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A Short and Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

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Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3, 6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2, 3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (3)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)) - i \sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (8)$$

The integral is absolutely convergent. We take the imaginary part of both sides of the above equation and define the function G as following:

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (9)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (10)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (11)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (12)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (13)$$

Before we move forward, we need to calculate the following functions $I(\sigma, \tau)$ and $I(1 - \sigma, \tau)$ for $\sigma > 0$ and $\tau > 0$:

$$I(\sigma, \tau) = \int_1^{+\infty} \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (14)$$

$$= \frac{\tau}{(\sigma)^2 + \tau^2} \quad (15)$$

And

$$I(1 - \sigma, \tau) = \int_1^{+\infty} \frac{\sin(\tau \ln(x))}{x^{2-\sigma}} dx \quad (16)$$

$$= \frac{\tau}{(1 - \sigma)^2 + \tau^2} \quad (17)$$

To continue, we will need to prove the following lemma:

Lemma 1.1. *Let's consider two variables σ and τ such that $\sigma > 0$ and $\tau > 0$. Let's define two integrals I_ϵ and I as follows:*

$$I_\epsilon(\sigma, \tau) = \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x) \quad (18)$$

$$I(\sigma, \tau) = \int_1^{+\infty} dx \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \quad (19)$$

$$\epsilon(x) = \{x\} \quad (20)$$

Therefore, $I_\epsilon \leq I$.

i.e $I_\epsilon(\sigma, \tau) \leq I(\sigma, \tau)$, $\forall \sigma > 0, \forall \tau > 0$.

In the context of this lemma, the variables σ and τ are arbitrary and not necessarily related to the zeros of the zeta function.

Proof. Without loss of generality, let's consider $\sigma > 0, \tau > 0$. The functions $x \rightarrow \frac{\sin(\tau \ln(x))}{x^{1+\sigma}}$ and $x \rightarrow \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} \epsilon(x)$ are integrable and we can write the following:

$$I_\epsilon = \sum_{k=0}^{+\infty} \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} \epsilon(x) \quad (21)$$

$$= \sum_{k=0}^{+\infty} I_\epsilon(k) \quad (22)$$

Where

$$I_\epsilon(k) = \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} \epsilon(x) \quad (23)$$

And

$$I = \sum_{k=0}^{+\infty} \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} \quad (24)$$

$$= \sum_{k=0}^{+\infty} I(k) \quad (25)$$

Where

$$I(k) = \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} \quad (26)$$

And

$$J_\epsilon = I - I_\epsilon \quad (27)$$

$$= \sum_{k=0}^{+\infty} \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} (1 - \epsilon(x)) \quad (28)$$

$$= \sum_{k=0}^{+\infty} J_\epsilon(k) \quad (29)$$

Where

$$J_\epsilon(k) = \int_{e^{\frac{2k\pi}{\tau}}}^{e^{\frac{2(k+1)\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} (1 - \epsilon(x)) \quad (30)$$

To prove the lemma, we will prove that for each integer $k \geq 0$:

$$J_\epsilon(k) \geq 0 \quad (31)$$

We define the function f_k over the interval $[e^{\frac{2k\pi}{\tau}}, +\infty)$ as following:

$$f_k(x) = \int_{e^{\frac{2k\pi}{\tau}}}^x dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} (1 - \epsilon(x)) \quad (32)$$

Let's prove the case of $k = 0$. Exactly the same technique below can be applied to prove that for each $k \geq 0$:

$$J_\epsilon(k) = f_k(e^{\frac{(2k+2)\pi}{\tau}}) \geq 0 \quad (33)$$

Let's now write $J_\epsilon(0)$ as following:

$$J_\epsilon(0) = \int_1^{e^{\frac{2\pi}{\tau}}} dx \frac{\sin(\tau \ln x)}{x^{1+\sigma}} (1 - \epsilon(x)) \quad (34)$$

$$= \int_0^{2\pi} dx \sin(x) \frac{(1 - \epsilon(e^{\frac{x}{\tau}}))}{\tau} e^{-\frac{\sigma}{\tau}x} \quad (35)$$

$$= \int_0^{2\pi} dx \sin(x) g(x) \quad (36)$$

Where the function $g(x)$ defined over $[0, +\infty)$ as follows:

$$g(x) = \frac{(1 - \epsilon(e^{\frac{x}{\tau}}))}{\tau} e^{-\frac{\sigma}{\tau}x} \quad (37)$$

Let's define the primitive function G of g as follows:

$$G(x) = \int_0^x du \frac{(1 - \epsilon(e^{\frac{u}{\tau}}))}{\tau} e^{-\frac{\sigma}{\tau}u} \quad (38)$$

