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► **To cite this version:**

Anis Matoussi, Rym Salhi. Generalized BSDE with jumps and stochastic quadratic growth. 2020. hal-03091716

HAL Id: hal-03091716

<https://hal.science/hal-03091716>

Preprint submitted on 31 Dec 2020

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Generalized BSDE with jumps and stochastic quadratic growth ^{*}

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December 31, 2020

Abstract

In this paper, we study a doubly Reflected Backward Stochastic Differential Equation with Jumps (DRBSDEs in short) when the driver have general quadratic growth. We extend the result of Essaky and Hassani [14] to the jump setting and a generator with general exponential quadratic growth.

Keywords: Doubly reflected BSDE with jumps, exponential stochastic quadratic growth, unbounded terminal condition.

1 Introduction

Motivation Our aim interest is to study backward stochastic differential equation with jumps (BSDEJs) and unbounded terminal condition where the driver has stochastic quadratic growth. We were inspired by the approach developed by Bahlali et al [2]. This approach is fundamentally different and consists in deriving the existence of the solution via quadratic BSDEs with two reflecting barriers. More precisely, the upper and lower bound in the structure condition of the driver will play the role of obstacles processes. In this regard, there is no longer any need to impose stronger integrability condition on the terminal condition ξ . The solution is then obtained by proving that the constrained processes are equal to zero.

Unfortunately, as far as we know, there is no existing result on this subject. Hence, as a first step, we concerns our self to the well-posedness of doubly reflected BSDEJ's where the driver has a general stochastic growth and unbounded terminal condition.

Literature review on doubly reflected BSDEs The notion of doubly reflected BSDEs (DRBSDE in short) were first introduced and studied by Cvitanic and Karatzas [7] in the Brownian setting. However the theory of BSDEs with constrains started with the seminal paper of El Karoui, Kapudjian, Pardoux, Peng and Quenez [11] as the generalization of the work of Pardoux and Peng [28]. Intuitively, the solution of these equation are constrained to stay in the region surrounded by two given obstacles. This is achieved via two processes K^- and K^+ under a minimality condition that is the process K^+ and K^- only act when Y reaches the obstacle L and U .

Several works have been obtained in this subject since they are encountered in various fields such as financial problem and mixed game problem. In [16], Hamadene Lepeltier and Matoussi

^{*}Research supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon.

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investigated the existence of solution for DRBSDE when the driver has linear growth by assuming regularity condition on the one barrier and also the existence of a positive semimartingale between the barriers constraints. Later Hamadene and Hassani [17] generalize the result of [16] and proved the existence of the solution when the barriers are completely separated. Under the Mokobodski condition, Bahlali et al. [3] treated the case where the driver has linear growth in y and quadratic in z . This result was followed by Hdhiri and Hamadene [20], they establish an existence result by assuming that the barriers are completely separated. Essaki and Hassani [14] proved existence and uniqueness of the solution of doubly reflected BSDE under more general assumption on the data by considering a driver with quadratic stochastic growth i.e $|f(t, y, z)| \leq \zeta_s(w) + \frac{C_s(w)}{2}|z|^2$ without imposing any integrability condition on the terminal condition. This result was generalized later by Essaky, Hassani and Ouknine to the case of RCLL obstacles.

More recently, Baadi and Ouknine [1] have studied doubly reflecting barrier BSDEs when the noise is driven by a Brownian motion. They showed existence and uniqueness of the solution when the reflecting barrier don't satisfy any regularity assumption.

Several authors have been regarding to extend these results to the discontinuous setting. Crepey and Matoussi [6] considered the case when the driver f is Lipschitz the Obstacles satisfy the Mokobodski's condition.

When the obstacles are completely separated, Hamadene and Hassani [18] showed that the DRBSDE driven by a Brownian motion and a Poisson noise has a unique solution under weaker integrability assumption. Let us mention that, in their study the proces Y has only inaccessible jumps. This result was generalized later by Hamadene and Wang [22].

Doubly reflected BSDE is further developed to different framework, such work includes that of [19, 5, 9, 10] among others.

Main contributions The main objective of this paper is to deal with generalized doubly reflected BSDEs when the noise comes from a Brownian motion and an independent jump random measure.

$$\left\{ \begin{array}{l} Y_t = \xi + \int_t^T f(Y_s, Z_s, V_s)ds + dR_s + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) \\ \quad + \int_t^T dK_s^+ - \int_t^T dK_s^-, \quad \mathbb{P}\text{-a.s.} \\ L_t \leq Y_t \leq U_t \text{ and } \int_0^T (Y_{s-} - U_s)dK_s^+ = \int_0^T (Y_{s-} - L_s)dK_s^- = 0, \quad \mathbb{P}\text{-a.s.} \end{array} \right.$$

where $t \in [0, T]$, B is a standard Brownian motion, $\tilde{\mu}$ is a random jump measure, A is a non decreasing process and R is positive measure. In our study we consider the case when the driver has stochastic quadratic growth, unbounded terminal condition and that the two barriers are completely separated. We extend the result of [14] and show existence and uniqueness of the solution.

The main motivation of our work is that the assumptions on the driver that we consider here are much involved in finance. If we look for the simplest example of a pricing problem in a Black and Scholes market problem we can see that the fair price of an European options is given by the solution of a linear BSDE where $f(t, y, z) := r(t)y_t + \theta(t)z_t$, where θ is the risk premium. Unfortunately, the interest rate and the risk premium are in general not bounded in the market. Indeed, we want to investigated the solvability of generalized DRBSDE when the noise is governed by a Brownian motion and random jump measure. Similar to [14], our approach combines exponential transformation and monotone approximation by mean of sup convolution. The first transformation enables us to obtain the doubly reflected BSDEJs with bounded terminal condition and a driver with uniform structure condition. However the exponential transformation as

it has been used in Kobylanski [24] cannot be applied singularly since we can not get a Lipschitz one.

We are thus led naturally to a truncature procedure. We construct an approximation sequence of globally Lipschitz doubly reflected BSDEJs by mean of sup convolution for which the solution exists. The solution of the DRBSDE with jumps is constructed by proving a monotone limit theorem and achieving the convergence of the Lipschitz DRBSDEs constructed by the above truncation procedure. Then we show that the limit of those solution indeed solves the original DRBSDEJ. As a final step we comeback to our initial system and prove that admit a solution using a logarithmic transformation.

The paper is organized as follow. In section 2, we describe our setting and we recall briefly some preliminary notations. In section 3 we give a precise definition of the solution of BRBSDEJ and show by an exponential transformation how they are connected to another general doubly reflected BSDEJ with more tractable coefficients. Then, we prove that these equations admit a unique solution. Last section contains technical results. Furthermore, the special case of generalized DRBSDEjs when the driver is Lipschitz is established .

2 Framework

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t < T}$ satisfies the usual conditions of completeness and right continuity and that $\mathcal{F}_T = \mathcal{F}$ and \mathcal{F}_0 be trivial. Due to these usual conditions, we can take all semimartingales to have right continuous paths with left limits.

On this stochastic basis, let W a d-dimensional Brownian motion and $\mu(\omega, dt, de)$ an independent integer valued random measure defined on $([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{B}(E))$, with compensator $\nu(\omega, dt, de)$.

For a σ -finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ and a bounded $\mathcal{P} \otimes \mathcal{B}(E)$ -measurable non negative density function ζ , we will assume that the compensator ζ is absolutely continuous with respect to $\lambda \otimes dt$ such that

$$\nu(\omega, dt, de) = \zeta(\omega, t, e) \lambda(de) dt, \quad 0 \leq \zeta \leq C_\eta, \quad \text{for some constant } C_\nu. \quad (2.1)$$

Finally, we will denote by $\tilde{\mu}$ the compensated measure of μ as

$$\tilde{\mu}(\omega, dt, de) = \mu(\omega, dt, de) - \nu(\omega, dt, de). \quad (2.2)$$

Let f be a $\mathcal{P} \otimes \mathcal{E}$ -measurable function, the integral with respect to the random measure and the compensator are defined as follow

$$(f \star \mu)_t = \int_0^t \int_E f(s, e) \mu(ds, de) \quad , \quad (f \star \nu)_t = \int_0^t \int_E f(s, e) \nu(ds, de).$$

In particular, the stochastic integral $U \star \tilde{\mu} = \int U_s(e) \tilde{\mu}(ds, de)$ is a local square integrable martingale, for any predictable locally integrable process U .

We will assume that W and $\tilde{\mu}$ satisfies the following weak representation property with respect to $(\mathcal{F}_t)_{0 \leq t \leq T}$

$$M = M_0 + \int_0^\cdot Z_s \cdot dW_s + \int_0^\cdot \int_E U_s(e) \tilde{\mu}(de, ds).$$

We will now introduce the specific spaces corresponding to our framework.

- $\mathcal{L}^{2,d}$ is the spaces of \mathbb{R}^d -valued and \mathcal{P} -measurable processes such that

$$\|Z\|_{\mathcal{L}^2}^2 := \int_0^T |Z_s|^2 ds < +\infty, \quad \mathbb{P}\text{-a.s.}$$

- $\mathcal{L}_\nu^{2,d}$ is the space of $\tilde{\mathcal{P}}$ -measurable processes such that

$$\|U\|_{\mathcal{L}_\nu^2}^2 := \int_0^T \int_E |U_s(e)|^2 \nu(de, ds) < +\infty, \quad \mathbb{P}\text{-a.s.}$$

- \mathcal{K} the space of \mathcal{P} -measurable continuous non decreasing process such that $K_0 = 0$.
- \mathcal{D} (respectively \mathcal{D}^c) the space of \mathbb{R} -valued \mathcal{P} -measurable càdlàg processes resp. ($\Delta Y_t = 0$).

We also introduce the following classical spaces

- \mathbb{H}^2 the set of all \mathcal{P} -measurable processes Z such that $\mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty$, \mathbb{P} -a.s.
- For u, \bar{u} in the space $\mathbb{L}^0(\mathcal{B}(E), \nu)$ of all $\mathcal{B}(E)$ -measurable functions with the topology of convergence in measure, we define

$$|u - \bar{u}|_t = \left(\int_E |u - \bar{u}|^2 \zeta(t, e) \lambda(de) \right)^{\frac{1}{2}}.$$

- \mathbb{H}_ν^2 the set of all $\tilde{\mathcal{P}}$ -measurable processes Z such that

$$\mathbb{E}\left[\int_0^T \int_E |V_s(e)|^2 \nu(de, ds)\right] < +\infty, \quad \mathbb{P}\text{-a.s.}$$

3 Generalized Doubly reflected BSDE with jumps

3.1 Formulation

In this section, we aim to prove existence of solution of generalized doubly reflected BSDE with jumps given by $(f, ds + g dA_s + dR_s, \xi, L, U)$ under weaker assumption. Let us first introduce the following definition of *generalized doubly reflected backward stochastic differential equation with jumps*.

We are given the following objects:

- ξ an \mathcal{F}_T -measurable real valued random variable.
- A function $f : \Omega \times [0, T] \times \mathbb{R}^{1+d} \times \mathbb{L}^0(\mathcal{B}(E), \nu) \rightarrow \mathbb{R}$ such that f is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d}) \otimes \mathcal{B}(\mathbb{L}^0(\mathcal{B}(E), \nu))$ -measurable.
- A function $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that g is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable.
- Two continuous \mathbb{R} -valued processes L_t and U_t such that $L_t \leq U_t$ satisfying $L_t \leq \xi \leq U_t$.
- A positive random measure dR and a non-decreasing continuous process A .

Definition 3.1. *We say that a quintuple (Y, Z, V, K^+, K^-) is a solution to the generalized doubly reflected BSDE with jumps associated to $(f, ds + g dA_s + dR_s, \xi, L, U)$, if*

$$(\mathcal{E}) \left\{ \begin{array}{l} Y \in \mathcal{D}, \quad Z \in \mathcal{L}^{2,d}, \quad V \in \mathcal{L}_\nu^{2,d}, \quad K^\pm \in \mathcal{K}. \\ Y_t = \xi + \int_t^T f_s(Y_s, Z_s, V_s) ds + \int_t^T dR_s + \int_t^T g_s(Y_s) dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e) \tilde{\mu}(ds, de) \\ \quad + \int_t^T dK_s^+ - \int_t^T dK_s^-, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \\ L_t \leq Y_t \leq U_t \quad \text{and} \quad \int_0^T (Y_{s-} - U_s) dK_s^+ = \int_0^T (Y_{s-} - L_s) dK_s^- = 0, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \end{array} \right.$$

The last condition is called the Skorohod condition. It requires that the processes K^+ and K^- are minimal in the sense that they only act when Y reaches the obstacles L and U . This condition is crucial to obtain the wellposedness of generalized doubly reflected BSDEs with jumps. Note that when there is no barrier, the system becomes an ordinary BSDE with jumps.

Definition 3.2. We say that dM_t^1 and dM_t^2 are two singular measure and we denote $dM_s^1 \perp dM_s^2$ if

$$\int_0^T 1_{\{A_s(w)\}} dM_t^1 = \int_0^T 1_{\{A_s^c(w)\}} dM_s^2 = 0, \quad \forall A \in \mathcal{P}. \quad (3.3)$$

We shall make the following standing assumptions on the maps under consideration.

Assumption 3.1 (Assumptions on the drivers).

- The first assumption characterize the growth of the driver f with a lower and an upper bound: For every $(y, z, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$, there exist two positives processes η and C respectively in $\mathbb{L}^1(\Omega, [0, T])$ and \mathcal{D}^c such that

$$\begin{aligned} \underline{q}_s(y, z, v) &= -\eta_s(w) - \frac{C_s(w)}{2} |z|^2 - \frac{1}{\delta} J(v) \\ &\leq f_s(w, y, z, v) \leq \bar{q}_s(y, z, v) = \eta_s(w) + \frac{C_s(w)}{2} |z|^2 + \frac{1}{\delta} J(v), \end{aligned}$$

$dt \otimes d\mathbb{P}$ -a.s, $(w, t) \in \Omega \times [0, T]$, where

$$J(v) = \int_E \left(e^{\delta v(e)} - \delta v(e) - 1 \right) \nu(de).$$

- The second assumption consists in specifying a lower and upper bound for g : For all $y \in [L_s(w), U_s(w)]$,

$$|g_s(w, y)| \leq 1 \quad A(ds) \otimes \mathbb{P}(dw)\text{-a.s.}$$

- The last assumption known as the " A_γ -condition " deals with the increments of the driver f with respect to the jump component: For all $(y, z, u, \bar{u}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$ there exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+2}) \otimes \mathcal{B}(E)$ -measurable function γ where $C_1 \leq \gamma \leq C_2$ with $-1 < C_1 \leq 0$, $C_2 \geq 0$,

$$f_t(y, z, v) - f_t(y, z, \bar{v}) \leq \int_E \gamma_t(e) [v(e) - \bar{v}(e)] \nu(de), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

Remark 3.1. We emphasize that, usually the above structure condition is uniform that is the constants in front of z and u are constants. However in our context, we look for solution of generalized doubly reflected BSDEs where the driver has stochastic growth i.e. η and C are no longer constants but predictable processes.

