, D. *Théorèmes limites pour les sommes de Birkhoff de fonctions d'intégrale nulle en théorie ergodique en mesure infinie.* PhD Thesis, Université de Rennes 1, 2013 version (in French).
- [48] Thomine, D. *A generalized central limit theorem in infinite ergodic theory.* Probab. Theory Related Fields **158** (2014), no. 3-4 , 597–636.
- [49] Weiss B. *The isomorphism problem in ergodic theory*, Bull. A.M.S. **78** (1972), 668–684.
- [50] Xiao, Y. *The Hausdorff dimension of the level sets of stable processes in random scenery.* Acta Sci. Math. (Szeged) **65**, (1999), 385–395.

- [51] Zweimüller, R. *Mixing limit theorems for ergodic transformations*, Journal of Theoretical Probability, 20 (2007), 1059–1071.

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