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Charaf Ech-Chatbi

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A Short and Simple Proof of the Riemann's Hypothesis

Charaf ECH-CHATBI *

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

*One Raffles Quay, North Tower Level 35. 048583 Singapore. Email: charaf.chatbi@gmail.com. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (3)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)) - i \sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (8)$$

The integral is absolutely convergent. We take the imaginary part of both sides of the above equation and define the function G as following:

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (9)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (10)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (11)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (12)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (13)$$

Before we move forward, we need to calculate the following function $I(a, \sigma, \tau)$ for $a > 0$:

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (14)$$

$$= \int_0^{\ln(a)} \sin(\tau x) e^{(1-\sigma)x} dx \quad (15)$$

$$= K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma - 1) \sin(\tau \ln(a))}{\tau a^{\sigma-1}} \right) \quad (16)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (17)$$

Now, let's write $G(\sigma, \tau)$ as the limit of a sequence as follows:

$$G(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_{e^{-\frac{1}{2\pi\tau N}}}^{e^{\frac{2\pi N}{\tau}}} \frac{\{x\} \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (18)$$

$$= \lim_{N \rightarrow +\infty} \int_{\tau \ln(e^{-\frac{1}{2\pi\tau N}})}^{\tau \ln(e^{\frac{2\pi N}{\tau}})} dx \sin(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (19)$$

$$= \lim_{N \rightarrow +\infty} \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \sin(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (20)$$

$$= \lim_{N \rightarrow +\infty} \int_{-\frac{1}{2\pi N}}^{2\pi N} dx g(x) \sin(x) \quad (21)$$

$$= \lim_{N \rightarrow +\infty} U(N, \sigma) \quad (22)$$

Where

$$U(N, \sigma) = \int_{-\frac{1}{2\pi N}}^{2\pi N} dx g(x) \sin(x) \quad (23)$$

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau} x} \quad (24)$$

$$\epsilon(x) = \{x\} \quad (25)$$

Method One

Let's study the function g over \mathbb{R} . We have the function g piecewise continuous and its derivatives are also piecewise continuous and:

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau} x} \quad (26)$$

$$g'(x) = \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) - \frac{\sigma}{\tau} g(x) \quad (27)$$

And the second derivative defined as follows:

$$g''(x) = \frac{1-2\sigma}{\tau} \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) + \frac{e^{\frac{2-\sigma}{\tau} x}}{\tau^3} \epsilon''(e^{\frac{x}{\tau}}) + \frac{\sigma^2}{\tau^2} g(x) \quad (28)$$

Thanks to the integration by parts we can write:

$$U(N, \sigma) = \int_{-\frac{1}{2\pi N}}^{2\pi N} dx g(x) \sin(x) \quad (29)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) - \int_{-\frac{1}{2\pi N}}^{2\pi N} dx g''(x) \sin(x) \quad (30)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) - \frac{\sigma^2}{\tau^2} \int_{-\frac{1}{2\pi N}}^{2\pi N} dx g(x) \sin(x) \quad (31)$$

$$- \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \left(\frac{1-2\sigma}{\tau} \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) + \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon''(e^{\frac{x}{\tau}}) \right) \sin(x) \quad (32)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) - \frac{\sigma^2}{\tau^2} U(N, \sigma) \quad (33)$$

$$- \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \left(\frac{1-2\sigma}{\tau} \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) + \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon''(e^{\frac{x}{\tau}}) \right) \sin(x) \quad (34)$$

Therefore

$$\left(1 + \frac{\sigma^2}{\tau^2}\right) U(N, \sigma) = g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) \quad (35)$$

$$- \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \left(\frac{1-2\sigma}{\tau} \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) + \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon''(e^{\frac{x}{\tau}}) \right) \sin(x) \quad (36)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) - \frac{1-2\sigma}{\tau} \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) \sin(x) \quad (37)$$

$$- \int_{-\frac{1}{2\pi N}}^{2\pi N} \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon''(e^{\frac{x}{\tau}}) \sin(x) \quad (38)$$

