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# A novel phasor control design method: application to MEMS gyroscopes

Fabrice Saggin, Gérard Scorletti and Anton Kornienko

**Abstract**—In several applications, the main objective of the controllers is to ensure some process variable to track (and/or reject) a sinusoidal reference (disturbance) signal. To that end, two control approaches are defined: those based on the sinusoidal signals, and those based on the envelope (amplitude and phase) of these signals. The former one, which we name direct approach, corresponds to the classical architectures used in control engineering. This approach offers a broad range of methods to design linear controllers with guarantees of stability and performance. In general, envelope-based approaches are nonlinear and do not provide those guarantees. However, they allow obtaining controllers with a bandwidth much smaller than it would have in the direct approach. In this paper, we use time-varying complex phasors to describe the envelopes of the signals in the system. Then, we show that with a suitable reformulation, the system remains linear. Hence, links between these approaches are established under the assumption that a phasor can be instantaneously defined from a modulated signal (ideality). We propose thus two methods to design a phasor-based controller: one considering the ideal case and another where nonidealities are taken into account. Numerical examples, based on the operation of MEMS gyroscopes, show the effectiveness of these methods.

## I. INTRODUCTION

In numerous applications, the primary objective of a controller is to compute a control signal such that the output  $y$  of a (linear) process follows a sinusoidal reference trajectory  $y_r$  with a given frequency and/or reject sinusoidal disturbances with the same frequency. An emerging application illustrating this is the control of micro-electro-mechanical systems (MEMS) gyroscopes, where a proof mass shall oscillate close to the resonance frequency with a controlled amplitude, such that the angular rate can be accurately estimated [1], [2]. Similar examples are found in other fields, as in electrical machines and systems, inertial sensors, radio-frequency accelerators, see *e.g.* [3], [4], [5] and references therein.

In the literature, we may distinguish two main approaches to achieve the goals above for linear time-invariant (LTI) systems: the direct control, and the envelope or phasor-based one. The direct control corresponds to the general practice in control engineering: the control signal  $u$  is computed from the measurements of the plant output  $y$  and a sinusoidal reference signal  $y_r$ , as in [6], [7]. In this case, the controller can be designed by different methods, *e.g.* the  $H_\infty$  synthesis.

In the phasor-based approach, the control signal is often based on the envelope, *i.e.*, the amplitude and phase shift,

of the signals of interest. The main advantage of this approach is that the envelope of the signals of interest are constant in steady-state. Hence, one can use, for instance, PID controllers to track the reference amplitude and phase shift, as in [8]. Another advantage is that the controller may work at a lower frequency than it would have in the classical architecture. Thus, in case of digital implementation, smaller sampling frequencies can be used. For these (implementation) reasons, this approach is widely used, for instance, in the control of MEMS gyroscopes [9]. Nevertheless, an important drawback of the envelope-based approach is that nonlinearities are included into the loop to compute the amplitude and phase shift of the signals. The conventional approach is then to linearize the system around operating points [1], [10], leading to controllers whose performance is not guaranteed.

In this paper, we show that by changing the signal representation, a complex-valued linear model is obtained. This modification is made by using the real and imaginary parts of a time-varying complex phasor, described in [3], instead of amplitude and phase shift representation. Even if the model becomes complex-valued, this reformulation allows us to design linear controllers with guarantees on the performance of the closed-loop system.

In this paper, we also reveal that there exists a link between the classical approach and the phasor one. This link allows transforming a controller designed for the direct approach into a controller for the phasor one, yielding the same performance level. Hence, linear controllers from the direct approach may be transposed to the phasor one. The only condition for this property to hold is that the phasor construction block is ideal, *i.e.*, a phasor can be instantaneously defined from a modulated/sinusoidal signal.

In practice, however, nonidealities are present in that construction block and may yield to a significant performance discrepancy and even instability of the closed-loop system. Another contribution of this paper is, therefore, a control design method, based on the  $H_\infty$  synthesis [11], that takes into account these nonidealities to compute a phasor-based controller, ensuring the stability and an appropriate performance level.