To conclude this part we introduce the following requirement on the obstacle processes.

Assumption 3.2 (Assumptions on the Obstacle).

(i) There exists a semimartingale S with the following decomposition

$$S = S_0 + V^+ - V^- + \int_0^\cdot \alpha_s dB_s, \quad \text{where } S_0 \in \mathbb{R}, V^+, V^- \in \mathcal{K} \text{ and } \alpha_s \in \mathcal{L}^{2,d}, \quad \mathbb{P}\text{-a.s.}$$

(ii) $L_t \leq S_t \leq U_t$, $L_t \leq 0 \leq U_t$, $t \in [0, T]$, \mathbb{P} -a.s.

(iii) For all $R \in \mathcal{K}$, $dR_t \geq 0$, \mathbb{P} -a.s.

Theorem 3.1. Under Assumptions 3.1 and 3.2, there exists a maximal solution (Y, Z, V, K^+, K^-) for the doubly reflected BSDE with jumps associated to (f, ξ, L, U) satisfying the system (\mathcal{E}) .

3.2 Exponential transformation and estimates

To achieve our main result, that is the existence of solutions of generalized doubly reflected BSDEJs, we first introduce an auxiliary BSDEJ which is explicitly given in terms of exponential transformation of the original one. We then establish a correspondence between solutions of the auxiliary one and those of BSDEJs given by $(f ds + g dA_s + dR_s, \xi, L, U)$. To do so, we consider the following \mathcal{F}_t -adapted continuous increasing process

$$m_t = 2 \sup_{0 \leq s \leq t} |C_s| + \sup_{0 \leq s \leq t} |U_s| + |R_t| + A_t + \frac{1}{\delta} + 1. \quad (3.4)$$

Then we have the following result.

Proposition 3.1. *There exists a solution $(Y, Z, V, K^+, K^-) \in \mathcal{D} \times \mathcal{L}^{2,d} \times \mathcal{L}_\nu^{2,d} \times \mathcal{K}^2$ to the system (\mathcal{E}) if and only if $(\bar{Y}, \bar{Z}, \bar{V}, \bar{K}^+, \bar{K}^-)$ is solution of (\mathcal{E}) with data $(\bar{f}, \bar{\xi}, \bar{L}, \bar{U})$ where*

$$\begin{aligned} \bar{Y}_t &= e^{m_t(Y_t - m_t)}, \quad \bar{Z}_t = m_t \bar{Y}_t Z_t, \quad \bar{V}_t = \bar{Y}_t [e^{m_t V_t(e)} - 1], \quad \bar{K}_s^\pm = m_s \bar{Y}_s K_s^\pm. \\ \bar{f} &= \tilde{f}_s((\bar{y} \wedge \bar{L}_s) \vee \bar{U}_s, \bar{z}, \bar{v}) - \eta_s m_s \quad \text{and} \quad \bar{g} = \frac{1}{8m_s} \tilde{g}_s((\bar{y} \wedge \bar{L}_s) \vee \bar{U}_s) - \frac{1}{2} \\ \bar{U}_t &= e^{m_t(U_t - m_t)}, \quad \bar{L}_t = e^{m_t(L_t - m_t)}, \quad d\bar{R}_s = \frac{1}{2} d\bar{A}_s + \eta_s m_s ds, \quad d\bar{A}_s = 8m_s dm_s. \end{aligned} \quad (3.5)$$

with

$$\begin{cases} \tilde{g}_s(\bar{y}) = \bar{y} [m_s g_s(\frac{\ln(\bar{y})}{m_s} + m_s) \frac{dA_s}{dm_s} + m_s \frac{dR_s}{dm_s} + m_s - \frac{\ln(\bar{y})}{m_s}], \\ \tilde{f}_s(\bar{y}, \bar{z}, \bar{v}) = m_s \bar{y} [f_s(\frac{\ln(\bar{y})}{m_s} + m_s, \frac{\bar{z}}{m_s \bar{y}}, \frac{1}{m_s} \ln(\frac{\bar{v}}{\bar{y}} + 1)) \\ \quad + m_s \bar{y} - \frac{|\bar{z}|^2}{2m_s \bar{y}^2} - \frac{1}{m_s} \int_E (e^{\ln(1 + \frac{\bar{v}}{\bar{y}})} - \ln(1 + \frac{\bar{v}}{\bar{y}}) - 1) \nu(de)]. \end{cases}$$

Proof. Applying Itô's formula to $\bar{Y}_t = e^{m_t(Y_t - m_t)}$, we obtain for all $t \in [0, T]$, \mathbb{P} -a.s

$$\begin{aligned} \bar{Y}_t &= \bar{Y}_T + \int_t^T \bar{Y}_s [m_s f_s(Y_s, Z_s, V_s) - \frac{1}{2} |m_s Z_s|^2 - \int_E [e^{m_s V_s(e)} - m_s V_s(e) - 1] \nu(de)] ds \\ &\quad + \int_t^T m_s \bar{Y}_s dK_s^+ - \int_t^T m_s \bar{Y}_s dK_s^- - \int_t^T m_s \bar{Y}_s - Z_s dB_s - \int_t^T \int_E \bar{Y}_s [e^{m_s V_s(e)} - 1] \tilde{\mu}(de, ds) \\ &\quad + \int_t^T \bar{Y}_s [2m_s - Y_s] dm_s + \int_t^T [\bar{Y}_s - m_s g_s(Y_s) \frac{dA_s}{dm_s}] dm_s + \int_t^T [\bar{Y}_s - m_s \frac{dR_s}{dm_s}] dm_s. \end{aligned}$$

Thus, taking \bar{f} , \bar{g} , \bar{A} and $d\bar{R}$ as in (3.5) yield to

$$\begin{aligned} \bar{Y}_t &= \bar{Y}_T + \int_t^T \bar{f}_s(\bar{Y}_s, \bar{Z}_s, \bar{V}_s) ds + \int_t^T \bar{g}_s(\bar{Y}_s) d\bar{A}_s + \int_t^T d\bar{R}_s - \int_t^T \bar{Z}_s dB_s - \int_t^T \int_E \bar{V}_s(e) \tilde{\mu}(ds, de) \\ &\quad + \int_t^T d\bar{K}_s^+ - \int_t^T d\bar{K}_s^-. \end{aligned}$$

We then deduce that if (Y, Z, V, K^+, K^-) is a solution of (\mathcal{E}) then $(\bar{Y}, \bar{Z}, \bar{V}, \bar{K}^+, \bar{K}^-)$ is a solution of the generalized doubly reflected BSDEJs associated to $(\bar{f} ds + \bar{g} d\bar{A}_s + d\bar{R}_s, \bar{\xi}, \bar{L}, \bar{U})$. Conversely, let $(\bar{Y}, \bar{Z}, \bar{V}, \bar{K}^+, \bar{K}^-)$ be the solution of (\mathcal{E}) with data $(\bar{f}, \bar{\xi}, \bar{L}, \bar{U})$. Applying Itô's formula to $Y_t = \frac{\ln(\bar{Y}_t)}{m_t} + m_t$, we obtain that (Y, Z, V, K^+, K^-) is a solution of (\mathcal{E}) . \square

The following lemma summarizes the properties satisfied by the transformed data.

Lemma 3.1. *The data $(\bar{f}ds + d\bar{R}_s + \bar{g}d\bar{A}_s, \bar{\xi}, \bar{L}, \bar{U})$ obtained by the above exponential transformation satisfy the following properties*

$$(i) \ 0 \leq \bar{L}_t \leq e^{-m_t^2} \leq \bar{U}_t \leq e^{-1} < 1 \text{ and } \bar{L}_T \leq \bar{\xi} \leq \bar{U}_T, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

(ii) \bar{f} is a \mathcal{P} -measurable function such that for all $(s, y, z, v) \in [0, T] \times [\bar{L}_s, \bar{U}_s] \times \mathbb{R}^{1+d} \times L^0(B(E), \nu)$:

$$-2m_s\eta_s - \frac{|z|^2}{L_s} - \frac{1}{\delta}J(\ln(\frac{\bar{v}}{L_s} + 1)) \leq \bar{f}_s(w, y, z, v) \leq 0, \quad \mathbb{P}\text{-a.s.}$$

$$(iii) \ -1 \leq g_s(w, y) \leq 0, \quad \forall y \in [L_s, U_s].$$

(iv) $d\bar{R}$ is a random positive measure .

Proof. (i) Since $U_t \leq m_t - 1$, it follows directly that $0 \leq \bar{L}_t \leq e^{-m_t^2} \leq \bar{U}_t \leq e^{-1} < 1$.

Let us now prove that \bar{f} satisfies the property(ii). Recalling the expression of \bar{f} and using Assumption (3.1), we aim to determine an upper bound to \bar{f}

$$\begin{aligned} & \bar{f}_s(w, \bar{y}, \bar{z}, \bar{v}) \\ & \leq m_s\bar{y}[\eta_s + \frac{C_s}{2} \frac{|\bar{z}|^2}{m_s^2\bar{y}^2} + \frac{1}{\delta}J(\frac{1}{m_s}\ln(1 + \frac{\bar{v}}{\bar{y}}))] - \frac{|\bar{z}|^2}{2m_s\bar{y}^2} - \frac{1}{m_s} \int_E (e^{\ln(1+\frac{\bar{v}}{\bar{y}})} - \ln(1 + \frac{\bar{v}}{\bar{y}}) - 1)\nu(de) \\ & \leq \bar{y}m_s\eta_s + \left(\frac{C_s}{2m_s} - \frac{1}{2}\right) \frac{|z|^2}{\bar{y}} + \frac{m_s}{\delta}\bar{y}J(\frac{1}{m_s}\ln(1 + \frac{\bar{v}}{\bar{y}})) - \int_E (e^{\ln(1+\frac{\bar{v}}{\bar{y}})} - \ln(1 + \frac{\bar{v}}{\bar{y}}) - 1)\nu(de). \end{aligned}$$

Using the following inequality $J(ku) \geq kJ(u)$, $\forall k \geq 1$ and $\bar{y} \leq \bar{U} \leq e^{-1}$, we obtain

$$\bar{f}_s(w, \bar{y}, \bar{z}, \bar{v}) \leq e^{-1}m_s\eta_s + \left(\frac{C_s}{2m_s} - \frac{1}{2}\right) \frac{|z|^2}{\bar{y}}.$$

Since $C_s \leq m_s$ yield to $\frac{C_s}{2m_s} - \frac{1}{2} \leq 0$, we see that $\bar{f}_s(w, \bar{y}, \bar{z}, \bar{v}) \leq \eta_s m_s$.

Now we aim to find a lower bound of \bar{f} . In fact

$$\begin{aligned} \bar{f}_s(w, \bar{y}, \bar{z}, \bar{v}) & \geq m_s\bar{y} \left[\left(-\eta_s - \frac{C_s}{2} \frac{|\bar{z}|^2}{m_s^2\bar{y}^2} - \frac{1}{\delta}J_s\left(\frac{1}{m_s}\ln(1 + \frac{\bar{v}}{\bar{y}})\right) \right) \right. \\ & \quad \left. - \frac{|\bar{z}|^2}{2m_s\bar{y}^2} - \frac{1}{m_s} \int_E (e^{\ln(1+\frac{\bar{v}}{\bar{y}})} - \ln(1 + \frac{\bar{v}}{\bar{y}}) - 1)\nu(de) \right] \\ & \geq -e^{-1}m_s\eta_s - \left(\frac{C_s}{2m_s} + \frac{1}{2}\right) \frac{|z|^2}{\bar{y}} - \int_E (e^{\ln(1+\frac{\bar{v}}{\bar{y}})} - \ln(1 + \frac{\bar{v}}{\bar{y}}) - 1)\nu(de). \end{aligned}$$

Using once again $J(ku) \geq kJ(u)$ together with $\bar{L}_s \leq \bar{y} \leq \bar{U}_s$ and $\frac{C_s}{2m_s} + \frac{1}{2} \leq 1$, we finally get

$$\begin{aligned} \bar{f}_s(w, \bar{y}, \bar{z}, \bar{v}) & \geq -e^{-1}m_s\eta_s - \left(\frac{C_s}{2m_s} + \frac{1}{2}\right) \frac{|z|^2}{\bar{y}} - \frac{1}{\delta}J_s(\ln(1 + \frac{\bar{v}}{\bar{y}})) \\ & \geq m_s\eta_s - \frac{|z|^2}{L_s} - \frac{1}{\delta}J_s(\ln(1 + \frac{\bar{v}}{L_s})). \end{aligned}$$

Similar arguments can be used to prove that g satisfy $-1 \leq g_s(w, y) \leq 0$.

The property (iv) follows from Assumption 3.2 and the definition 3.4. \square

3.3 Auxiliary generalized doubly reflected BSDE with jumps: Existence and uniqueness results

Our problem is then reduced to find a maximal solution for generalized doubly reflected BSDEJs under the following weaker assumptions.

Assumption 3.3.

1. There exist two positives processes $\eta \in \mathbb{L}^1(\Omega, [0, T])$, $C \in \mathcal{D}^c$ such that

$$\underline{q} := -\eta_t(w) - \frac{C_t(w)}{2}|z|^2 - \frac{1}{\delta}j(v) \leq f_t(w, y, z, v) \leq 0, \quad \forall t \in [0, T].$$

2. For all $y \in \mathbb{R}$, $-1 \leq g_t(w, y) \leq 0$, $0 \leq t \leq T$.

3. There exists a continuous non decreasing process $S = S_0 - V$, where $S_0 \in \mathbb{R}$, $V \in \mathcal{K}$ such that $L_t \leq S_t \leq U_t$, \mathbb{P} -a.s.

