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GROUPOIDS, GEOMETRIC INDUCTION AND GELFAND MODELS

ANNE-MARIE AUBERT, ANTONIO BEHN, AND JORGE SOTO-ANDRADE

Abstract. In this paper we introduce an intrinsic version of the classical induction of representations for a subgroup H of a (finite) group G, called here *geometric induction*, which associates to any, not necessarily transitive, G-set X and any representation of the action groupoid $A(G, X)$ associated to G and X, a representation of the group G. We show that geometric induction, applied to one dimensional characters of the action groupoid of a suitable G-set X affords a Gelfand Model for G in the case where G is either the symmetric group or the projective general linear group of rank 2.

1. Introduction

A groupoid is a natural extension of a group, considered as a category: it is a category where each morphism is invertible. A groupoid with only one object is a group.

Mackey's theory of induced representations from a subgroup H of a group G to G itself, may be described in intrinsic, more geometric terms. The idea is to associate a representation of G not to a representation of a subgroup H of G but to a G-set X and a representation $\sigma_{G,X}$ of the so called *action groupoid* $A(G, X)$, also denoted $X \rtimes G$, associated to X.

Recall that the one dimensional characters of $A(G, X)$ play a key role in the geometric construction of linear representations of G. Indeed, we can twist the natural representation of G in the function space $L^2(X)$ by those characters, to obtain multiplicity-free representations of G in many cases, even if the original action is not transitive.

We show below that in this way we can obtain a geometric construction of Gel'fand Models for symmetric groups and the projective general linear group of rank 2.

On the other hand, groupoids also play a significant role in group representation theory when we look at representations of a whole family of groups instead of just one. For instance, when we construct linear representations of the "field" of symmetric groups S_n, each sitting on a set of size n, for all n. More precisely, we consider then the groupoid $\mathbf{S}$ whose objects are finite sets, and whose arrows are bijections among finite sets. We have a q-analogue of this, i.e. the linear groupoid $\mathbf{GL}(q)$ whose objects are finite dimensional vector spaces over $\mathbb{F}_q$ and whose arrows are linear isomorphisms.

2. Induced representations: a geometric approach

Let G be a finite group. If G acts on a finite set X, we can construct the associated action groupoid $A(G, X)$ in the following way:

- the objects of $A(G, X)$ are the points of X.
- the arrows of $A(G, X)$ are the triples (x, g, y) such that $y = g \cdot x$, with the obvious composition.

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Given an action of a group G on a set X, the action groupoid $A(G,X) = X \rtimes G$ is closely related to the quotient set X/G (the set of G-orbits). But, instead of taking elements of X in the same G-orbit as being equal in X/G, in the action groupoid they are just isomorphic.

A *representation* of the groupoid $A(G,X)$ is an action of G on a finite dimensional vector bundle $p\colon W \to X$. A representation gives a functor σ from $A(G,X)$ to the category of finite dimensional complex vector spaces such that $\sigma(x)$ is the fiber W_x of p at x and $\sigma(x,g,y)\colon W_x \to W_y$ is an isomorphism for each $g \in G$ such that $g \cdot x = y$.

Now, to any representation (σ, W) of $A(G,X)$ we can associate the induced representation (ρ, V) of σ to G, which we call *the natural representation ρ of G associated to the G-set X twisted by σ*, denoted $\mathrm{Ind}^G_{A(X,G)}(\sigma)$, defined as follows:

(a) the space of ρ is the complex vector space $V = L^2_W(X)$ of all sections of p;
(b) $(\rho_g f)(x) = \sigma(g^{-1} \cdot x, g, x)[f(g^{-1} \cdot x)]$ for all $g \in G$, $x \in X$, $f \in V$.