This paper is structured as follows. Section II states the phasor control problem. Section III presents the (linear) complex phasor model of an LTI system and some of its properties. In Section IV, links between the direct and the phasor approach are established, allowing to design a phasor-based controller from a direct one. Section V discusses how to implement the phasor construction block and the

nonidealities entailed by it. Then, a novel phasor control design approach is presented, allowing one to take into consideration those nonidealities. Numerical examples, based on MEMS gyroscope operation, illustrate our main results in Section VI and conclusions are drawn in Section VII.

*Notation:* Subscripts  $R$  and  $I$  indicate respectively the real and imaginary parts of a complex variable;  $j^2 = -1$ .  $I_n$  is the identity matrix of  $\mathbb{R}^{n \times n}$  and  $0_{n \times m}$  is the zero matrix of  $\mathbb{R}^{n \times m}$  (subscripts are omitted if obvious from context). For two matrices  $A, B$ ,  $\text{diag}(A, B)$  is their diagonal concatenation.  $A^T$  is the transpose of  $A$ .  $M_\perp$  denotes the orthogonal complement of  $M$ . In linear matrix inequalities (LMI),  $\diamond$  represents terms that can be deduced from symmetry. For a complex matrix  $M$ ,  $\overline{\sigma}(M)$  denotes its maximum singular value. The  $\star$  denotes the Redheffer (star) product [11]. For a given LTI system  $F$ ,  $\|F\|_\infty$  denotes its  $H_\infty$ -norm. The  $L_2$ -norm of a signal  $v$  from  $\mathbb{R}^+$  to  $\mathbb{R}^{n_v}$  is defined as  $\|v\|_2^2 = \int_0^\infty v(t)^T v(t) dt$  and the set of signals for which the  $L_2$ -norm is bounded is denoted  $L_2$ . The  $L_2$ -gain of an operator  $\Sigma$  is defined as  $\|\Sigma\|_{l_2} = \sup_{v \in L_2, v \neq 0} \|\Sigma v\|_2 / \|v\|_2$ .

## II. PHASOR CONTROL PROBLEM

Let us consider the linear time-invariant (LTI) plant

$$G: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (1)$$

with  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^{n_u}$ ,  $y(t) \in \mathbb{R}^{n_y}$ , and real matrices  $A, B, C$  and  $D$  of appropriate dimensions. The original control objective is to compute  $u$  such that  $y$  tracks a sinusoidal reference  $y_r$  with frequency  $\omega_0$  and/or reject sinusoidal disturbances with the same frequency.

In control theory, this problem is usually solved by designing an LTI controller using, for instance, the  $H_\infty$  synthesis, see [11]. This controller computes  $u$  from the measurement of  $y$  and the reference  $y_r$ . We name this approach **direct control problem**.

An alternative approach, named **phasor control problem**, is considered. In this case, the control signal has the form:

$$u(t) = U(t) \cos(\phi_{exc}(t) + \phi_u(t)), \quad (2)$$

where  $\phi_{exc}(t)$ , called the excitation phase, is a given differentiable function. The couple  $(U, \phi_u)$  is referred to as a phasor of the signal  $u$ , from where a complex phasor can be similarly defined:  $\underline{u}(t) = U(t)e^{j\phi_u(t)}$ . Then, instead of computing  $u$ , the controller  $K_{ph}$  computes  $U$  and  $\phi_u$  from the measurement of  $Y$  and  $\phi_y$ , similarly defined as:

$$y(t) = Y(t) \cos(\phi_{exc}(t) + \phi_y(t)), \quad (3)$$

that is the measure of the phasor of  $y$ . The control problem is then recast as the tracking and/or the rejection of phasor signals: compute the phasor  $(U, \phi_u)$  such that the defined phasor  $(Y, \phi_y)$  tracks a reference phasor  $(Y_r, \phi_{y_r})$ .