4. $\forall t \in [0, T]$, $\forall R \in \mathcal{K}$, $dR_t \geq 0$ and $0 \leq L_t \leq U_t < 1$, \mathbb{P} -a.s.

Theorem 3.2. Assume that Assumption 3.3 and A_γ -condition are fulfilled then the generalized DRBSDE with jumps (\mathcal{E}) associated to $(f, ds + dR_s + gdA_s, \xi, L, U)$ has a maximal solution.

Classically, when we want to manage a quadratic BSDE, it seems natural to start by an exponential change of variable to obtain a Lipschitz BSDE. However in general, this method may fail as it can be seen in lemma (3.1). A possible way to do so is to approximate the BSDE by mean of sup-convolution. This technique was introduced by Lepeltier and San Martin [26] in the backward theory.

To be a little bit more precise, the scheme of our proof is the following.

- The first step consists on introducing an auxiliary generator $(f^n)_n$ globally Lipschitz with respect to (y, z, u) as follows

$$f_t^n(y, z, u) := \sup_{(p, q, r) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu)} \{f_t(p, q, r) + n|y - p| + n|z - q| + n|v - r|\}$$

Moreover, since the integrability conditions on the data are weaker, we will introduce a family of stopping times $(\tau_i)_{i \geq 0}$. Hence, using the existence results of Appendix, we justify the existence of a unique processes $(Y^{n,i}, Z^{n,i}, U^{n,i}, K^{-,n,i}, K^{+,n,i})$ solution for the truncated generalized doubly reflected BSDEJs.

In the last step, we prove a stability result for the approximating sequence of this type of BSDEJs and hence we deduce from it that the limit exists and solves the original one.

Before proceeding in the proof, we will need the following lemma which provides essential properties of the truncated drivers. Define

$$f^n(t, y, z, v) = \sup_{(p, q, r) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu)} \{f(t, p, q, r) - n|p - y| - n|q - z| + n|r - v|\}.$$

$$\underline{q}^n(t, y, z, v) = \sup_{(p, q, r) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu)} \{\underline{q}(t, p, q, r) - n|p - y| - n|q - z| + n|r - v|\}.$$

$$g^n(t, y) = \sup_{p \in \mathbb{R}} \{g(t, p) - n|p - y|\}.$$

Lemma 3.2. [27] Under Assumption 3.3 and A_γ -condition, we have

- The sequences $(f^n)_n$, $(g^n)_n$ are Lipschitz with respect to (y, z, u) .
- For all $(t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu)$, $\forall n \in \mathbb{N}$

$$\underline{q}_t(y, z, u) \leq \underline{q}_t^n(y, z, u) \leq f_t^n(y, z, u) \leq 0.$$

- For all $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$, $\forall n \in \mathbb{N}$, $-1 \leq g_t^n(w, y) \leq 0$.
- The sequences $(f^n)_n$ and $(g^n)_n$ are increasing and converges uniformly in every compact set respectively to f and g \mathbb{P} -a.s.

proof of Theorem 3.2. The proof falls naturally into four steps

First step: Construction of the sequence of generalized doubly reflected BSDE with jumps:

Let $j, i, p \in \mathbb{N}$ such that $j \leq i \leq p$ and $t \in [0, \tau_j]$ where τ_j is a stationary family of stopping times defined as follow

$$\tau_j = \inf \left\{ t \geq 0; A_t + R_t + C_t + \int_0^t \eta_s ds \geq j \right\} \wedge T.$$

Let us now introduce the doubly reflected BSDEJ associated to the truncated driver $(f^n)_n$

$$(\mathcal{E}^1) \left\{ \begin{array}{l} (i) \ Y_t^{n,i} = \xi + \int_t^T f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i}) ds + \int_t^T dR_s^i + \int_t^T g_s^n(Y_s^{n,i}) dA_s^n + \int_t^T dK_s^{n,i,+} \\ \quad - \int_0^T dK_s^{n,i,-} - \int_t^T Z_s^{n,i} dB_s - \int_t^T \int_E V_s^{n,i}(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \\ (ii) \ L_t^{n,i} \leq Y_t^{n,i} \leq U_t^{n,i}, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \\ (iii) \ \int_0^T (Y_{s^-}^{n,i} - L_s^{n,i}) dK_s^{n,i,+} = \int_0^T (U_s^{n,i} - Y_{s^-}^{n,i}) dK_s^{n,i,-} = 0, \quad \mathbb{P}\text{-a.s.} \\ (iv) \ dK_s^{n,i,+} \perp dK_s^{n,i,-}. \end{array} \right. .$$

where $dR^i = 1_{\{s \leq \tau_i\}} dR_s$ and $dA^n = 1_{\{s \leq \tau_n\}} dA_s$.

First, we have to justify the existence of the solution for the system (\mathcal{E}^1) . Using both Lemma (3.2) and the associate A_γ -condition, it follows from Theorem 4.4, the existence of a unique solution $(Y^{n,i}, Z^{n,i}, V^{n,i}, K^{n,i,-}, K^{n,i,+})$. Moreover, we have the following estimate: $\forall n, i \in \mathbb{N}$

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^{n,i}|^2 + \int_0^T |Z_t^{n,i}|^2 dt + \int_0^T \int_E |V_t^{n,i}(e)|^2 \nu(de, dt) + (K_T^{n,i,+})^2 + (K_T^{n,i,-})^2 \right] < +\infty. \quad (3.6)$$

In the other hand, since $(\tau_i)_{i \geq 0}$ is an increasing family of stopping times, we deduce from the comparison Theorem 4.3 and (3.6), that the solution satisfy the following properties

$$\text{For all } i, n \in \mathbb{N}, \ dR^i \leq dR^{i+1}, \ L_t \leq Y_t^{n+1,i} \leq Y_t^{n,i} \leq Y_t^{n,i+1} \leq U_t, \quad \forall t \in [0, T], \ \mathbb{P}\text{-a.s} \quad (3.7)$$

and

$$dK_t^{n,i+1,+} \leq dK_t^{n,i,+} \leq dK_t^{n+1,i,+}, \ dK_t^{n+1,i,-} \leq dK_t^{n,i,-} \leq dK_t^{n,i+1,-}, \ \forall t \in [0, T], \ \mathbb{P}\text{-a.s.} \quad (3.8)$$

For a fixed n , since $(Y^{n,i})_i$ is increasing we can define Y^n as follows

$$Y_t^n = \lim_{i \rightarrow +\infty} \nearrow Y_t^{n,i}, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

From (3.6), the sequences $(Z^{n,i})$ and $(V^{n,i})$ are bounded which entails the weak convergence. We denote respectively (Z^n) and (V^n) their weak limits .

In the next step we prove a stability result for the approximating sequence of generalized doubly reflected BSDEJs. We define an order for the convergence: first we will send i to infinity and then in the third step we let n goes to infinity.

Step2: The convergence of the approximating generalized doubly reflected BSDEJ.

In this part we shall freeze $n \in \mathbb{N}$ and let i goes to ∞ . For simplicity, we shall make the following notations $\delta Y = Y^{n,i} - Y^{n,p}$, $\delta Z = Z^{n,i} - Z^{n,p}$ and $\delta V = V^{n,i} - V^{n,p}$.

Consider $e_t^n := e^{2nA_t^n + (2n^2 + 2n)t}$, we obtain by applying Itô's formula to $e_s^n (Y_s^{n,i} - Y_s^{n,p})^2$ between t and τ_j , the following

$$\begin{aligned}
e_t^n (\delta Y)_t^2 &= e_{\tau_j}^n (\delta Y)_{\tau_j}^2 + \int_t^{\tau_j} 2e_s^n \delta Y_s [f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i}) - f_s^n(Y_s^{n,p}, Z_s^{n,p}, V_s^{n,p})] ds - \int_t^{\tau_j} dR_s^p \\
&+ 2 \int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^+ - 2 \int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^- - 2 \int_t^{\tau_j} e_s^n \delta Y_s [g_s^n(Y_s^{n,i}) - g_s^n(Y_s^{n,p})] dA_s^n \\
&- 2 \int_t^{\tau_j} e_s^n \delta Y_s \delta Z_s dB_s - \int_t^{\tau_j} e_s^n |\delta Z_s|^2 ds - 2 \int_t^{\tau_j} \int_E e_s^n \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds) + \int_t^{\tau_j} dR_s^i \\
&- \int_t^{\tau_j} \int_E e_s^n |\delta V_s(e)|^2 \nu(de, ds) - \int_t^{\tau_j} e_s^n \delta Y_s^2 (2ndA_s^n + (2n + n^2)ds).
\end{aligned} \tag{3.9}$$

Before going any further, we need to estimate the following difference

$$\int_t^{\tau_j} 2e_s^n \delta Y_s [f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i}) - f_s^n(Y_s^{n,p}, Z_s^{n,p}, V_s^{n,p})] ds.$$

We first rely on the classical inequality: $\forall \epsilon > 0, a.b \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$ and the fact that both $(f^n)_n$ and $(g^n)_n$ are uniformly Lipschitz in (y, z, v) . In this way we obtain what follows

$$\begin{aligned}
&\int_t^{\tau_j} 2e_s^n \delta Y_s [f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i}) - f_s^n(Y_s^{n,p}, Z_s^{n,p}, V_s^{n,p})] ds \\
&\leq \int_t^{\tau_j} 2.n.e_s^n \delta Y_s [|\delta Y_s| + |\delta Z_s| + |\delta V_s|] ds. \\
&\leq \epsilon \int_t^{\tau_j} n e_s^n |\delta Y_s|^2 ds + \frac{1}{\epsilon} \int_t^{\tau_j} n e_s^n [|\delta Y_s|^2 + |\delta Z_s|^2 + |\delta V_s|^2] ds.
\end{aligned} \tag{3.10}$$

Besides, we have

$$\int_t^{\tau_j} 2e_s^n \delta Y_s (g_s^n(Y_s^{n,i}) - g_s^n(Y_s^{n,p})) dA_s^n \leq \int_t^{\tau_j} 2.n.e_s^n |\delta Y_s|^2 dA_s^n. \tag{3.11}$$

On the other hand, it follow from (3.8)

$$\int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^+ \leq 0 \quad \text{and} \quad \int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^- \geq 0. \tag{3.12}$$

Besides, Notice that if we use the standard localization procedure we can prove that the local martingale $\int_t^{\tau_j} e_s^n \delta Y_s \delta Z_s dB_s - \int_t^{\tau_j} \int_E e_s^n \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds)$ is in fact a true $(\mathcal{F}, \mathbb{P})$ -martingale. Now notice that since the family of stopping time τ_j is increasing then,

$$\begin{aligned}
\int_t^{\tau_j} dR_s^i - \int_t^{\tau_i} dR_s^p &= \int_t^{\tau_j \wedge \tau_i} dR_s - \int_t^{\tau_j \wedge \tau_p} dR_s \\
&= \int_t^{\tau_j} dR_s - \int_t^{\tau_j} dR_s = 0
\end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13) and putting all terms containing δZ and δV in the left-hand side, we can rewrite (3.9) as follow

$$\mathbb{E}[e_t^n |\delta Y_t|^2 + \int_t^{\tau_j} e_s^n [|\delta Z_s|^2 + |\delta V_s|^2] ds] \leq \mathbb{E}[e_{\tau_j}^n |\delta Y_{\tau_j}|^2].$$

Now in order to justify the passage to the limit in the right hand side as i goes to $+\infty$, we apply Lebesgue's Dominated convergence theorem for a fixed n , since we know that the process $Y^{n,i}$

is bounded $\mathbb{E}[|\delta Y_{\tau_j}|^2]$ goes to 0 as i goes to infinity.

Hence there exists $Z^n \in \mathcal{L}^{2,d}$ and $V^n \in \mathcal{L}_\nu^{2,d}$ such that for any $n \in \mathbb{N}$ we have

$$\lim_{i \rightarrow +\infty} \mathbb{E} \left[\int_0^{\tau_n} (|Z_s^{n,i} - Z_s^n|^2 + \int_E |V_s^{n,i}(e) - V_s^n(e)|^2 \nu(de)) ds \right] = 0.$$

Now apply Itô formula to $|Y_t^{n,i} - Y_t^{n,p}|^2$ and taking first the supremum over $t \in [0, \tau_j]$ then the conditional expectation we get using the Burkholder-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq \tau_j} |Y_t^{n,i} - Y_t^{n,p}|^2 \right] &\leq \mathbb{E} [|Y_{\tau_j}^{n,i} - Y_{\tau_j}^{n,p}|^2] + 2c \mathbb{E} \left[\int_0^{\tau_j} |Y_s^{n,i} - Y_s^{n,p}|^2 |Z_s^{n,i} - Z_s^{n,p}|^2 ds \right]^{\frac{1}{2}} \\ &\quad + 2c \mathbb{E} \left[\int_0^{\tau_j} \int_E |Y_s^{n,i} - Y_s^{n,p}|^2 |V_s^{n,i}(e) - V_s^{n,p}(e)|^2 \nu(de, ds) \right]^{\frac{1}{2}}. \end{aligned}$$

Letting n goes to infinity in the above inequality, we can deduce from the monotone convergence theorem and dominated convergence

$$\lim_{i \rightarrow +\infty} \mathbb{E} \left[\sup_{t \leq \tau_j} |Y_t^{n,i} - Y_t^{n,p}|^2 \right] = 0.$$

To conclude that the process Y^n is càdlàg and hence belongs to \mathcal{D} , the idea is to define $Y^{\bar{p}}$ the projection of Y as the unique predictable process such that $Y_{\bar{\tau}} = \mathbb{E}_{\bar{\tau}-}[Y_{\bar{\tau}}]$ on $\{\bar{\tau} < \infty\}$ for all predictable time $\bar{\tau}$ and then $(Y^{n,i})^{\bar{p}} = Y_-^{n,i}$.

Putting $Y^n = \lim_{i \rightarrow \infty} Y^{n,i}$ together with the fact that $Y^{n,i}$ is càdlàg from [4] we deduce by the weak convergence of $Z^{n,i}$ and $V^{n,i}$ that $Y_-^{n,i} = (Y^{n,i})^{\bar{p}} \uparrow (Y^n)^{\bar{p}}$ as $i \rightarrow \infty$.