Proposition 2.1. *With the above notations, if $X = \bigcup_{i\in I} X_i$ is the G-orbit decomposition of X and H_i is the isotropy group of a chosen point $x_i \in X_i$ for each $i \in I$, then we have*

$$(\rho, V) \cong \bigoplus_{i\in I} \mathrm{Ind}^G_{H_i}(\sigma|_{H_i}),$$

where $\sigma|_{H_i}$ stands for the restriction of the representation σ to the isotropy group H_i, defined by $\sigma|_{H_i}(h) := \sigma(x_i, h, x_i)$ for $h \in H_i$.

□

Notice that it follows from this proposition that the induced representation (ρ, V) depends only on the restriction of the inducing character σ to the isotropy groups H_i.

3. Gelfand models

A Gel'fand model for a finite group G is a complex linear representation M of G that contains each of its irreducible representations with multiplicity one. The representation M is called an *involution model* if there exists a set of representatives $\{\omega_i\}$ of the distinct conjugacy classes of involutions in G and a set $\{\lambda_i\colon \mathrm{C}_G(\omega_i) \to \mathbb{C}\}$ of linear characters of the centralizers $\mathrm{C}_G(\omega_i)$ in G of the ω_i, such that

$$M \simeq \bigoplus_i \mathrm{Ind}^G_{\mathrm{C}_G(\omega_i)}(\lambda_i)$$

Klyachko [12], Inglis, Richardson and Saxl [10], Kodiyalam and Verma [13], Adin, Postnikov and Roichman [1] constructed *involution models* for the symmetric group. Baddeley classified which irreducible Weyl groups admit involution models: in particular only those of type D_{2n} ($n > 1$); F_4, E_6, E_7 and E_8 do not have involution models. Vinroot extended this classification to finite Coxeter groups, in particular showing that the Coxeter groups of type $\mathrm{I}_2(n) = \mathrm{G}_2(n)$ (the dihedral groups) and H_3 do have involution models while the group H_4 does not.

Bump and Ginzburg introduced in [7] the more flexible notion of *generalized involution model*. Marberg proved [17] that a finite complex reflection group has a generalized involution model if and only if each of its irreducible factors is one of the following: $\mathrm{G}(r,p,n)$ with $\gcd(p,n) = 1$, $\mathrm{G}(r,p,2)$ with r/p odd, or G_{23}, the Coxeter group of type H_3.

On the other hand, Garge and Oesterlé constructed in [9] for any finite group G with a faithful representation V, a representation which they called the *polynomial model for G associated to V*. Araujo and others have proved that the polynomial models for certain irreducible Weyl groups associated to their canonical representations are Gel'fand models. In [9] it is proved that

a polynomial model for a finite Coxeter group G is a Gel'fand model if and only if G has no direct factor of the type D_{2n}, E_7 or E_8.

Combining the above results of [17] and [9] shows that

- the groups F_4, H_4 and E_6 (which in the Shephard-Todd classification are G_{28}, G_{30} and G_{35}, respectively) do not have a generalized involution model, however they have a polynomial model which is a Gel'fand model;
- the groups E_7 and E_8 (which in the Shephard-Todd classification are G_{36} and G_{37}, respectively) do not have a generalized involution model, have a polynomial model, but this polynomial model is not a Gel'fand model.

3.1. **A geometric Gelfand model for the symmetric group.** Let $G = S_n$ be the symmetric group realized as the permutation group of $\{1, 2, \ldots, n\}$ and take for X the set of all involutions of G.

In order to illustrate the general picture, we start by the example of the group S_4.

