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COMPARISON OF THE DEPTHS ON BOTH SIDES OF THE LOCAL LANGLANDS CORRESPONDENCE FOR WEIL-RESTRICTED GROUPS

ANNE-MARIE AUBERT AND ROGER PLYMEN,
WITH AN APPENDIX BY JESSICA FINTZEN

Abstract. Let E/F be a finite and Galois extension of non-archimedean local fields. Let G be a connected reductive group defined over E and let $M := \mathfrak{R}_{E/F}\, G$ be the reductive group over F obtained by Weil restriction of scalars. We investigate depth, and the enhanced local Langlands correspondence, in the transition from $G(E)$ to $M(F)$. We obtain a depth-comparison formula for Weil-restricted groups.

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1. Introduction

Let E/F be a finite Galois extension of non-archimedean local fields. Let G be a connected reductive group defined over E and let $M := \mathfrak{R}_{E/F}\, G$ be the reductive group over F obtained by Weil restriction of scalars from G. We have an isomorphism of locally compact totally disconnected topological groups $\iota\colon G(E) \to M(F)$ between the E-points of G and the F-points of M.

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We denote by $\Pi(G,E)$ the smooth dual of $G(E)$, the set of equivalence classes of irreducible smooth representations of $G(E)$ and by $\Phi(G,E)$ the set of $G^\vee$-conjugacy classes of Langlands parameters for $G(E)$, where $G^\vee$ is the complex dual group of G. Similarly for $\Pi(M,F)$ and $\Phi(M,F)$.

A local Langlands correspondence (LLC) for (G,E) is a surjective map

$$\lambda_G \colon \Pi(G,E) \to \Phi(G,E)$$

which satisfies the conditions (desiderata) laid down in [Bor, §10]. If such a map λ_G exists then the map λ_M is defined to be the unique map for which the following diagram commutes

$$\begin{array}{ccc} \Pi(M,F) & \xrightarrow{\lambda_M} & \Phi(M,F) \\ {\scriptstyle \iota^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Sh}} \\ \Pi(G,E) & \xrightarrow{\lambda_G} & \Phi(G,E) \end{array} \tag{1.1}$$

where Sh is the restriction to $\Phi(M,F)$ of the Shapiro isomorphism, as in [Bor, 8.4].

An *enhanced* local Langlands correspondence for (G,E) includes additional (refined) data that pin down the smooth irreducible representations of $G(E)$, i.e. divide each L-packet into singletons. It is a bijection

$$\begin{array}{rccc} \lambda^{\mathrm{e}}_G \colon & \Pi(G,E) & \to & \Phi^{\mathrm{e}}(G,E) \\ & \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array},$$

such that $\phi_\pi = \lambda_G(\pi)$, and where $\Phi^{\mathrm{e}}(G,E)$ is the set of $G^\vee$-conjugacy classes of enhanced L-parameters (see Definition 3.7), which satisfies several stringent conditions. These extra conditions, which are made precise in Definition 3.24, comprise Whittaker data, the HII conjecture for square-integrable modulo center representations, extended endoscopic triples, transfer properties.

This leads to our first main result:

Theorem 1.2. *Consider a local Langlands correspondence for* (G,E)

$$\lambda_G \colon \Pi(G,E) \longrightarrow \Phi(G,E).$$

Then the map λ_M *defined above induces an enhanced* LLC *for* (M,F) *if and only if* λ_G *induces an enhanced* LLC *for* (G,E).

Each side of the LLC admits a numerical invariant, namely the *depth*, that is defined in quite different ways on each side and, *a priori*, there is no reason to expect that this numerical invariant will be preserved by the LLC. The depth $\mathrm{dep}(\pi)$ of $\pi \in \Pi(G,E)$, that is due to Moy and Prasad, will be recalled in Definition 1.9.

The map λ_M is the composition of three maps, and there are therefore three separate opportunities for a change of depth. In this article, for each of these maps, we record how the depth can change.

In section 2, we forge a new definition of depth for L-parameters of $G(E)$, Definition 2.11, based on the Galois cohomology group $\mathrm{H}^1(W_E, G^\vee)$. We show that this definition is consistent with the existing definitions and prove the following depth-comparison result (Theorem 2.16) for the right vertical map Sh:

$$\mathrm{dep}(\phi) = \varphi_{E/F}(\mathrm{dep}(\mathrm{Sh}(\phi)) \quad \text{for any } \phi \in \Phi(M,F), \tag{1.3}$$

where $\varphi_{E/F}$ is the classical Herbrand function.

In the bottom horizontal map, the depth change varies from group to group:

- for $\mathrm{GL}_n(E)$ and its inner forms, we do have preservation of depth under the LLC for any representation, see [ABPS1];
- for $\mathrm{SL}_n(E)$ and its inner forms, we have preservation of depth under the LLC for any essentially tame representation, see [ABPS1, Theorem 3.8];
- in large residual characteristic, we have preservation of depth under the LLC for quasi-split classical groups and for arbitrary unitary groups, [Oi1], [Oi2];
- for tamely ramified tori, we have preservation of depth under the LLC for any character, see Yu [Yu1];
- for $\mathrm{SL}_2(E)$ and its inner form, the depth *changes* under the LLC for any representation π_E of $G(E)$ of positive depth that is not essentially tame in the following way: $\mathrm{dep}(\lambda_G(\pi_E)) < \mathrm{dep}(\pi_E)$, see [AMPS].

The Appendix, due to Jessica Fintzen, is devoted to a depth-comparison result (Corollary A.13) for the left vertical map ι^*. The depth change will depend on the ramification index $e = e(E/F)$.

These results are summarized in the following.

Theorem 1.4. *Consider a local Langlands correspondence*

$$\lambda_G \colon \Pi(G, E) \longrightarrow \Phi(G, E).$$

Let $\pi \in \Pi(M, F)$. *If* $\mathrm{dep}(\lambda_G(\iota^*\pi)) = \kappa_\pi \cdot \mathrm{dep}(\iota^*\pi)$ *then we have*

$$\mathrm{dep}(\lambda_M(\pi)) = \varphi_{E/F}(\kappa_\pi \cdot e \cdot \mathrm{dep}(\pi)),$$

where $e = e(E/F)$ *is the ramification index of* E/F *and* $\varphi_{E/F}$ *is the classical Herbrand function.*

As a special case, we have

Theorem 1.5. *If* λ_G *is depth-preserving, then, for all* $\pi \in \Pi(M, F)$, *we have*

$$\mathrm{dep}(\lambda_M(\pi)) = \varphi_{E/F}(e \cdot \mathrm{dep}(\pi)).$$

In particular, we have

- $\mathrm{dep}(\lambda_M(\pi))/\mathrm{dep}(\pi) \to 1$ *as* $\mathrm{dep}(\pi) \to \infty$
- λ_M *is depth-preserving if* E/F *is tamely ramified,*
- *if* E/F *is wildly ramified then, for each* π *such* $\mathrm{dep}(\pi) \neq 0$, *we have*

$$\mathrm{dep}(\lambda_M(\pi)) > \mathrm{dep}(\pi).$$

When $G(E) = \mathrm{GL}_1(E)$, Theorem 1.5 strengthens the main result of [MiPa] for induced tori. For tamely ramified induced tori, we recover the depth-preservation theorem of Yu [Yu1].

Theorem 1.5 provides numerous new instances of *non-preservation of depth* by the LLC:

Corollary 1.6. *Let p denote the residual characteristic of F. If E/F is wildly ramified, and G is either an inner form of $\mathrm{GL}_n(E)$ (with no restriction on p) or a quasi-split classical group or a unitary groups (with p large), then the depth of a representation π of $M(F)$ with $M = \mathfrak{R}_{E/F}(G)$ such that $\mathrm{dep}(\pi) \neq 0$ is never preserved by the* LLC.

Section 4 contains some applications to automorphic induction and the Asai lift.

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Notation and conventions.

Let E/F be a finite Galois extension of non-archimedean local fields, with ramification index denoted by $e = e(E|F)$ and residual index $f = f(E|F)$. Let $\mathrm{N}_{E/F}$ be the norm map. We fix a separable closure $F^{\mathrm{sep}} = E^{\mathrm{sep}}$ of both F and E. From now on all field extensions will be assumed to be contained in F^{sep}. Let F^{ur} denote the maximal unramified extension of F.

Let K/F be any Galois extension of F and let v^K denote the unique extension to F^{sep} of a normalized valuation on K (i.e. such that $v^K(K^\times) = \mathbb{Z}$). Let Γ_K be the absolute Galois group of K, let W_K be the absolute Weil group of K, and let W'_K denote the Weil-Deligne group $W_K \times \mathrm{SL}_2(\mathbb{C})$.

Let H be a connected reductive group defined over K and let $\Pi(H, K)$ denote the set of isomorphism classes of irreducible admissible representations of $H(K)$. Let $H^\vee$ be the Langlands dual of H. Write ${}^LH := H^\vee \rtimes W_K$. Homomorphisms $\phi\colon W'_K \to {}^LH$ which are admissible as defined in [Bor, 8.2] are called L-parameters for (H, K). We denote by $\Phi(H, K)$ the set of $H^\vee$-conjugacy classes of (resp. bounded) L-parameters for (H, K).

Let $\mathscr{B}(H, K)$ be the reduced Bruhat–Tits building of H over K. We denote by $H(K)_{x,r}$ the Moy–Prasad filtration group of [MP1], [MP2], defined with respect to the valuation v^K, where $x \in \mathscr{B}(H, K)$ and $r \in \mathbb{R}_{\geq 0}$, and write $H(K)_{x,r+} := \bigcup_{s>r} H(K)_{x,s}$.

Example 1.7. The Moy-Prasad filtration of $\mathrm{GL}_1(K) = K^\times$ is

$$\mathrm{GL}_1(K)^{\mathrm{MP}}_r = K^\times_r, \tag{1.8}$$

in particular, $\mathrm{GL}_1(K)^{\mathrm{MP}}_n = U^n_K$ for any non-negative integer n.

Definition 1.9. *Let (π, V_π) be an irreducible smooth representation of $H(K)$. The depth $\mathrm{dep}(\pi)$ of π the smallest nonnegative real number r for which there exists an $x \in \mathscr{B}(H, K)$ such that $V_\pi^{H(K)_{x,r+}} \neq 0$.*

The definition of depth given in Definition 1.9 makes sense, see [DeB, Lemma 5.2.1].

Let G be a connected reductive algebraic group defined over E and let $M := \mathfrak{R}_{E/F}\, G$ be the reductive group over F obtained by Weil restriction of scalars from G. Let $G(E)$ denote the group of E-points of G, and $M(F)$ the F-points of M.

The set $M(F)$ has the structure of a variety defined over F, and the set $G(E)$ has the structure of a variety defined over E. The sets $M(F)$ and $G(E)$ are in

bijective correspondence. They are homeomorphic as topological spaces. Once the group structure on $G(E)$ is transported to $M(F)$, we have an isomorphism of locally compact totally disconnected topological groups:

$$\iota\colon G(E) \to M(F). \tag{1.10}$$

Therefore, $G(E)$ and $M(F)$ have the same representation theory. Let

$$\iota^*\colon \Pi(G, M) \to \Pi(G, E) \tag{1.11}$$

be the canonical bijection.

Given $F \subset E \subset F^{\mathrm{sep}}$. Denote by $v = v^F$ the unique extension to F^{sep} of a normalized valuation on F. Let v^E denote the unique extension from E to F^{sep} of the normalized valuation $e \cdot v|_E$, where $e = e(E/F)$ is the ramification index of E/F. So we have

$$v^E(x) = e \cdot v(x) \tag{1.12}$$

for all $x \in E$.

Definition 1.13. *Let w be a valuation on E. Define*

$$E^\times_{w,r} := \{x \in E^\times : w(x-1) \geq r\}.$$

The Moy-Prasad filtration of $\mathrm{GL}_1(E)$ is given by

$$\mathrm{GL}_1(E)^{\mathrm{MP}}_r = E^\times_{v^E,r}.$$

Example 1.14. Let $T(F)$ be an induced torus of the form $\mathfrak{R}_{E/F}(\mathbb{G}_m)$. The Moy-Prasad filtration of $T(F)$ is given, for $r > 0$, by

$$T(F)^{\mathrm{MP}}_r := E^\times_{v,r}$$

as in [Yu2, §4.2].

We then have

$$\begin{aligned} T(F)^{\mathrm{MP}}_r &= E^\times_{v,r} \qquad \text{by definition} \\ &= E^\times_{v^E,er} \qquad \text{by} \quad (1.12) \\ &= \mathrm{GL}_1(E)^{\mathrm{MP}}_{er}. \end{aligned}$$

Let $\pi \in \Pi(T, F)$. Therefore, we have

$$\mathrm{dep}(\iota^*\pi) = e \cdot \mathrm{dep}(\pi)$$

as in [MiPa, 7.2].

2. Depth-comparison under the Shapiro isomorphism

Let $\mathcal{I}_F$ be the inertia subgroup of W_F and $\mathcal{P}_F$ the wild inertia subgroup. Attached to a real number $r \geq -1$ is the ramification subgroup W^r_F of W_F. We use the upper numbering convention of [Ser, Chap. IV], so that $W^{-1}_F = W_F$. We have the semi-continuity property

$$W^r_F = \bigcap_{s<r} W^s_F$$

for all $r > 0$.

One also forms the subgroup $\bigcup_{s>r} W_F^s$ and its closure

$$W_F^{r+} = \mathrm{cl}(\bigcup_{s>r} W_F^s)$$

in W_F. One says that r is a *jump* of $\overline{F}/F$ if $W_F^{r+} \neq W_F^r$. In particular,

$$W_F^0 = \mathcal{I}_F, \qquad W_F^{0+} = \mathcal{P}_F.$$

Each of the groups W_F^r, W_F^{r+} with $r \geq 0$ is profinite, closed and normal in W_F.

The classical convex Herbrand function. Let $e = e(E/F)$. Let Γ denote the Galois group $\mathrm{Gal}(E/F)$ and consider the ramification groups in the upper numbering $\Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^n = \Gamma^{n+1} = \cdots = \{1\}$, where Γ^0 is the inertia subgroup of Γ. Let $\psi = \psi_{E/F}$ denote the classical convex Herbrand function [Ser, IV §3]. We recall that $\psi_{E/F}$ is the inverse of $\varphi_{E/F}$. We record the following elementary lemma.

Lemma 2.1. *Let $x \geq 0$. We have $\psi_{E/F}(x) \leq ex$ with equality if and only if E/F is tamely ramified. Moreover, $\psi_{E/F}(x)/x \to e$ as $x \to \infty$.*

Proof. We have

$$\psi_{E/F}(x) := \int_0^x (\Gamma^0 : \Gamma^t) dt.$$

Let $e_j = (\Gamma^0 : \Gamma^j)$ with $1 \leq j \leq n$. We then have $e_1 < e_2 < \cdots < e_n = e(E/F)$. The graph of $y = \psi_{E/F}(x)$ is piecewise linear, with successive gradients

$$e_1 < e_2 < \ldots < e_n = e.$$

It is then immediate that $\psi(x) \leq ex$ with equality if and only if $n = 1$ if and only if E/F is tamely ramified. If j is the largest jump of Γ then we have

$$x \geq j \implies \psi(x) = \psi(j) + (x - j)e$$

as required.

□

Lemma 2.2. *The comparison lemma. If $r \geq 0$, then*

$$W_F^{r+} \cap W_E \quad = \quad W_E^{\psi_{E/F}(r)+}.$$

Proof. If $r > 0$ then this is the second statement in [BH, Proposition, §1.4]. To deal with the case $r = 0$, we proceed as follows. Let X be a topological space, let A be an open and closed subset of X. Then, for every subset B of X, according to [Bou, §1.6 prop. 5] we have

$$\overline{B} \cap A = \overline{B \cap A}$$

where the overline denotes closure in X. If $r > 0$ then, by the first statement in [BH, Proposition, §1.4], we have

$$W_F^r \cap W_E = W_E^{\psi_{E/F}(r)}.$$

Now let $X = W_F, A = W_E$, $B = \bigcup_{r>0} W_F^r$. We obtain

$$W_F^{0+} \cap W_E = W_E^{0+}.$$

□

Induction. We require the continuous noncommutative cohomology as developed in Borel-Serre [BS]. So, let $\mathbf{g}$ denote a topological group. A $\mathbf{g}$-set is a discrete topological space X on which $\mathbf{g}$ operates on the left in a continuous fashion (i.e. the isotropy subgroup of each point of X is open in $\mathbf{g}$). A $\mathbf{g}$-group A is a group in the category of $\mathbf{g}$-sets, as in [BS, §1.2]. The cohomology set $\mathrm{H}^1(\mathbf{g}, A)$ is defined in [BS, §1.2]: it is constructed from continuous cocycles of $\mathbf{g}$ with values in A. Then $\mathrm{H}^1(\mathbf{g}, A)$ is a pointed set - the distinguished point is the class of the unit cocycle.