The interval $(0, 2\pi N)$ is the union of the intervals $(\tau \ln(n), \tau \ln(n+1))$ where $1 \leq n \leq n_N$ and the interval $(\tau \ln(n_N + 1), 2\pi N)$ where n_N is the integer part of $\left[e^{\frac{2\pi N}{\tau}} - 1\right]$:

$$n_N = \left[e^{\frac{2\pi N}{\tau}} - 1 \right] \quad (39)$$

We have $\epsilon'(x) = 1$ and $\epsilon''(x) = 0$ for each $x \in (\tau \log(n), \tau \log(n+1))$ for each $1 \leq n \leq n_N + 1$. And for each $x < 0$, we have $\epsilon'(e^{\frac{x}{\tau}}) = 1$ and

$\epsilon H(e^{\frac{x}{\tau}}) = 0$. Therefore, we can write the following:

$$\left(1 + \frac{\sigma^2}{\tau^2}\right)U(N, \sigma) = g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) - \frac{1-2\sigma}{\tau} \sum_{n=1}^{n_N-1} \int_{\tau \ln(n)}^{\tau \ln(n+1)} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) \quad (40)$$

$$- \frac{1-2\sigma}{\tau} \int_{\tau \ln(n_N)}^{2\pi N} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) \quad (41)$$

$$- \sum_{n=1}^{n_N-1} \int_{\tau \ln(n)}^{\tau \ln(n+1)} \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) - \int_{\tau \ln(n_N)}^{2\pi N} \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) \quad (42)$$

$$- \frac{1-2\sigma}{\tau} \int_{-\frac{1}{2\pi N}}^0 dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) - \int_{-\frac{1}{2\pi N}}^0 \frac{e^{\frac{2-\sigma}{\tau}x}}{\tau^3} \epsilon H\left(e^{\frac{x}{\tau}}\right) \sin(x) \quad (43)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) - \frac{1-2\sigma}{\tau^2} \int_{-\frac{1}{2\pi N}}^{2\pi N} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau} \sin(x) \quad (44)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) - \frac{1-2\sigma}{\tau^2} I\left(e^{\frac{2\pi N}{\tau}}, \sigma, \tau\right) \quad (45)$$

$$- \frac{1-2\sigma}{\tau^2} \int_{-\frac{1}{2\pi N}}^0 dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau} \sin(x) \quad (46)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) + \frac{1-2\sigma}{\tau^2} \int_0^{\frac{1}{2\pi N}} dx \frac{e^{-\frac{(1-\sigma)}{\tau}x}}{\tau} \sin(x) \quad (47)$$

$$- \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{\sigma-1}} - \frac{(\sigma-1) \sin(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{\sigma-1}}\right) \quad (48)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) - \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) \quad (49)$$

$$+ \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) e^{\frac{2\pi N}{\tau}(1-\sigma)} + \frac{1-2\sigma}{\tau^2} I\left(e^{\frac{1}{2\pi\tau N}}, 2-\sigma, \tau\right) \quad (50)$$

Therefore

$$\left(1 + \frac{\sigma^2}{\tau^2}\right)U(N, \sigma) = g\left(-\frac{1}{2\pi N}\right)^+ - \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) \quad (51)$$

$$+ \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) e^{\frac{2\pi N}{\tau}(1-\sigma)} + \frac{1-2\sigma}{\tau^2} I\left(e^{\frac{1}{2\pi\tau N}}, 2-\sigma, \tau\right) \quad (52)$$

$$= g\left(-\frac{1}{2\pi N}\right)^+ - \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) - g(2\pi N^-) + g'\left(-\frac{1}{2\pi N}\right)\sin\left(\frac{1}{2\pi N}\right) + \frac{1-2\sigma}{\tau^2} I\left(e^{\frac{1}{2\pi\tau N}}, 2-\sigma, \tau\right) \quad (53)$$

$$+ V(N, \sigma) \quad (54)$$

Where

$$V(N, \sigma) = \frac{1-2\sigma}{\tau^2} K(\sigma, \tau) e^{\frac{2\pi N}{\tau}(1-\sigma)} \quad (55)$$