Example 3.1. Let $G = S_4$. Let I denote the identity in G. We have

$$X = \{\mathrm{I}, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)\}.$$

Then $|X| = 10$ and G has three orbits in X, to wit, the orbit that is reduced to the identity I, the orbit consisting of all 6 transpositions (ij) and the orbit consisting of all 3 double transpositions $(ij)(kl)$. Let $\sigma = \sigma_{X,S_4}$ denote the one dimensional representation of $A(X, G)$ defined by

$$\sigma(x, g) := \begin{cases} 1 & \text{if } x = \mathrm{I}, \text{ for all } g \in G; \\ 1 & \text{if } x = (ij) \text{ with } i < j \text{ and } g(i) < g(j); \\ -1 & \text{if } x = (ij) \text{ with } i < j \text{ and } g(i) > g(j); \\ 1 & \text{if } x = (ij)(kl) \text{ with } i < j, k < l \text{ and } g(i) < g(j) \text{ and } g(k) < g(l) \text{ or } g(i) > g(j); \\ & \text{and } g(k) > g(l) \\ -1 & \text{otherwise.} \end{cases}$$

Then the isotropy group G_{I} at I is G, the isotropy group $G_{(12)}$ at (12) is $\{\mathrm{I}, (12), (34), (12)(34)\} \simeq C_2 \times C_2$, where $C_2 \simeq \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2, and the isotropy group $G_{(12)(34)}$ at $(12)(34)$ is

$$\{\mathrm{I}, (12), (34), (12)(34), (14)(23), (13)(24), (1324), (1423)\} \simeq (C_2 \times C_2) \rtimes C_2.$$

Notice that these isotropy groups appear as automorphisms groups of the trees naturally associated to the given partitions.

The restriction of the representation σ_{X,S_4} to G_{I} is the unit character $\mathbf{1}$ of G, the restriction of σ_{X,S_4} to $G_{(12)}$ is the tensor product $\mathrm{sgn} \otimes \mathbf{1}$ of the non trivial character sgn of C_2 and the trivial character $\mathbf{1}$ of C_2, the restriction of σ_{X,S_4} to $G_{(12)(34)}$ is the restriction of the signature character sign of G to $G_{(12)(34)}$. So we obtain that the representation (ρ, V) of G induced from σ_{X,S_4} is isomorphic to

$$\mathbf{1} \oplus \mathrm{Ind}_{C_2 \times C_2}^{G}(\mathrm{sgn} \otimes \mathbf{1}) \oplus \mathrm{Ind}_{(C_2 \times C_2) \rtimes C_2}^{G}(\mathrm{sign}).$$

Recall now that G has 5 irreducible representations, to wit:

- the unit representation $\mathbf{1}$;
- the signature representation sign;
- a 2-dimensional representation, that we denote by $\sigma^{(2)}$;

- a 3-dimensional representation, denoted $\sigma^{(3)}$, such that $\mathbf{1} \oplus \sigma^{(3)} \simeq \tau$, where τ stands for the natural (permutation) representation of G associated to its canonical action on a 4-point set;
- the 3-dimensional representation, $\mathrm{sign} \otimes \sigma^{(3)}$.

It is easy to check that:

$$\mathrm{Ind}_{C_2\times C_2}^{G}(\mathrm{sgn}\otimes 1) \simeq \sigma^{(3)} \oplus (\mathrm{sign}\otimes\sigma^{(3)}) \quad \text{and} \quad \mathrm{Ind}_{(C_2\times C_2)\rtimes C_2}^{G}(\mathrm{sign}) \simeq \mathrm{sign} \oplus \sigma^{(2)}.$$

It follows that the induced representation (ρ, V) is indeed a Gel'fand Model for $G = S_4$.

We will now consider the case of $G = S_n$ with n arbitrary.

For any positive integer t, we will denote by $C_2 \wr S_t$ the *wreath product* of C_2 by S_t, that is, the semidirect product $(C_2)^t \rtimes S_t$, where the symmetric group S_t acts on $(C_2)^t$ by permuting the t factors C_2. In other words $C_2 \wr S_t$ is the *hyperoctahedral group* $W(B_t)$ (the Weyl group of type B_t).

For $0 \le t \le \lfloor n/2 \rfloor$, let X_t denote the conjugacy class of involutions of G that are product of t disjoint transpositions. We have

$$X = \bigsqcup_{t=0}^{\lfloor n/2 \rfloor} X_t.$$

Let $x \in X_t$, with $0 \le t \le \lfloor n/2 \rfloor$. The involution x fixes precisely $f = n - 2t$ points.