Similar to the arguments used in [13] we can prove that the processes K^+ and K^- belongs to \mathcal{K} . From (3.8), we deduce that $K^{n,i,+}$ converges weakly to the continuous increasing process $K^{n,+}$. Furthermore, using Fatou lemma :for fixed $n \in \mathbb{N}$ and for all $i \in \mathbb{N}$

$$\mathbb{E} [(K^{n,+})^2] \leq \mathbb{E} [(K^{n,i,+})^2] \leq \mathbb{E} [(K^{n,0,+})^2] < +\infty.$$

Hence, we get that $\mathbb{E} [(K^{n,+})^2] < \infty$ which proves that $K^{n,+}$ belongs to \mathcal{K} .

Now since τ_j is a stationary family of stopping times , it follows from the system (\mathcal{E}^1) with monotone convergence theorem that for a fixed $n \in \mathbb{N}$, $K_T^{n,-} < +\infty$, \mathbb{P} -a.s.

Finally, letting i goes to infinity for fixed $n \in \mathbb{N}$ in the system (\mathcal{E}^1) we get that the quintuple $(Y^n, Z^n, V^n, K^{n,+}, K^{n,-})$ solves the following system

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f_s^n(Y_s^n, Z_s^n, U_s^n) ds + dR_s^n + g_s^n(Y_s^n) dA_s^n - \int_t^T Z_s^n dB_s \\ &\quad + \int_t^T dK_s^{n,+} - \int_t^T dK_s^{n,-} - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e) \tilde{\mu}(ds, de), \end{aligned} \quad (3.14)$$

$$\text{with } L_t^n \leq Y_t^n \leq U_t^n, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (3.15)$$

Note that Y^n satisfies the above system for all $t \in [0, \tau_j]$. However , since the family of stopping time τ_j satisfies $\mathbb{P} [\cup_{i=1}^\infty (\tau_j = T)] = 1$, we have immediately that Y^n satisfies the system (3.14) for all t in $[0, T]$.

To complete this step, it remains to prove the Skorohod condition of Y^n . Since $0 \leq U_t - Y_t^n \leq U_t - Y_t^{n,i}$, we clearly have

$$\int_0^T (U_t - Y_t^n) dK_t^{n,i,-} = 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore, since the process $K^{n,i,-}$ converges to the continuous increasing process K^n , we obtain the weak convergence of the measure $dK^{n,i}$. Hence it follows that

$$\int_0^T (U_t - Y_{t^-}^n) dK_t^{n,-} = 0, \quad \mathbb{P}\text{-a.s.}$$

The proof of $\int_0^T (Y_{t^-}^n - L_t) dK_t^{n,+} = 0$ is in the same spirit, we only have to notice that we have the weak convergence of the measure $dK^{n,i}$ to obtain that

$$0 \leq \int_0^T (Y_{t^-}^{n,i} - L_t) dK_t^{n,i,+} \leq \int_0^T (Y_{t^-}^{n,i} - L_t) dK_t^{n,+} = 0, \quad \mathbb{P}\text{-a.s.}$$

With the help of Fatou lemma we get the desired result.

The 5-uplet $(Y^n, Z^n, V^n, K^{n,+}, K^{n,-})$ is solution of the following generalized doubly reflected BSDEs with jumps.

$$(\mathcal{E}^2) \left\{ \begin{array}{l} (i) \ Y_t^n = \xi + \int_t^T f_s^n(Y_s^n, Z_s^n, V_s^n) ds + \int_t^T dR_s^n + \int_t^T g_s^n(Y_s^n) dA_s^n - \int_t^T Z_s^n dB_s \\ \quad + \int_t^T dK_s^{n,+} - \int_0^T dK_s^{n,-} - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ (ii) \ L_t^n \leq Y_t^n \leq U_t^n, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ (iii) \ \int_0^T (Y_s^n - L_s^n) dK_s^{n,+} = \int_0^T (Y_s^n - U_s^n) dK_s^{n,-} = 0, \quad \mathbb{P}\text{-a.s.} \end{array} \right.$$

Moreover the processes $Z^n, V^n, K^{n,+}$ and $K^{n,-}$ inherits what follows : for all $n \in \mathbb{N}$

$$\mathbb{E} \left[\int_0^{\tau_j} |Z_s^n|^2 ds + \int_0^{\tau_j} \int_E |V_s^n(e)|^2 \nu(de, ds) + (K_T^{n,+})^2 + (K_T^{n,-})^2 \right] < +\infty.$$

To conclude this step, since $dK^{n,+} = \inf_i dK^{n,i,+}$ and $dK^{n,-} = \sup_i dK^{n,i,-}$ we have that $dK^{n,+}$ and $dK^{n,-}$ are singular.

Step 3 : In this part, we will derive a stability result of the system (\mathcal{E}^2) . We proceed exactly as in the second step. To this end, since we know that the sequence $(Y^n)_n$ is decreasing and uniformly bounded, we only need to prove that there exists a Z and V in $\mathcal{L}^{2,d}$ and $\mathcal{L}_\nu^{2,d}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T \int_E |V_s^n(e) - V_s(e)|^2 \nu(de, ds) \right] = 0. \quad (3.16)$$

As in [25, 14], we first consider the function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $\phi(x) = \frac{1}{4j}(e^{4jx} - 4jx - 1)$ with the following properties

$$\left\{ \begin{array}{l} 0 \leq x \leq 1, \quad \phi(0) = 0, \quad \phi'(0) = 0, \quad \phi'(x), \quad \phi''(x) \geq 0, \\ \phi'(x) = e^{4jx} - 1, \quad \phi''(x) = 4je^{4jx}, \\ \phi''(x) = 4j\phi'(x) + 4j. \end{array} \right. \quad (3.17)$$

For the sake of clarity we define the processes \bar{Y}, \bar{Z} and \bar{V}

$$\forall n \leq m, \quad \bar{Y} := Y^n - Y^m, \quad \bar{Z} := Z^n - Z^m, \quad \bar{V} := V^n - V^m.$$

Since ϕ is C^2 , by Itô's formula we get

$$\begin{aligned}
\phi(\bar{Y}_t) &= \phi(\bar{Y}_{\tau_j}) - \int_t^{\tau_j} \phi'(\bar{Y}_s) d\bar{K}_s^- + \int_t^{\tau_j} \phi'(\bar{Y}_s) d\bar{K}_s^+ - \int_t^{\tau_j} \phi'(\bar{Y}_s) \bar{Z}_s dB_s \\
&\quad - \frac{1}{2} \int_t^{\tau_j} \phi''(\bar{Y}_s) |\bar{Z}_s|^2 ds - \int_t^{\tau_j} \int_E \phi'(\bar{Y}_s) \bar{V}_s(e) \tilde{\mu}(de, ds) + \int_t^{\tau_j} \phi'(\bar{Y}_s) g_s^n(Y_s^n) dA_s^n \\
&\quad - \int_t^{\tau_j} \phi'(\bar{Y}_s) g_s^m(Y_s^m) dA_s^m - \int_t^T \int_E (\phi(\bar{Y}_s + \bar{V}_s) - \phi(\bar{Y}_s) - \phi'(\bar{Y}_s) \bar{V}_s) \mu(de, ds) \\
&\quad + \int_t^{\tau_j} \phi'(\bar{Y}_s) [(f_s^n(Y_s^n, Z_s^n, V_s^n) - f_s^m(Y_s^m, Z_s^m, V_s^m))] ds
\end{aligned} \tag{3.18}$$

Clearly $f^n \leq f$. By the growth assumption on f we get for all $n \in \mathbb{N}$

$$-\eta_s - \frac{C_s}{2} |Z_s^n|^2 - \frac{1}{\delta} J_s(V_s^n) \leq f_s^n(Y_s^n, Z_s^n, V_s^n) \leq 0.$$

Then we have

$$0 \leq -f_s^m(Y_s^m, Z_s^m, V_s^m) \leq \eta_s + \frac{C_s}{2} |Z_s^m|^2 + \frac{1}{\delta} J(V_s^m).$$

Now the elementary inequality $|Z^m|^2 = |Z^m - Z^n + Z^n|^2 \leq 2|\bar{Z}|^2 + 2|Z^n|^2$ yield to an upper bound of $f^n - f^m$.

$$\begin{aligned}
-\eta_s - \frac{C_s}{2} |Z_s^n|^2 - \frac{1}{\delta} J_s(V_s^n) &\leq f_s^n(Y_s^n, Z_s^n, V_s^n) - f_s^m(Y_s^m, Z_s^m, V_s^m) \\
&\leq \eta_s + C_s(w) (|\bar{Z}_s|^2 + |Z_s^n|^2) + \frac{1}{\delta} J_s(V_s^n) \\
&\leq \eta_s + C_s(w) (|\bar{Z}_s|^2 + |Z_s^n|^2) + \epsilon |V_s^m|^2 \\
&\leq \eta_s + C_s(w) (|\bar{Z}_s|^2 + |Z_s^n|^2) + 2\epsilon |V_s^n|^2 + 2\epsilon |\bar{V}_s|^2.
\end{aligned} \tag{3.19}$$

In the other hand

$$\begin{aligned}
\int_t^{\tau_j} g_s^n(Y_s^n) dA_s^n - \int_t^{\tau_j} g_s^m(Y_s^m) dA_s^m &= \int_t^{\tau_j} g_s^n(Y_s^n) 1_{\{s \leq \tau_n\}} dA_s - \int_t^{\tau_j} g_s^m(Y_s^m) 1_{\{s \leq \tau_m\}} dA_s \\
&= \int_t^{\tau_j} (g_s^n(Y_s^n) - g_s^m(Y_s^m)) 1_{\{s \leq \tau_n\}} dA_s - \int_t^{\tau_j} \phi'(\bar{Y}_s) g_s^m(Y_s^m) 1_{\{\tau_n \leq s \leq \tau_m\}} dA_s.
\end{aligned}$$

By lemma (3.2), we have

$$\begin{aligned}
\int_t^{\tau_j} \phi'(\bar{Y}_s) (g_s^n(Y_s^n) - g_s^m(Y_s^m)) 1_{\{s \leq \tau_n\}} dA_s &\leq - \int_t^{\tau_j} \phi'(\bar{Y}_s) g_s^m(Y_s^m) 1_{\{s \leq \tau_n\}} dA_s \\
&\leq \int_t^{\tau_j} \phi'(\bar{Y}_s) 1_{\{s \leq \tau_n\}} dA_s \\
&\leq \int_t^{\tau_j} 4j e^{4j} \bar{Y}_s dA_s.
\end{aligned} \tag{3.20}$$

Hence, reporting (3.19) and (3.20) in (3.18) yield to

$$\begin{aligned}
\phi(\bar{Y}_t) &+ \int_t^T \int_E (\phi(\bar{Y}_s + \bar{V}_s) - \phi(\bar{Y}_s) - \phi'(\bar{Y}_s) \bar{V}_s) \mu(de, ds) \\
&\leq \phi(\bar{Y}_{\tau_j}) - \int_t^{\tau_j} \int_E \bar{V}_s(e) \phi'(\bar{Y}_s) \tilde{\mu}(de, ds) - \int_t^{\tau_j} \phi'(\bar{Y}_s) \bar{Z}_s dB_s \\
&\quad - \frac{1}{2} \int_t^{\tau_j} \phi''(\bar{Y}_s) |\bar{Z}_s|^2 ds - \int_t^{\tau_j} \phi'(\bar{Y}_s) d\bar{K}_s^- + \int_t^{\tau_j} \phi'(\bar{Y}_s) d\bar{K}_s^+ \\
&\quad + \int_t^{\tau_j} \phi'(\bar{Y}_s) [\eta_s + C_s(w) (|\bar{Z}_s|^2 + |Z_s^n|^2) + 2\epsilon |V_s^n|^2 + 2\epsilon |\bar{V}_s|^2] ds \\
&\quad + 4j \int_t^{\tau_j} e^{4j} \bar{Y}_s dA_s + 4j \int_t^{\tau_j} e^{4j} \bar{Y}_s 1_{\{\tau_n \leq s \leq \tau_m\}} dA_s.
\end{aligned} \tag{3.21}$$

Now, since the process $C_s(w)$ is bounded by j for all $s \in [0, \tau_j]$ it follows

$$\begin{aligned}
& \int_t^{\tau_j} \phi'(\bar{Y}_s) C_s(w) |\bar{Z}_s|^2 ds - \frac{1}{2} \int_t^{\tau_j} \phi''(\bar{Y}_s) |\bar{Z}_s|^2 ds \\
& \leq \int_t^{\tau_j} j \phi'(\bar{Y}_s) |\bar{Z}_s|^2 ds - \frac{1}{2} \int_t^{\tau_j} (4j \phi'(\bar{Y}_s) + 4j) |\bar{Z}_s|^2 ds \\
& \leq j \int_t^{\tau_j} \phi'(\bar{Y}_s) |\bar{Z}_s|^2 ds - 2j \int_t^{\tau_j} \phi'(\bar{Y}_s) |\bar{Z}_s|^2 ds - 2j \int_t^{\tau_j} |\bar{Z}_s|^2 ds \\
& \leq -j \int_t^{\tau_j} |\bar{Z}_s|^2 ds - 2j \int_t^{\tau_j} \phi'(\bar{Y}_s) |\bar{Z}_s|^2 ds.
\end{aligned} \tag{3.22}$$

Using the same argument as in [15], we obtain

$$\tilde{C} |\bar{V}_s(e)|^2 \leq \phi(\bar{Y}_{s-} + \bar{V}_s(e)) - \phi(\bar{Y}_{s-}) - \phi'(\bar{Y}_{s-}) \bar{V}_s(e), \quad d\nu a.e. \tag{3.23}$$