If $\mathbf{h}$ is a subgroup of $\mathbf{g}$ and A is a $\mathbf{g}$-group, then the induced group A^* is defined in [BS, §1.28]:

$$A^* := \mathrm{Ind}_{\mathbf{h}}^{\mathbf{g}}(A)$$

It comprises all continuous $\mathbf{h}$-equivariant maps from $\mathbf{g}$ to A which are constant on left cosets modulo an open subgroup of $\mathbf{g}$. Then A^* becomes a $\mathbf{g}$-group via

$$({}^g f)(x) = f(xg)$$

for all $f \in A^*$ and $g, x \in \mathbf{g}$.

Theorem 2.3. [BS, Proposition 1.29]. *Let $\mathbf{h}$ be a subgroup of $\mathbf{g}$, let A be an $\mathbf{h}$-group, and let A^* be the corresponding induced $\mathbf{g}$-group. Suppose that the open normal subgroups of $\mathbf{g}$ form a fundamental system of neighbourhoods of $1 \in \mathbf{g}$. Then we have a canonical isomorphism of pointed sets:*

$$\mathrm{H}^1(\mathbf{g}, A^*) \simeq \mathrm{H}^1(\mathbf{h}, A). \tag{2.4}$$

This is the *Shapiro isomorphism*, denoted Sh.

Here is an important application of the Shapiro isomorphism. Let $G^\vee$ (resp. $M^\vee$) denote the complex Langlands dual of G (resp. M). Let $\mathbf{g} = W_F, \mathbf{h} = W_E, A = G^\vee$. Then $M^\vee$ is the induced group $\mathrm{Ind}_{W_E}^{W_F}(G^\vee)$. The Weil groups W_E and W_F are locally profinite topological groups. The inertia subgroup $\mathcal{I}_E$ (resp. $\mathcal{I}_F$) contains open normal subgroups forming a fundamental system of neighbourhoods of the identity in W_E (resp. W_F).

We will take $G^\vee$ in its discrete topology, so that $G^\vee$ becomes a W_E-group; and $M^\vee$ in its discrete topology, so that $M^\vee$ becomes a W_F-group. We can now apply Theorem 2.3, and obtain the canonical isomorphism of pointed sets

$$\mathrm{H}^1(W_F, M^\vee) \simeq \mathrm{H}^1(W_E, G^\vee). \tag{2.5}$$

Here is a much more specialized application, which we will need in the proof of Theorem 2.15. Let

$$\mathbf{h} = W_E/W_E^{\psi(r)+}, \quad \mathbf{g} = W_F/W_F^{r+}, \quad A = (G^\vee)^{W_E^{\psi(r)+}}$$

To simplify notation, set $G_1 = W_E$, $G_2 = W_F$, $H_1 = W_E^{\psi(r)+}$, $H_2 = W_F^{r+}$, so that $G_1 \subset G_2$, $H_1 \subset H_2$, $H_1 = H_2 \cap G_1$ by the comparison lemma 2.2. We need to show that the map

$$\eta\colon \mathbf{h} \to \mathbf{g}, \quad xH_1 \mapsto xH_2$$

is injective. To prove this, note that, for all $x \in G_1$ we have

$$xH_2 = H_2 \implies x \in H_2 \implies x \in H_2 \cap G_1 \implies x \in H_1 \implies xH_1 = H_1.$$

Setting $x = z^{-1}y$ with $y, z \in G_1$ we infer that $yH_2 = zH_2 \implies yH_1 = zH_1$ as required. We identify $\mathbf{h}$ with its image $\eta(\mathbf{h}) \subset \mathbf{g}$, and view $\mathbf{h}$ as a subgroup of $\mathbf{g}$.

We can therefore apply Theorem 2.3 and obtain the canonical isomorphism

$$\mathrm{H}^1\left(W_F/W_F^{r+}, \mathrm{Ind}_{W_E/W_E^{\psi(r)+}}^{W_F/W_F^{r+}}(G^\vee)^{W_E^{\psi(r)+}}\right) \simeq \mathrm{H}^1\left(W_E/W_E^{\psi(r)+}, (G^\vee)^{W_E^{\psi(r)+}}\right). \tag{2.6}$$

The following lemma is observed in [MiPa, Lemma 3].

Lemma 2.7. *The submodule lemma. Let J be a group, H and A subgroups of J with A being normal in J. Let $B = H \cap A$, let M be an H-module. Then there is a canonical isomorphism of J/A-modules:*

$$(\mathrm{Ind}_H^J M)^A \simeq \mathrm{Ind}_{H/B}^{J/A} M^B.$$

We shall need this lemma in the proof of Theorem 2.15.

Lemma 2.8. *We have a canonical isomorphism of pointed sets:*

$$H^1(W_F, M^\vee) = \bigcup_{r\in\mathbb{R}_{\geq 0}} H^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}}).$$

Proof. The group W_F^{r+} is a normal subgroup of W_F for $r \geq 0$. According to [BS, Proposition 1.27], we have a canonical injective map

$$\varinjlim_{r\in\mathbb{R}_{\geq 0}} \mathrm{H}^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}}) \to \mathrm{H}^1(W_F, M^\vee). \tag{2.9}$$

We check the surjectivity of this map.

The ramification filtration $\{W_F^r\}_{r\geq 0}$ is descending and satisfies

$$\bigcap_{r\geq 0} W_F^r = \{1\} \tag{2.10}$$

The dual group $M^\vee$ is equipped with the discrete topology, and is an W_F^0-group in the terminology of [BS]. That is, there is a continuous action of W_F^0 on $M^\vee$, i.e. we have a continuous homomorphism $\rho : W_F^0 \to \mathrm{Aut}(M^\vee)$. Since $\mathrm{Aut}(M^\vee)$ is also discrete, the kernel of ρ is an open normal subgroup $U \subset W_F^0$. By (2.10), we will have $W_F^{r+} \subset U$ for sufficiently large r, say $r \geq r_0$. In other words, given $\alpha \in Z^1(W_F, M^\vee)$, the image of α is contained in $M^\vee = (M^\vee)^{W_F^{r+}}$ for $r \geq r_0$.

By the continuity (smoothness) of α, α is trivial on some open subgroup H of W_F^0, thus H is of finite index in W_F^0. Again by (2.10), W_F^{r+} is contained in H for sufficiently large r, say $r \geq r_1$. Therefore, for $r \geq \max(r_0, r_1)$, α belongs to $Z^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}})$.

To complete the proof, we combine this data with the injective map (2.9).

□

Lemma 2.8 allows us to present a new definition of depth. Our definition of depth of an L-parameter ϕ will depend only on the restriction $\phi|_{W_F}$. With this in mind, we have

$$w \in W_F \implies \phi(w) = (\alpha_\phi(w), w) \in M^\vee \rtimes W_F$$

where α_ϕ is a cocycle in $Z^1(W_F, M^\vee)$. The cohomology class of the 1-cocycle α_ϕ will be denoted $[\alpha_\phi]$. If we denote by $\Phi^1(M, F)$ the subset of $M^\vee$-conjugacy classes

of L-parameters which are trivial on $\mathrm{SL}_2(\mathbb{C})$ then we obtain an injective map of pointed sets:

$$\Phi^1(M,F) \hookrightarrow \mathrm{H}^1(W_F, M^\vee), \qquad \phi \mapsto \alpha_\phi$$

Definition 2.11. *For $\phi \in \Phi(M,F)$, we define the depth of ϕ as*

$$\mathrm{dep}(\phi) := \inf\{r \in \mathbb{R}_{\geq 0} : [\alpha_\phi] \in \mathrm{H}^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}})\}.$$

If $M(F)$ is tamely ramified, then the wild inertia group $\mathcal{P}_F = W_F^{0+}$ acts trivially on $M^\vee$. In particular, we may regard the restriction $\alpha_\phi|_{W_F^{0+}}$ of α_ϕ to W_F^{0+} as a continuous homomorphism from W_F^{0+} to $M^\vee$.

For tamely ramified groups, the customary definition is as follows.

Definition 2.12. *For tamely ramified groups, the usual depth of ϕ is defined as follows:*

$$\mathrm{dep}(\phi) := \inf\{r \geq 0 : \alpha_\phi(W_F^{r+}) \textit{ has trivial image in } M^\vee\}.$$

Note that this is well-defined, i.e. independent of the choice of the representative α_ϕ of $[\alpha_\phi]$. Write $\alpha = \alpha_\phi$ and choose another cocycle β representing $[\alpha_\phi]$. Then, since α and β are cohomologous in $Z^1(W_F, M^\vee)$, there exists $m \in M^\vee$ such that

$$\beta(w) = m^{-1} \cdot \alpha(w) \cdot {}^w m$$

for all $w \in W_F$. By the triviality of the action of W_F^{0+} on $M^\vee$, we have

$$\beta(w) = m^{-1} \cdot \alpha(w) \cdot m$$

for all $w \in W_F^{0+}$. Therefore, for $r \geq 0$, $\alpha(W_F^{r+})$ has trivial image in $M^\vee$ if and only if so does $\beta(W_F^{r+})$.

We need to reconcile these two definitions of depth.

Lemma 2.13. *For tamely ramified groups, these two definitions are equivalent:*

Proof. It suffices to check that the following are equivalent for $r \geq 0$:

- $\alpha_\phi(W_F^{r+})$ has trivial image in $M^\vee$,
- $[\alpha_\phi]$ belongs to $\mathrm{H}^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}})$.

Since $M(F)$ is tamely ramified, the wild inertia group $\mathcal{P}_F$ acts trivially on $M^\vee$ and so the fixed set $(M^\vee)^{W_F^{r+}}$ equals $M^\vee$ for any $r \geq 0$. Therefore $\mathrm{H}^1(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}})$ is nothing but $\mathrm{H}^1(W_F/W_F^{r+}, M^\vee)$ which is the subset of $\mathrm{H}^1(W_F, M^\vee)$ consisting of cohomology classes which can be represented by a 1-cocycle whose restriction to W_F^{r+} is trivial. Thus the above two conditions are equivalent. □

We note that definition 2.11 is well-adapted to the proofs in [MiPa].

Lemma 2.13 shows that we now have a consistent definition of depth.

Remark 2.14. In the special case when M is F-split, the group W_F acts trivially on $M^\vee$, and α_ϕ is a homomorphism, which, by definition, coincides with the restriction of ϕ to W_F. Hence Lemma 2.13 shows that $\mathrm{dep}(\phi)$ coincides with the definition of the depth of ϕ, as defined for instance in [ABPS1, §2.3].

Our next result is a refinement of the isomorphism (2.5).

Theorem 2.15. *If $r \geq 0$, then we have a canonical isomorphism*

$$\mathrm{H}^1\left(W_F/W_F^{r+}, (M^\vee)^{W_F^{r+}}\right) \;\cong\; \mathrm{H}^1\left(W_E/W_E^{\psi(r)+}, (G^\vee)^{W_E^{\psi(r)+}}\right)$$

where $\psi = \psi_{E/F}$.

Proof. We have the following isomorphisms of complex reductive groups:

$$\begin{aligned}(M^\vee)^{W_F^{r+}} &\cong (\mathrm{Ind}_{W_E}^{W_F} G^\vee)^{W_F^{r+}} \\ &\cong \mathrm{Ind}_{W_E/W_E\cap W_F^{r+}}^{W_F/W_F^{r+}} (G^\vee)^{W_F^{r+}\cap W_E} \\ &\cong \mathrm{Ind}_{W_E/W_E^{\psi(r)+}}^{W_F/W_F^{r+}} (G^\vee)^{W_E^{\psi(r)+}}.\end{aligned}$$

In this proof, we have used, successively

- the construction of $M^\vee$ as an induced group,
- the submodule lemma 2.7 with $H = W_E$, $J = W_F$, $M = G^\vee$, $A = W_F^{r+}$,
- the comparison lemma 2.2.

Now apply the canonical isomorphism (2.6). □

Theorem 2.16. *We have* $\mathrm{dep}(\mathrm{Sh}(\phi)) = \psi_{E/F}(\mathrm{dep}(\phi))$. *In particular, ϕ has depth* 0 *if and only if* $\mathrm{Sh}(\phi)$ *has depth* 0.

Proof. This follows immediately from Theorem 2.15 and definition 2.11. □

3. Depth-comparison under the local Langlands correspondence for Weil-restricted groups

We assume that the K-group H is quasi-split. Let $\mathrm{Z}(H)$ and $\mathrm{Z}(H^\vee)$ denote the center of H and $H^\vee$, respectively.

Definition 3.1. *A local Langlands correspondence (or* LLC*) for (H, K) is a surjective map*

$$\lambda_H \colon \Pi(H, K) \to \Phi(H, K),$$

which satisfies the conditions laid down by Langlands in [Lan, §3].

These conditions are the desiderata of Borel [Bor]. We will recall them now. The parameter ϕ determines a character χ_ϕ as in [Lan]. Given $\pi \in \Pi(H, K)$, an element $\alpha \in \mathrm{H}^1(W_K, \mathrm{Z}(H^\vee))$ determines an element $\pi_\alpha \in \Pi(H, K)$, see [Lan, p.20].

To every ϕ in $\Phi(H, K)$, the pre-image of ϕ via λ_H is a finite but nonempty set Π_ϕ in $\Pi(H, K)$ such that the following conditions are satisfied.

(i) If $\phi \neq \phi'$ then Π_ϕ and $\Pi_{\phi'}$ are disjoint.

(ii) If $\pi \in \Pi_\phi$ then

$$\pi(z) = \chi_\phi(z) I, \qquad z \in \mathrm{Z}_H(K).$$

(iii) If $\phi' = \alpha\phi$ with $\alpha \in \mathrm{H}^1(W_K, \mathrm{Z}(H^\vee)$, then

$$\Pi_{\phi'} = \{\pi_\alpha \otimes \pi | \pi \in \Pi_\phi\}.$$

(iv) If $\eta\colon H' \to H$ has abelian kernel and cokernel, if $\phi \in \Phi(H,K)$ and $\phi' = \eta^*(\phi)$ then the pullback of any $\pi \in \Pi_\phi$ to $G'(K)$ is the direct sum of finitely many irreducible, quasi-simple representations, all of which lie in $\Pi_{\phi'}$.

(v) If $\phi \in \Phi(H,K)$ and one element of Π_ϕ is square integrable modulo $(\mathrm{Z}(H))(K)$ then all elements are.

(vi) If $\phi \in \Phi(H,K)$ and one element of Π_ϕ is tempered then all elements are. With respect to a distinguished splitting, write $\phi(w) = (a(w), w)$. The elements of Π_ϕ are tempered if and only if $\{a(w) : w \in W_K\}$ is relatively compact in $H^\vee$.

Remark 3.2. Note that, although Langlands in [Lan] is working primarily with the extension $\mathbb{C}/\mathbb{R}$, he explicitly writes that many of his results hold more generally for finite extensions E/F of local fields, see [Lan, p.7]. In this generality, he proves that the map $\phi \mapsto \chi_\phi$ respects restriction of scalars, see [Lan, Lemma 2.11]. The map $\alpha \mapsto \pi_\alpha$ also respects restriction of scalars, see [Lan, Lemma 2.12].

Definition 3.3. *Let E/F be a finite and Galois extension. Let G be a connected reductive group defined over E which admits a* LLC*, say λ_G. Then the map λ_M is defined to be the unique map for which the following diagram commutes*

$$\begin{array}{ccc} \Pi(M,F) & \xrightarrow{\lambda_M} & \Phi(M,F) \\ {\scriptstyle \iota^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Sh}} \\ \Pi(G,E) & \xrightarrow{\lambda_G} & \Phi(G,E) \end{array} \tag{3.4}$$

where Sh *is the restriction to $\Phi(M,F)$ under the injection* (**??**) of the Shapiro isomorphism, as in [Bor, 8.4].