We write in the cycle notation:

$$x = (i_1 j_1)(i_2 j_2)\cdots(i_t j_t),$$

where $i_k < j_k$ for each $k \in \{1, 2, \ldots, t\}$. Then the isotropy group $G_x = \mathrm{C}_G(x)$ at x is generated by permutations of following forms:

- $(i_k i_{k+1})(j_k j_{k+1})$, for $1 \le k \le t$;
- $(i_k j_k)$, for $1 \le k \le t$;
- any permutation of S_n which fixes the $2t$ points i_1, j_1, …, i_t, j_t, in other words, any permutation in S_f.

It follows that G_x is isomorphic to the group $W(B_t) \times S_f$, where $W(B_t) = C_2 \wr S_t$ is embedded in S_n so that the subgroup $(C_2)^t$ is generated by the 2-cycles of x.

As in [2] and [16], we attach to each permutation $g \in G$ its *inversion set* $\mathrm{Inv}(g)$ and the set $\mathrm{Pair}(g)$ of the 2-cycles in g, that is:

$$\mathrm{Inv}(g) := \{(i,j) \,:\, 1 \le i < j \le n,\ g(i) > g(j)\},$$

$$\mathrm{Pair}(g) := \{(i,j) \,:\, 1 \le i < j \le n,\ g(i) = j,\ g(j) = i\}.$$

Let $x \in X$ and $g \in G$. We set

$$\mathrm{Inv}_x(g) := \mathrm{Inv}(g) \cap \mathrm{Pair}(x),$$

and we will denote by $\mathrm{inv}_x(g)$ the cardinality of the set $\mathrm{Inv}_x(g)$.

Definition 3.2. Let $\sigma_{G,X} \colon A(G,X) \to \mathbb{C}$ be the map defined by

$$\sigma_{G,X}(x,g) := (-1)^{\mathrm{inv}_x(g)}, \quad \text{for } x \in X \text{ and } g \in G.$$

Remark 3.3. Let x be a involution in S_n which is the product of t disjoint transpositions. If we write x as $x = (i_1 i_2)(i_3 i_4)\cdots(i_{2t-1} i_{2t})$, where $i_{2j-1} < i_{2j}$ for each $j = 1, 2, \ldots, t$. Then $\mathrm{inv}_x(g)$ equals the cardinality of

$$\{j \in [1,t] : g(i_{2j-1}) > g(i_{2j})\},$$

for any $g \in G$. Hence the sign which occurs is the definition of $\sigma_{G,X}$ coincides with the sign which was introduced by Kodiyalam and Verma in [13]. It is also the sign which occurs in [1] and [16].

Lemma 3.4. *The map $\sigma_{G,X}$ defines a one dimensional representation of $A(X,G)$.*

Proof. Let $x_1, x_2 \in X$ and let $g_1, g_2 \in G$ such that $x_1 = g_2 \cdot x_2$. Let $P_n := \{(i,j) \,:\, 1 \leq i < j \leq n\}$. For any subset Q of P_n, we will denote by Q^{c} its complement in P_n. Let i and j two elements of $\{1, \ldots, n\}$. We have

– $(i,j) \in \mathrm{Inv}_{x_2}(g_1 g_2) \cap \mathrm{Inv}(g_2)^{\mathrm{c}}$ if and only if $i < j$ and $(g_2(i), g_2(j)) \in \mathrm{Inv}_{x_1}(g_1)$;

– $(j,i) \in \mathrm{Inv}_{x_2}(g_2) \cap \mathrm{Inv}(g_1 g_2)^{\mathrm{c}}$ if and only if $i > j$ and $(g_2(i), g_2(j)) \in \mathrm{Inv}_{x_1}(g_1)$.