The above description is illustrated in Fig. 1. The nonlinear block p2s transforms a phasor into a modulated signal, as in (2), and s2p transforms a modulated signal into a phasor (amplitude and phase shift) with respect to  $\phi_{exc}$ .

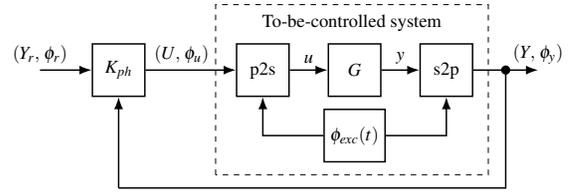


Fig. 1. Block-diagram of the phasor-based control architecture.

In the phasor control problem, the to-be-controlled plant is defined by (1), (2) and s2p, *i.e.*, (dropping time dependency)

$$\begin{cases} \dot{x} = Ax + BU \cos(\phi_{exc} + \phi_u) \\ \begin{bmatrix} Y \\ \phi_y \end{bmatrix} = \text{s2p}(Cx + DU \cos(\phi_{exc} + \phi_u)) \end{cases} \quad (4)$$

Since s2p and (2) are nonlinear functions of the phasor, the design of the controller is, a priori, a difficult problem. In the next section, we reveal that if the signals are replaced by their *complex* phasor, the plant (4) becomes linear.

## III. COMPLEX PHASOR MODELING

### A. Complex phasor model

The following theorem introduces a (linear) model of the plant based on the complex phasors.

**Theorem 1.** *Given a differentiable function  $\phi_{exc}(t)$  and a couple  $(U, \phi_u)$ , the output  $y$  of the system (1), for  $u(t) = U(t) \cos(\phi_{exc}(t) + \phi_u(t))$ , is given by*

$$y(t) = Y(t) \cos(\phi_{exc}(t) + \phi_y(t)),$$

where  $(Y, \phi_y)$  is such that  $Y(t) = |y(t)|$ ,  $\phi_y(t) = \arg(y(t))$  and  $\underline{y}$  is the output of the system  $\underline{G}$ , defined by

$$\underline{G}: \begin{cases} \dot{\underline{x}}(t) = (A - j\dot{\phi}_{exc}(t)I_n)\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \end{cases} \quad (5)$$

for the input  $\underline{u}(t) = U(t)e^{j\phi_u(t)}$ .

*Proof.* Let  $\underline{y}(t)$  and  $\underline{x}(t)$  be solutions of (5) for the input  $\underline{u}(t) = U(t)e^{j\phi_u(t)}$ . By multiplying (5) by  $e^{j\phi_{exc}(t)}$ , we have, after simplifications

$$\begin{cases} \frac{d}{dt} \left( \underline{x}(t)e^{j\phi_{exc}(t)} \right) = A\underline{x}(t)e^{j\phi_{exc}(t)} + B\underline{u}(t)e^{j\phi_{exc}(t)} \\ \underline{y}(t)e^{j\phi_{exc}(t)} = C\underline{x}(t)e^{j\phi_{exc}(t)} + D\underline{u}(t)e^{j\phi_{exc}(t)} \end{cases}$$

By taking the real part, since  $A, B, C$  and  $D$  are real-valued matrices, we obtain

$$\begin{cases} \frac{d}{dt} \left( \Re(\underline{x}e^{j\phi_{exc}}) \right) = A\Re(\underline{x}e^{j\phi_{exc}}) + B\Re(\underline{u}e^{j\phi_{exc}}) \\ \Re(\underline{y}e^{j\phi_{exc}}) = C\Re(\underline{x}e^{j\phi_{exc}}) + D\Re(\underline{u}e^{j\phi_{exc}}) \end{cases}$$

This system corresponds to (1) when the input signal is given by  $u(t) = U(t) \cos(\phi_{exc}(t) + \phi_u(t))$ .  $\square$

Note that the system  $\underline{G}$ , see (5), computes the complex phasors of  $x$  and  $y$  from the complex phasor of the input  $u$ . Thus,  $\underline{G}$  is referred to as the Complex Phasor Model (CPM) of  $G$ . Furthermore,  $\underline{G}$  is linear parameter varying (LPV). However, when  $\phi_{exc}(t) = \omega_0 t$ , it is actually an LTI system.