The pre-image via λ_M of $\mathrm{Sh}^{-1}(\phi)$ will be denoted $\Pi_{\mathrm{Sh}^{-1}\phi}$. Since the two vertical maps are bijective, it is clear that we have equality of cardinalities:

$$\mathrm{card}(\Pi_{\mathrm{Sh}^{-1}\phi}) = \mathrm{card}(\Pi_\phi).$$

The map λ_M satisfies all the above conditions, and hence it qualifies as a local Langlands correspondence.

A LLC for (H,K) can be enhanced in the following way. Let $H^\vee_{\mathrm{sc}}$ be the simply connected covering of the derived group of $H^\vee$, and $\mathrm{Z}(H^\vee_{\mathrm{sc}})$ be the center of $H^\vee_{\mathrm{sc}}$. Let $H^\vee_{\mathrm{ad}}$ be the adjoint group of $H^\vee$. Let $\phi \in \Phi(H,K)$. We denote by $\mathrm{Z}_{H^\vee}(\phi)$ denote the centralizer of $\phi(W'_K)$ in $H^\vee$. Since $\mathrm{Z}_{H^\vee}(\phi) \cap \mathrm{Z}(H^\vee) = \mathrm{Z}(H^\vee)^{W_K}$, we get $\mathrm{Z}_{H^\vee}(\phi)/\mathrm{Z}(H^\vee)^{W_K} \cong \mathrm{Z}_{H^\vee}(\phi)\mathrm{Z}(H^\vee)/\mathrm{Z}(H^\vee)$. The latter can be considered as a subgroup of the adjoint group $H^\vee_{\mathrm{ad}}$. Let $\mathrm{Z}^1_{H^\vee_{\mathrm{sc}}}(\phi)$ be its inverse image under the quotient map $H^\vee_{\mathrm{sc}} \to H^\vee_{\mathrm{ad}}$.

Following Arthur [Ar, (3.2)], we consider the component group of $\mathrm{Z}^1_{H^\vee}(\phi)$:

$$\mathscr{S}_\phi := \pi_0\big(\mathrm{Z}^1_{H^\vee_{\mathrm{sc}}}(\phi)\big). \tag{3.5}$$

An *enhancement* of ϕ is an irreducible representation ρ of $\mathscr{S}_\phi$. Via the canonical map $\mathrm{Z}(H^\vee_{\mathrm{sc}}) \to \mathrm{Z}(\mathscr{S}_\phi)$, every enhancement ρ determines a character ζ_ρ of $\mathrm{Z}(H^\vee_{\mathrm{sc}})$.

On the other hand, the group H is an inner twist of a unique quasi-split K-group H^*. The parametrization of equivalence classes of inner twists of H^* by

$$\mathrm{H}^1_c(W_K, H_{\mathrm{ad}}) \simeq \mathrm{Irr}\left(\mathrm{Z}(H^\vee_{\mathrm{ad}})^{W_K}\right) \tag{3.6}$$

provides a character ζ_H of $\mathrm{Z}(H^\vee_{\mathrm{ad}})^{W_K}$. We choose an extension to a character ζ_H of $\mathrm{Z}(H^\vee_{\mathrm{sc}})$. (Such an extension is determined by an explicit construction of H is inner twist of H^*.) Then we say that (ϕ,ρ) (or ρ) is $H(K)$-relevant if $\zeta_\rho = \zeta_H^+$.

Definition 3.7. *A pair (ϕ,ρ), where ϕ is a Langlands parameter for $H(K)$ and ρ is an $H(K)$-relevant irreducible representation of the group $\mathscr{S}_\phi$ defined in* (3.5)*, is called an enhanced L-parameter for (H,K). We denote by $\Phi^{\mathrm{e}}(H,K)$ the set of $H^\vee$-orbits of enhanced L-parameters for (H,K) for the following action of $H^\vee$:*

$$h\cdot(\phi,\rho) = (h\phi h^{-1}, \rho\circ \mathrm{Ad}(h^{-1})) \quad \textit{for } h\in H^\vee.$$

Remark 3.8. A notion of *cuspidality* for enhanced L-parameters was defined in [AMS, Definition 6.9]. Cuspidal $H(K)$-relevant enhanced L-parameters are expected to parametrize the supercuspidal smooth irreducible representations of $H(K)$ (see [AMS, Conjecture 6.10]).

It is natural to request that a LLC

$$\begin{array}{rccc} \lambda\colon & \Pi(H,K) & \to & \Phi(H,K) \\ & \pi & \mapsto & \phi_\pi \end{array},$$

may be *enhanced* so that we obtain a bijection

$$\begin{array}{rccc} \lambda^{\mathrm{e}}\colon & \Pi(H,K) & \to & \Phi^{\mathrm{e}}(H,K) \\ & \pi & \mapsto & (\phi_\pi,\rho_\pi) \end{array}. \tag{3.9}$$

For any $\phi\in\Phi(H,K)$, the elements in the L-packet Π_ϕ will then be parametrized by the set of isomorphism classes of $H(K)$-relevant irreducible representations of the finite group $\mathscr{S}_\phi$.

Feng, Opdam and Solleveld proved that the map $\mathrm{Sh}\colon \Phi(M,F)\to\Phi(G,E)$ extends naturally to a bijection

$$\mathrm{Sh}^{\mathrm{e}}\colon \Phi^{\mathrm{e}}(M,F)\to\Phi^{\mathrm{e}}(G,E), \tag{3.10}$$

and that Sh^{e} preserves cuspidality (see [FOS, Lemma A4]).

Definition 3.11. *If there exists a bijection $\lambda^{\mathrm{e}}_G\colon \Pi(G,E)\to\Phi^{\mathrm{e}}(G,E)$, then the map λ^{e}_M is defined to be the unique map for which the following diagram commutes*

$$\begin{array}{ccc} \Pi(M,F) & \xrightarrow{\lambda^{\mathrm{e}}_M} & \Phi^{\mathrm{e}}(M,F) \\ {\scriptstyle\iota^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Sh}^{\mathrm{e}}} \\ \Pi(G,E) & \xrightarrow{\lambda^{\mathrm{e}}_G} & \Phi^{\mathrm{e}}(G,E). \end{array} \tag{3.12}$$

By construction λ^{e}_M is a bijection and enhances the map λ_M defined in Definition 3.3.

We assume that $G(E)$ is quasi-split, that is, there is a Borel subgroup of G defined over E. Recall that a Whittaker datum for $G(E)$ is a $G(E)$-conjugacy class of pairs (B,θ), where B is a Borel subgroup of G defined over E with unipotent radical U, and θ is a non-degenerate character $U(E)\to\mathbb{C}^\times$. Whittaker datum $\mathfrak{w}=(B,\theta)$, an admissible representation $\pi\in\Pi(G(E))$ is called $\mathfrak{w}$-generic if $\mathrm{Hom}_{U(E)}(\pi,\theta)\neq 0$.

We attempt to lift $\mathfrak{w}$ from $G(E)$ to $M(F)$. Note first that the Weil-restricted group $(\mathfrak{R}_{E/F}B)$ is a Borel subgroup of M, see [Bor, §5.2]. Thus M is a quasi-split F-group. We know that $B(E)$ and $(\mathfrak{R}_{E/F}B)(F)$ are isomorphic topological groups.

We have an exact sequence $1 \to U \to B \to T \to 1$, with T a maximal torus. It gives an exact sequence

$$1 \to \mathfrak{R}_{E/F}U \to \mathfrak{R}_{E/F}B \to \mathfrak{R}_{E/F}T \to 1,$$

as checked for instance in [Oes, A.3.2]. Since $\mathfrak{R}_{E/F}T$ is maximal torus of $\mathfrak{R}_{E/F}B$, we have an exact sequence

$$1 \to V \to \mathfrak{R}_{E/F}B \to \mathfrak{R}_{E/F}T \to 1,$$

where V denotes the unipotent radical of $\mathfrak{R}_{E/F}B$. It follows that

$$V = \mathfrak{R}_{E/F}U, \tag{3.13}$$

and hence we have $V(F) = U(E)$.

Let $\Phi(G,T)$ denote the root system of G with respect to T, and let $\Phi(G,T)^+$ be the set of positive roots corresponding to the choice of B. Non-trivial minimal closed unipotent subgroups of G normalized by T are isomorphic to $\mathbb{G}_a$; the conjugation action by T is mapped by this isomorphism to an action of T on $\mathbb{G}_a$ of the form $x \mapsto \alpha(t)x$, where $\alpha \in X^*(T)$. Then $\alpha \in \Phi(G,T)$, see [DM, Theorem 0.31]. Such unipotent subgroups are called root subgroups.

Let U_α be a root subgroup of G. Let P_α be the unique parabolic subgroup containing U_α and no other root group [DM, §1.11, 1.20]. Let L_α denote the Levi subgroup of P_α. The unipotent radical of P_α equals

$$\prod_{\beta \in \Phi(G,T)^+ \setminus \Phi(L_\alpha,T)} U_\beta. \tag{3.14}$$

On the other hand, we have

$$L_\alpha = \langle T, U_\beta, U_{-\beta} \,:\, \text{for all } \beta \in \Delta - \{\alpha\}\rangle,$$

where Δ is the set of simple roots in $\Phi(G,T)$. Hence the unipotent radical of P_α is U_α.

Under the identification of $X^*(T)$ with $X^*(\mathfrak{R}_{E/F}T)$ the relative root system $\Phi(G,T)$ gets identified with the relative root system $\Phi(M, \mathfrak{R}_{E/F}T)$. Note that $\mathfrak{R}_{E/F}P_\alpha$ is a parabolic subgroup of $\mathfrak{R}_{E/F}G$ by [Bor, §5.2]. Denote by V_α its unipotent radical. The group L_α can also be described as the centralizer in G of the identity component of

$$\bigcap_{\beta \in \Delta - \{\alpha\} \subset \Phi(G,T)} \ker \beta.$$

It follows from the functorial properties of $\mathfrak{R}_{E/F}$ that $\mathfrak{R}_{E/F}L_\alpha$ is the centralizer in $\mathfrak{R}_{E/F}G$ of the identity component of

$$\bigcap_{\beta \in \Delta - \{\alpha\} \subset \Phi(\mathfrak{R}_{E/F}G, \mathfrak{R}_{E/F}T)} \ker \beta$$

hence is a Levi subgroup of $\mathfrak{R}_{E/F}P_\alpha$.

Then, a parallel argument as for proving (3.13), with U replaced by U_α, B replaced by P_α and T replaced by L_α demonstrates that

$$V_\alpha = \mathfrak{R}_{E/F}U_\alpha$$

and hence we have $V_\alpha(F) = U_\alpha(E)$.

Similarly as above, the root systems $\Phi(L_\alpha, T)$ and $\Phi(\mathfrak{R}_{E/F}L_\alpha, \mathfrak{R}_{E/F}T)$ can be identified. Then the analog of (3.14) for the description of the unipotent radical of $\mathfrak{R}_{E/F}P_\alpha$ shows that V_α is the root subgroup of M associated with α.

Suppose given a character $\theta\colon U(E) \to \mathbb{C}^\times$. This character, with domain $V(F)$ instead of $U(E)$, will be denoted $\theta_{E/F}$. We have

$$\theta_{E/F}(V_\alpha(F)) = \theta(U_\alpha(E)),$$

so that θ is non-trivial on root subgroups of $G(E)$ if and only if $\theta_{E/F}$ is non-trivial on root subgroups of $M(F)$. Therefore θ is non-degenerate if and only if $\theta_{E/F}$ is non-degenerate. This leads to the following definition:

$$\mathfrak{w}_{E/F} := (\mathfrak{R}_{E/F}B, \theta_{E/F}).$$

Then $\mathfrak{w}_{E/F}$ is a Whittaker datum for the Weil-restricted group $M(F)$.

Since $G(E)$ and $M(F)$ are isomorphic as topological groups, we have

$$\mathrm{Hom}_{V(F)}(\pi, \theta_{E/F}) = \mathrm{Hom}_{U(E)}(\iota^*\pi, \theta).$$

Therefore, π is $\mathfrak{w}_{E/F}$-generic if and only if $\iota^*\pi$ is $\mathfrak{w}$-generic. In particular, the set Π_ϕ contains a unique $\mathfrak{w}$-generic constituent if and only if the set $\Pi_{\mathrm{Sh}^{-1}(\phi)}$ contains a unique $\mathfrak{w}_{E/F}$-generic constituent, in conformity with Conjecture C in Kaletha's article [Kal].

We write $\mathscr{R}_\phi := \pi_0\big(\mathrm{Z}_{H^\vee}(\phi)/\mathrm{Z}(H^\vee)^{W_K}\big)$. The map $H^\vee_{\mathrm{sc}} \to H^\vee_{\mathrm{ad}}$ induces a homomorphism $\mathscr{S}_\phi \to \mathscr{R}_\phi$ and $\mathscr{S}_\phi$ is a central extension of $\mathscr{R}_\phi$ by $\mathrm{Z}(H^\vee_{\mathrm{sc}})/\mathrm{Z}(H^\vee_{\mathrm{sc}}) \cap \mathrm{Z}_{H^\vee_{\mathrm{sc}}}(\phi)^\circ$ (see [ABPS2, Lemma 1.7]).

From now on we assume that $G(E)$ is quasi-split, and that a Whittaker datum $\mathfrak{w} = (B, \theta)$ for $G(E)$ is fixed. Then the expected parametrization reduces to bijections

$$\mathscr{L}_\phi\colon \begin{array}{ccc} \Pi_\phi & \to & \mathrm{Irr}(\mathscr{R}_\phi) \\ \pi & \mapsto & \rho_\pi \end{array}, \tag{3.15}$$

for all $\phi \in \Phi(H, K)$, where $\mathrm{Irr}(\mathscr{R}_\phi)$ denotes the set irreducible characters of $\mathscr{R}_\phi$.

Then we can form for any $\phi \in \Phi_{\mathrm{bd}}(H, K)$ and $r \in \mathscr{R}_\phi$ the virtual character

$$\Theta^r_\phi := \sum_{\pi \in \Pi_\phi} (\mathscr{L}_\phi(\pi))(r)\, \Theta_\pi, \tag{3.16}$$

where Θ_π is the Harish-Chandra distribution character of π.

As observed in [FOS, (A.22)], for any $\phi \in \Phi(M, F)$, we have a canonical isomorphism

$${}^L\iota\colon \mathscr{R}_{\mathrm{Sh}(\phi)} \xrightarrow{\sim} \mathscr{R}_\phi. \tag{3.17}$$

We define a bijection

$${}^L\iota^*\colon \mathrm{Irr}(\mathscr{R}_\phi) \xrightarrow{\sim} \mathrm{Irr}(\mathscr{R}_{\mathrm{Sh}(\phi)}), \tag{3.18}$$

by setting

$$({}^L\iota^*(\rho))(r') := \rho({}^L\iota(r')), \quad \text{for any } \rho \in \mathrm{Irr}(\mathscr{R}_\phi) \text{ and any } r' \in \mathscr{R}_{\mathrm{Sh}(\phi)}. \tag{3.19}$$

Proposition 3.20. *Let $\phi \in \Phi(M,F)$. We assume that there exists a bijection $\mathscr{L}_\phi$ as in (3.15). Then, for any $f \in \mathscr{C}_c^\infty(M(F))$, we have*

$$\Theta_\phi^r(f) = \Theta_{\mathrm{Sh}(\phi)}^{{}^L\iota(r)}(\iota^* f), \quad \textit{for any } r \in \mathscr{R}_\phi,$$

where $\iota^* f \colon G(E) \to \mathbb{C}$ is the function defined by $(\iota^* f)(g) := f(\iota(g))$ for $g \in G(E)$.