It follows that

$$\mathrm{inv}_{x_2}(g_1 g_2) \equiv \mathrm{inv}_{x_2}(g_2) \,+\, \mathrm{inv}_{x_1}(g_1) \pmod 2.$$

We obtain

$$\sigma_{G,X}(x_2, g_2) \circ \sigma_{G,X}(x_1, g_1) = \sigma_{G,X}(x_2, g_1 g_2),$$

that is, $\sigma_{G,X}$ is a representation of $A(X,G)$. □

There is a unique character $\mathrm{sgn}_{\mathrm{CD}} \colon W(B_t) \to \{\pm 1\}$ whose restriction to the normal subgroup $(C_2)^t$ of $W(B_t)$ is the product of the sign characters of C_2 and that is trivial on the subgroup S_t. The kernel of $\mathrm{sgn}_{\mathrm{CD}}$ is isomorphic to the Weyl group $W(D_t)$. In the parametrization of the irreducible representations of $W(B_t)$ by the pairs of partitions of t (see [14]) the representation afforded by the character $\mathrm{sgn}_{\mathrm{CD}}$ corresponds to $(\emptyset, (t))$.

Let $x \in X$. Recall that

$$G_x = W(B_t) \times S_f.$$

The restriction of $\sigma_{G,X}$ to G_x is the tensor product of $\mathrm{sgn}_{\mathrm{CD}}$ by the unit representation of S_f:

$$(\sigma_{G,X})_{|G_x} = \mathrm{sgn}_{\mathrm{CD}} \otimes \mathbf{1}_{S_f}.$$

By transitivity of induction, we get

$$\mathrm{Ind}_{G_x}^{G}\big((\sigma_{G,X})_{|G_x}\big) = \mathrm{Ind}_{S_{2t} \times S_f}^{S_n}(\mathrm{Ind}_{G_x}^{S_{2t} \times S_f}(\mathrm{sgn}_{\mathrm{CD}} \otimes \mathbf{1}_{S_f})) = \mathrm{Ind}_{S_{2t} \times S_f}^{S_n}\left(\mathrm{Ind}_{W(B_t)}^{S_{2t}}(\mathrm{sgn}_{\mathrm{CD}}) \,\otimes \mathbf{1}_{S_f}\right).$$

The induced representation $\mathrm{Ind}_{W(B_t)}^{S_{2t}}(\mathrm{sgn}_{\mathrm{CD}})$ is the mutiplicity free sum of the irreducible characters of S_{2t} corresponding to partitions of $2t$ with even columns only (see, for instance, [15, Ch.I §8 and Ch. VII(2.4)]). By using the Littlewood-Richardson rule, it follows that $\mathrm{Ind}_{G_x}^{G}\big((\sigma_{G,X})_{|G_x}\big)$ is the multiplicity free sum of all irreducible Specht modules indexed by partitions with exactly f odd columns. It follows that the induced representation (V, ρ) is indeed a Gel'fand Model for $G = S_n$.

3.2. **A geometric Gel'fand Model for the group $G = \mathrm{PGL}_2(q)$, q odd.** In [7] a Gelfand Model for $G = \mathrm{PGL}(2,q)$ is constructed as a sum of three induced representations, taking advantage of the results of [18]. More precisely, we have the natural action of G on the set X consisting of all symmetric matrices in G, which may be identified with non degenerate symmetric bilinear forms mod centre or non-degenerate quadratic forms mod centre on the finite plane over $k = \mathbb{F}_q$. This action has 3 orbits:

- the orbit O_1 of the split isotropic hyperbolic form H mod centre, given by $(x,y) \mapsto xy$;
- the orbit O_2 of the non-split anisotropic normic form mod centre given by the norm N of the unique quadratic extension of k.
- the orbit O_3 of the alternating form det mod centre.