Plugging (3.22) and (3.23) in the previous Itô equation (3.21) yields

$$\begin{aligned}
& \phi(\bar{Y}_t) + \tilde{C} \int_t^T \int_E |\bar{V}_s(e)|^2 \nu(de, ds) \leq \\
& \phi(\bar{Y}_{\tau_j}) - 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |\bar{V}_s|_s^2 ds - 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |V_s^n|_s^2 ds - 2j \int_t^{\tau_j} |\bar{Z}_s|^2 ds - j \int_t^{\tau_j} \phi'(\bar{Y}_s) |\bar{Z}_s|^2 ds \\
& - j \int_t^{\tau_j} \phi'(\bar{Y}_s) |Z_s^n|^2 ds + \int_t^{\tau_j} (e^{j\bar{Y}_s} - 1) dA_s + \int_t^{\tau_j} (e^{j\bar{Y}_s} - 1) d\bar{K}_s^- + \int_t^{\tau_j} (e^{j\bar{Y}_s} - 1) d\bar{K}_s^+ \\
& - \int_t^{\tau_j} (e^{j\bar{Y}_s} - 1) \bar{Z}_s dB_s + dR_s^{n,m} + 4j \int_t^{\tau_j} \eta_s e^{4j\bar{Y}_s} ds - \int_t^{\tau_j} \int_E \bar{V}_s(e) \phi'(\bar{Y}_s) \tilde{\mu}(de, ds).
\end{aligned} \tag{3.24}$$

Let us underline that the process \bar{K} acts only when \bar{Y} reaches the obstacles L and U , it is easy to see that $K^{m,+}$ only increases when $Y_t^n = S_t$ and $\bar{K}^{n,-}$ only increases when $Y_t^m = L_t$. Therefore, we have

$$\begin{aligned}
& \phi(\bar{Y}_t) + \tilde{C} \int_t^T \int_E |\bar{V}_s(e)|^2 \nu(de, ds) \leq \phi(\bar{Y}_{\tau_j}) - 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |\bar{V}_s|_s^2 ds - 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |V_s^n|_s^2 ds \\
& - 2j \int_t^{\tau_j} [1 + \phi'(\bar{Y}_s)] |\bar{Z}_s|^2 ds - j \int_t^{\tau_j} \phi'(\bar{Y}_s) |Z_s^n|^2 ds + \int_t^{\tau_j} 4j e^{4j} (\bar{Y}_s + 1_{\{\tau_n \leq s \leq \tau_m\}}) dA_s \\
& + \int_t^{\tau_j} j (Y_{s-}^n - L_s) dK_s^{m,+} + dR_s^{n,m} + \int_t^{\tau_j} j (-Y_{s-}^m + U_s) dK_s^{n,-} - \int_t^{\tau_j} \int_E \bar{V}_s(e) \phi'(\bar{Y}_s) \tilde{\mu}(de, ds) \\
& - \int_t^{\tau_j} (e^{j\bar{Y}_s} - 1) \bar{Z}_s dB_s + 4j \int_t^{\tau_j} \eta_s e^{4j\bar{Y}_s} ds.
\end{aligned}$$

where

$$d\tilde{R}_s^{n,m} = dR_s^{n,m} + \left(e^{4j(Y_{s-}^n - L_s)} - 4j(Y_{s-}^n - L_s) \right) dK_s^{m,+} - \left(e^{4j(-Y_{s-}^m + U_s)} - 4j(-Y_{s-}^m + U_s) \right) dK_s^{n,-}.$$

Taking in the left sides all the terms containing either \bar{Z} or \bar{V} , we get

$$\begin{aligned}
& \phi(\bar{Y}_t) + \tilde{C} \int_t^T \int_E |\bar{V}_s(e)|^2 \nu(de, ds) + 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |\bar{V}_s|_s^2 ds + 2j \int_t^{\tau_j} [1 + \phi'(\bar{Y}_s)] |\bar{Z}_s|^2 ds \\
& + 4j \int_t^{\tau_j} j (Y_{s-}^n - L_s) dK_s^{m,+} + 4j \int_t^{\tau_j} j (-Y_{s-}^m + U_s) dK_s^{n,-} + \int_t^{\tau_j} d\tilde{R}_s^{n,m}. \\
& \leq \phi(\bar{Y}_{\tau_j}) + j \int_t^{\tau_j} \phi'(\bar{Y}_s) |Z_s^n|^2 ds - 2\epsilon \int_t^{\tau_j} \phi'(\bar{V}_s) |V_s^n|_s^2 ds + \int_t^{\tau_j} 4j e^{4j} (\bar{Y}_s + 1_{\{\tau_n \leq s \leq \tau_m\}}) dA_s \\
& + 4j \int_t^{\tau_j} \eta_s e^{4j\bar{Y}_s} ds - \int_t^{\tau_j} \phi'(\bar{Y}_s) \bar{Z}_s dB_s - \int_t^{\tau_j} \int_E \bar{V}_s(e) \phi'(\bar{Y}_s) \tilde{\mu}(de, ds).
\end{aligned} \tag{3.25}$$

Then using the fact that $Y^n - Y^m \leq 1$ we obtain for $t = 0$,

$$\begin{aligned} \mathbb{E}[2j(1 + \phi'(1)) \int_0^{\tau_j} |Z_s^n - Z_s^m|^2 ds + (2\epsilon\phi'(1) + \tilde{C}) \int_0^{\tau_j} \int_E |V_s^n - V_s^m|^2 \nu(de, ds)] \\ \leq \phi(1) + \mathbb{E}[4je^{4j} \int_t^{\tau_j} \eta_s ds + \phi'(1) \int_0^{\tau_j} dA_s + \int_0^{\tau_j} d\tilde{R}_s^{n,m}] \\ + \mathbb{E}[\int_0^{\tau_j} j \cdot \phi'(\bar{Y}) |Z_s^n|^2 - 2\epsilon\phi'(1) \mathbb{E} \int_t^{\tau_j} |V_s^n|^2 ds]. \end{aligned} \quad (3.26)$$

Let us underline that since $\phi(Y^{n,m})$ is uniformly bounded process, the conditional expectation of the martingale part of equation (3.25) vanishes. Now, taking $n = 0$ we get from the apriori estimates of $(Y^n, Z^n, K^{n,+}, K^{n,-})$ that

$$\mathbb{E}[\int_0^{\tau_j} |Z_s^0 - Z_s^m|^2 ds + \int_0^{\tau_j} \int_E |V_s^0 - V_s^m(e)|^2 \nu(de, ds)] \leq C'_j. \quad (3.27)$$

and

$$\sup_{m \in \mathbb{N}} \mathbb{E}[\int_0^{\tau_j} |Z_s^m|^2 ds + \int_0^{\tau_j} \int_E |V_s^m(e)| \nu(ds, de)] < +\infty. \quad (3.28)$$

For all $t \in [0, \tau_j]$, we can then extract a subsequence $(m_k^j)_{k \in \mathbb{N}}$ such that $Z_t^{m_k^j}$ and $V_t^{m_k^j}$ converges weakly respectively to an \mathcal{F}_t -adapted processes \hat{Z}_t and \hat{V}_t in \mathbb{H}^2 and \mathbb{H}_ν^2 . It is obvious that $(\phi'(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}} 1_{[0, \tau_j]}(Z_s^n - Z_s^{m_k^j})$ respectively $(\phi'(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}} 1_{[0, \tau_j]}(V_s^n - V_s^{m_k^j})$ converge weakly in \mathbb{H}^2 and in \mathbb{H}_ν^2 to $(\phi'(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}} 1_{[0, \tau_j]}(Z_s^n - Z_s)$ and $(\phi'(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}} 1_{[0, \tau_j]}(V_s^n - V_s)$. Hence, recalling the Itô equation (3.25) we obtain

$$\begin{aligned} \mathbb{E}[\int_0^{\tau_j} 2j(\frac{1}{2} + \phi'(\bar{Y}_s)) |Z_s^n - Z_s^{m_k^j}|^2 ds + \int_0^{\tau_j} \int_E (2\epsilon\phi'(\bar{Y}_s) + \tilde{C}) |V_s^n - V_s^{m_k^j}(e)|^2 \nu(de, ds)] \\ \leq \mathbb{E}[\phi(\bar{Y}_{\tau_j}) + 4je^{4j} \int_0^{\tau_j} \eta_s ds + \int_0^{\tau_j} 4je^{4j} (\bar{Y}_s + 1_{\{\tau_n \leq s \leq \tau_m\}}) dA_s + \int_0^{\tau_j} j \cdot \phi'(\bar{Y}_s) |Z_s^n|^2 ds] \\ - 2\epsilon \mathbb{E}[\int_0^{\tau_j} \phi'(\bar{Y}_s) |V_s^n|^2 ds]. \end{aligned}$$

Notice that

$$j \mathbb{E}[\int_0^{\tau_j} \phi'(\bar{Y}_s) |Z_s^n|^2 ds] \leq 2j \mathbb{E}[\int_0^{\tau_j} \phi'(\bar{Y}_s) |Z_s^n - \hat{Z}_s|^2 ds + \int_0^{\tau_j} \phi'(\bar{Y}_s) |\hat{Z}_s|^2 ds],$$

and

$$\begin{aligned} 2\epsilon \mathbb{E}[\int_t^{\tau_j} \phi'(Y_s^n - \hat{Y}_s) |V_s^n|^2 ds] \leq 2\epsilon \mathbb{E}[\int_t^{\tau_j} \phi'(Y_s^n - \hat{Y}_s) |V_s^n - \hat{V}_s|^2 ds] \\ + 2\epsilon \mathbb{E}[\int_t^{\tau_j} \phi'(Y_s^n - \hat{Y}_s) |\hat{V}_s|^2 ds]. \end{aligned}$$

Now since we have

$$\begin{aligned} \mathbb{E}[2j(1 + \phi'(Y_s^n - \hat{Y}_s)) \int_0^{\tau_j} |Z_s^n - \hat{Z}_s|^2 ds + \int_0^{\tau_j} \int_E (2\epsilon\phi'(Y_s^n - \hat{Y}_s) + \tilde{C}) |V_s^n(e) - \hat{V}_s(e)|^2 \nu(de, ds)] \\ \leq \liminf_k \mathbb{E}[2j(1 + \phi'(Y_s^n - Y_s^{m_k^j})) \int_0^{\tau_j} |Z_s^n - Z_s^{m_k^j}|^2 ds] \\ + \liminf_k \mathbb{E}[(2\epsilon\phi'(Y_s^n - Y_s^{m_k^j}) + \tilde{C}) \int_0^{\tau_j} \int_E |V_s^n(e) - V_s^{m_k^j}(e)|^2 \nu(de, ds)], \end{aligned}$$

letting k goes to infinity allows us to obtain

$$\begin{aligned}
& \mathbb{E}\left[2j \int_0^{\tau_j} |Z_s^n - \hat{Z}_s|^2 ds + \tilde{C} \int_0^{\tau_j} \int_E |V_s^n(e) - \hat{V}_s(e)|^2 \nu(de, ds)\right] \\
& \leq \liminf_k \left[\mathbb{E}\left[2j(1 + \phi'(Y_s^n - Y_s^{m_k^j})) \int_0^{\tau_j} |Z_s^n - Z_s^{m_k^j}|^2 ds\right.\right. \\
& \quad \left.\left. + (2\epsilon\phi'(Y_s^n - Y_s^{m_k^j}) + \tilde{C}) \int_0^{\tau_j} \int_E |V_s^n(e) - V_s^{m_k^j}(e)|^2 \nu(de, ds)\right]\right] \\
& \leq \mathbb{E}[\phi(\bar{Y}_{\tau_j}) + 4je^{4j} \int_t^{\tau_j} \eta_s ds + \int_0^{\tau_j} \phi'(\bar{Y}_s) dA_s + 2j \int_0^{\tau_j} \phi'(\bar{Y}_s) |\hat{Z}_s|^2 ds] \\
& \quad + 2\epsilon \mathbb{E}\left[\int_t^{\tau_j} \phi'(Y_s^n - \hat{Y}_s) |V_s^n|^2 ds\right].
\end{aligned}$$

Since $\phi'(Y^n - \hat{Y}_s)$ goes to zero as k goes to infinity, we conclude by the dominated convergence theorem, that

$$\mathbb{E}\left[\int_0^{\tau_j} |Z_s^n - \hat{Z}_s^j|^2 ds + \int_0^{\tau_j} \int_E |V_s^n(e) - \hat{V}_s^j(e)|^2 \nu(de, ds)\right] \xrightarrow{n \rightarrow \infty} 0. \quad (3.29)$$

Therefore by the uniqueness of the limit we have , $\hat{Z}_s^j(w) = \hat{Z}_s^{j+1}(w)$ and $\hat{V}_s^j(w) = \hat{V}_s^{j+1}(w)$, \mathbb{P} -a.s.

We then denote by Z and V their respective limits when j goes to infinity. Finally since $\tau_j = T$, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T \int_E |V_s^n(e) - V_s(e)|^2 \nu(de, ds)\right] = 0. \quad (3.30)$$

Hence, it remains to prove that the limit process Y is càdlàg.

$$\begin{aligned}
|Y_s - Y_s^n| & \leq \int_t^{\tau_j} |(f_s(Y_s, Z_s, V_s) - f_s^n(Y_s^n, Z_s^n, V_s^n))| ds + \int_t^{\tau_j} |g_s(Y_s) - g_s^n(Y_s^n)| dA_s^n \\
& \quad + \int_t^{\tau_j} |g_s(Y_s) 1_{\{s \geq \tau_n\}}| dA_s + \left| \int_t^{\tau_j} (Z_s - Z_s^m) dB_s \right| + \left| \int_t^{\tau_j} \int_E (V_s(e) - V_s^m(e)) \tilde{\mu}(de, ds) \right|.
\end{aligned}$$

Taking the supremum over t and the conditional expectation yields to

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq t \leq \tau_j} |Y_s - Y_s^n|\right] & \leq \mathbb{E}\left[\int_0^{\tau_j} |(f_s(Y_s, Z_s, V_s) - f_s^n(Y_s^n, Z_s^n, V_s^n))| ds\right] \\
& \quad + \mathbb{E}\left[\int_0^{\tau_j} |g_s(Y_s) 1_{\{s \geq \tau_n\}}| dA_s\right] + \mathbb{E}\left[\int_0^{\tau_j} |g_s(Y_s) - g_s^n(Y_s^n)| dA_s^n\right] \\
& \quad + \mathbb{E}\left[\sup_{0 \leq t \leq \tau_j} \left| \int_t^{\tau_j} (Z_s - Z_s^m) dB_s \right| + \sup_{0 \leq t \leq \tau_j} \left| \int_t^{\tau_j} \int_E (V_s(e) - V_s^m(e)) \tilde{\mu}(de, ds) \right|\right].
\end{aligned} \quad (3.31)$$

Furthermore, the Burkholder-Davis-Gundy inequality allows us to deduce that

$$\mathbb{E}\left[\sup_{0 \leq t \leq \tau_j} \left| \int_0^{\tau_j} (Z_s - Z_s^m) dB_s \right|\right] \leq 2\mathbb{E}\left[\int_0^{\tau_j} |Z_s - Z_s^m|^2 ds\right]^{\frac{1}{2}}. \quad (3.32)$$

and

$$\mathbb{E}\left[\sup_{0 \leq t \leq \tau_j} \int_0^{\tau_j} \int_E (V_s(e) - V_s^n(e)) \tilde{\mu}(de, ds)\right] \leq 2\mathbb{E}\left[\int_0^{\tau_j} \int_E |V_s(e) - V_s^n(e)|^2 \nu(de, ds)\right]^{\frac{1}{2}}. \quad (3.33)$$

Reporting (3.32), (3.33) in the previous inequality yields to

$$\mathbb{E}\left[\sup_{0 \leq t \leq \tau_j} |Y_t - Y_t^n|\right] \rightarrow_n 0.$$

Step 4: Identification of the limit

In this part we aim to prove that quintuple (Y, Z, V, K^-, K^+) is the solution of the generalized doubly reflected BSDEJ (\mathcal{E}) . To do so we first prove that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau_j} |f_t^n(Y_t^n, Z_t^n, U_t^n) - f_t(Y_t, Z_t, U_t)| dt \right] = 0 \text{ and } \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau_j} |g_t^n(Y_t^n) - g_t(Y_t)| dt \right] = 0.$$

By Lemma 3.2, $f_s^n(y, z, v)$ converges to $f_s(y, z, v)$, $\forall s \in [0, T]$, \mathbb{P} -a.s. Moreover, from the càdlàg version of Dini's theorem, $f_s^n(y, z, v)$ converges uniformly on every compact set. Thus, $f_s^n(y, z, v)1_{[0, \tau_j]}$ converges uniformly to $f_s(y, z, v)1_{[0, \tau_j]}$ as n goes to infinity, $dt \otimes d\mathbb{P} \otimes d\nu$ a.s.