Proof. Let $\mathscr{L}_{\mathrm{Sh}(\phi)} \colon \Pi_{\mathrm{Sh}(\phi)} \to \mathrm{Irr}(\mathscr{R}_{\mathrm{Sh}(\phi)})$ denote the unique map which makes the following diagram commutative:

$$\begin{array}{ccc} \Pi_\phi & \xrightarrow{\mathscr{L}_\phi} & \mathrm{Irr}(\mathscr{R}_\phi) \\ {\scriptstyle \iota^*}\big\downarrow & & \big\downarrow{\scriptstyle {}^L\iota^*} \\ \Pi_{\mathrm{Sh}(\phi)} & \xrightarrow{\mathscr{L}_{\mathrm{Sh}(\phi)}} & \mathrm{Irr}(\mathscr{R}_{\mathrm{Sh}(\phi)}) \end{array} . \tag{3.21}$$

Let $r \in \mathscr{R}_\phi$. We write $r' := {}^L\iota^{-1}(r)$. Then we obtain that

$$\Theta_{\mathrm{Sh}(\phi)}^{r'} = \sum_{\pi' \in \Pi_{\mathrm{Sh}(\phi)}} (\mathscr{L}_{\mathrm{Sh}(\phi)}(\pi'))(r')\, \Theta_{\pi'} = \sum_{\pi \in \Pi_\phi} (\mathscr{L}_{\mathrm{Sh}(\phi)}(\iota^*\pi))(r')\, \Theta_{\iota*\pi}.$$

We observe that, for any $\pi \in \Pi(M,F)$, we have, for any $f \in \mathscr{C}_c^\infty(M(F))$:

$$\Theta_\pi(f) = \Theta_{\iota^*\pi}(\iota^* f). \tag{3.22}$$

Using the commutativity of the diagram (3.21), we get

$$\Theta_{\mathrm{Sh}(\phi)}^{r'}(\iota^* f) = \sum_{\pi \in \Pi_\phi} ({}^L\iota^*(\mathscr{L}_\phi(\pi)))(r')\, \Theta_{\iota^*\pi}(\iota^* f).$$

By using (3.19) and (3.22), we finally get that $\Theta_{\mathrm{Sh}(\phi)}^{{}^L\iota(r)}(\iota^* f) = \Theta_\phi^r(f)$, for any $r \in \mathscr{R}_\phi$. □

The Hiraga Ichino Ikeda conjecture [HII]. We fix an additive character $\psi \colon K \to \mathbb{C}^\times$ which is trivial on the ring of integers $\mathfrak{o}_K$ and endow K with the Haar measure that gives its ring of integers volume 1.

Definition 3.23. *A* LLC *correspondence λ_H satisfies the* HII *conjecture for ψ if for any square-integrable modulo centre representation ω of $L(K)$, the formal degree of ω is*

$$\mathrm{fdeg}(\omega) = \dim(\rho_\omega)\, |\mathscr{R}_{\phi_\omega}|^{-1}\, \gamma(0, \mathrm{Ad}_{H^\vee, L^\vee} \circ \phi_\omega, \psi),$$

where $\lambda_H^{\mathrm{e}}(\omega) = (\phi_\omega, \rho_\omega) \in \Phi^{\mathrm{e}}(L,K)$, and where $\mathrm{Ad}_{H^\vee,L^\vee}$ is the adjoint representation of the group ${}^LL$ on the quotient of the Lie algebra of $H^\vee$ by that of $\mathrm{Z}(L^\vee)^{W_K}$ and $\gamma(0, \mathrm{Ad}_{H^\vee,L^\vee} \circ \phi_\omega, \psi)$ is the corresponding adjoint γ-factor.

Transfer. We assume from now that H is a quasi-split K-group and that K has characteristic zero. A semisimple element in $H(K)$ is called *strongly regular* if its centralizer is a torus. We denote by $H(K)_{\mathrm{sr}}$ the open subvariety of $H(K)$ fsting of the strongly regular semisimple elements.

Let $\gamma \in H(K)_{\mathrm{sr}}$ and let $H(K)_\gamma$ denote its centralizer in $H(K)$. Let $f \in \mathscr{C}_c^\infty(H(K))$. The orbital integral $\mathrm{O}_\gamma(f)$ of (f,γ) is

$$\mathrm{O}_\gamma(f) := \int_{H_\gamma(K)\backslash H(K)} f(x^{-1}\gamma x)\, d\dot{x},$$

where $d\dot{x}$ is an invariant measure on the quotient $H(K)_\gamma \backslash H(K)$.

The stable orbital integral $\mathrm{SO}_\gamma(f)$ of (f,γ) is

$$\mathrm{SO}_\gamma(f) := \sum_{\gamma' \in S(\gamma)} \mathrm{O}_{\gamma'}(f),$$

where $S(\gamma)$ is a set of representatives for the $H(K)$-conjugacy classes of γ in its $H(K^{\mathrm{sep}})$-conjugacy class (so-called the *stable conjugacy* class of γ).

We recall from [Kal, Def. 2] that an *extended endoscopic triple* for (H,K) is a triple $\mathfrak{e} = (H_\mathfrak{e}, s, {}^L\eta)$, where:

- $H_\mathfrak{e}$ is a quasi-split connected reductive K-group,
- s is a semisimple element in $H^\vee$,
- ${}^L\eta\colon {}^LH_\mathfrak{e} \to {}^LH$ is an L-homomorphism of L-groups (as in [Bor, §15.1]) that restricts to an isomorphism of complex reductive groups $H_\mathfrak{e}^\vee \xrightarrow{\sim} \mathrm{Z}_{H^\vee}(s)^\circ$, such that ${}^L\eta(h)$ commutes with s, for any $h \in {}^LH_\mathfrak{e}$.

Let $\mathfrak{e} = (H_\mathfrak{e}, s, {}^L\eta)$ be an extended endoscopic triple for (H,K). We have $\phi(W_K') \subset {}^L\eta(W_K')$, since ${}^L\eta\colon {}^LH_\mathfrak{e} \to {}^LH$. It follows that $s \in \mathrm{Z}_{H^\vee}(\phi)$. We denote by $\overline{s}$ the image of s in $\mathscr{R}_\phi$.

Let $\mathfrak{w}$ be a Whittaker datum for $H(K)$. We recall that $f_\mathfrak{e} \in \mathscr{C}_c^\infty(H_\mathfrak{e}(K))$ is called a *transfer* of $f \in \mathscr{C}_c^\infty(H(K))$ if for all $\gamma \in H_\mathfrak{e}(K)_{\mathrm{sr}}$ we have

$$\mathrm{SO}_\gamma(f_\mathfrak{e}) = \sum_\delta \Delta[\mathfrak{w},\mathfrak{e}](\gamma,\delta)\, \mathrm{O}_\delta(f),$$

where δ runs over the set of conjugacy classes in $H(K)_{\mathrm{sr}}$, and where

$$\Delta(\mathfrak{w},\mathfrak{e})\colon H_\mathfrak{e}(K)_{\mathrm{sr}} \times H(K)_{\mathrm{sr}} \to \mathbb{C}$$

is the Langlands-Shelstad *transfer factor* associated to $\mathfrak{w}$.

Definition 3.24. *An enhanced* LLC *for* (H,K) *is a bijection*

$$\begin{array}{rccc} \lambda_H^{\mathrm{e}}\colon & \Pi(H,K) & \to & \Phi^{\mathrm{e}}(H,K) \\ & \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array}$$

such that the map $\lambda_H\colon \pi \mapsto \phi_\pi$ *is a* LLC*, and the following extra properties hold:*

(1) λ_H *satisfies the* HII *conjecture for square integrable modulo center representations.*
(2) *For any Whittaker datum* $\mathfrak{w}$ *for* (H,K)*, and all* $\phi \in \Phi_{\mathrm{bd}}(H,K)$ *the* L*-packet* $\Pi_\phi := \lambda_H^{-1}(\phi)$ *contains a unique* $\mathfrak{w}$*-generic constituent.*
(3) λ_H^{e} *restricts to a bijection from the set of isomorphism classes of supercuspidal irreducible representations of* $H(K)$ *to the set of* $H^\vee$*-conjugacy classes of cuspidal enhanced* L*-parameters.*
(4) *For any* $\phi \in \Phi(H,K)$*, the map*

$$\begin{array}{rccc} \mathscr{L}_\phi\colon & \Pi_\phi & \to & \mathrm{Irr}(\mathscr{S}_\phi) \\ & \pi & \mapsto & \rho_\pi \end{array}$$

is a bijection,

(5) *When H is quasi-split over K, for any extended endoscopic triple $\mathfrak{e} = (H_{\mathfrak{e}}, s, {}^L\eta)$ for (H, K), there exists a bijection*

$$\lambda^{\mathrm{e}}_{H_{\mathfrak{e}}} \colon \Pi(H_{\mathfrak{e}}, K) \to \Phi^{\mathrm{e}}(H_{\mathfrak{e}}, K)$$

which satisfies the analogs of (1)–(4) *for $(H_{\mathfrak{e}}, K)$, and a Whittaker datum $\mathfrak{w}_{\mathfrak{e}}$ for $(H_{\mathfrak{e}}, K)$, such that*

(a) *for $\phi_{\mathfrak{e}} \in \Phi_{\mathrm{bd}}(H_{\mathfrak{e}}, K)$, the character $\rho_{\pi^{\mathrm{g}}_{\mathfrak{e}}}$ is trivial if $\pi^{\mathrm{g}}_{\mathfrak{e}}$ is the $\mathfrak{w}_{\mathfrak{e}}$-generic constituent of $\Pi_{\phi_{\mathfrak{e}}}$,*

(b) *for any pair $(f_{\mathfrak{e}}, f) \in \mathscr{C}^{\infty}_c(H_{\mathfrak{e}}(K)) \times \mathscr{C}^{\infty}_c(H(K))$ of functions such that $f_{\mathfrak{e}}$ is a transfer of f, we have the equality*

$$\Theta^1_{\phi_{\mathfrak{e}}}(f_{\mathfrak{e}}) = \Theta^{\bar{s}}_{{}^L\eta\circ\phi_{\mathfrak{e}}}(f).$$

Theorem 3.25. *Consider a bijection*

$$\lambda^{\mathrm{e}}_G \colon \Pi(G, E) \longrightarrow \Phi^{\mathrm{e}}(G, E).$$

Then

(i) *The map λ^{e}_M defined in* (3.11) *is an enhanced* LLC *for (M, F) if and only if λ^{e}_G is an enhanced* LLC *for (G, E).*

(ii) *Furthermore, if $\pi \in \Pi(M, F)$ and $\mathrm{dep}(\lambda_G(\iota^*\pi)) = \kappa_\pi \cdot \mathrm{dep}(\iota^*\pi)$ then we have*

$$\mathrm{dep}(\lambda_M(\pi)) = \varphi_{E/F}(\kappa_\pi \cdot e \cdot \mathrm{dep}(\pi)),$$

where $\varphi_{E/F}$ is the Hasse-Herbrand function.

Proof. It is proved in [FOS, Proposition A.7] that, for any finite separable field extension E/F, the HII conjecture holds for ω a square-integrable modulo center irreducible representation of $M(F)$ if and only if its holds for $\iota^*(\omega)$: it shows that λ_G satisfies Definition 3.24 (1) if and only if λ_M satisfies it.

We have already seen that Definition 3.24 (2) is satisfied by λ_G if and only if it is satisfied by λ_M. Since Sh^{e} preserves the cuspidality, Definition 3.24 (3) is satisfied by λ^{e}_G if and only if it is satisfied by λ^{e}_M.

We write $\lambda^{\mathrm{e}}_M(\pi) = (\phi_\pi, \rho_\pi)$ for $\pi \in \Pi(M, F)$ and $\mathscr{L}_\phi(\pi) = \rho_\pi$ for $\phi = \phi_\pi$. Then Definition 3.24 (4) is satisfied by $\mathscr{L}_\phi$ if and only if it is satisfied by $\mathscr{L}_{\mathrm{Sh}(\phi)}$, where

$$\mathscr{L}_{\mathrm{Sh}(\phi)}(\iota^*\pi) := {}^L\iota^*(\rho_\pi),$$

and ${}^L\iota^*$ is the natural bijection between $\mathrm{Irr}(\mathscr{S}_\phi)$ and $\mathrm{Irr}(\mathscr{S}_{\mathrm{Sh}(\phi)})$.

When G is quasi-split over E, let $\mathfrak{w}$ be a Whittaker datum for (G, E), and let $\mathfrak{w}_{E/F}$ be its lift to (M, F) as in (**??**). Let $\phi \in \Phi_{\mathrm{bd}}(M, F)$. Then the diagram (3.21) shows that the existence of a bijection $\mathscr{L}_\phi \colon \Pi_\phi \to \mathrm{Irr}(\mathscr{R}_\phi)$ satisfying Definition 3.24 (5).a is equivalent to the existence of a bijection $\mathscr{L}_{\mathrm{Sh}(\phi)} \colon \Pi_{\mathrm{Sh}(\phi)} \to \mathrm{Irr}(\mathscr{R}_{\mathrm{Sh}(\phi)})$ satisfying Definition 3.24 (5).a. Indeed, as already observed, $\pi^{\mathrm{g}} \in \Pi_\phi$ is $\mathfrak{w}_{E/F}$-generic if and only if $\iota^*\pi^{\mathrm{g}}$ is $\mathfrak{w}$-generic. On the other hand, the diagram (3.21) implies that $\mathscr{L}_\phi(\pi^{\mathrm{g}})$ is the trivial character of $\mathscr{R}_\phi$ if an only if $\mathscr{L}_{\mathrm{Sh}(\phi)}(\iota^*\pi^{\mathrm{g}})$ is the trivial character of $\mathscr{R}_{\mathrm{Sh}(\phi)}$, since (3.19) shows that ${}^L\iota^*$ maps the trivial character of $\mathscr{R}_\phi$ to that of $\mathscr{R}_{\mathrm{Sh}(\phi)}$.

We fix a Γ_E-stable Borel pair $(B^\vee, T^\vee)$ in $G^\vee$ and an extended endoscopic triple $\mathfrak{e} = (G_{\mathfrak{e}}, s, {}^L\eta)$ for (G, E) such that $s' \in T^\vee$. The following construction is based on [LW, § 1.2]. Every Borel pair in $M^\vee$ can be written as $(I^{\Gamma_F}_{\Gamma_E}(B^\vee), I^{\Gamma_F}_{\Gamma_E}(T^\vee))$ for

a well-determined Borel pair $(B^\vee, T^\vee)$ in $G^\vee$. It is Γ_F-stable if and only if the pair $(B^\vee, T^\vee)$ is Γ_E-stable. Setting $\mathcal{W}^\vee := \mathrm{N}_{G^\vee}(T^\vee)/T^\vee$, we have

$$\mathrm{N}_{M^\vee}(\mathfrak{R}_{E/F}(T^\vee))/\mathfrak{R}_{E/F}(T^\vee) = I_{\Gamma_E}^{\Gamma_F}(\mathcal{W}^\vee).$$

We denote by $\tau \mapsto \tau_G$ the natural action of Γ_E on $G^\vee$ and we set $B_{\mathfrak{e}}^\vee := B^\vee \cap G_{\mathfrak{e}}^\vee$. Then $(B_{\mathfrak{e}}^\vee, T^\vee)$ is a Borel pair in $G_{\mathfrak{e}}$. Let $\tau \in \Gamma_E$. For each $v' \in W_E$ in the inverse image of τ under the natural map $W_E \to \Gamma_E$, we choose an element $(g(v'), v')$ of $G_{\mathfrak{e}}^\vee \rtimes W_E = {}^L G_{\mathfrak{e}}$ such that the automorphism $\mathrm{Int}_{g(v')} \circ \tau_G$ preserves the Borel pair $(B_{\mathfrak{e}}^\vee, T^\vee)$. The coset $T^\vee g(v')$ being well-determined, there is a well-determined element $w_{G_{\mathfrak{e}}}(\tau)$ of $\mathcal{W}^\vee$ such that the conjugacy action of $(g(v'), v')$ on $T^\vee$ is given by $w_{G_{\mathfrak{e}}}(\tau)\tau_G$, where we identify $w_{G_{\mathfrak{e}}}(\tau)$ with the automorphism $\mathrm{Int}_{w_{G_{\mathfrak{e}}}(\tau)}$ of $T^\vee$. The map $\tau \mapsto w_{G_{\mathfrak{e}}}(\tau)$ is a 1-cocycle. Let $\alpha \in \mathrm{H}^1(\Gamma_E, \mathcal{W}^\vee)$ denote its cohomology class, and let $\mathrm{Sh}^{-1}(\alpha)$ be the inverse image of α under the Shapiro isomorphism

$$\mathrm{Sh}\colon \mathrm{H}^1(\Gamma_F, I_{\Gamma_E}^{\Gamma_F}(\mathcal{W}^\vee)) \to \mathrm{H}^1(\Gamma_E, \mathcal{W}^\vee).$$

