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A Short and Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

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Where $\{x\} = x - [x]$ and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (3)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)) - i \sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (8)$$

The integral is absolutely convergent. We take the imaginary part of both sides of the above equation and define the function G as following:

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (9)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (10)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (11)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (12)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (13)$$

Therefore:

$$\frac{G(\sigma, \tau)}{G(1 - \sigma, \tau)} = \frac{\sigma^2 + \tau^2}{(1 - \sigma)^2 + \tau^2} \quad (14)$$

Before we move forward, we need to calculate the following functions $I(a, \sigma, \tau)$ and $J(a, \sigma, \tau)$ for $a > 0$:

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (15)$$

$$= \int_0^{\ln(a)} \sin(\tau x) e^{(1-\sigma)x} dx \quad (16)$$

$$= K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma - 1) \sin(\tau \ln(a))}{\tau a^{\sigma-1}} \right) \quad (17)$$

And

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (18)$$

$$= \int_0^{\ln(a)} \cos(\tau x) e^{(1-\sigma)x} dx \quad (19)$$

$$= K(\sigma, \tau) \left(\frac{(\sigma - 1)}{\tau} - \frac{(\sigma - 1) \cos(\tau \ln(a))}{\tau a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (20)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (21)$$

Now, let's write $G(\sigma, \tau)$ as the limit of a series as follows:

$$G(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_1^{N+1} \frac{\{x\} \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (22)$$

$$= \lim_{N \rightarrow +\infty} \sum_{n=1}^N \int_n^{n+1} \frac{\{x\} \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (23)$$

$$= \lim_{N \rightarrow +\infty} \sum_{n=1}^N \int_n^{n+1} \frac{(x - n) \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (24)$$

$$= \lim_{N \rightarrow +\infty} \sum_{n=1}^N \int_n^{n+1} \frac{\sin(\tau \ln(x))}{x^\sigma} dx - \sum_{n=1}^N n \int_n^{n+1} \frac{\sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (25)$$

$$= \lim_{N \rightarrow +\infty} \int_1^{N+1} \frac{\sin(\tau \ln(x))}{x^\sigma} dx - \sum_{n=1}^N n \left(I(n+1, \sigma+1, \tau) - I(n, \sigma+1, \tau) \right) \quad (26)$$

$$= \lim_{N \rightarrow +\infty} I(N+1, \sigma, \tau) - \sum_{n=1}^N n \left(I(n+1, \sigma+1, \tau) - I(n, \sigma+1, \tau) \right) \quad (27)$$

$$= \lim_{N \rightarrow +\infty} U(N, \sigma) \quad (28)$$

Where

$$U(N, \sigma) = I(N+1, \sigma, \tau) - \sum_{n=1}^N n \left(I(n+1, \sigma+1, \tau) - I(n, \sigma+1, \tau) \right) \quad (29)$$

Now let's study the series $\sum_{n \geq 1} n(I(n+1, \sigma+1, \tau) - I(n, \sigma+1, \tau))$. For that purpose, we will use the asymptotic expansion. For simplification, let's denote $U_n = I(n, \sigma+1, \tau)$. Let's do now the asymptotic expansion of $U_{n+1} - U_n$. For this we need the asymptotic expansion of $\cos(\tau \ln(n+1))$ and $\sin(\tau \ln(n+1))$.

$$\cos(\tau \ln(n+1)) = \cos\left(\tau \ln(n) + \tau \ln\left(1 + \frac{1}{n}\right)\right) \quad (30)$$

$$= \cos(\tau \ln(n)) \cos\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) - \sin(\tau \ln(n)) \sin\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) \quad (31)$$

we have the asymptotic expansion of $\ln\left(1 + \frac{1}{n}\right)$ in order two as follow:

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (32)$$

Using the asymptotic expansion of the functions \sin and \cos that I will speare you the details here, we have

$$\cos\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) = 1 - \frac{\tau^2}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (33)$$

And

$$\sin\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) = \frac{\tau}{n} - \frac{\tau}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (34)$$

Hence

$$\cos(\tau \ln(n+1)) = \cos(\tau \ln(n)) - \frac{\tau}{n} \sin(\tau \ln(n)) + \frac{\tau}{2n^2} \left(\sin(\tau \ln(n)) - \tau \cos(\tau \ln(n)) \right) + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (35)$$

Also the Asymptotic expansion of $\frac{1}{(1+n)^\sigma}$:

$$\frac{1}{(1+n)^\sigma} = \frac{1}{n^\sigma} - \frac{\sigma}{n^{1+\sigma}} + \frac{\sigma(1+\sigma)}{2n^{2+\sigma}} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (36)$$

Hence

$$\frac{\cos(\tau \ln(n+1))}{(1+n)^\sigma} = \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\tau \sin(\tau \ln(n)) + \sigma \cos(\tau \ln(n))}{n^{\sigma+1}} \quad (37)$$

$$+ \frac{1}{n^{\sigma+2}} \left(\left(\tau\sigma + \frac{\tau}{2} \right) \sin(\tau \ln(n)) + \frac{\sigma(\sigma+1) - \tau^2}{2} \cos(\tau \ln(n)) \right) + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (38)$$

$$= \frac{\cos(\tau \ln(n))}{n^\sigma} + \frac{c_n}{n^{\sigma+1}} + \frac{d_n}{n^{\sigma+2}} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (39)$$