The CPM is a system with complex-valued parameters. Nonetheless, by splitting the signals into real and imaginary parts,  $\underline{G}$  is equivalently represented by the real-valued system below, denoted Complex Phasor Real Model (CPRM):

$$G_{cp} : \begin{cases} \dot{x}_{cp}(t) = A_{cp}(\dot{\phi}_{exc}(t))x_{cp}(t) + B_{cp}u_{cp}(t) \\ y_{cp}(t) = C_{cp}x_{cp}(t) + D_{cp}u_{cp}(t) \end{cases} \quad (6)$$

with  $x_{cp} = [\underline{x}_R^T \ \underline{x}_I^T]^T$ ,  $u_{cp} = [\underline{u}_R^T \ \underline{u}_I^T]^T$ ,  $y_{cp} = [\underline{y}_R^T \ \underline{y}_I^T]^T$ ,

$$A_{cp}(\dot{\phi}_{exc}(t)) = \begin{bmatrix} A & \dot{\phi}_{exc}(t)I_n \\ -\dot{\phi}_{exc}(t)I_n & A \end{bmatrix},$$

$B_{cp} = \text{diag}(B, B)$ ,  $C_{cp} = \text{diag}(C, C)$  and  $D_{cp} = \text{diag}(D, D)$ . Note that this model corresponds to

$$\begin{cases} \dot{x} = Ax + B[\cos(\phi_{exc}) & -\sin(\phi_{exc})]u_{cp} \\ y_{cp} = \text{s2cp}(Cx + D[\cos(\phi_{exc}) & -\sin(\phi_{exc})]u_{cp}) \end{cases}, \quad (7)$$

where  $x$  is the state vector of (4) and  $\text{s2cp}$  is an operator that associates  $y_{cp}$  with  $y$ .

Based on the CPRM, we define a new control problem, named **complex phasor control problem**. In this problem, the plant is defined by  $G_{cp}$  of (6); the controller, denoted  $K_{cp}$ , has to compute  $u_{cp}$  from  $y_{cp}$  and  $y_{rcp}$  such that  $y_{cp}$  tracks the reference signal  $y_{rcp}$ .

For the sake of simplicity, in this work, we focus on the case where  $\dot{\phi}_{exc}(t) = \omega_0 t$ , i.e.,  $G_{cp}$  is an LTI system. The general case can be similarly discussed using the LPV control approach, see e.g. [12].

Before to proceed with the solution for the complex phasor control problem, we present some relevant properties of the complex phasor model in the LTI case.

### B. Properties of the complex phasor model in the LTI case

The systems  $\underline{G}$  and  $G_{cp}$  have poles with the same real part that the poles of  $G$ . Then, the stability of one is equivalent to the stability of the others. Moreover, since  $\underline{G}$  can be defined as  $\underline{G}(s) = G(s + j\omega_0)$ , the frequency response of  $\underline{G}$  corresponds to a frequency shift of the frequency response of  $G$ . Thus,  $\|\underline{G}(s)\|_\infty = \|\underline{G}(s)\|_\infty = \|G_{cp}(s)\|_\infty$  [5].

## IV. CONTROL DESIGN BASED ON THE DIRECT PROBLEM

In this section, we intend to answer two key questions.

(i) Besides the implementation-related advantages of phasor control approach over the direct control one [9], does any of them ensure a better performance level?

(ii) Is it possible to compute a direct controller and transform it into a complex phasor controller?

We define the performance level as an upper bound on the  $H_\infty$ -norm of the interconnection of the augmented plant, i.e., the plant augmented with weighting functions, and the controller [11].