In fact as $Z^{n_k^j}$ and $U^{n_k^j}$ are unbounded, one can decompose the expression above in the following way

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_j} |f_t^n(Y_t^n, Z_t^n, U_t^n) - f_t(Y_t, Z_t, U_t)| dt \right] &= \mathbb{E} \left[\int_0^{\tau_j} |f_t^n(Y_t^n, Z_t^n, U_t^n) - f_t(Y_t, Z_t, U_t)| \mathbf{1}_{\{|Z_s^{n_k^j}| + |U_s^{n_k^j}| \leq C\}} dt \right] \\ &\quad + \mathbb{E} \left[\int_0^{\tau_j} |f_t^n(Y_t^n, Z_t^n, U_t^n) - f_t(Y_t, Z_t, U_t)| \mathbf{1}_{\{|Z_s^{n_k^j}| + |U_s^{n_k^j}| \geq C\}} dt \right] \end{aligned}$$

The first term in the right-hand side goes to zero as k goes to infinity since $Y^{n_k^j}$ is bounded over $[0, \tau_j]$ and

$$|f_t^{n_k^j}(Y_t^{n_k^j}, Z_t^{n_k^j}, V_t^{n_k^j}) - f_t(Y_t, Z_t, V_t)| \leq \eta_t + \frac{j}{2} |Z_t^{n_k^j}|^2 + \frac{1}{\delta} J(V_t^{n_k^j}).$$

For the last one, using Markov inequality we have

$$\mathbb{E} \left[\mathbf{1}_{\{|Z^{n_k^j}| + |V^{n_k^j}| \geq C\}} \right] \leq \frac{2}{C^2} \mathbb{E} [|Z^{n_k^j}|^2 + |V^{n_k^j}|^2].$$

Hence, using the dominated convergence theorem we obtain that $f_t^{n_k^j}(Y_t^{n_k^j}, Z_t^{n_k^j}, U_t^{n_k^j})$ converge to $f_t(Y_t, Z_t, U_t)$ in $\mathbb{L}^1(dt \otimes d\mathbb{P} \otimes d\nu)$ for all $t \leq \tau_j$.

Using the same argument we can prove that $\mathbb{E} \left[\int_0^{\tau_j} |g_t^n(Y_t^n) - g_t(Y_t)| dt \right]$ goes to zero as n goes to infinity.

Now, notice from the system (\mathcal{E}^2) that $\sup_n \mathbb{E}[K_{\tau_j}^{n,+}] < \infty$. By Fatou lemma we deduce that $\mathbb{E}[K_{\tau_j}^+] < \infty$. Thus, we obtain that $K_T^+ < \infty$, \mathbb{P} -a.s. We can show similarly that $K_T^- < \infty$.

In order to finish the proof we need to show that the limit process Y satisfies the following minimality condition

$$\int_0^T (Y_{s-} - L_s) dK_s^+ = \int_0^T (U_s - Y_{s-}) dK_s^- = 0, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

This is deduced from the following facts :

- From the system (4.1) we have $\int_0^T (Y_s^n - L_s^n) dK_s^{n,+} = \int_0^T (U_s^n - Y_s^n) dK_s^{n,-} = 0$, \mathbb{P} -a.s.
- $K^{n,+}$ respectively $K^{n,-}$ converges uniformly to K^+ , K^- .

The only point remaining concerns the singularity of the measures dK^+ and dK^- . The result follows from the singularity of $dK^{n,-}$ and $dK^{n,+}$ with $dK^- = \inf_n dK^{n,-}$, $dK^+ = \sup_n dK^{n,+}$. We can hence conclude that the 5-uplet (Y, Z, V, K^+, K^-) is a solution of the system (\mathcal{E}^1) . \square

Proof of Theorem 3.1. As already explained, the existence is obtained directly from a logarithmic change of variable. Let $Y_t = \frac{\ln(\bar{Y}_t)}{m_t} + m_t$, $Z_t = \frac{\bar{Z}_t}{m_s Y_t}$ and $V_t = \frac{1}{m_t} \ln(\frac{\bar{V}_t}{Y_t} + 1)$. Then applying Itô's formula, we obtain

$$\begin{aligned} Y_t &= \xi + \int_t^T f_s(Y_s, Z_s, V_s) ds + \int_t^T dR_s + \int_t^T g_s(Y_s) dA_s + \int_t^T dK_s^+ - \int_t^T dK_s^- \\ &\quad - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \end{aligned}$$

In addition, since $\int_0^T (\bar{Y}_{s-} - \bar{L}_s) d\bar{K}_s^+ = \int_0^T (\bar{Y}_{s-} - \bar{U}_s) d\bar{K}_s^- = 0$. we have $\int_0^T (Y_{s-} - L_s) dK_s^+ = \int_0^T (Y_{s-} - U_s) dK_s^- = 0$, \mathbb{P} -a.s. \square

4 Appendix

4.1 Existence and uniqueness result: The Lipschitz case.

In this section, we extend some of the results of Pardoux and Zhang [29] concerning generalized BSDEs with no reflection to the case of doubly reflected BSDEJs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of [29], [21] [30] and [12] with some minor modifications due to jumps and reflection. However, we still provide the proof of existence since it will be needed in the construction of stochastic quadratic BSDEJ's solution. To the best of our knowledge, they do not appear anywhere else in the literature.

We look for the solution of the following generalized doubly reflected BSDE with jumps,

$$(S) \begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s, V_s) ds + \int_t^T g_s(Y_s) dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e) \tilde{\mu}(ds, de) \\ \quad + \int_t^T dK_s^+ - \int_t^T dK_s^- \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ L_t \leq Y_t \leq U_t \quad \text{and} \quad \int_0^T (Y_{s-} - U_s) dK_s^+ = \int_0^T (Y_{s-} - L_s) dK_s^- = 0, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \end{cases}$$

under the following assumption

$$(H1) \begin{cases} (i) \text{ There exists a positive constant } L_f \text{ such that } \forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, u, u' \in \mathbb{L}^0(\mathcal{B}(E), \nu), \\ \quad |f_t(y, z, u) - f_t(y', z', u')| \leq L_f(|y - y'| + |z - z'| + |k - k'|_t), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists a positive constant } L_g \text{ such that } \forall y, y' \in \mathbb{R}, \\ \quad |g_t(y) - g_t(y')| \leq L_g|y - y'|, \text{ and } -1 \leq g_t(y) \leq 0, \quad \forall t \in [0, T]. \\ (iii) \forall R \in \mathcal{K}, \quad dR_t \geq 0 \text{ and } 0 \leq L_t \leq U_t < 1, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \end{cases}$$

Comparison result

Theorem 4.3. *Let $(Y^1, Z^1, V^1, K^{1,-}, K^{1,+})$ and $(Y^2, Z^2, V^2, K^{2,-}, K^{2,+})$ be to two solutions of (S) associated to $(f^i, \xi^i, L^i, U^i)_{i=1,2}$, such that, for $(i = 1, 2)$ Assumption (H1) is satisfied. Assume moreover that*

$$\begin{cases} \bullet \xi^1 \leq \xi^2, \quad \mathbb{P}\text{-a.s.} \\ \bullet L_t^1 \leq L_t^2 \text{ and } S_t^1 \leq S_t^2, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ \bullet f_s^1(Y_s^2, Z_s^2, V_s^2) \leq f_s^2(Y_s^2, Z_s^2, V_s^2), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ \bullet g_s^1(Y_s^2) \leq g_s^2(Y_s^2), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \end{cases}$$

then we have $Y_t^1 \leq Y_t^2, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$

Furthermore, if $U_t^1 = U_t^2, L_t^1 = L_t^2, L_t \leq U_t, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$, then $K_t^{-,1} \leq K_t^{-,2}$ and $K_t^{+,2} \leq K_t^{+,1}, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$

Proof. The proof follows the lines of the proof of the theorem (1.3) in [?] in the continuous setting . For simplicity, we shall make the following notations.

$$\begin{aligned} (\delta Y_t, \delta Z_t, \delta V_t) &= (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2, V_t^1 - V_t^2), \quad \delta \xi = \xi^1 - \xi^2 \\ \delta f_t &= f_t^1(Y_t^2, Z_t^2, V_t^2) - f_t^2(Y_t^2, Z_t^2, V_t^2) \quad \delta g_t = g_t^1(Y_t^2) - g_t^2(Y_t^2) \end{aligned}$$

Let us define the following bounded processes

$$\begin{aligned} \alpha_s &= \frac{f_s^2(Y_s^1, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^1, V_s^1)}{Y_s^1 - Y_s^2} 1_{\{Y_s^1 \neq Y_s^2\}}, \quad \tilde{\alpha}_s = \frac{g_s^2(Y_s^1) - g_s^2(Y_s^2)}{Y_s^1 - Y_s^2} 1_{\{Y_s^1 \neq Y_s^2\}}. \\ \beta_s &= \frac{f_s^2(Y_s^2, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^2, V_s^1)}{\|Z_s^1 - Z_s^2\|^2} (Z_s^1 - Z_s^2) 1_{\{Z_s^1 \neq Z_s^2\}}. \end{aligned}$$

Consider the following stopping times

$$\tau_k = \inf \left\{ t \geq 0, \int_0^t (|Z_s^1|^2 + |Z_s^2|^2) ds + \int_0^t \int_E (|V_s^1(e)|^2 + |V_s^2(e)|^2) \nu(de, ds) \geq k \right\} \wedge T.$$

We start by applying Itô formula to $R_t \delta Y_t = e^{\alpha t} (Y_t^1 - Y_t^2)^+$ where $R_t = e^{\alpha t}$.

$$\begin{aligned} R_{t \wedge \tau_k} (\delta Y_{t \wedge \tau_k})^+ &= R_{\tau_k} (\delta Y_{\tau_k})^+ + \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s \delta Y_s [f_s^1(Y_s^1, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^2, V_s^2)] ds \\ &+ \int_{t \wedge \tau_k}^{\tau_k} R_s \delta Y_s [g_s(Y_s^1) - g_s(Y_s^2)] dA_s - \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} \delta Y_s R_s \delta Z_s dB_s \\ &- \int_{t \wedge \tau_k}^{\tau_k} \int_E R_s \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds) - \int_{t \wedge \tau_k}^{\tau_k} \alpha_s R_s \delta Y_s ds \\ &+ \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s (dK_s^{+,1} - dK_s^{-,1}) - \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s (dK_s^{+,2} - dK_s^{-,2}). \end{aligned}$$

Notice that when $Y^1 \geq Y^2$ we have $U_t^2 \geq Y_t^2$ and $Y_t^1 \geq L_t^1$, we obtain

$$\int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_{s-}^1 \geq Y_{s-}^2\}} \delta Y_s (dK_s^{+,1} - dK_s^{+,2}) - \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_{s-}^1 \geq Y_{s-}^2\}} \delta Y_s (dK_s^{-,1} - dK_s^{-,2}) \leq 0.$$

Hence, using Assumption (H1), we get

$$\begin{aligned} R_{t \wedge \tau_j} (\delta Y_{t \wedge \tau_k})^+ &\leq R_{\tau_k} (\delta Y_{\tau_k})^+ + \int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} \alpha_s R_s |\delta Y_s| ds \\ &- \underbrace{\int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s \delta Z_s dB_s - \int_{t \wedge \tau_k}^{\tau_k} \int_E 1_{\{Y_s^1 \geq Y_s^2\}} R_s \delta V_s(e) \tilde{\mu}(de) ds + \int_{t \wedge \tau_k}^{\tau_k} R_s \tilde{\alpha}_s 1_{\{Y_s^1 \geq Y_s^2\}} |\delta Y_s| dA_s}_{M} \\ &+ \underbrace{\int_{t \wedge \tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s |\delta Z_s| \beta_s ds + \int_{t \wedge \tau_k}^{\tau_k} \int_E 1_{\{Y_s^1 \geq Y_s^2\}} R_s \gamma_s(e) \delta V_s(e) \nu_s(de, ds)}_{M - \langle M, N \rangle}. \end{aligned} \quad (4.34)$$

Now we define the probability measure $\tilde{\mathbb{P}}$ such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(N)_T$. Since $-1 + \delta_K \leq \gamma(U_s^1, U_s^2) \leq C_K$ and $|\beta_t(Z_s^1, Z_s^2)| \leq \bar{C}$ since Z^1 and Z^2 are of BMO types.