The isotropy groups H_1, H_2 and H_3 in G for these orbits are just the corresponding two projective similarity orthogonal groups and the projective similarity symplectic group; more precisely:

- H_1 is the normalizer of the split torus T_1 in G;
- H_2 is the normalizer of the non-split torus T_2 in G;
- H_3 is the whole group G.

Bump and Ginzburg make an *ad hoc* choice of characters ψ_1, ψ_2 and ψ_3 to induce from H_1, H_2 and H_3:

- ψ_1 is the trivial character $\mathbf{1}$ for H_1;
- ψ_2 is the order 2 character of H_2 that is trivial on the non-split torus
- ψ_3 is the non unique trivial linear character of G, i. e. the " sign character" given by the determinant mod squares.

Theorem 1. *A model M for G is given by*

$$M = \mathrm{Ind}_{H_1}^G \psi_1 \oplus \mathrm{Ind}_{H_2}^G \psi_2 \oplus \mathrm{Ind}_{H_3}^G \psi_3$$

□

Our viewpoint is to obtain the model M as a geometric induced representation from a " sign character" ε of the action groupoid $A(X, G)$ naturally associated to the G-space X.

The sign character ε, which takes values ± 1 only, is defined first on the holonomy (isotropy) groups G_x associated to any quadratic form $x \in X$ as follows:

Definition 3.5. Let $\mathrm{Isotr}(x)$ denote the set of all isotropic 1-dimensional vector subspaces of $x \in X$. Then G_x acts naturally on $\mathrm{Isotr}(x)$ and we define

$$\varepsilon(g) := \text{sign of the permutation of } \mathrm{Isotr}(x) \text{ defined by } g \in G_x.$$

Denote by ε_i the restriction of ε to H_i, for $i = 1, 2, 3$. Then

- ε_1 is the order 2 character of H_1 which is trivial on T_1;
- ε_2 is the trivial character of H_2;
- ε_3 is the sign character of G.

Indeed, the determinant mod squares character at $g \in G$ may be obtained as well as the signature of the permutation induced by g on the set of all 1-dimensional subspaces of the finite plane (the isotropic lines for det mod centre).

Then for any extension $\tilde{\varepsilon}$ of the character ε to the whole action groupoid $A(X, G)$, we have

Theorem 2. *The Model M may be obtained as*

$$M = \mathrm{Ind}_{A(X,G)}^G(\tilde{\varepsilon}) \simeq \mathrm{Ind}_{H_1}^G \varepsilon_1 \oplus \mathrm{Ind}_{H_2}^G \varepsilon_2 \oplus \mathrm{Ind}_{H_3}^G \varepsilon_3.$$

□

Recall that the induced representation $\mathrm{Ind}_{A(X,G)}^G(\tilde{\varepsilon})$ depends only on ε.

Notice also that we may extend ε to a full-fledged character of the action grupoid $A(X, G)$, by choosing first a point x_i in each orbit O_i, then for each $x \in O_i$, $x \neq x_i$ an element $g_x \in G$ such that $g_x \cdot x_i = x$ and assigning arbitrarily a non-zero value α_x to each arrow (x_i, g_x, x) in $A(X, G)$.

3.3. **Gelfand-Graev representations as geometrically induced representations: The case of $\mathrm{GL}_2(q)$.** To construct the classical Gel'fand-Graev representation of $G = \mathrm{GL}_2(q) = \mathrm{GL}_2(\mathbb{F}_q)$ as a geometrically induced representation, we consider first the G-set X consisting of all pairs (u, ω), where u is a non zero vector in k^2 and ω is a non zero 2-vector in $\bigwedge^2 k^2$. The elements of X may be called Grassmann chains. The group G acts in a natural way on X.

So we have a G-set X and the associated action groupoid $M = A(X, G)$.