### A. Solving the direct control problem and the complex phasor control problem

The controller for the direct control problem can be computed by solving the standard  $H_\infty$  problem. We consider an

augmented plant  $P$ , composed of the plant  $G$  and weighting functions, usually defined by a state-space representation

$$P : \begin{cases} \dot{x}_P = A_P x_P + B_u u_P + B_w w \\ y_P = C_y x_P + D_{y_w} w \\ z = C_z x_P + D_{z_u} u_P + D_{z_w} w \end{cases} \quad (8)$$

with  $x_P(t) \in \mathbb{R}^{n_P}$ ,  $u_P(t) \in \mathbb{R}^{n_{u_P}}$ ,  $w(t) \in \mathbb{R}^{n_w}$ ,  $y_P(t) \in \mathbb{R}^{n_{y_P}}$ ,  $z(t) \in \mathbb{R}^{n_z}$ , and real matrices of appropriate dimensions. The problem is: given  $\gamma > 0$ , compute a controller

$$K : \begin{cases} \dot{x}_K = A_K x_K + B_K y_P \\ u_P = C_K x_K + D_{K y_P} \end{cases}, \quad (9)$$

where  $x_K(t) \in \mathbb{R}^{n_P}$ , if there is any, such that  $\|P \star K\|_\infty < \gamma$ , i.e., the closed-loop system defined by (8) and (9) has an  $H_\infty$  norm smaller than  $\gamma$ . The feasibility of this problem is assessed by the following theorem.

**Theorem 2** ([13]). *Consider the system (8). Then, there is a dynamic output feedback, in the form of (9), such that  $\|P \star K\|_\infty < \gamma$  if and only if there exist  $R, S \in \mathbb{R}^{n_P \times n_P}$  such that  $R = R^T$ ,  $S = S^T$ ,*

$$\begin{aligned} [\diamond]_\perp^T & \begin{bmatrix} SA_P^T + A_P S & B_w & SC_z^T \\ B_w^T & -\gamma I & D_{z_w}^T \\ C_z S & D_{z_w} & -\gamma I \end{bmatrix} \begin{bmatrix} B_u^T & 0 & D_{z_u}^T \end{bmatrix}_\perp \prec 0, \\ [\diamond]_\perp^T & \begin{bmatrix} A_P^T R + RA_P & RB_w & C_z^T \\ B_w^T R & -\gamma I & D_{z_w}^T \\ C_z & D_{z_w} & -\gamma I \end{bmatrix} \begin{bmatrix} C_y & D_{y_w} & 0 \end{bmatrix}_\perp \prec 0 \\ & \text{and } \begin{bmatrix} R & I \\ I & S \end{bmatrix} \succ 0. \end{aligned}$$

The complex phasor control problem can be addressed as a special case of the  $H_\infty$  control problem, but with an augmented plant  $P_{cp}$ , which is the CPRM associated to  $P$ . Nevertheless, the solution  $\underline{K}$  of this  $H_\infty$  control problem does not necessarily admit a state-space representation with the structure of a CPRM.

Since the direct control problem and the complex phasor one can be formulated as  $H_\infty$  control problems, we consider this formalism to investigate the links between them.

### B. Connections between the direct control problem and the complex phasor real control problem

We now investigate the connections between the direct control problem and the complex phasor one.

**Theorem 3.** *Let  $P$  be the plant defined in (8) and  $K$  the controller defined by (9). Let  $P_{cp}$  and  $K_{cp}$  be respectively the CPRM of  $P$  and  $K$  for  $\dot{\phi}_{exc}(t) = \omega_0 t$ . Hence, given  $\gamma > 0$ ,*

- 1) *if  $K$  is such that  $\|P \star K\|_\infty < \gamma$ , then  $\|P_{cp} \star K_{cp}\|_\infty < \gamma$ ;*
- 2)  *$(P \star K)_{cp} = P_{cp} \star K_{cp}$ ;*
- 3) *there exists a controller  $\underline{K}$  such that  $\|P_{cp} \star \underline{K}\|_\infty < \gamma$  if and only if there exists  $K$  such that  $\|P \star K\|_\infty < \gamma$ .*

*Proof.* Here, we just present the sketch of the proof. Details for the property 3 are developed in [14], and, for the sake of brevity, the details based on routine algebra are omitted.