Hence $\mathcal{E}(M)$ is a BMO-martingale with $M = \int_0^t R_s \tilde{\alpha}_s \delta Y_s dA_s + \int_0^t R_s (Z_s^1 - Z_s^2) dB_s + \int_0^t R_s (V_s^1 - V_s^2) \tilde{\mu}(de)$ and $N = \int_0^t \beta_s dB_s + \int_0^t \int_E \gamma(e) \tilde{\mu}(de, ds)$. Using Girsanov theorem, we obtain that $M - \langle M, N \rangle$ are locale \tilde{P} -martingale.

Hence, taking the conditional expectation in (4.1) between t and τ_n when τ_k converges to T as long as k goes to infinity yield

$$R_t (Y_t^1 - Y_t^2)^+ \leq \mathbb{E}^{\tilde{\mathbb{P}}} [R_{\tau_k} (Y_{\tau_k}^1 - Y_{\tau_k}^2)^+ | \mathcal{F}_t].$$

Sending k to ∞ we get $\delta Y_t = Y_t^1 - Y_t^2 = 0, \forall t \in [0, T], \mathbb{P}$ -a.s.

• Let as now prove that

$$K_t^{-,1} \leq K_t^{-,2}, \quad K_t^{+,2} \leq K_t^{+,1}, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

Exactly as (Theorem (1.3), [21]), we define the following family of stopping times τ

$$\tau = \inf \left\{ t \geq 0, \quad K_t^{-,1} \geq K_t^{-,2} \right\} \wedge T.$$

Suppose that $\mathbb{P}(\tau < T) > 0$. Hence, $K_{\tau}^{-,1} = K_{\tau}^{-,2}$ on $\{\tau < T\}$. Moreover

$$Y_{\tau}^1 = Y_{\tau}^2 = U_{\tau} \quad \text{on } \{\tau < T\}.$$

if $Y_{\tau(w)}^1(w) \neq U_{\tau(w)}(w)$ then $Y_t(w) < U_t(w)$ for all $t \in]\tau(w) - p(w), \tau + p(w)[$ where $p(w)$ is a positive real number.

Thus, it follows that for all $t \in]\tau(w) - p(w), \tau + p(w)[$, $K_{t(w)}^{-,1}(w) = K_{t(w)}^{-,2}(w) = Y_{t(w)}^2(w)$ which contradicts the definition of the stopping family $\tau(w)$. Hence

$$Y_{t(w)}^1(w) = U_{t(w)}(w) = Y_{t(w)}^2(w).$$

In the other hand, we consider the family of stopping times $\delta = \inf \{t \geq \tau; Y_t^1 = L_t\} \wedge T$ such that $\{\tau < T\} \subset \{\delta < T\}$. Notice that

- if $\tau_w < T$ then $Y_{\tau(w)}^1 = U_{\tau(w)}(w)$.

- if $\delta(w) = \tau(w)$ then $Y_{\delta(w)}(w) = L_{\delta(w)} = U_{t(w)} = L_{t(w)}(w)$ which contradicts the fact that the process U_t remain above the process L_t . We can deduce that $\mathbb{P}[\delta < \tau] > 0$.

This implies that $K_{\delta}^{+,1} = K_t^{+,1}$ and $K_{\delta}^{+,2} = K_t^{+,2}$, $\forall t \in [\tau, \delta]$, \mathbb{P} -a.s, since $Y^1 \leq Y^2$ and K^+ (resp. K^-) moves when Y^1 (resp. Y^2) reaches the lower obstacles L . Henceforth, we have

$$\begin{aligned} Y_t^1 &= Y_{\delta}^1 + \int_t^{\delta} f_s^1(Y_s^1, Z_s^1, V_s^1)ds + \int_t^T g_s^1(Y_s^1)dA_s - K_{\delta}^{1,-} + K_t^{1,+} - \int_t^{\delta} Z_s^1 dB_s - \int_t^{\delta} \int_E V_s^1(e)\tilde{\mu}(de, ds). \\ Y_t^2 &= Y_{\delta}^2 + \int_t^{\delta} f_s^2(Y_s^2, Z_s^2, V_s^2)ds + \int_t^T g_s^2(Y_s^2)dA_s - K_{\delta}^{2,-} + K_t^{2,+} - \int_t^{\delta} Z_s^2 dB_s - \int_t^{\delta} \int_E V_s^2(e)\tilde{\mu}(de, ds). \end{aligned}$$

Now in order to conclude, we define $(\bar{Y}_t^1, \bar{Z}_t^1, \bar{V}_t^1, \bar{K}_t^1)_{t \leq \delta}$ (resp. $(\bar{Y}_t^2, \bar{Z}_t^2, \bar{V}_t^2, \bar{K}_t^2)_{t \leq \delta}$) solution of the reflected BSDE with jumps in the upper obstacle L associated to $(\bar{f}^1 ds + \bar{g}^1 dA_s, Y_{\delta}^1)$ (resp. $(\bar{f}^2 ds + \bar{g}^2 dA_s, Y_{\delta}^2)$). Then by the comparison theorem given in [30], we have

$$\bar{Y}_t^1 \leq \bar{Y}_t^2, \quad \text{and} \quad \bar{K}_t^{1,-} - \bar{K}_s^{1,-} \geq \bar{K}_t^{2,+} - \bar{K}_s^{2,+}, \quad \forall t \in [s, \delta], \quad \mathbb{P}\text{-a.s.}$$

Using the growth property of \bar{f}^1 (resp. \bar{g}^1) and \bar{f}^2 (resp. \bar{g}^2), we obtain that $\bar{Y}_t^1 = Y_t^1$, $\bar{Y}_t^2 = Y_t^2$, $\bar{Z}_t^1 = Z_t^1$, $\bar{Z}_t^2 = Z_t^2$, $\bar{V}_t^1 = V_t^1$ and $\bar{V}_t^2 = V_t^2$, $\forall t \in [\tau, \delta]$, \mathbb{P} -a.s. Hence, we immediately get

$$\bar{K}_{\delta}^1 - \bar{K}_t^1 = K_{\delta}^1 - K_t^1, \quad \bar{K}_{\delta}^2 - \bar{K}_t^2 = K_{\delta}^2 - K_t^2, \quad \forall t \in [\tau, \delta], \quad \mathbb{P}\text{-a.s.}$$

which contradicts the definition of the stopping time τ . Therefore, $\mathbb{P}[\tau < T]$ which implies that $K^{-,1} \leq K^{-,2}$, \mathbb{P} -a.s. To conclude the proof, we can show similarly that $K^{+,2} \leq K^{+,1}$, \mathbb{P} -a.s.. \square

Theorem 4.4. *Under Assumption (H1), there exists a unique solution (Y, Z, V, K^+, K^-) to the generalized doubly reflected backward stochastic differential equation with jumps associated to $(f ds + g dA_s, \xi)$. Moreover, it satisfies*

$$\mathbb{E} \left[\sup_{t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt + \int_0^T \int_E |V_t(e)|^2 \nu(de) dt + |K_T^+|^2 + |K_T^-|^2 \right] < +\infty. \quad (4.35)$$

Proof. The uniqueness is a simple consequence of the above comparison theorem. Let us prove the existence of the solution. We consider the following penalized generalized BSDEJ: for any $n, m \in \mathbb{N}^*$

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T f_s(Y_s^{n,m}, Z_s^{n,m}, V_s^{n,m})ds + \int_t^T g_s(Y_s^{n,m})dA_s + m \int_t^T (Y_s^{n,m} - L_s)^- ds \\ &\quad - n \int_t^T (U_s - Y_s^{n,m})^- ds - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_E V_s^{n,m} \tilde{\mu}(de, ds), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned} \quad (4.36)$$

were $f^{n,m}(s, y, z, v) = f(s, y, z, v) + m(y - L_s)^- - n(U_s - y)^-$. Referring to the results of [23], we obtain existence and uniqueness for a solution $(Y^{n,m}, Z^{n,m}, V^{n,m})$ to the BSDEs given by

$(f^{n,m}, \xi)$. We set $K_t^{n,m+} = m \int_0^t (Y_{s^-}^{n,m} - L_s)^- ds$ and $K_t^{n,m-} = n \int_0^t (U_s - Y_{s^-}^{n,m})^- ds$.

Step1. we aim to prove the following estimate. There exists a constant C such that

$$\sup_{n,m \in \mathbb{N}^*} \mathbb{E} \left[\sup_{t \leq T} |Y_t^{n,m}|^2 + \int_0^T |Z_t^{n,m}|^2 dt + \int_0^T \int_E |V_t^{n,m}(e)|^2 \nu(de, dt) + |K_T^{n,m,+}|^2 + |K_T^{n,m,-}|^2 \right] < +\infty. \quad (4.37)$$

As usual we start by applying Itô formula to $e^{\lambda A_t} |Y_t^{n,m}|^2$, relying on the Lipschitz property of f and g and using Young's inequality we obtain

$$\begin{aligned} e^{\lambda A_t} |Y_t^{n,m}|^2 &\leq e^{\lambda A_T} |Y_T^{n,m}|^2 - \int_t^T \lambda e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s + \int_t^T e^{\lambda A_s} [g_s(0) dA_s + f_s(0, 0, 0) ds] \\ &\quad - \frac{1}{2} \int_t^T e^{\lambda A_s} [|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2] ds + (1 + 6L_f^2) \int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \\ &\quad + (1 + 2L_g) \int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s + 2 \int_t^T e^{\lambda A_s} Y_s^{n,m} dK_s^{n,m+} - 2 \int_t^T e^{\lambda A_s} Y_s^{n,m} dK_s^{n,m-} \\ &\quad - 2 \int_t^T e^{\lambda A_s} Y_s^{n,m} [Z_s^{n,m} dB_s + \int_E V_s^{n,m}(e) \tilde{\mu}(de, ds)]. \end{aligned}$$

From the Skorokhod condition and Young inequality we have

$$2 \int_t^T Y_s dK_s^- = \int_t^T L_s dK_s^- \leq \frac{1}{\epsilon} \sup_{0 \leq s \leq T} |L_s|^2 + \epsilon |K_T^- - K_0^-|^2. \quad (4.38)$$

$$2 \int_t^T Y_s dK_s^+ = \int_t^T U_s dK_s^+ \leq \frac{1}{\tilde{\epsilon}} \sup_{0 \leq s \leq T} |U_s|^2 + \tilde{\epsilon} |K_T^+ - K_0^+|^2. \quad (4.39)$$

Therefore taking the conditional expectation we get

$$\begin{aligned} &\mathbb{E}[e^{\lambda A_t} |Y_t^{n,m}|^2 + \frac{1}{2} \int_t^T e^{\lambda A_s} [|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2] ds] \\ &\leq \mathbb{E}[e^{\lambda A_T} |Y_T^{n,m}|^2] + \int_t^T e^{\lambda A_s} [g_s(0) dA_s + f_s(0, 0, 0) ds] + (-\lambda + 1 + 2L_g) \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s \right] \\ &\quad + (1 + 6L_f^2) \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \right] + \frac{1}{\epsilon} (\mathbb{E} \sup_{0 \leq s \leq T} |L_s|^2 + \mathbb{E} \sup_{0 \leq s \leq T} |U_s|^2) + \epsilon (\mathbb{E} |K_T^-|^2 + \mathbb{E} |K_T^+|^2) \\ &\quad - 2 \mathbb{E} \left[\int_t^T e^{\lambda A_s} Y_s^{n,m} [Z_s^{n,m} dB_s + \int_E V_s^{n,m}(e) \tilde{\mu}(de, ds)]. \right] \end{aligned} \quad (4.40)$$

In the other hand, we consider a sequence of stopping time τ_n such that

$$\begin{aligned} \tau_{n+1} &= \inf \{t \geq \tau_n, \quad Y_t^{n,m} \leq L_t\} \wedge T \\ \tau_{n+2} &= \inf \{t \geq \tau_{n+1}, \quad Y_t^{n,m} \geq U_t\} \wedge T. \end{aligned}$$

Using the same argument as in [18] or [22] we can rewrite the penalized generalized BSDEJ as follow

$$\begin{aligned} e^{\lambda A_{\tau_n}} Y_{\tau_n}^{n,m} &= e^{\lambda A_{\tau_{n+1}}} Y_{\tau_{n+1}}^{n,m} + \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} [f_s(Y_s^{n,m}, Z_s^{n,m}, V_s^{n,m}) ds + g_s(Y_s^{n,m}) dA_s] \\ &\quad - n \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (U_s - Y_s^{n,m})^- ds - \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} Z_s^{n,m} dW_s - \int_{\tau_n}^{\tau_{n+1}} \int_E e^{\lambda A_s} V_s^{n,m} \tilde{\mu}(de, ds). \end{aligned} \quad (4.41)$$

Relying on assumption **(H1)** we get

$$\begin{aligned} Y_{\tau_n}^{n,m} &\geq S_{\tau_n} \text{ on } \{\tau_n < T\}; \quad Y_{\tau_n}^{n,m} = S_{\tau_n} \text{ on } \{\tau_n = T\}. \\ Y_{\tau_{n+1}}^{n,m} &\leq S_{\tau_{n+1}} \text{ on } \{\tau_{n+1} < T\}; \quad Y_{\tau_{n+1}}^{n,m} = S_{\tau_{n+1}} \text{ on } \{\tau_{n+1} = T\}. \end{aligned}$$

Hence from (4.41) with the help of the Lipschitz assumption of f and g we derive the following estimate

$$\begin{aligned} n \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (U_s - Y_s^{n,m})^- ds &\leq - \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (\alpha_s - Z_s^{n,m}) dW_s - \int_{\tau_n}^{\tau_{n+1}} \int_E e^{\lambda A_s} V_s^{n,m} \tilde{\mu}(de, ds) \\ &\quad + \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} [f_s(0, 0, 0) + L_f(|Y_s^{n,m}| + |Z_s^{n,m}| + |V_s^{n,m}|_s)] ds \\ &\quad + \int_{\tau_n}^{\tau_{n+1}} (e^{\lambda A_s} [g_s(0) + L_g |Y_s^{n,m}| dA_s] + e^{\lambda A_s} d\tilde{V}_s^+ - e^{\lambda A_s} d\tilde{V}_s^-). \end{aligned}$$