We proceed now to define a linear character $\widetilde{\psi}$ of the groupoid M associated to any character ψ of the additive group of the finite field $k = \mathbb{F}_q$. Let ψ be a non trivial (complex) character of the additive group of the finite field $k = \mathbb{F}_q$. Choose arbitrarily, for each $(u, \omega) \in X$, a vector $v \in k^2$, such that $\omega = u \wedge v$.

Notice that v is well defined modulo the line $< u >$ generated by u. So we are choosing in fact an element in the fiber $k^2/ < u >$ above u. We have then a natural $G-$fiber bundle $\mathcal{E}$ with the set of all lines through the origin in the finite plane k^2 as basis.

Then, for $(x, g, y) \in M$, if we write $x = (u, \omega) = (u, u \wedge v)$, $y = (u', \omega') = (u', u' \wedge v')$, there is a unique $b \in k$ such that $g(v) = v' + bu'$. Define

$$\widetilde{\psi}(x, g, y) = \psi(b)$$

with b as above.

Proposition 3.3.1. *The mapping $\widetilde{\psi}$ just defined is a linear character of the groupoid M.*

Proof. We have to check that

$$\widetilde{\psi}((x, g, y)(y, h, z)) = \widetilde{\psi}(x, g, y) \cdot \widetilde{\psi}(y, h, z),$$

i. e.

$$\widetilde{\psi}((x, h \circ g, z)) = \widetilde{\psi}(x, g, y) \cdot \widetilde{\psi}(y, h, z).$$

Let us write

$$x = (u, \omega), y = (u', \omega'), (z = (u'', \omega'')$$

and denote by v, v', v'' the chosen vectors such that

$$\omega = u \wedge v,\ \omega' = u' \wedge v',\ \omega'' = u'' \wedge v''.$$

Then, $g(v) = v' + bu'$ for a suitable $b \in k$, since $g(u) \wedge g(v) = u' \wedge g(v) = u' \wedge v'$ and $h(v') = v'' + b'u''$ for a suitable $b' \in k$, since $h(u') \wedge h(v') = u'' \wedge h(v') = u'' \wedge v''$. But then

$$(h \circ g)(v) = h(v' + bu') = h(v') + bh(u') = v'' + b'u'' + bu'' = v'' + (b + b')u''.$$

Hence

$$\widetilde{\psi}((x, h \circ g, z)) = \psi(b + b') = \psi(b) \cdot \psi(b') = \widetilde{\psi}(x, g, y) \cdot \widetilde{\psi}(y, h, z).$$

□

Proposition 3.6. *The classical Gelfand-Graev representation of G is isomorphic to the geometrically induced representation of G from the linear character $\widetilde{\psi}$ of the action groupoid $A(X, G)$ as above.*

□

3.4. **Gelfand Models as top cohomology spaces, à la Solomon-Tits-Lehrer.**

We also suspect that a Gelfand Model is quite often accessible to a cohomological construction à la Solomon-Tits (just like the Steinberg representation is). This is the case for the symmetry groups of regular polygons, for instance. In the simplest example, of $G = \mathrm{GL}_2(q)$ for $q = 2$, where the geometry associated to the G-affine plane is just the geometry of the equilateral triangle, we see that a Gelfand Model for G may be obtained as $H^1(\mathcal{C})$ where $\mathcal{C} = \{C_n\}_n$ is the following cochain complex:

$C_n = L^2(X_n)$, where X_{-1} is a point, X_0 is the set of vertices of the equilateral triangle, and X_1 is the set of oriented edges of the equilateral triangle, with the usual coboundary operators $\delta_n \colon C_n \to C_{n+1}$, given by $(\delta_{-1}\lambda)(x) = \lambda$, for all $x \in X_0$ and $\lambda \in L^2(X_1) = \mathbb{C}$; $(\delta_0 f)(x, y) = f(y) - f(x)$, for $x, y \in X_0$, and $\delta_1 = 0$.