We choose a 1-cocycle $w_{G_{\mathfrak{e}},M}$ of Γ_F with values in $I_{\Gamma_E}^{\Gamma_F}(\mathcal{W}^\vee)$ which belongs to the cohomology class of α. Up to replacing $w_{G_{\mathfrak{e}},M}$ by a cohomologous 1-cocycle, we may, and do, assume that

$$(w_{G_{\mathfrak{e}},M}(\tau))(1) = w_{G_{\mathfrak{e}}}(\tau) \quad \text{for any } \tau \in \Gamma_E. \tag{3.26}$$

Let $\sigma \mapsto \sigma_{G_{\mathfrak{e}},M}$ be the action of Γ_F on the torus $I_{\Gamma_E}^{\Gamma_F}(T^\vee)$ of $M^\vee$ defined by

$$\sigma_{G_{\mathfrak{e}},M} := w_{G_{\mathfrak{e}},M}(\sigma)\,\sigma_M, \tag{3.27}$$

where $\sigma \mapsto \sigma_M$ is the natural action of Γ_F on $M^\vee$. It allows to define an application $s_{E/F}$ from Γ_F to $T^\vee$ by sending σ to

$$s_{E/F}(\sigma) := w_{G_{\mathfrak{e}},M}(\sigma)(1)^{-1}\,(s). \tag{3.28}$$

In particular, $s_{E/F}$ belongs to $I_{\Gamma_E}^{\Gamma_F}(T^\vee)$, is fixed by the action $\sigma \mapsto \sigma_{G_{\mathfrak{e}},M}$ and we have $s_{E/F}(1) = s$. We set

$$M_{\mathfrak{e}}^\vee := \mathrm{Z}_{M^\vee}(s_{E/F})^\circ. \tag{3.29}$$

Then $M_{\mathfrak{e}}^\vee \cap I_{\Gamma_E}^{\Gamma_F}(B^\vee)$ is a Borel subgroup of $M_{\mathfrak{e}}^\vee$. For each $v \in W_F$ with image σ in Γ_F, we choose a representative $\tilde{w}_{G_{\mathfrak{e}},M}(v) = \tilde{w}_{G_{\mathfrak{e}},M}(\sigma)$ of $w_{G_{\mathfrak{e}},M}(\sigma)$ in $\mathrm{N}_{M^\vee}(I_{\Gamma_E}^{\Gamma_F}(T^\vee))$.

The automorphism $\mathrm{Int}_{\tilde{w}_{G_{\mathfrak{e}},M}(\sigma)} \circ \tau_M$ preserves the Borel pair $(M_{\mathfrak{e}}^\vee \cap I_{\Gamma_E}^{\Gamma_F}(B^\vee), I_{\Gamma_E}^{\Gamma_F}(T^\vee))$ of $M_{\mathfrak{e}}^\vee$. We define

$$\mathscr{M}_{\mathfrak{e}} := \left\{(m\,\tilde{w}_{G_{\mathfrak{e}},M}(\sigma)(v), v) \,:\, m \in M_{\mathfrak{e}}^\vee, v \in W_F\right\} \subset {}^L M. \tag{3.30}$$

The set $\mathscr{M}_{\mathfrak{e}}$ is a group which normalizes $M_{\mathfrak{e}}^\vee$. Thus we can deduce an L-action of Γ_F on $M_{\mathfrak{e}}^\vee$, and hence form the semidirect product $M_{\mathfrak{e}}^\vee \rtimes W_F$. Then let $M_{\mathfrak{e}}$ be a quasi-split connected reductive F-group which has $M_{\mathfrak{e}}^\vee \rtimes W_F$ as L-group. We have $M_{\mathfrak{e}} = \mathfrak{R}_{E/F}(G_{\mathfrak{e}})$. Let $\lambda_{G_{\mathfrak{e}}}$ be a LLC for $(G_{\mathfrak{e}}, E)$. Then the arguments above, applied to $(G_{\mathfrak{e}}, E)$, show that $\lambda_{G_{\mathfrak{e}}}$ satisfies the properties (1), (2) and (3).

Let ${}^L\eta_{E/F}\colon {}^LM_{\mathfrak{e}} \to {}^LM$ be an L-homomorphism of L-groups such that $\mathfrak{e}_{E/F} = (M_{\mathfrak{e}}, s_{E/F}, {}^L\eta_{E/F})$ is an extended endoscopic triple for (M, F), then the diagram

$$\begin{tikzcd} \Phi(M_{\mathfrak{e}}, F) \arrow[r, "\Phi({}^L\eta_{E/F})"] \arrow[d, "\mathrm{Sh}"'] & \Phi(M, F) \arrow[d, "\mathrm{Sh}"] \\ \Phi(G_{\mathfrak{e}}, E) \arrow[r, "\Phi({}^L\eta)"] & \Phi(G, E) \end{tikzcd} \tag{3.31}$$

where

$$\Phi({}^L\eta)\colon \phi_{\mathfrak{e}} \mapsto {}^L\eta \circ \phi_{\mathfrak{e}} \quad \text{and} \quad \Phi({}^L\eta_{E/F})\colon \phi'_{\mathfrak{e}} \mapsto {}^L\eta_{E/F} \circ \phi'_{\mathfrak{e}},$$

is commutative, that is, we have ${}^L\eta_{E/F} \circ \mathrm{Sh}(\phi_{\mathfrak{e}}) = \mathrm{Sh}({}^L\eta \circ \phi_{\mathfrak{e}})$ for any $\phi_{\mathfrak{e}} \in \Phi(M_{\mathfrak{e}}, F)$.

Let $\iota_{\mathfrak{e}}\colon G_{\mathfrak{e}}(E) \xrightarrow{\sim} M_{\mathfrak{e}}(F)$. The maps ι and $\iota_{\mathfrak{e}}$ induce bijections $G(E)_{\mathrm{sr}} \xrightarrow{\sim} M(F)_{\mathrm{sr}}$ and $G_{\mathfrak{e}}(E)_{\mathrm{sr}} \xrightarrow{\sim} M_{\mathfrak{e}}(F)_{\mathrm{sr}}$, respectively. Let $\delta \in G(E)_{\mathrm{sr}}$ and $\gamma \in G_{\mathfrak{e}}(E)_{\mathrm{sr}}$. We have

$$\iota(G(E)_\delta) \simeq M(F)_{\iota(\delta)} \quad \text{and} \quad \iota_{\mathfrak{e}}(G_{\mathfrak{e}}(E)_\gamma) \simeq M_{\mathfrak{e}}(F)_{\iota_{\mathfrak{e}}(\gamma)}.$$

Let $(f, f_{\mathfrak{e}}) \in \mathscr{C}_c^\infty(M(F)) \times \mathscr{C}_c^\infty(M_{\mathfrak{e}}(F))$ such that $f_{\mathfrak{e}}$ is a transfer of f. We denote by $\iota^* f\colon G(E) \to \mathbb{C}$ and $\iota_{\mathfrak{e}}^* f_{\mathfrak{e}}\colon G_{\mathfrak{e}}(E) \to \mathbb{C}$ the functions defined by

$$(\iota^* f)(g) := f(\iota(g)) \quad \text{and} \quad (\iota_{\mathfrak{e}}^* f_{\mathfrak{e}})(g) := f_{\mathfrak{e}}(\iota_{\mathfrak{e}}(g_{\mathfrak{e}})) \quad \text{for } g \in G(E) \text{ and } g_{\mathfrak{e}} \in G_{\mathfrak{e}}(E).$$

We have $\iota^* f \in \mathscr{C}_c^\infty(G(E))$ and $\iota_{\mathfrak{e}}^* f_{\mathfrak{e}} \in \mathscr{C}_c^\infty(G_{\mathfrak{e}}(E))$. The transfer factors

$$\Delta(\mathfrak{w}, \mathfrak{e})\colon G_{\mathfrak{e}}(E)_{\mathrm{sr}} \times G(E)_{\mathrm{sr}} \to \mathbb{C} \quad \text{and} \quad \Delta(\mathfrak{w}_{E/F}, \mathfrak{e}_{E/F})\colon M_{\mathfrak{e}}(F)_{\mathrm{sr}} \times M(F)_{\mathrm{sr}} \to \mathbb{C}$$

coincide (it was observed in [Wal, Lemme 5.4] in the Lie algebras case). It follows that $f_{\mathfrak{e}}$ is a transfer of f if and only if $\iota_{\mathfrak{e}}^* f_{\mathfrak{e}}$ is a transfer of $\iota^* f$.

Let $\phi_{\mathfrak{e}} \in \Phi(M_{\mathfrak{e}}, F)$. Then the combination of Proposition 3.20 with the commutative diagram (3.31) implies that

$$\Theta^1_{\phi_{\mathfrak{e}}}(f_{\mathfrak{e}}) = \Theta^{\overline{s}_{E/F}}_{{}^L\eta_{E/F} \circ \phi_{\mathfrak{e}}}(f) \quad \text{if and only if} \quad \Theta^1_{\mathrm{Sh}(\phi_{\mathfrak{e}})}(\iota_{\mathfrak{e}}^* f_{\mathfrak{e}}) = \Theta^{\overline{s}}_{{}^L\eta \circ \mathrm{Sh}(\phi_{\mathfrak{e}})}(\iota^* f),$$

that is, λ_M satisfies Definition 3.24 (5).b if and only if λ_G satisfies it.

The assertion (ii) follows from Corollary A.13, Theorem 2.16 and the commutativity of the diagram (3.12). □

As a special case of Theorem 3.25, we have

Theorem 3.32. *If λ_G is depth-preserving, then, for all $\pi \in \Pi(M, F)$, we have*

$$\mathrm{dep}(\lambda_M(\pi)) = \varphi_{E/F}(e \cdot \mathrm{dep}(\pi))$$

In particular, we have

- $\mathrm{dep}(\lambda_M(\pi))/\mathrm{dep}(\pi) \to 1$ *as* $\mathrm{dep}(\pi) \to \infty$
- *λ_M is depth-preserving if and only if E/F is tamely ramified,*
- *if E/F is wildly ramified then, for each π with $\mathrm{dep}(\pi) > 0$, we have*

$$\mathrm{dep}(\lambda_M(\pi)) > \mathrm{dep}(\pi).$$

Proof. This follows from Theorem 3.25 and Lemma 2.1. □

When $G(E) = \mathrm{GL}_1(E)$, Theorem 3.32 strengthens the main result of [MiPa] for induced tori. For tamely ramified induced tori, we recover the depth-preservation theorem of Yu [Yu1].

4. Applications

4.1. An inequality between depths.

Lemma 4.1. *Let M and $\widetilde{M}$ be two reductive F-groups such that $\widetilde{M}$ is F-split and there exist an L-embedding $u\colon {}^LM \to {}^L\widetilde{M}$ which satisfies the following property: if $v \in W_F$ acts trivially on $M^\vee$, then we have $u(1,v) = (1,v)$.*

Then, for a given element $(m,v) \in {}^LM$: if $u(m,v) = (1,v)$, then we have $m = 1$. In particular, for any $\phi \in \Phi(M,F)$, we have $u \circ \phi \in \Phi(\widetilde{M},F)$ and

$$\mathrm{dep}(u \circ \phi) \leq \mathrm{dep}(\phi).$$

Proof. The conjugation isomorphism $\mathrm{Int}(1,v)$ of ${}^L\widetilde{M} = M^\vee \rtimes W_F$ is trivial on the first factor $M^\vee$. On the other hand, since we have $u(m,v) = (1,v)$, it should restricts to $\mathrm{Int}(m,v)$ on $M^\vee$. This implies that the isomorphism $\mathrm{Int}(m) \circ [v]$ of $M^\vee$ is the identity, where $[v]$ denotes the action of $v \in W_F$ on $M^\vee$. Here we recall that the action of W_F on $M^\vee$ is defined by using a fixed pinning. More precisely, we have a canonical exact sequence

$$1 \to \mathrm{Int}(M^\vee) \to \mathrm{Aut}(M^\vee) \to \mathrm{Out}(M^\vee) \to 1.$$

As M is defined over F, we get an action of W_F on its root data, hence we have a homomorphism $W_F \to \mathrm{Out}(M^\vee)$. ($\mathrm{Out}(M^\vee)$ is nothing but the automorphisms of the root data). As we also have a splitting $\mathrm{Out}(M^\vee) \to \mathrm{Aut}(M^\vee)$ coming from a fixed pinning, we can get a homomorphism $W_F \to \mathrm{Out}(M^\vee) \to \mathrm{Aut}(M^\vee)$ by sending v to $[v]$. This was nothing but the definition of the action of W_F on $M^\vee$. Therefore the equality $\mathrm{Int}(m) \circ [v] = \mathrm{Id}_{M^\vee}$ says that $[v] = \mathrm{Int}(m)^{-1} = \mathrm{Id}_{M^\vee}$. Hence, by our assumption, we get $u(1,v) = (1,v)$. As we have $u(m,v) = u(m,1) \cdot u(1,v)$, the equality $u(m,v) = (1,v)$(and the injectivity of the restriction of u to $M^\vee$) implies $m = 1$. Then the inequality between the depths of ϕ and $u \circ \phi$ follows from Definition 2.11. □

4.2. Automorphic induction.

Let $n \geq 1$ be an integer and $d = [E : F]$. Let G be the E-group GL_n and let $\widetilde{M}$ be the F-group GL_{nd}. Both groups G and $\widetilde{M}$ admits a local Langlands correspondence (see [HT], [He1] or [Sch]), and the corresponding maps λ_G and $\lambda_{\widetilde{M}}$ are bijective.

Let $\pi_E \in \Pi(G,E)$. We will denote by $\phi_E \in \Phi(G,E)$ the L-parameter of π_E, that is, $\phi_E := \lambda_G(\pi_E)$. It is proved in [He2, §7.3, Proposition 2] that the L-parameter $\tilde{\phi}$ of the representation $\tilde{\pi}$ of $\widetilde{M}$ obtained from π_E by *automorphic induction* (when the latter exists, see [HH]) satisfies

$$\tilde{\phi} = \mathrm{Ind}_{W'_E}^{W'_F}(\phi_E). \tag{4.2}$$

Lemma 4.3. *Let $\phi_E \in \Phi(\mathrm{GL}_n, E)$. We have*

$$\mathrm{dep}(\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)) = \varphi_{E/F}(\mathrm{dep}(\phi_E)).$$

Proof. The restriction to W_E of the L-parameter $\phi_E\colon W'_E \to \mathrm{GL}_n(\mathbb{C})$ is a representation of W_E of space V with $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. By using Lemma 2.7 and Lemma 2.2, we obtain

$$(\mathrm{Ind}_{W_E}^{W_F}(V))^{W_F^{r+}} \simeq \mathrm{Ind}_{W_E/W_E\cap W_F^{r+}}^{W_F/W_F^{r+}}(V^{W_E\cap W_F^{r+}}) \simeq \mathrm{Ind}_{W_E/W_E^{\psi_{E/F}(r)+}}^{W_F/W_F^{r+}}(V^{W_E^{\psi_{E/F}(r)+}}).$$

It follows that

$$(\mathrm{Ind}_{W_E}^{W_F}(V))^{W_F^{r+}} \neq \{0\} \iff V^{W_E^{\psi_{E/F}(r)+}} \neq \{0\}.$$

Then the result follows from Remark 2.14. □

Theorem 4.4. *We have*

$$\mathrm{dep}(\tilde{\pi}) = \varphi_{E/F}(\mathrm{dep}(\pi_E)).$$

Proof. Since $\lambda_{\widetilde{M}}$ and λ_G are depth preserving (see [ABPS1, Theorem 2.9]), using Lemma 4.3, we get

$$\mathrm{dep}(\tilde{\pi}) = \mathrm{dep}(\tilde{\phi}) = \mathrm{dep}(\mathrm{Ind}_{W'_E}^{W'_F}(\phi_E)) = \varphi_{E/F}(\mathrm{dep}(\phi_E)) = \varphi_{E/F}(\mathrm{dep}(\pi_E)).$$

□

4.3. **Asai lift.** We take for G the group GL_n and $M = \mathfrak{R}_{E/F}(G)$. We assume that $[E : F] = 2$. We denote by V the 2-dimensional $\mathbb{C}$-vector space $\mathbb{C}^2$. Hence we have ${}^LG = G^\vee \times W_F$, with $G^\vee = \mathrm{GL}_2(\mathbb{C}) = \mathrm{GL}(V)$, and ${}^LM = M^\vee \rtimes W_F$, where $M^\vee = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and W_F permutes the two factors $\mathrm{GL}_2(\mathbb{C})$ among themselves. Let $\widetilde{G}$ denote the group GL_{n^2}. We have $\widetilde{G}^\vee = \mathrm{GL}_{n^2}(\mathbb{C}) = \mathrm{GL}(V^{\otimes 2})$ and ${}^L\widetilde{G} = \widetilde{G}^\vee \times W_F$.