Hence, since τ_n is stationary sequence [18] we obtain from assumption **(H1)**

$$\begin{aligned} \mathbb{E} \left[n \int_t^T e^{\lambda A_s} (U_s - Y_s^{n,m})^- ds \right]^2 &\leq C \{ 1 + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Z_s^{n,m}|^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |V_s^{n,m}|_s^2 ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s \right] \right\}. \end{aligned} \quad (4.42)$$

Similarly we obtain

$$\begin{aligned} \mathbb{E} \left[m \int_t^T (Y_s^{n,m} - L_s)^- ds \right]^2 &\leq C \{ 1 + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Z_s^{n,m}|^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |V_s^{n,m}|_s^2 ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s \right] \right\}. \end{aligned} \quad (4.43)$$

where C is a generic constant. Plugging (4.42) and (4.43) in (4.40) we obtain

$$\begin{aligned} &\mathbb{E} [e^{\lambda A_t} |Y_t^{n,m}|^2 + \frac{1}{2} \int_t^T e^{\lambda A_s} (|Z_s^{n,m}|^2 + \int_E |V_s^{n,m}(e)|^2 \nu(de)) ds] \quad (4.44) \\ &\leq \mathbb{E} [e^{\lambda A_T} |Y_T^{n,m}|^2 + \int_t^T e^{\lambda A_s} [g_s(0) dA_s + f_s(0, 0, 0) ds]] + \frac{1}{\epsilon} (\mathbb{E} [\sup_{0 \leq s \leq T} |L_s|^2] + \mathbb{E} [\sup_{0 \leq s \leq T} |U_s|^2]) \\ &\quad + \mathbb{E} \left[\int_t^T (-\lambda + 1 + 2L_g) e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s \right] + (1 + 6L_f^2) \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \right] \\ &\quad + 2\epsilon C \left[1 + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} (|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2) ds \right] + \mathbb{E} \left[\int_t^T e^{\lambda A_s} |Y_s^{n,m}|^2 dA_s \right] \right] \end{aligned}$$

Hence, using Gronwall inequality we get

$$\sup_{0 \leq t \leq T} \mathbb{E} |Y_t^{n,m}|^2 + \mathbb{E} \int_0^T |Z_s|^2 ds + \mathbb{E} \int_0^T \int_E |V_s(e)|^2 \nu(de, ds) \leq C,$$

which implies the desired result.

Step 2: There exists a constant C such that for any $n \in \mathbb{N}^*$ we have $\mathbb{E} [\sup_t |Y_t^n|^2] \leq C$ and

there exists an \mathcal{F}_t -adapted process $(Y_t)_t$ such that $\lim_{n \rightarrow +\infty} \mathbb{E} \left[\int_0^T |Y_t^n - Y_t|^2 dt \right] = 0$.

We know that all the requirements of existence result of [30] are fulfilled. Thus we know that for fixed $m \in \mathbb{N}$ $Y^{n,m}$, $Z^{n,m}$ and $V^{n,m}$ converge respectively to Y^n , Z^n and V^n as m goes to infinity. Moreover the limit process (Y^n, Z^n, V^n) is the unique solution of the generalized reflected BSDEJs associated to (ξ, f^n, g, K^{n+}) where $f^n(y, z, v) = f(y, z, v) - n(U_s - y)^-$. Moreover the limit process inherits the property (4.37).

Notice that for all $n \in \mathbb{N}$ and $\forall (s, y, z, v)$

$$f_s^n(y, z, v) \leq f_s^{n+1}(y, z, v).$$

Therefore by comparison Theorem [21] we have, $Y_t^n \leq Y_t^{n+1}$, $\forall t \in [0, T]$, \mathbb{P} -a.s. Hence, $Y_t^n \nearrow Y_t$ \mathbb{P} -a.s. Now from the property (4.37) and Fatou's lemma we deduce that $\mathbb{E}[|Y_t|^2] \leq C$ and then by Dominated convergence theorem we obtain that

$$\lim_n \mathbb{E} \left[\int_0^T |Y_t^n - Y_t|^2 \right] = 0.$$

Step3 In this step, we aim to prove $\lim_{n \rightarrow +\infty} \left[\sup_{0 \leq t \leq T} |(Y_t^n - U_t)^+|^2 \right] = 0$.

Let $(\bar{Y}_t^n, \bar{Z}_t^n, \bar{V}_t^n, \bar{K}_t^n)$ be the solution of the following reflected generalized BSDE with jumps

$$\begin{cases} \bar{Y}_t^n = \xi + \int_t^T [f_s(\bar{Y}_s^n, \bar{Z}_s^n, \bar{V}_s^n) + n(U_s - Y_s^n)] ds + \int_t^T g_s(\bar{Y}_s^n) dA_s + \bar{K}_T^{n+} - \bar{K}_t^{n-} \\ \quad - \int_t^T \bar{Z}_s^n dW_s - \int_t^T \int_E \bar{V}_s^n(e) \tilde{\mu}(de, ds), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.}, \\ \bar{Y}_t^n \geq L_t, \quad \int_0^T (\bar{Y}_t^n - L_t) d\bar{K}_t^{n+} = 0, \quad \mathbb{P}\text{-a.s.} \end{cases} \quad (4.45)$$

In addition, we consider $(\bar{Y}_t^{n,m}, \bar{Z}_t^{n,m}, \bar{V}_t^{n,m})$ the solution of the penalized generalized BSDEJ associated to the system (4.45). By comparison, since

$$f_t^{n,m}(y, z, v) \leq f_t(y, z, v) + n(U_t - y) + m(y - L_t)^-.$$

We have, for any $n, m \in \mathbb{N}^*$, $Y_t^{n,m} \leq \bar{Y}_t^{n,m}$, \mathbb{P} -a.s. Letting m goes to infinity we obtain that $Y_t^n \leq \bar{Y}_t^n$. Now, applying Itô's formula to $\bar{Y}_t^n e^{-nt}$, yield to

$$\begin{aligned} \bar{Y}_\tau^n &= \text{ess sup}_\tau \mathbb{E}_\tau [e^{-n(T-t)} \xi 1_{\{\tau=T\}} + e^{-n(\tau-t)} L_\tau 1_{\{\tau < T\}} \\ &\quad + \int_\tau^T e^{-n(s-t)} [f_s(\bar{Y}_s^n, \bar{Z}_s^n, \bar{V}_s^n) ds + g_s(\bar{Y}_s^n) dA_s] + n \int_\tau^T e^{-n(s-t)} U_s ds], \end{aligned} \quad (4.46)$$

where τ is an \mathcal{F}_t -family of stopping times $\tau \leq T$. Now since U is continuous then

$$e^{-n(T-t)} \xi 1_{\{\tau=T\}} + n \int_\tau^T e^{-n(s-t)} U_s ds \xrightarrow{n \rightarrow \infty} \xi 1_{\{\tau=T\}} + U_\tau 1_{\{\tau \leq T\}} \quad \mathbb{P}\text{-a.s. and in } \mathbb{H}^2.$$

In addition, we have

$$\mathbb{E}_\tau \left[\int_\tau^T e^{-n(s-\tau)} |f_s(\bar{Y}_s^n, \bar{Z}_s^n, \bar{V}_s^n)| ds \right] \leq \frac{1}{\sqrt{2n}} \mathbb{E} \left[\int_0^T |f_s(\bar{Y}_s^n, \bar{Z}_s^n, \bar{V}_s^n)|^2 ds \right]^{\frac{1}{2}} \rightarrow 0. \quad (4.47)$$

$$\mathbb{E}_\tau \left[\int_\tau^T e^{-n(s-\tau)} |g_s(\bar{Y}_s^n)| dA_s \right] \leq \frac{1}{\sqrt{2n}} \mathbb{E} \left[\int_0^T |g_s(\bar{Y}_s^n)|^2 dA_s \right]^{\frac{1}{2}} \rightarrow 0. \quad (4.48)$$

Besides, since we have that $L_t \leq S_t \leq U_t$ and

$$\mathbb{E}_\tau \left[\int_\tau^\nu e^{-n(s-\tau)} (dV_s^+ - dV_s^-) \right] \leq \frac{1}{\sqrt{2n}} \mathbb{E} [|V_T^+|^2 + |V_T^-|^2] \rightarrow 0. \quad (4.49)$$

where ν is a stopping time with $\tau \leq \nu \leq T$. Combining (4.47), (4.48) and (4.49), we finally obtain

$$\bar{Y}_\tau^n \leq \bar{Y}_\tau^n \xrightarrow{n} \xi 1_{\{\tau=T\}} + U_{\{\tau \leq T\}} \text{ in } \mathbb{H}^2. \quad (4.50)$$

Henceforth, we deduce from (Theorem 86, [8]) that $\forall t \in [0, T]$ $Y_t \leq U_t$ \mathbb{P} -a.s. Consequently, $(Y_t^n - U_t)^- \searrow 0$, $\forall t \in [0, T]$, \mathbb{P} -a.s. Then, from Dini's theorem [8], we deduce that $\sup_t (Y_t^n - U_t)^- \searrow 0$.

Therefore, since for any $n \in \mathbb{N}$, $(Y_t^n - U_t) \leq Y_t^0 - S_t$ and $(Y^n - U_t)^+ \leq |Y^0| + |U_t|$, by dominated convergence theorem, we obtain that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |(Y_t^n - U_t)^+|^2 \right] = 0, \text{ a.s.}$$

Step 4 In this step, we aim to prove that $\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 \right] = 0$ and there exists Z , V , K^+ and K^- such that

$$\lim_n \mathbb{E} \left[\int_0^T |Z_t|^2 dt + \int_0^T \int_E |V_t(e)|^2 \nu(de, dt) + |K_T^+|^2 + |K_T^-|^2 \right] = 0 \quad (4.51)$$

Let $n, p \in \mathbb{N}$ and consider the following processes $\delta Y_t = Y_t^n - Y_t^p$, $\delta Z_t = Z_t^n - Z_t^p$ and $\delta V_t = V_t^n - V_t^p$. Applying Itô's formula to $|\Delta Y|^2$ between t and T

$$\begin{aligned} |\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 ds + \int_t^T \int_E |\delta V_s(e)|^2 \nu(de, ds) &= |\delta Y_T|^2 + 2 \int_t^T \delta Y_s [g_s(Y_s^n) - g_s(Y_s^p)] dA_s \\ &+ 2 \int_t^T \int_E \delta Y_s [f_s(Y_s^n, Z_s^n, V_s^n) - f_s(Y_s^p, Z_s^p, V_s^p)] ds - 2 \int_t^T \int_E \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds) \\ &- 2 \int_t^T \delta Y_s \delta Z_s dB_s + 2 \int_t^T \int_E \delta Y_s (dK_s^{+,n} - dK_s^{+,p}) - 2 \int_t^T \delta Y_s (dK_s^{-,n} - dK_s^{-,p}). \end{aligned} \quad (4.52)$$

Since $n \leq p$ we have $2 \int_t^T \int_E \delta Y_t (dK_s^{+,n} - dK_s^{+,p}) \leq 0$ and

$$\begin{aligned} 2 \int_t^T \int_E \delta Y_s (dK_s^{-,n} - dK_s^{-,p}) &= 2 \int_t^T \int_E \delta Y_s dK_s^{-,n} - 2 \int_t^T \int_E \delta Y_t dK_s^{-,p} \\ &\leq 2 \sup_{0 \leq t \leq T} |Y_t^p - U_t| |K_T^{-,n}| + 2 \sup_{0 \leq t \leq T} |U_t - Y_t^n| |K_T^{-,p}|. \end{aligned}$$

In the other hand, from the growth assumption of g we have

$$2 \int_t^T \delta Y_s [g_s(Y_s^n) - g_s(Y_s^p)] dA_s \leq 2C \int_t^T |\delta Y_s|^2 dA_s.$$

Using step 2 and 3, we can deduce that

$$\begin{aligned} \mathbb{E}[|\delta Y_t|^2] - 2L_g \mathbb{E} \left[\int_t^T |\delta Y_s|^2 dA_s \right] + 2 \mathbb{E} \left[\int_t^T |\delta Z_s|^2 ds + \int_t^T \int_E |\delta V_s(e)|^2 \nu(de, ds) \right] \\ \leq C \mathbb{E} \left[\int_0^T |\delta Y_s|^2 ds + 2 \sup_{0 \leq t \leq T} |Y_t^p - U_t| |K_T^{-,n}| + 2 \sup_{0 \leq t \leq T} |U_t - Y_t^n| |K_T^{-,p}| \right] \xrightarrow{n,p} 0. \end{aligned}$$

Therefore (Z^n) and (V^n) are a Cauchy sequences in a complete spaces. Consequently there exists two processes Z and V such that (Z^n) and (V^n) converge respectively to Z and V .

Since $\mathbb{E}[\int_0^T |\delta Y_s|^2 dA_s] \leq 2L_g \sqrt{T} (\mathbb{E}[\int_0^T |\delta Y_s|^2 ds])^{\frac{1}{2}}$, using BDG inequality we can rewrite (4.52) as follows,

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] &\leq 2L_g \sqrt{T} (\mathbb{E}[\int_0^T |\delta Y_s|^2 ds])^{\frac{1}{2}} \\ &+ C \mathbb{E} \left[\int_0^T |\delta Y_s|^2 ds + 2 \sup_{0 \leq t \leq T} |Y_t^p - U_t| |K_T^{-,n}| + 2 \sup_{0 \leq t \leq T} |U_t - Y_t^n| |K_T^{-,p}| \right]. \end{aligned}$$

Therefore, $\lim_{p \rightarrow +\infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] = 0$ and then $\lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] = 0$.

Moreover, since $K^{n,+}$ is an increasing sequence then $K^{n,+}$ converges to the process K^+ . In addition, $\mathbb{E}[K_T^{n,+}] \leq C$ we deduce that $\mathbb{E}[|K_T^+|^2] < +\infty$.

Besides, for any $n \in \mathbb{N}$ we have $L_t \leq Y_t^n \leq Y_t \leq U_t$ and $\lim_n \mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_t^n - U_t)^+ \right]^2 = 0$.

Then $L_t \leq Y_t \leq U_t$. In the other hand, $\int_0^T (Y_t^n - L_t) dK_t^{n,+} = 0$ and $\lim_n \int_0^T (Y_t^n - L_t) dK_t^{n,+} = \int_0^T (Y_t - L_t) dK_t^+$. Thus the Skorohod condition is satisfied and the proof of the Theorem (4.4) is complete. \square

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