So we may expect for more general groups G, to find a cochain complex $\mathcal{C}$, naturally associated to some canonical geometric space X for G, whose top cohomology space (with complex coefficients) realizes the model M we are after. This geometric space X should be naturally endowed with a poset structure and the cohomology theory involved in our construction should be analogous to, but different from, the usual cohomology of posets [6], the difference coming from the choice of the associated complex $\mathcal{C}$, as in the very simple example above where $\mathcal{C}$ is not the cochain complex which gives the usual cohomology of the triangle.

3.5. **Properties of Gelfand Models.**

We state three properties that should be satisfied in many cases by Gel'fand Models.

Property 1. $M = L^2(X) - L^2(Y)$ in the ring $\mathrm{Green}(G)$ of natural (permutation) representations of G, for suitable G-sets x and Y. Equivalently, M is an integer linear combination of natural representations.

Property 2. M is a "twisted" $L^2(X)$, i.e. the induced representation from a suitable linear character of the isotropy group of a point in a transitive G-space X.

Property 3. M may be realized as the top cohomology space of a G-poset complex associated to a suitable G-set X.

3.6. **Positive results.**

- Property 1 is satisfied for symmetric groups. It holds in fact already for the irreducible representations of the symmetric groups (see [11], 2.3).
- Property 1 is satisfied for $\mathrm{GL}_n(q)$. This follows from [19] and [12].
- Property 2 is satisfied for $\mathrm{PGL}_2(q)$.
- Property 3 is satisfied for S_3.

3.7. **Counterexamples.**

(1) **The quaternion group** $\mathrm{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ is the smallest case where Property 1 is not satisfied.

It has 5 conjugacy classes,

its Gelfand character takes values [6 0 0 2 0],

it has a four 1-dimensional irreducible representations and one 2-dimensional irreducible representation that appears only in its regular representation among natural representations.

So M cannot be an integer linear combination of natural representations of H.

(2) **The group** $G(96) = \mathrm{smallgroup}\,(96, 3)$ in the GAP Small Groups Library, of order 96, whose Gelfand character takes values [30 2 -2 6 6 2 2 2] on its 8 conjugacy classes, is a counterexample to the non-negativity of the Gelfand Character. This group, which is in fact isomorphic to the semidirect product $((C_2 \times C_4) \rtimes C_4) \rtimes C_3$ is also a counterexample

to Property 1. Indeed, it may be checked by SAGE that no integer linear combination of its permutation characters can afford its Gel'fand character.

(3) **The binary icosahedral group** $G = \mathrm{SL}_2(5)$ as a counter example to Property 1.

The list of the dimensions of the irreducible representations of G is :
1 (unit representation),
2, 2 (half dimensional cuspidal representations),
3, 3 (half dimensional principal series representations),
4, 4 (cuspidal representations),
5 (Steinberg representation),
6 (principal series representation).
Then $\dim M = 30$.

We give below the list of the natural representations of G, ordered by decreasing dimension, described by the multiplicities in their decomposition into irreducibles, in the same order as in our list above, followed by their dimensions.

[1 2 2 3 3 4 4 5 6]	120
[1 0 0 3 3 4 0 5 0]	60
[1 0 0 1 1 2 2 1 2]	40
[1 0 0 1 1 2 0 3 0]	30
[1 0 0 1 1 0 0 1 2]	24
[1 0 0 1 1 2 0 1 0]	20
[1 0 0 0 0 1 0 2 0]	15
[1 0 0 1 1 0 0 1 0]	12
[1 0 0 0 0 1 0 1 0]	10
[1 0 0 0 0 0 0 1 0]	6
[1 0 0 0 0 1 0 0 0]	5

Notice that the natural representation of dimension $20 = 25^2 - 5$ is afforded by the (double cover) of the finite analogue of Poincaré's half plane over $\mathbb{F}_5$, endowed by the homographic action of G, which is not multiplicity free, contrary to the case of $\mathrm{PGL}_2(5)$). Clearly M cannot be obtained as a linear integer combination of these natural representations.

□

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