*Properties 1) & 2):* By building the state-space realizations of  $(P \star K)_{cp}$  and  $P_{cp} \star K_{cp}$  from those of  $P$  and  $K$  and by

observing that they are equal, we prove the property 2. From section III-B,  $\|(P \star K)_{cp}\|_\infty = \|P \star K\|_\infty$ , then  $\|P \star K\|_\infty = \|P_{cp} \star K_{cp}\|_\infty$ , which proves property 1.

*Property 3):* The proof consists in first applying the  $H_\infty$  problem associated to the complex phasor control problem. The existence of  $\underline{K}$  such that  $\|P_{cp} \star \underline{K}\|_\infty < \gamma$  is equivalent to the fact that the feasibility problem defined by Theorem 2 has a solution. The second step consists in using the solution of this feasibility problem to construct the solution of the feasibility problem defined by Theorem 2 when this theorem is applied to the direct control problem, which implies that there exists  $K$  such that  $\|P \star K\|_\infty < \gamma$ .  $\square$

The first property claims that if a direct controller  $K$  achieves a performance level  $\gamma$ , then  $K_{cp}$  is a solution of the complex phasor control problem, ensuring the same performance level. The second property states that the complex phasor model of the interconnection  $P \star K$  is equal to the interconnection of each complex phasor model.

The third property reveals that if a solution  $\underline{K}$  of the complex phasor control problem achieves a given performance level, then, necessarily, the same level of performance can be obtained by a direct controller and vice-versa. In other words, even with an augmented degree of freedom (number of variables), the complex phasor control cannot ensure better performance level than that of the direct control.

All these results are based on the fact that (7) is exactly described by  $G_{cp}$ . However, the (ideal) s2cp block cannot be implemented. To extract amplitude and phase of a signal (or real and imaginary parts), additional nonlinear operators and filters are introduced into the loop. These elements, that were not considered in this section, may deteriorate the performance of the closed-loop system or even make it unstable. In the sequel, we discuss how to implement the s2cp block and how to model its nonidealities. Then, we propose an approach to take into account these nonidealities during the control design.

## V. IMPLEMENTATION OF S2CP AND CONTROL DESIGN

Up to this point, we have considered the s2cp block as an operator that allows to extract  $y_{cp}$  from a signal  $y$ . Nonetheless, by observing (3), one can notice that, for a given  $\phi_{exc}$  and  $y$ , an infinity number of couples  $(Y, \phi_y)$ , and consequently  $(\underline{y}_R, \underline{y}_I)$ , satisfies the equation. This ambiguity problem is recurrent in communication theory and signal processing, and can be solved by means of the Hilbert transform [15]. In practice, this operation is performed by a synchronous demodulation, introducing constraints on the signals and nonidealities into the s2cp block. In this section, we discuss the implementation of the s2cp block and how to model its nonidealities. Then, we propose a control design method that takes them into account.

### A. Implementation of the s2cp block

In practice (with  $\phi_{exc}(t) = \omega_0 t$ ), the operator s2cp is implemented by a synchronous demodulation if the power spectrum of  $y$  is in the interval  $(0, 2\omega_0)$  [15]. Fig. 2 shows

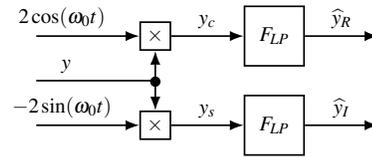


Fig. 2. Block-diagram of the synchronous demodulation.