From the equations (18-22), we have the sequence $(U(N, \sigma))_{N \geq 1}$ is

convergent. We also have:

$$\lim_{N \rightarrow +\infty} g(2\pi N^-) = 0 \quad (56)$$

$$\lim_{N \rightarrow +\infty} g\left(-\frac{1}{2\pi N}\right) \sin\left(\frac{1}{2\pi N}\right) = 0 \quad (57)$$

$$\lim_{N \rightarrow +\infty} g\left(\left(-\frac{1}{2\pi N}\right)^+\right) = g(0^-) = \frac{1}{\tau} \quad (58)$$

$$\lim_{N \rightarrow +\infty} I\left(e^{\frac{1}{2\pi\tau N}}, 2 - \sigma, \tau\right) = 0 \quad (59)$$

Therefore, the sequence $(V(N, \sigma)_{N \geq 1})$ is also convergent.

Since $0 < \sigma < 1$, from the equation (55), the sequence $(V(N, \sigma)_{N \geq 1})$ converges only and only if $\sigma = \frac{1}{2}$.

Therefore

$$\sigma = \frac{1}{2} \quad (60)$$

Hence, the Riemann's Hypothesis is true. \blacksquare

Method Two

Let's define the function $J(a, \sigma, \tau)$ for $a > 0$:

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (61)$$

$$= \int_0^{\ln(a)} \cos(\tau x) e^{(1-\sigma)x} dx \quad (62)$$

$$= K(\sigma, \tau) \left(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau} \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (63)$$

Where $K(\sigma, \tau)$ is defined above in the equation (17).

We will use the same notation as in Method One, with a small modification: we will use $g(x, \sigma)$ instead of $g(x)$. We will also work with a slightly modified $U(N, \sigma)$ as follows:

$$U(N, \sigma) = \int_0^{2\pi N} dx g(x, \sigma) \sin(x) \quad (64)$$

$$= g(0^+, \sigma) - g(2\pi N^-, \sigma) + \int_0^{2\pi N} dx g(x, \sigma) \cos(x) \quad (65)$$

$$= g(0^+, \sigma) - g(2\pi N^-, \sigma) - \frac{\sigma}{\tau} \int_0^{2\pi N} dx g(x, \sigma) \cos(x) + \int_0^{2\pi N} dx \left(\frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} e^{\frac{x}{\tau}} \cos(x) \right) \quad (66)$$

With the same arguments like in the equations (37-44), we can drop $e^{\frac{x}{\tau}}$ and write the following:

$$U(N, \sigma) = g(0^+, \sigma) - g(2\pi N^-, \sigma) - \frac{\sigma}{\tau} \int_0^{2\pi N} dx g(x, \sigma) \cos(x) + \int_0^{2\pi N} dx \frac{e^{\frac{1-\sigma}{\tau}x}}{\tau^2} \cos(x) \quad (67)$$

$$= g(0^+, \sigma) - g(2\pi N^-, \sigma) - \frac{\sigma}{\tau} \int_0^{2\pi N} dx g(x, \sigma) \cos(x) + \frac{1}{\tau} J\left(e^{\frac{2\pi N}{\tau}}, \sigma, \tau\right) \quad (68)$$

We have $g(0^+, \sigma) = g(0^+, 1 - \sigma)$. Therefore, we can write the following:

$$U(N, \sigma) - U(N, 1 - \sigma) = g(2\pi N^-, 1 - \sigma) - g(2\pi N^-, \sigma) + \frac{1 - \sigma}{\tau} \int_0^{2\pi N} dx g(x, 1 - \sigma) \cos(x) - \frac{\sigma}{\tau} \int_0^{2\pi N} dx g(x, \sigma) \cos(x) \quad (69)$$

$$+ \frac{1}{\tau} J(e^{\frac{2\pi N}{\tau}}, \sigma, \tau) - \frac{1}{\tau} J(e^{\frac{2\pi N}{\tau}}, 1 - \sigma, \tau) \quad (70)$$

$$= W(N, \sigma) + \frac{1}{\tau} J(e^{\frac{2\pi N}{\tau}}, \sigma, \tau) - \frac{1}{\tau} J(e^{\frac{2\pi N}{\tau}}, 1 - \sigma, \tau) \quad (71)$$