We can write $J_\epsilon(0)$ in function of G as follows:

$$J_\epsilon(0) = \int_0^{2\pi} dx \sin(x) g(x) \quad (39)$$

$$= - \int_0^{2\pi} dx \cos(x) G(x) \quad (40)$$

$$= - \left(\int_0^{\frac{\pi}{2}} dx G(x) \cos(x) + \int_{\frac{\pi}{2}}^{\pi} dx G(x) \cos(x) \right) \quad (41)$$

$$+ \left(\int_{\pi}^{\frac{3\pi}{2}} dx G(x) \cos(x) + \int_{\frac{3\pi}{2}}^{2\pi} dx G(x) \cos(x) \right) \quad (42)$$

$$= - \left(\int_0^{\frac{\pi}{2}} dx G(x) \cos(x) - \int_0^{\frac{\pi}{2}} dx G(\pi - x) \cos(x) \right) \quad (43)$$

$$- \left(\int_0^{\frac{\pi}{2}} dx G(\pi + x) \cos(x) + \int_0^{\frac{\pi}{2}} dx G(2\pi - x) \cos(x) \right) \quad (44)$$

$$= \int_0^{\frac{\pi}{2}} dx \cos(x) \left(G(\pi - x) + G(\pi + x) - G(x) - G(2\pi - x) \right) \quad (45)$$

Let's now study the function G over the interval $[0, +\infty)$. We have the function G continuous and its first derivative is a peicewise continuous function over $[1, +\infty)$ and:

$$G'(x) = g(x) = \frac{(1 - \epsilon(e^{\frac{x}{\tau}}))}{\tau} e^{-\frac{x}{\tau}} \quad (46)$$

And the second derivative defined as follows:

$$G''(x) = -\frac{e^{-\frac{x}{\tau}}}{\tau^2} \left(e^{\frac{x}{\tau}} \epsilon'(e^{\frac{x}{\tau}}) + \sigma(1 - \epsilon(e^{\frac{x}{\tau}})) \right) \quad (47)$$

We have $\epsilon'(x) = 1$ for each $x \in (n, n+1)$ and n a positive integer. And $G''(x) < 0$ for each $x \in (n, n+1)$ and n a positive integer. Therefore, the function G is concave over $(0, +\infty)$.

Now let's take $x \in (0, \frac{\pi}{2})$. We have:

$$0 < x < \frac{\pi}{2} < \pi - x < \pi < \pi + x < \frac{3\pi}{2} < 2\pi - x < 2\pi \quad (48)$$

We can write the following:

$$\pi - x = \frac{\pi}{2\pi - 2x} x + \frac{\pi - 2x}{2\pi - 2x} (2\pi - x) \quad (49)$$

$$\pi + x = \frac{\pi - 2x}{2\pi - 2x} x + \frac{\pi}{2\pi - 2x} (2\pi - x) \quad (50)$$

We use Jensen's inequality thanks to the concavity of the function G and get:

$$G(\pi - x) \geq \frac{\pi}{2\pi - 2x} G(x) + \frac{\pi - 2x}{2\pi - 2x} G(2\pi - x) \quad (51)$$

$$G(\pi + x) \geq \frac{\pi - 2x}{2\pi - 2x} G(x) + \frac{\pi}{2\pi - 2x} G(2\pi - x) \quad (52)$$

Therefore for each $x \in (0, \frac{\pi}{2})$, we have:

$$G(\pi + x) + G(\pi - x) \geq G(x) + G(2\pi - x) \quad (53)$$

Therefore

$$J_\epsilon(0) = \int_0^{\frac{\pi}{2}} dx \cos(x) \left(G(\pi - x) + G(\pi + x) - G(x) - G(2\pi - x) \right) \geq 0 \quad (54)$$

Therefore for each $k \geq 0$: $J_\epsilon(k) \geq 0$. Therefore $J_\epsilon = \sum_{k=0}^{+\infty} J_\epsilon(k) \geq 0$. ■ □

Let's now go back to finish our proof of the Riemann's Hypothesis. The variables σ and τ are here again the real part and the imaginary part of a non-trivial zeta zero as stated in the begining of the proof. i.e $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

From the lemma 1.1 above, we have that

$$I_\epsilon(\sigma, \tau) \leq I(\sigma, \tau) \quad (55)$$

$$I_\epsilon(1 - \sigma, \tau) \leq I(1 - \sigma, \tau) \quad (56)$$

From the equations (9) and (12) we get that:

$$I_\epsilon(\sigma, \tau) = G(\sigma, \tau) \quad (57)$$

$$I_\epsilon(1 - \sigma, \tau) = G(1 - \sigma, \tau) \quad (58)$$

And from the equation (15) and (17) we have:

$$I(\sigma, \tau) = G(1 - \sigma, \tau) \quad (59)$$

$$I(1 - \sigma, \tau) = G(\sigma, \tau) \quad (60)$$

Therefore

$$G(\sigma, \tau) \leq G(1 - \sigma, \tau) \quad (61)$$

$$G(1 - \sigma, \tau) \leq G(\sigma, \tau) \quad (62)$$

Therefore

$$\boxed{G(1 - \sigma, \tau) = G(\sigma, \tau)} \quad (63)$$

Therefore

$$\sigma = \frac{1}{2} \quad (64)$$

Hence, the Riemann's Hypothesis is true. ■

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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