the scheme of the demodulator, which includes **ideal** low-pass filters  $F_{LP}$  with cutoff frequency  $\omega_c = \omega_0$ . This scheme is motivated by the following point. Let  $y$  be the complex phasor of  $y$ , i.e.,  $y(t) = \underline{y}_R(t) \cos(\omega_0 t) - \underline{y}_I(t) \sin(\omega_0 t)$ . Then,  $y_c(t) = 2y(t) \cos(\omega_0 t)$  and  $y_s(t) = -2y(t) \sin(\omega_0 t)$  are recast:

$$\begin{aligned} y_c(t) &= \underline{y}_R(t) + \overbrace{\left( \underline{y}_R(t) \cos(2\omega_0 t) - \underline{y}_I(t) \sin(2\omega_0 t) \right)}^{\delta \underline{y}_R(t)} \\ y_s(t) &= \underline{y}_I(t) - \overbrace{\left( \underline{y}_R(t) \sin(2\omega_0 t) + \underline{y}_I(t) \cos(2\omega_0 t) \right)}^{\delta \underline{y}_I(t)} \end{aligned} \quad (10)$$

As the power spectrum support of the signal  $y$  is assumed to be included in the frequency interval  $(0, 2\omega_0)$ , the power spectrum support of  $\underline{y}_R(t)$  and  $\underline{y}_I(t)$  is in  $(0, \omega_0)$ , and the power spectrum support of  $\delta \underline{y}_R(t)$  and  $\delta \underline{y}_I(t)$  is in  $(\omega_0, 3\omega_0)$ . As a consequence, since  $F_{LP}$  is an ideal filter with  $\omega_c = \omega_0$ ,  $\widehat{y}_R(t) = \underline{y}_R(t)$  and  $\widehat{y}_I(t) = \underline{y}_I(t)$ .

In practice, the low-pass filters are not ideal. Indeed, they present a transition band between the pass band and the stop band. Since the power spectrum support of  $\delta \underline{y}_R(t)$  and the power spectrum support of  $\underline{y}_R(t)$  can be very close, the existence of the transition band can dramatically change the behavior of the closed-loop system. Then, it is crucial to:

- 1) evaluate the effect of this nonideal filter when the controller was computed as described in section IV; this *a posteriori* analysis is investigated in [9];
- 2) consider this nonideality when designing the control law; this point is developed in the sequel.

### B. Modeling the nonidealities of s2cp

The question now is how to model the nonideal s2cp. From (10),  $y_c$  and  $y_s$  can be rewritten in matrix form as

$$\begin{aligned} \begin{bmatrix} y_c(t) \\ y_s(t) \end{bmatrix} &= (I + \Delta(\omega_0)) \begin{bmatrix} \underline{y}_R(t) \\ \underline{y}_I(t) \end{bmatrix} \\ \text{with } \Delta(\omega_0) &= \begin{bmatrix} \cos(2\omega_0 t) & -\sin(2\omega_0 t) \\ -\sin(2\omega_0 t) & -\cos(2\omega_0 t) \end{bmatrix}. \end{aligned}$$

Then, the nonideal s2cp (synchronous demodulation) is modeled as the series connection of an ideal s2cp, the block  $(I + \Delta(\omega_0))$  and the nonideal filters  $F_{LP}$ , as in Fig. 3. In the sequel, this modeling is used to compute a CPRM controller.

### C. Solution to the complex phasor control problem with nonideal s2cp

Here, we briefly present how to compute a controller which ensures performance and robust stability with respect to the nonidealities of s2cp using  $H_\infty$  criterion.



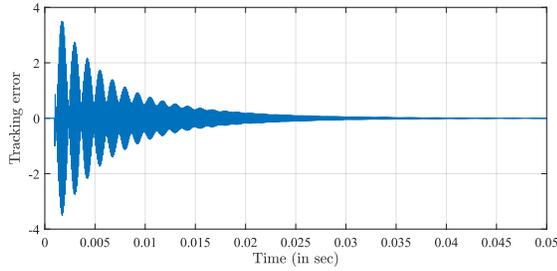


Fig. 6. Case 1 - tracking error of the nonideal CPRM ( $G$ , nonideal s2cp,  $K_{cp}$ , cp2s) with  $\omega_c = 2\pi \cdot 100 \text{ rad s}^{-1}$ .