Let $r_{\mathrm{A}}\colon {}^LM \to {}^L\widetilde{G}$ denote the map defined by

$$r_{\mathrm{A}}(g_1, g_2, b) := \begin{cases} (g_1 \otimes g_2, b) & \text{if } b \in W_E, \\ (g_2 \otimes g_1, b) & \text{if } b \notin W_E, \end{cases} \tag{4.5}$$

where $g_1, g_2 \in \mathrm{GL}_n(\mathbb{C})$ and $a \in W_F$. If $\phi \in \Phi(M, F)$, then $r_{\mathrm{A}} \circ \phi \in \Phi(\mathrm{GL}_{n^2}, F)$.

Lemma 4.6. *We have*

$$\mathrm{dep}(r_{\mathrm{A}} \circ \phi) = \mathrm{dep}(\phi),$$

for any $\phi \in \Phi(M, F)$.

Proof. Let $\phi = (a_\phi, \nu) \in \Phi(M, F)$. From (4.5) we get

$$(r_{\mathrm{A}} \circ \phi)(w) = \begin{cases} (g_1(w) \otimes g_2(w), \nu(w)) & \text{if } \nu(w) \in W_E, \\ (g_2(w) \otimes g_1(w), \nu(w)) & \text{if } \nu(w) \notin W_E. \end{cases}$$

Since $H^1(W_F/W_F^{r+}, M^\vee)$ are the cohomology classes which can be represented by a 1-cocycle whose restriction to W^{r+} is trivial, we have $\alpha_\phi \in H^1(W_F/W_F^{r+}, M^\vee)^{W_F^{r+}}$ if and only if

$$g_1(w) \otimes g_2(w) = \mathrm{I}_n \otimes \mathrm{I}_n \text{ for every } w \in W_F^{r+}, \tag{4.7}$$

where I_n denotes the identity matrix in $\mathrm{GL}_n(\mathbb{C})$.

But (4.7) is satisfied if and only if we have $W_F^{r+} \subset \ker(r_{\mathrm{A}} \circ \phi)$. Then the result follows by Definition 2.11 . □

The *Asai lift* of $\pi_E \in \Pi(\mathrm{GL}_n, E)$ is the representation $\mathrm{As}(\pi_E) \in \Pi(\mathrm{GL}_{n^2}, F)$ with L-parameter $r_{\mathrm{A}} \circ \phi$, where $\phi = (\lambda_M \circ \iota^{-1})(\pi_E)$, that is,

$$\mathrm{As}(\pi_E) := \lambda_{\widetilde{G}}^{-1}(r_{\mathrm{A}} \circ \phi) = (\lambda_{\widetilde{G}}^{-1} \circ r_{\mathrm{A}} \circ \lambda_M \circ \iota^{-1})(\pi_E) = (\lambda_{\widetilde{G}}^{-1} \circ r_{\mathrm{A}} \circ \mathrm{Sh}^{-1} \circ \lambda_G)(\pi_E). \tag{4.8}$$

Theorem 4.9. *Let $\pi_E \in \Pi(\mathrm{GL}_n, E)$. We have*

$$\mathrm{dep}(\mathrm{As}(\pi_E)) = \varphi_{E/F}(\mathrm{dep}(\pi_E)).$$

Proof. Since $\lambda_{\widetilde{G}}$ is depth preserving [ABPS1, Theorem 2.9], it follows from (4.8), that

$$\mathrm{dep}(\mathrm{As}(\pi_E)) = \mathrm{dep}(r_{\mathrm{A}} \circ \mathrm{Sh}^{-1} \circ \lambda_G)(\pi_E).$$

By combining Theorem 2.15 and Lemma 4.6, we obtain

$$\mathrm{dep}(\mathrm{As}(\pi_E)) = \mathrm{dep}((\mathrm{Sh}^{-1} \circ \lambda_G)(\pi_E)) = \varphi_{E/F}(\mathrm{dep}(\lambda_G(\pi_E))).$$

Since λ_G is depth preserving [ABPS1, Theorem 2.9], we have

$$\mathrm{dep}(\mathrm{As}(\pi_E)) = \varphi_{E/F}(\mathrm{dep}(\pi_E)).$$

□

Let $\mathfrak{o}_E$ denote the ring of integers of E, let $\mathfrak{p}_E$ be the maximal ideal of $\mathfrak{o}_E$, and let q_E be the order of $\mathfrak{o}_E/\mathfrak{p}_E$. Let ψ_E be a continuous nontrivial additive character of E and let $c(\psi_E)$ denote the largest integer c such that ψ is trivial on $\mathfrak{p}_E^{-c}$. Let π_E an essentially square-integrable irreducible representation of $\mathrm{GL}_n, E)$. Its Godement-Jacquet local constant $\epsilon(s, \pi_E, \psi_E)$ takes the form

$$\epsilon(s, \pi_E, \psi_E) = \epsilon(0, \pi_E, \psi_E) \cdot q_E^{-f(\pi_E, \psi_E)s},$$

where $s \in \mathbb{C}$ and $\epsilon(0, \pi_E, \psi_E) \in \mathbb{C}^\times$. The integer $f(\pi_E) := f(\pi_E, \psi_E) - nc(\psi_E)$ is called the *conductor* of π_E. We recall that $f(\pi_E) - n$ is the Swan conductor of π_E (see for instance [Bus, § 4.3.2]). We write (as in [Bus, § 5.3.2]):

$$\varsigma(\pi_E) := \frac{f(\pi_E) - n}{n}. \tag{4.10}$$

Corollary 4.11. *We assume $n \geq 2$. For any essentially square-integrable irreducible representation of $\mathrm{GL}_n, E)$, we have*

$$\varsigma(\mathrm{As}(\pi_E)) = \varphi_{E/F}(\varsigma(\pi_E)).$$

Proof. From [ABPS1, Theorem 2.7], we have (since $n \geq 2$)

$$f(\pi_E) = n\,\mathrm{dep}(\pi_E)) + n.$$

It gives

$$\varsigma(\pi_E) = \mathrm{dep}(\pi_E).$$

Similarly, we have

$$\varsigma(\mathrm{As}(\pi_E)) = \mathrm{dep}(\mathrm{As}(\pi_E)).$$

Then the result follows from Theorem 4.9. □

Appendix A. Moy–Prasad filtration of Weil–restricted groups by Jessica Fintzen

Let E/F be a finite Galois extension of non-archimedean local fields with ramification index e. Let G be a connected reductive group defined over E and set $M := \mathfrak{R}_{E/F}\, G$ the Weil restriction of scalars of G for the field extension E/F. We denote by $\iota : G(E) \xrightarrow{\simeq} M(F)$ the isomorphism arising from the defining adjunction property of M.

In this appendix we are going to prove (Proposition A.12) that for every $x \in \mathscr{B}(G, E)$ and $r \in \mathbb{R}_{\geq 0}$, we have

$$\iota(G(E)_{x,er}) = M(F)_{i_{\mathscr{B}}(x),r}, \tag{A.1}$$

where $i_{\mathscr{B}} : \mathscr{B}(G, E) \xrightarrow{\simeq} \mathscr{B}(M, F)$ is an identification of the (reduced) Bruhat–Tits building $\mathscr{B}(G, E)$ of G over E with the (reduced) Bruhat–Tits building $\mathscr{B}(M, F)$ of M over F that we will define in Definition A.11.

We are using the notation from the main part of the paper "Comparison of the depths on both sides of the local Langlands correspondence for Weil-restricted groups", i.e. if H is a (connected) reductive group over a non-archimedean local field K, x a point in the (reduced) Bruhat–Tits building $\mathscr{B}(H, K)$ of H over K, and $r \in \mathbb{R}_{\geq 0}$, then we denote by $H(K)_{x,r}$ the corresponding Moy–Prasad filtration subgroup ([MP1, MP2]), and we write $H(K)_{x,r+}$ for the subgroup $\bigcup_{s>r} H(K)_{x,s}$ of $H(K)$.

We also fix a separable closure F^{sep} of F and view all separable field extensions of F inside F^{sep}. For any finite separable extension K of F, we write K^{ur} for its maximal unramified extension (contained in F^{sep}).

In order to define and prove (A.1), we will first work over maximal unramified extensions and then combine the results with étale descent. We write $\mathbf{G} := G_{E^{\mathrm{ur}}}$ and define $\mathbf{M} := \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{G}$. Note that if E/F is not totally ramified, then $\mathbf{M} \not\simeq M_{F^{\mathrm{ur}}}$.

For a torus T defined over F^{ur}, we denote by T^{ft} the ft-Néron model of T ([CY], see also [Bra]) and by $T^{ft,0}$ the connected component of T^{ft} that contains the identity.

Lemma A.2. *Let T be a torus defined over F^{ur}. Then we have $(\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}T)^{ft,0} \simeq \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(T^{ft,0})$.*

Proof. By [Bra, 3.1.4 Satz] we have $(\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}T)^{ft} \simeq \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(T^{ft})$. Since T^{ft} is smooth and affine, we have by [CGP, Proposition A.5.2(4)] that $\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(T^{ft,0})$ is an open subgroup scheme of $\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(T^{ft}) \simeq (\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}T)^{ft}$, and by [CGP, Proposition A.5.11(3)] the open subgroup scheme $\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(T^{ft,0})$ has geometrically connected fibers, hence it is the identity component of $(\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}T)^{ft}$. □

Let $\mathbf{T}^{\mathbf{G}}$ be a maximally split, maximal torus of $\mathbf{G} = G_{E^{\mathrm{ur}}}$ defined over E^{ur}, and let $\mathbf{T}^{\mathbf{M}} := \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{T}^{\mathbf{G}}$. Then $\mathbf{T}^{\mathbf{M}}$ is a maximally split, maximal torus of $\mathbf{M} = \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{G}$, and by [CGP, Proposition A.5.15] all maximally split, maximal tori of $\mathbf{M}$ arise in this way.

Let $\mathbf{S}^{\mathbf{G}}$ be the maximal split subtorus of $\mathbf{T}^{\mathbf{G}}$ and $\mathbf{S}^{\mathbf{M}}$ the maximal split subtorus of $\mathbf{T}^{\mathbf{M}}$. Then $\mathbf{S}^{\mathbf{M}}$ is contained in $\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{S}^{\mathbf{G}} \subset \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{T}^{\mathbf{G}} = \mathbf{T}^{\mathbf{M}}$. We obtain a

map

$$i_S : X^*(\mathbf{S}^{\mathbf{G}}) \to X^*(\mathbf{S}^{\mathbf{M}})$$

by sending $f \in \mathrm{Hom}_{E^{\mathrm{ur}}}(\mathbf{S}^{\mathbf{G}}, \mathbb{G}_m) = X^*(\mathbf{S}^{\mathbf{G}})$ to the element $i_S(f) \in \mathrm{Hom}_{F^{\mathrm{ur}}}(\mathbf{S}^{\mathbf{M}}, \mathbb{G}_m) = X^*(\mathbf{S}^{\mathbf{M}})$ whose composition with $\mathbb{G}_m \hookrightarrow \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbb{G}_m$ yields $\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(f)|_{\mathbf{S}^{\mathbf{M}}}$. Note that the map i_S is an isomorphism. We use this isomorphism i_S to identify $X^*(\mathbf{S}^{\mathbf{G}})$ and $X^*(\mathbf{S}^{\mathbf{M}})$. Under this identification the restricted root system $\Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ of $\mathbf{G}$ with respect to $\mathbf{S}^{\mathbf{G}}$ gets identified with the restricted root system $\Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$ of $\mathbf{M}$ with respect to $\mathbf{S}^{\mathbf{M}}$.

Let $a \in \Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}}) = \Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$, and let $\mathbf{U}_a^{\mathbf{G}}$ be the corresponding root subgroup of $\mathbf{G}$, i.e., the connected unipotent (closed) subgroup of $\mathbf{G}$ normalized by $\mathbf{S}^{\mathbf{G}}$ whose Lie algebra is the sum of the root spaces corresponding to the roots that are a positive integral multiple of a. Similarly, we denote by $\mathbf{U}_a^{\mathbf{M}}$ the root subgroup of $\mathbf{M}$ corresponding to a. Then we have

$$\mathbf{U}_a^{\mathbf{M}} = \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{U}_a^{\mathbf{G}} \subset \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}\mathbf{G}. \tag{A.3}$$

Let K be a finite Galois extension of F^{ur} containing E^{ur} and such that $\mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}} K$ is split. We fix a Chevalley–Steinberg system

$$\{x_\alpha^K : \mathbb{G}_a \to \mathbf{U}_\alpha^K\}_{\alpha \in \Phi}$$

of $\mathbf{G}$ with respect to $\mathbf{T}^{\mathbf{G}}$, where we write $\Phi := \Phi(\mathbf{G}_K, \mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}} K)$ and $\mathbf{U}_\alpha^K$ denotes the root subgroup of $\mathbf{G}_K$ corresponding to α, see [Fin, §2.1] for the notion of a Chevalley–Steinberg system, which is based on [BT]. Recall that if we write K_α for the fixed subfield of K of the stabilizer $\mathrm{Stab}_{\mathrm{Gal}(K/E^{\mathrm{ur}})}(\alpha)$ of α in $\mathrm{Gal}(K/E^{\mathrm{ur}})$ (for $\alpha \in \Phi$), then x_α^K is by definition of a Chevalley–Steinberg system defined over K_α.

We will now show how this Chevalley–Steinberg system of $\mathbf{G}$ with respect to $\mathbf{T}^{\mathbf{G}}$ yields a Chevalley–Steinberg system of $\mathbf{M}$ with respect to $\mathbf{T}^{\mathbf{M}}$.

First, note that

$$\mathbf{M} \times_{F^{\mathrm{ur}}} K \simeq \prod_{f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)} \mathbf{G} \times_{E^{\mathrm{ur}}, f} K,$$

which contains the split torus

$$\mathbf{T}^{\mathbf{M}} \times_{F^{\mathrm{ur}}} K \simeq \prod_{f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)} \mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}, f} K.$$

For later use, we fix for every $f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$ an element $\widetilde{f} \in \mathrm{Gal}(K/F^{\mathrm{ur}})$ such that $\widetilde{f}|_{E^{\mathrm{ur}}} = f$. We write $id : E^{\mathrm{ur}} \hookrightarrow K$ for the inclusion of E^{ur} into K arising from our convention to view both fields within the same fixed separable closure, and we choose $\widetilde{id}$ to be the identity element in $\mathrm{Gal}(K/F^{\mathrm{ur}})$. Let $\alpha \in \Phi = \Phi(\mathbf{G}_K, \mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}} K)$ and $f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$. Then we write α_f for the root in $\Phi^{\mathbf{M}} := \Phi(\mathbf{M} \times_{F^{\mathrm{ur}}} K, \mathbf{T}^{\mathbf{M}} \times_{F^{\mathrm{ur}}} K)$ obtained by composing the projection

$$\mathbf{T}^{\mathbf{M}} \times_{F^{\mathrm{ur}}} K \simeq \prod_{f' \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)} \mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}, f'} K \twoheadrightarrow \mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}, f} K$$

that sends $\mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}, f'} K$ to the identity for $f' \neq f$, with the composition of the following K-group scheme homomorphisms

$$\mathbf{T}^{\mathbf{G}} \times_{E^{\mathrm{ur}}, f} K \simeq \mathbf{T}_K^{\mathbf{G}} \times_{K, \widetilde{f}} K \xrightarrow{\alpha \times id} \mathbb{G}_m \times_{K, \widetilde{f}} K \xrightarrow{\simeq} \mathbb{G}_m.$$

Note that

$$\Phi^{\mathbf{M}} = \Phi(\mathbf{M} \times_{F^{\mathrm{ur}}} K, \mathbf{T}^{\mathbf{M}} \times_{F^{\mathrm{ur}}} K) = \{\alpha_f \mid \alpha \in \Phi, f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)\}.$$

For $f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$, we write

$$i_f : \mathbf{G} \times_{E^{\mathrm{ur}},f} K \hookrightarrow \prod_{f' \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}},K)} \mathbf{G} \times_{E^{\mathrm{ur}},f'} K \simeq \mathbf{M}_K$$

for the inclusion whose image is the factor corresponding to f, and we define the K-group scheme homomorphism

$$\begin{aligned} x^K_{\alpha_f} \quad : \quad & \mathbb{G}_a \simeq \mathbb{G}_a \times_{K,\widetilde{f}} K \xrightarrow{x_\alpha \times id} \mathbf{U}^K_\alpha \times_{K,\widetilde{f}} K \subset \mathbf{G} \times_{E^{\mathrm{ur}},f} K \\ & \xrightarrow{i_f} \prod_{\mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}},K)} \mathbf{G} \times_{E^{\mathrm{ur}},f'} K \simeq \mathbf{M}_K. \end{aligned}$$

Note that the image of $\mathbf{U}^K_\alpha \times_{K,\widetilde{f}} K$ via i_f in $\mathbf{M}_K$ is the root subgroup $\mathbf{U}^K_{\alpha_f}$ of $\mathbf{M}_K$ attached to the root α_f. Thus $x^K_{\alpha_f}$ factors through the root subgroup $\mathbf{U}^K_{\alpha_f}$.