Where

$$W(N, \sigma) = g(2\pi N^-, 1 - \sigma) - g(2\pi N^-, \sigma) + \frac{1 - \sigma}{\tau} \int_0^{2\pi N} dx g(x, 1 - \sigma) \cos(x) - \frac{\sigma}{\tau} \int_0^{2\pi N} dx g(x, \sigma) \cos(x) \quad (72)$$

Using the equation (53), we can write the following:

$$J(e^{\frac{2\pi N}{\tau}}, \sigma, \tau) - J(e^{\frac{2\pi N}{\tau}}, 1 - \sigma, \tau) = K(\sigma, \tau) \left(\frac{\sigma - 1}{\tau} - \frac{\sigma - 1}{\tau} \frac{\cos(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{\sigma-1}} + \frac{\sin(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{\sigma-1}} \right) \quad (73)$$

$$- K(1 - \sigma, \tau) \left(\frac{-\sigma}{\tau} - \frac{-\sigma}{\tau} \frac{\cos(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{-\sigma}} + \frac{\sin(\tau \ln(e^{\frac{2\pi N}{\tau}}))}{(e^{\frac{2\pi N}{\tau}})^{-\sigma}} \right) \quad (74)$$

$$= K(1 - \sigma, \tau) \frac{\sigma}{\tau} - K(\sigma, \tau) \frac{1 - \sigma}{\tau} + K(\sigma, \tau) \frac{1 - \sigma}{\tau} e^{\frac{2\pi N}{\tau}(1-\sigma)} - K(1 - \sigma, \tau) \frac{\sigma}{\tau} e^{\frac{2\pi N}{\tau}\sigma} \quad (75)$$

Therefore

$$U(N, \sigma) - U(N, 1 - \sigma) = W(N, \sigma) + K(1 - \sigma, \tau) \frac{\sigma}{\tau^2} - K(\sigma, \tau) \frac{1 - \sigma}{\tau^2} + K(\sigma, \tau) \frac{1 - \sigma}{\tau^2} e^{\frac{2\pi N}{\tau}(1-\sigma)} - K(1 - \sigma, \tau) \frac{\sigma}{\tau^2} e^{\frac{2\pi N}{\tau}\sigma} \quad (76)$$

$$= W(N, \sigma) + K(1 - \sigma, \tau) \frac{\sigma}{\tau^2} - K(\sigma, \tau) \frac{1 - \sigma}{\tau^2} + Z(N, \sigma) \quad (77)$$

Where

$$Z(N, \sigma) = K(\sigma, \tau) \frac{1 - \sigma}{\tau^2} e^{\frac{2\pi N}{\tau}(1-\sigma)} - K(1 - \sigma, \tau) \frac{\sigma}{\tau^2} e^{\frac{2\pi N}{\tau}\sigma} \quad (78)$$

From the equations (18-22), we have the sequence $(U(N, \sigma))_{N \geq 1}$ is convergent. And therefore, the sequence $(U(N, \sigma) - U(N, 1 - \sigma))_{N \geq 1}$ is convergent. Also, it is straightforward to see that the sequence $(W(N, \sigma))_{N \geq 1}$ is convergent. Therefore, the sequence $(Z(N, \sigma))_{N \geq 1}$ is convergent. Since $0 < \sigma < 1$, from the equation (78), the sequence $(Z(N, \sigma))_{N \geq 1}$ converges only and only if $\sigma = 1 - \sigma$. Therefore

$$\sigma = \frac{1}{2} \quad (79)$$

Hence, the Riemann's Hypothesis is true. \blacksquare

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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