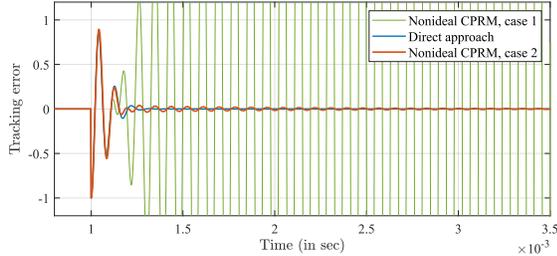


Fig. 7. Case 2 - comparison between direct approach and nonideal CPRM with  $\omega_c = 2\pi \cdot 8 \text{ rad s}^{-1}$  for case 1 ( $K_{cp}$ ) and case 2 ( $\tilde{K}$ ).

substantial differences with respect to the ideal CPRM are mainly due to  $F_{LP}$ , present in the nonideal s2cp.

Besides the important changes on the behavior of the closed-loop system, if we reduce the cutoff frequency of  $F_{LP}$ , the system may become unstable. This is the case when we make, e.g.,  $\omega_c = 2\pi \cdot 8 \text{ rad s}^{-1}$ , illustrated in Fig. 7.

### B. Case 2: design based on the CPRM and nonideal s2cp

Here, we illustrate the method presented in section V. We consider  $F_{LP}$  with  $\omega_c = 2\pi \cdot 8 \text{ rad s}^{-1}$ , which makes the closed-loop system of the previous case unstable.

With the  $H_\infty$  synthesis, we obtain a controller that ensures, in addition to the reference tracking,  $\|T_{n_{cp} \rightarrow y_{cp}}\|_\infty = 0.983$ . Hence, the small gain theorem conditions are satisfied and ensure the stability of the closed-loop system in presence of the time-varying matrix  $\Delta(\omega_0)$ .

The overall system ( $G$ , nonideal s2cp,  $\tilde{K}$ , cp2s) is simulated with  $\omega_c = 2\pi \cdot 8 \text{ rad s}^{-1}$ . We apply the same reference signal of the previous case. The tracking error is presented in Fig. 7. For the sake of comparison, we also plot the results with the controller designed in case 1 (instability) and direct approach. We observe that, despite the nonideal s2cp, at the transient-state, the phasor controller that takes  $F_{LP}$  into account (case 2) has a performance similar to the direct one. The phasor approach presents some small oscillations until  $t \approx 3 \text{ ms}$ . In steady-state, the tracking errors of both strategies converge to close to  $0.001Y_r$ , as required. Furthermore, note that the phasor controller is conceived to control the nonlinear system presented in Fig. 1 or Fig. 3, where a direct controller cannot be applied.

## VII. CONCLUSIONS

In some application, as the control of MEMS gyroscopes, it is usual to consider a phasor control architecture due to its implementation advantages. However, this strategy

introduces nonlinear elements into the loop. In this paper, we present the complex phasor modeling, which allows considering the to-be-controlled system as linear for the controller design.

Correspondences between classical control architectures and phasor-based ones were established. These equivalences allow one to design a linear controller in classical architecture and implement an equivalent controller in a phasor-based architecture, ensuring the same performance level if the s2cp block is ideal. In practical implementations, nonidealities appear and can quickly degrade the performance of the closed-loop system. The modeling of these nonidealities allows us to analyze the closed-loop system and to take them into account to design a controller. Simulation results emphasized the effectiveness of the proposed method.

The final message of this paper is that if there are practical constraints imposing the use of a phasor architecture, the nonidealities of the s2cp block have to be taken into consideration, guaranteeing that the design specifications are verified in the real implementation.

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