Lemma A.4. *The set $\{x^K_{\alpha_f} : \mathbb{G}_a \to \mathbf{U}^K_{\alpha_f}\}_{\alpha_f \in \Phi^{\mathbf{M}}}$ forms a Chevalley–Steinberg system of $\mathbf{M}$ with respect to $\mathbf{T}^{\mathbf{M}}$.*

Proof. Let $\alpha_f \in \Phi^{\mathbf{M}}$, i.e. $\alpha \in \Phi$ and $f \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$. For $\gamma \in \mathrm{Gal}(K/F^{\mathrm{ur}})$, we can write $\gamma\widetilde{f} = \widetilde{f'}\gamma_0$ for some $f' \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$ and $\gamma_0 \in \mathrm{Gal}(K/E^{\mathrm{ur}})$. Then we have $\gamma(\alpha_f) = (\gamma_0(\alpha))_{f'}$. Hence the fixed field K_{α_f} of the stabilizer $\mathrm{Stab}_{\mathrm{Gal}(K/F^{\mathrm{ur}})}(\alpha_f)$ of α_f is $\widetilde{f}K_\alpha\widetilde{f}^{-1}$. Since x_α is defined over K_α, we deduce from the construction of $x^K_{\alpha_f}$ that $x^K_{\alpha_f}$ is defined over K_{α_f}. We distinguish two cases.

Case 1: The restriction of α_f to $\Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$ is not divisible. In this case it remains to check that for all $\gamma \in \mathrm{Gal}(K/F^{\mathrm{ur}})$, we have $x^K_{\gamma(\alpha_f)} = \gamma \circ x^K_{\alpha_f} \circ \gamma^{-1}$. Write $\gamma = \widetilde{f'}\gamma_0 \circ \widetilde{f}^{-1}$ for some $f' \in \mathrm{Hom}_{F^{\mathrm{ur}}}(E^{\mathrm{ur}}, K)$ and $\gamma_0 \in \mathrm{Gal}(K/E^{\mathrm{ur}})$. Note that the restriction of α to $\Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ agrees with the restriction of α_f to $\Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$ under the above identification of $\Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ with $\Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$. Thus the restriction of α to $\Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ is non-divisible, and since $\{x^K_\alpha\}_{\alpha \in \Phi}$ form a Chevalley–Steinberg system, we have $x^K_{\gamma_0(\alpha)} = \gamma_0 \circ x^K_\alpha \circ \gamma_0^{-1}$. Thus we obtain

$$\gamma \circ x^K_{\alpha_f} \circ \gamma^{-1} = \widetilde{f'} \circ \gamma_0 \circ x^K_{\alpha_{id}} \circ \gamma_0^{-1}\widetilde{(f')}^{-1} = \widetilde{f'} \circ x^K_{\gamma_0(\alpha)_{id}} \circ (f')^{-1} = x^K_{\gamma_0(\alpha)_{f'}} = x^K_{\gamma(\alpha_f)}.$$

Case 2: The restriction of α_f to $\Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$ is divisible. Hence the restriction of α to $\Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ is divisible and there exist β and β' with $\alpha = \beta + \beta'$, $\beta|_{\mathbf{S}^{\mathbf{G}}} = \beta'|_{\mathbf{S}^{\mathbf{G}}}$, and $K_\beta = K_{\beta'}$ is a quadratic extension of K_α. Hence $\alpha_f = \beta_f + \beta'_f$ and $K_{\beta_f} = \widetilde{f}K_\beta\widetilde{f}^{-1}$ is a quadratic extension of $K_{\alpha_f} = \widetilde{f}K_\alpha\widetilde{f}^{-1}$. It remains to show that for $\gamma \in \mathrm{Gal}(K/K_{\alpha_f})$, we have

$$x^K_{\gamma(\alpha_f)} = \gamma \circ x^K_{\alpha_f} \circ \gamma^{-1} \circ \epsilon, \tag{A.5}$$

where $\epsilon \in \{\pm 1\}$ is 1 if and only if γ induces the identity on K_{β_f}. Note that if we write $\gamma = \widetilde{f}\gamma_0\widetilde{f}^{-1}$ with $\gamma_0 \in \mathrm{Gal}(K/K_\alpha)$, then γ induces the identity on K_{β_f} if and only if γ_0 induces the identity on K_β. Hence the desired identify (A.5) follows from the property $x^K_{\gamma_0(\alpha)} = \gamma_0 \circ x^K_\alpha \circ \gamma_0^{-1} \circ \epsilon$ of the Chevalley–Steinberg system $\{x^K_\alpha\}_{\alpha \in \Phi}$. $\square$

Recall that following [BT] we obtain a parametrization of root groups from our Chevalley–Steinberg systems. More precisely, let $a \in \Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$ and fix $\alpha \in \Phi$ such that $\alpha|_{\mathbf{S}^{\mathbf{G}}} = a$. Recall that x_α is defined over K_α by the properties of a Chevalley–Steinberg system. If a is not multipliable, then

$$x_a := \mathfrak{R}_{K_\alpha/E^{\mathrm{ur}}} x_\alpha^K : \mathfrak{R}_{K_\alpha/E^{\mathrm{ur}}} \mathbb{G}_a \xrightarrow{\simeq} \mathbf{U}_a^{\mathbf{G}}$$

is the parametrization of $\mathbf{U}_a^{\mathbf{G}}$ corresponding to the Chevalley–Steinberg system. If a is multipliable, then let $\widetilde{\alpha} \in \Phi$ such that $\widetilde{\alpha}|_{\mathbf{S}^{\mathbf{G}}} = a$ and $\alpha + \widetilde{\alpha} \in \Phi$. Using $x_\alpha^F, x_{\widetilde{\alpha}}^F$ and $x_{\alpha+\widetilde{\alpha}}^F$, following [BT, 4.1.9] (see also [Fin, Section 2.2] for an exposition) we obtain a parametrization

$$x_a : \mathfrak{R}_{K_{\alpha+\widetilde{\alpha}}/E^{\mathrm{ur}}} H_0(K_\alpha, K_{\alpha+\widetilde{\alpha}}) \xrightarrow{\simeq} \mathbf{U}_a^{\mathbf{G}}$$

of $\mathbf{U}_a^{\mathbf{G}}$, where $H_0(K_\alpha, K_{\alpha+\widetilde{\alpha}})$ is as defined in [BT, 4.1.9] (see also [Fin, Section 2.2]). Composing the inverse of x_a with the valuation on $(\mathfrak{R}_{K_\alpha/E^{\mathrm{ur}}} \mathbf{G}_a)(E^{\mathrm{ur}}) = K_\alpha$ or with a scaling of the valuation on the second factor of $K_\alpha \times K_\alpha \supset (H_0(K_\alpha, K_{\alpha+\widetilde{\alpha}}))(K_{\alpha+\widetilde{\alpha}}) = (\mathfrak{R}_{K_{\alpha+\widetilde{\alpha}}/E^{\mathrm{ur}}} H_0(K_\alpha, K_{\alpha+\widetilde{\alpha}}))(E^{\mathrm{ur}})$ as described in [BT, 4.2.2] (see also [Fin, Section 2.2]), we obtain a valuation

$$\varphi_a : \mathbf{U}_a^{\mathbf{G}}(E^{\mathrm{ur}}) \to \frac{1}{2[K_\alpha : E^{\mathrm{ur}}]} \mathbf{Z} \cup \{\infty\}$$

of $\mathbf{U}_a^{\mathbf{G}}(E^{\mathrm{ur}})$. These valuations $\{\varphi_a\}_{a \in \Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})}$ determine a point x_φ in the apartment $\mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$ corresponding to $\mathbf{S}^{\mathbf{G}}$, and all other points in the apartment correspond to valuations of the form $(\widetilde{\varphi}_a : \mathbf{U}_a^{\mathbf{G}}(E^{\mathrm{ur}}) \to \mathbb{R} \cup \{\infty\})_{a \in \Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})}$ with $\widetilde{\varphi}_a(u) = \varphi_a(u) + a(v)$ for some $v \in X_*(\mathbf{S}^{\mathbf{G}}) \otimes \mathbb{R}$ and for all $u \in \mathbf{U}_a^{\mathbf{G}}(E^{\mathrm{ur}})$, $a \in \Phi(\mathbf{G}, \mathbf{S}^{\mathbf{G}})$.

Similarly, the Chevalley–Steinberg system $\{x_{\alpha_f}^K\}_{\alpha_f \in \Phi^{\mathbf{M}}}$ yields valuations $\{\varphi_a^{\mathbf{M}}\}_{a \in \Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})}$ that determine a point $x_{\varphi^{\mathbf{M}}}$ in the apartment $\mathscr{A}(\mathbf{S}^{\mathbf{M}}, F^{\mathrm{ur}})$ corresponding to $\mathbf{S}^{\mathbf{M}}$, and all other points in the apartment correspond to valuations of the form $(\widetilde{\varphi}_a^{\mathbf{M}} : \mathbf{U}_a^{\mathbf{M}}(F^{\mathrm{ur}}) \to \mathbb{R} \cup \{\infty\})_{a \in \Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})}$ with $\widetilde{\varphi}_a^{\mathbf{M}}(u) = \varphi_a^{\mathbf{M}}(u) + a(v)$ for some $v \in X_*(\mathbf{S}^{\mathbf{M}}) \otimes \mathbb{R}$ and for all $u \in \mathbf{U}_a^{\mathbf{M}}(F^{\mathrm{ur}})$, $a \in \Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$.

Using the identification of $X^*(\mathbf{S}^{\mathbf{G}})$ with $X^*(\mathbf{S}^{\mathbf{M}})$ via i_S and the resulting identification of $X_*(\mathbf{S}^{\mathbf{G}})$ with $X_*(\mathbf{S}^{\mathbf{M}})$, we obtain a bijection

$$i_{\mathscr{A}} : \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}}) \xrightarrow{\simeq} \mathscr{A}(\mathbf{S}^{\mathbf{M}}, F^{\mathrm{ur}})$$

by sending $\varphi_a + a(v)$ to $\varphi_a^{\mathbf{M}} + \frac{1}{e} \cdot a(v)$, where $e = [E^{\mathrm{ur}} : F^{\mathrm{ur}}]$.

Let $\iota^{\mathrm{ur}} : \mathbf{G}(E^{\mathrm{ur}}) \xrightarrow{\simeq} \mathbf{M}(F^{\mathrm{ur}})$ denote the isomorphism arising from the defining adjunction property of $\mathbf{M}$.

Lemma A.6. *Let $x \in \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$ and $r \in \mathbb{R}_{\geq 0}$. Then*

$$\iota^{\mathrm{ur}}(\mathbf{G}(E^{\mathrm{ur}})_{x,er}) = \mathbf{M}(F^{\mathrm{ur}})_{i_{\mathscr{A}}(x),r}.$$

Proof. Let $a \in \Phi(\mathbf{M}, \mathbf{S}^{\mathbf{M}})$, and let $x_a^{\mathbf{M}}$ denote the parametrization of $\mathbf{U}_a^{\mathbf{M}}$ attached to the Chevalley–Steinberg system $\{x_{\alpha_f}^K\}_{\alpha_f \in \Phi^{\mathbf{M}}}$. If a is non-multipliable, let $\alpha_{id} \in \Phi^{\mathbf{M}}$ such that $\alpha_{id}|_{\mathbf{S}^{\mathbf{M}}} = a$. Then

$$x_a^{\mathbf{M}} = \mathfrak{R}_{K_{\alpha_{id}}/F^{\mathrm{ur}}} x_{\alpha_{id}}^K = \mathfrak{R}_{K_\alpha/F^{\mathrm{ur}}} x_{\alpha_{id}}^K = \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(\mathfrak{R}_{K_\alpha/E^{\mathrm{ur}}} x_{\alpha_{id}}^K)$$

and using (A.3) and the definition of $x^K_{\alpha_{id}}$ we obtain that

$$x^{\mathbf{M}}_a = \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}} x_a. \tag{A.7}$$

Similarly we observe that equation (A.7) also holds if a is non-multipliable. Hence

$$\varphi^{\mathbf{M}}_a \circ \iota^{\mathrm{ur}}|_{\mathbf{U}^{\mathbf{G}}_a(E^{\mathrm{ur}})} = \frac{1}{e} \cdot \varphi_a.$$

This implies that the bijection $i_{\mathscr{A}}$ of apartments induces a bijection $e \cdot i^*_{\mathscr{A}}$ between the set of affine roots $\Psi^{\mathbf{M}}_{F^{\mathrm{ur}}}$ of $\mathscr{A}(\mathbf{S}^{\mathbf{M}}, F^{\mathrm{ur}})$ and the affine roots $\Psi^{\mathbf{G}}_{E^{\mathrm{ur}}}$ of $\mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$. Hence we have

$$\iota^{\mathrm{ur}}\left(\left\langle \mathbf{U}^{\mathbf{G}}_\psi, \,|\, \psi \in \Psi^{\mathbf{G}}_{E^{\mathrm{ur}}}, \psi(x) \geq er \right\rangle\right) = \left\langle \mathbf{U}^{\mathbf{M}}_\psi \,|\, \psi \in \Psi^{\mathbf{M}}_{F^{\mathrm{ur}}}, \psi(i_{\mathscr{A}}(x)) \geq r \right\rangle,$$

where $\mathbf{U}^{\mathbf{G}}_\psi = \{u \in \mathbf{U}^{\mathbf{G}}_{\dot\psi}(E^{\mathrm{ur}}) \,|\, \varphi_{\dot\psi}(u) \geq \psi(x_\varphi)\}$ with $\dot\psi$ denoting the gradient of ψ, and similarly for $\mathbf{U}^{\mathbf{M}}_\psi$.

By Lemma A.2 we have $\iota^{\mathrm{ur}}(\mathbf{T}^{\mathbf{G}}_0) = \mathbf{T}^{\mathbf{M}}_0$, and hence $\iota^{\mathrm{ur}}(\mathbf{T}^{\mathbf{G}}_{er}) = \mathbf{T}^{\mathbf{M}}_r$ for $r \in \mathbb{R}_{\geq 0}$. Thus we obtain

$$\begin{aligned} \iota^{\mathrm{ur}}(\mathbf{G}(E^{\mathrm{ur}})_{x,er}) &= \iota^{\mathrm{ur}}\left(\left\langle \mathbf{T}^{\mathbf{G}}_{er}, \mathbf{U}^{\mathbf{G}}_\psi, \,|\, \psi \in \Psi^{\mathbf{G}}_{E^{\mathrm{ur}}}, \psi(x) \geq er \right\rangle\right) \\ &= \left\langle \mathbf{T}^{\mathbf{M}}_r, \mathbf{U}^{\mathbf{M}}_\psi \,|\, \psi \in \Psi^{\mathbf{M}}_{F^{\mathrm{ur}}}, \psi(i_{\mathscr{A}}(x)) \geq r \right\rangle = \mathbf{M}(F^{\mathrm{ur}})_{i_{\mathscr{A}}(x),r} \end{aligned}$$

□

Corollary A.8. *The bijection $i_{\mathscr{A}}$ extends to a bijection $i^{\mathrm{ur}}_{\mathscr{B}} : \mathscr{B}(\mathbf{G}, E^{\mathrm{ur}}) \xrightarrow{\simeq} \mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})$ that is compatible with the action of $\mathbf{G}(E^{\mathrm{ur}}) \xrightarrow{\iota^{\mathrm{ur}} \simeq} \mathbf{M}(F^{\mathrm{ur}})$ and such that for $x \in \mathscr{B}(\mathbf{G}, E^{\mathrm{ur}})$ and $r \in \mathbb{R}_{\geq 0}$ we have*

$$\iota^{\mathrm{ur}}(\mathbf{G}(E^{\mathrm{ur}})_{x,er}) = \mathbf{M}(F^{\mathrm{ur}})_{i^{\mathrm{ur}}_{\mathscr{B}}(x),r}.$$

It was pointed out to us that the isomorphisms between the buildings $\mathscr{B}(\mathbf{G}, E^{\mathrm{ur}})$ and $\mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})$ has already been observed by [HR, Proposition 4.6] without the statement about the comparison of the Moy–Prasad filtration subgroups.

Proof of Corollary A.8. We have a bijection $\iota^{\mathrm{ur}} \times i_{\mathscr{A}} : \mathbf{G}(E^{\mathrm{ur}}) \times \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}}) \to \mathbf{M}(F^{\mathrm{ur}}) \times \mathscr{A}(\mathbf{S}^{\mathbf{M}}, F^{\mathrm{ur}})$ and we will show that it descends to a bijection $i^{\mathrm{ur}}_{\mathscr{B}} : \mathscr{B}(\mathbf{G}, E^{\mathrm{ur}}) \xrightarrow{\simeq} \mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})$. Recall that the equivalence relation on $\mathbf{G}(E^{\mathrm{ur}}) \times \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$ that defines $\mathscr{B}(\mathbf{G}, E^{\mathrm{ur}})$ is given by $(g_1, x_1) \sim (g_2, x_2)$ if and only if there exists $n \in N_{\mathbf{G}}(\mathbf{S}^{\mathbf{G}})(E^{\mathrm{ur}})$ such that $x_2 = n.x_1$ and $g_1^{-1} g_2 n \in \mathbf{G}(E^{\mathrm{ur}})_{x_1,0}$, where $N_{\mathbf{G}}(\mathbf{S}^{\mathbf{G}})$ denotes the normalizer of $\mathbf{S}^{\mathbf{G}}$ in $\mathbf{G}$. We have analogous relations defining $\mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})$. Note that $i^{\mathrm{ur}}(N_{\mathbf{G}}(\mathbf{S}^{\mathbf{G}})(E^{\mathrm{ur}})) = N_{\mathbf{M}}(\mathbf{S}^{\mathbf{M}})(F^{\mathrm{ur}})$ and $\iota^{\mathrm{ur}}(\mathbf{G}(E^{\mathrm{ur}})_{x,0}) = \mathbf{M}(F^{\mathrm{ur}})_{i_{\mathscr{A}}(x),0}$ for $x \in \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$ by Lemma A.6. Thus $i_{\mathscr{A}}$ is equivariant under the action of $(N_{\mathbf{G}}(\mathbf{S}^{\mathbf{G}})(E^{\mathrm{ur}})) \xrightarrow{\simeq} N_{\mathbf{M}}(\mathbf{S}^{\mathbf{M}})(F^{\mathrm{ur}})$, and the equivalence relation on $\mathbf{G}(E^{\mathrm{ur}}) \times \mathscr{A}(\mathbf{S}^{\mathbf{G}}, E^{\mathrm{ur}})$ defining $\mathscr{B}(\mathbf{G}, E^{\mathrm{ur}})$ corresponds under $\iota^{\mathrm{ur}} \times i_{\mathscr{A}}$ to the equivalence relation on $\mathbf{M}(F^{\mathrm{ur}}) \times \mathscr{A}(\mathbf{S}^{\mathbf{M}}, F^{\mathrm{ur}})$ defining $\mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})$. The corollary follows. □

This concludes our study of the Moy–Prasad filtration subgroups over maximal unramified extensions. We will now employ étale descent to obtain the desired results over our local fields E and F. We write E_{ur} for the maximal unramified extension of

F contained in E. Then we have $E \otimes_{E_{\mathrm{ur}}} F^{\mathrm{ur}} = E^{\mathrm{ur}}$ and every $f \in \mathrm{Hom}_F(E_{\mathrm{ur}}, F^{\mathrm{ur}})$ yields an element of $\mathrm{Hom}_F(E, E^{\mathrm{ur}})$ that we also denote by f. Thus we obtain

$$M \times_F F^{\mathrm{ur}} = \mathfrak{R}_{E/F} G \times_F F^{\mathrm{ur}} \simeq \prod_{f \in \mathrm{Hom}_F(E_{\mathrm{ur}}, F^{\mathrm{ur}})} \mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(G \times_{E,f} E^{\mathrm{ur}}). \tag{A.9}$$

Hence

$$\mathscr{B}(M, F) = \mathscr{B}(M \times_F F^{\mathrm{ur}}, F)^{\mathrm{Gal}(F^{\mathrm{ur}}/F)}$$

with

$$\mathscr{B}(M \times_F F^{\mathrm{ur}}, F) = \prod_{f \in \mathrm{Hom}_F(E_{\mathrm{ur}}, F^{\mathrm{ur}})} \mathscr{B}(\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(G \times_{E,f} E^{\mathrm{ur}}), F^{\mathrm{ur}}).$$

By composing the latter product with the projection onto the factor corresponding to $f = 1$, we obtain a bijection

$$i_{\mathscr{B},M,\mathbf{M}} : \mathscr{B}(M, F) \xrightarrow{\simeq} \mathscr{B}(\mathfrak{R}_{E^{\mathrm{ur}}/F^{\mathrm{ur}}}(G \times_E E^{\mathrm{ur}}), F^{\mathrm{ur}})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})} = \mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})}.$$

Similarly, composing Equation (A.9) with the projection onto the factor corresponding to $f = 1$, we obtain an isomorphism

$$\iota_{M,\mathbf{M}} : M(F) \xrightarrow{\simeq} \mathbf{M}(F^{\mathrm{ur}})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})}$$

such that for $x \in \mathscr{B}(M, F)$ and $r \in \mathbb{R}_{\geq 0}$ we have

$$M(F)_{x,r} = ((M \times_F F^{\mathrm{ur}})(F^{\mathrm{ur}})_{x,r})^{\mathrm{Gal}(F^{\mathrm{ur}}/F)} = \iota_{M,\mathbf{M}}^{-1}\left((\mathbf{M}(F^{\mathrm{ur}})_{i_{\mathscr{B},M,\mathbf{M}}(x),r})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})}\right). \tag{A.10}$$

Definition A.11. *We let*

$$i_{\mathscr{B}} : \mathscr{B}(G, E) \xrightarrow{\simeq} \mathscr{B}(M, F)$$

denote the bijection obtained as the composition of the restriction of $i_{\mathscr{B}}^{\mathrm{ur}}$ to $\mathscr{B}(G, E)$:

$$i_{\mathscr{B}}^{\mathrm{ur}} : \mathscr{B}(G, E) = \mathscr{B}(\mathbf{G}, E^{\mathrm{ur}})^{\mathrm{Gal}(E^{\mathrm{ur}}/E)} \xrightarrow{\simeq} \mathscr{B}(\mathbf{M}, F^{\mathrm{ur}})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})}$$

with $i_{\mathscr{B},M,\mathbf{M}}^{-1}$.

Recall that $\iota : G(E) \xrightarrow{\simeq} M(F)$ denotes the isomorphism arising from the defining adjunction property of $M = \mathfrak{R}_{E/F} G$. Then we obtain the following result.

Proposition A.12. *Let $x \in \mathscr{B}(G, E)$ and $r \in \mathbb{R}_{\geq 0}$. Then*

$$\iota(G(E)_{x,er}) = M(F)_{i_{\mathscr{B}}(x),r}.$$

Proof. Combining Corollary A.8 and Equation (A.10) we obtain

$$\begin{aligned}
\iota(G(E)_{x,er}) &= \iota\left((\mathbf{G}(E^{\mathrm{ur}})_{x,er})^{\mathrm{Gal}(E^{\mathrm{ur}}/E)}\right) = \iota_{M,\mathbf{M}}^{-1}\iota^{\mathrm{ur}}\left((\mathbf{G}(E^{\mathrm{ur}})_{x,er})^{\mathrm{Gal}(E^{\mathrm{ur}}/E)}\right) \\
&= \iota_{M,\mathbf{M}}^{-1}\left((\mathbf{M}(F^{\mathrm{ur}})_{i_{\mathscr{B}}^{\mathrm{ur}}(x),r})^{\mathrm{Gal}(F^{\mathrm{ur}}/E_{\mathrm{ur}})}\right) = M(F)_{i_{\mathscr{B},M,\mathbf{M}}^{-1}(i_{\mathscr{B}}^{\mathrm{ur}}(x)),r} \\
&= M(F)_{i_{\mathscr{B}}(x),r}
\end{aligned}$$

□

As an immediate corollary we deduce the following result.

Corollary A.13. *Let (π, V_π) be an irreducible smooth complex representation of $M(F)$. Then*

$$\mathrm{dep}(\iota^*\pi) = e(\mathrm{dep}(\pi)), \tag{A.14}$$

where $\iota^*\pi$ denotes the composition of ι with π and $\mathrm{dep}(\cdot)$ denotes the depth of the corresponding representation.

Proof. This follows from Proposition A.12 and the fact that $i_{\mathscr{B}}$ is a bijection between $\mathscr{B}(G, E)$ and $\mathscr{B}(M, F)$. □

References

[Ar] J. Arthur, A note on L-packets, Pure Appl. Math. Quaterly **2.1** (2006), 199–217.

[ABPS1] A.-M. Aubert, P. Baum, R.J. Plymen, M. Solleveld, Depth and the local Langlands correspondence, Arbeitstagung Bonn 2013, Progress in Math., Birkhäuser 2016, `arxiv.org/abs/1311.1606`

[ABPS2] ———, Conjectures about p-adic groups and their noncommutative geometry, in *Around Langlands correspondences*, 15–51, Contemp. Math. **691**, Amer. Math. Soc., Providence, RI, 2017.

[AMPS] A.-M. Aubert, S. Mendes, R.J. Plymen, M. Solleveld, On L-packets and depth for SL_2 and its inner form, Int. J. Number Theory **13** (2017) 2545–2568.

[AMS] A.-M. Aubert, A. Moussaoui, M. Solleveld, Generalizations of the Springer correspondence and cuspidal Langlands parameters, Manuscripta Math. **157** (2018), no. 1-2, 121–192.

[Bor] A. Borel, Automorphic L-functions, Proc. Symp. Pure Math **33**, part 2 (1979), 27–61.

[BS] A. Borel, J.-P. Serre, Théorèmes de finitude en cohomologie galoisienne, Comment. Math. Helv. **39** (1964), 111–164.

[Bou] N. Bourbaki, *General Topology*, Springer 1989.

[Bra] B. Brahm, *Néron-Modelle algebraischer Tori*, available at https://miami.uni-muenster.de/Record/28d8877b-68e1-4136-9689-8c9bedd16975

[BT] F. Bruhat, J. Tits, Groupes réductifs sur un corps local II: Schémas en groupes. Existence d'une donnée radicielle valuée, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197–376.

[Bus] C. Bushnell, Arithmetic of cuspidal representations, in *Representations of reductive p-adic groups*, 39–126, Progr. Math. **328**, Birkhäuser/Springer, Singapore, 2019.

[BH] C. Bushnell, G. Henniart, Local Langlands correspondence and ramification for Carayol representations, Compos. Math. **155** (2019), no. 10, 1959–2038.

[CY] C.-L. Chai, J.-K. Yu, Congruences of Néron models for tori and the Artin conductor, With an appendix by Ehud de Shalit, Ann. Math. **154** (2001) 347–382.

[CGP] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, Second edition, New Mathematical Monographs **26**, Cambridge University Press, Cambridge, 2015.

[DeB] S. DeBacker, Some applications of Bruhat-Tits theory to harmonic analysis on a reductive p-adic group, Michigan Math. J. **50** (2002), no. 2, 241–261.

[DM] F. Digne, J. Michel, *Representations of Finite Groups of Lie type*, LMS Student Text **21**, 1991.

[FOS] Y. Feng, E. Opdam, M. Solleveld, Supercuspidal unipotent representations: L-packets and formal degrees, arXiv:1805.01888

[Fin] J. Fintzen, On the Moy-Prasad filtration, arXiv:1511.00726v4, to appear in J. Eur. Math. Soc.

[HR] T. Haines, T. Richarz, The test function conjecture for local models of Weil-restricted groups, Compositio Math. **156** (2020) 1348–1404.

[HT] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies **151**, Princeton University Press, 2001.

[He1] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, Invent. Math. **139**, 439–455, 2000.

[He2] ———, Sur la conjecture de Langlands locale pour GL_n, J. Théor. Nombres Bordeaux **13** (2001), no. 1, 167–187.

[HH] G. Henniart, R. Herb, Automorphic induction for $\mathrm{GL}(n)$ (over local non-archimedean fields), Duke Math. J. **78** (1995), 131–192.

[HII] K. Hiraga, A. Ichino, T. Ikeda, Formal degrees and adjoint γ-factors, J. Amer. Math. Soc. **21** (2008), no. 1, 283–304. Correction J. Amer. Math. Soc. **21** (2008), no. 4, 1211–1213.

[Kal] T. Kaletha, The local Langlands conjectures for non-quasi-split groups, in *Families of automorphic forms and the trace formula*, 217–257, Simons Symp., Springer, 2016.

[Lan] R.P. Langlands, On the classification of irreducible representations of real algebraic groups in *Representation theory and harmonic analysis on semisimple Lie groups*, 101–170, Math. Surveys Monogr. **31**, Amer. Math. Soc., Providence, RI, 1989.

[LW] B. Lemaire, J.-L. Waldspurger, Données endoscopiques d'un groupe réductif connexe: applications d'une construction de Langlands, arXiv:1911.03309.

[MiPa] M. Mishra, B. Pattanayak, A note on depth preservation, J. Ramanujan Math. Soc. **34** (2019) 393–400.

[MP1] A. Moy, G. Prasad, Unrefined minimal K-types for p-adic groups, Inv. Math. **116** (1994), 393–408.

[MP2] ———, Jacquet functors and unrefined minimal K-types, Comment. Math. Helv. **71** (1996), 98–121.

[Oes] J. Oesterlé, Nombres de Tamagawa et groupes unipotents en caractéristique p, Invent. Math. **78** (1984), no. 1, 13–88.

[Oi1] M. Oi, Depth preserving property of the local Langlands correspondence for quasi-split classical groups in a large residual characteristic, arXiv:1804.10901v2

[Oi2] ———, Depth preserving property of the local Langlands correspondence for non-quasi-split unitary groups, arXiv:1807.08232, to appear in Math. Res. Lett.

[Sch] P. Scholze, The local Langlands correspondence for GL_n over p-adic fields, Invent. Math. **192** (2013), 663–715.

[Ser] J.-P. Serre, *Local fields*, Springer-Verlag, Berlin 1979.

[Wal] J.-L. Waldspurger, Le lemme fondamental implique le transfert, Compositio Math. **105** (1997), no. 2, 153–236.

[Yu1] J.-K. Yu, On the local Langlands correspondence for tori in *Ottawa lectures on admissible representations of reductive p-adic groups*, Fields Institute Monograph **26** 177–183, Amer. Math. Soc., Providence, RI, 2009.

[Yu2] ———, Smooth models associated to concave functions in Bruhat-Tits theory, in *Autour des schémas en groupes* Vol. III, 227–258, Panor. Synth. **47**, Soc. Math. France, Paris, 2015.

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