Where

$$c_n = -\left(\tau \sin(\tau \ln(n)) + \sigma \cos(\tau \ln(n)) \right) \quad (40)$$

$$d_n = \left(\tau\sigma + \frac{\tau}{2} \right) \sin(\tau \ln(n)) + \frac{\sigma(\sigma+1) - \tau^2}{2} \cos(\tau \ln(n)) \quad (41)$$

For the asymptotic expansion of $\sin(\tau \ln(n+1))$.

$$\sin(\tau \ln(n+1)) = \sin\left(\tau \ln(n) + \tau \ln\left(1 + \frac{1}{n}\right)\right) \quad (42)$$

$$= \sin(\tau \ln(n)) \cos\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) + \cos(\tau \ln(n)) \sin\left(\tau \ln\left(1 + \frac{1}{n}\right)\right) \quad (43)$$

Using the asymptotic expansion of the functions \sin and \cos that I will spare you the details here, we have

Hence

$$\sin(\tau \ln(n+1)) = \sin(\tau \ln(n)) + \frac{\tau}{n} \cos(\tau \ln(n)) - \frac{\tau}{2n^2} \left(\cos(\tau \ln(n)) + \tau \sin(\tau \ln(n)) \right) + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (44)$$

Hence

$$\frac{\sin(\tau \ln(n+1))}{(1+n)^\sigma} = \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\tau \cos(\tau \ln(n)) - \sigma \sin(\tau \ln(n))}{n^{\sigma+1}} + \mathcal{O}\left(\frac{1}{n^{\sigma+2}}\right) \quad (45)$$

$$+ \frac{1}{n^{\sigma+2}} \left(\frac{\sigma(\sigma+1) - \tau^2}{2} \sin(\tau \ln(n)) - \left(\sigma\tau + \frac{\tau}{2} \right) \cos(\tau \ln(n)) \right) + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (46)$$

$$= \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{cc_n}{n^{\sigma+1}} + \frac{dd_n}{n^{\sigma+2}} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (47)$$

Where

$$cc_n = \tau \cos(\tau \ln(n)) - \sigma \sin(\tau \ln(n)) \quad (48)$$

$$dd_n = \frac{\sigma(\sigma+1) - \tau^2}{2} \sin(\tau \ln(n)) - \left(\tau\sigma + \frac{\tau}{2}\right) \cos(\tau \ln(n)) \quad (49)$$

Now we have:

$$U_n = K(\sigma+1, \tau) \left(1 - \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{(\sigma) \sin(\tau \ln(n))}{\tau n^\sigma}\right) \quad (50)$$

$$U_{n+1} = K(\sigma+1, \tau) \left(1 - \frac{\cos(\tau \ln(n+1))}{(n+1)^\sigma} - \frac{(\sigma) \sin(\tau \ln(n+1))}{\tau (n+1)^\sigma}\right) \quad (51)$$

Therefore

$$U_{n+1} - U_n = -K(\sigma+1, \tau) \left\{ \left(\frac{\cos(\tau \ln(n+1))}{(n+1)^\sigma} - \frac{\cos(\tau \ln(n))}{(n)^\sigma} \right) + \frac{\sigma}{\tau} \left(\frac{\sin(\tau \ln(n+1))}{(n+1)^\sigma} - \frac{\sin(\tau \ln(n))}{(n)^\sigma} \right) \right\} \quad (52)$$

$$= -K(\sigma+1, \tau) \left(\frac{c_n}{n^{1+\sigma}} + \frac{\sigma}{\tau} \frac{cc_n}{n^{1+\sigma}} + \frac{d_n}{n^{2+\sigma}} + \frac{\sigma}{\tau} \frac{dd_n}{n^{2+\sigma}} \right) + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (53)$$

$$= -K(\sigma+1, \tau) \left(-\frac{(\sigma^2 + \tau^2) \sin(\tau \ln(n))}{\tau n^{1+\sigma}} \right) - \frac{\delta_n}{n^{2+\sigma}} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (54)$$

$$= \frac{\sin(\tau \ln(n))}{n^{1+\sigma}} - \frac{\delta_n}{n^{2+\sigma}} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad (55)$$

Where

$$\delta_n = K(\sigma+1, \tau) \left(d_n + \frac{\sigma}{\tau} dd_n \right) \quad (56)$$

$$= \alpha \sin(\tau \ln(n)) + \beta \cos(\tau \ln(n)) \quad (57)$$

Where

$$\alpha = K(\sigma+1, \tau) \left(\frac{\tau(\sigma+1)}{2} + \frac{\sigma^2(\sigma+1)}{2\tau} \right) = \frac{1+\sigma}{2} \quad (58)$$

$$\beta = K(\sigma+1, \tau) \left(-\frac{\tau^2 + \sigma^2}{2} \right) = -\frac{\tau}{2} \quad (59)$$

Therefore

$$n(U_{n+1} - U_n) = \frac{\sin(\tau \ln(n))}{n^\sigma} - \frac{\delta_n}{n^{1+\sigma}} + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (60)$$

By definition of \mathcal{O} , there exist a bounded sequence (ε_n) such that for each $n \geq 1$ we can write:

$$n(U_{n+1} - U_n) = \frac{\sin(\tau \ln(n))}{n^\sigma} - \frac{\delta_n}{n^{1+\sigma}} + \frac{\varepsilon_n}{n^2} \quad (61)$$

Therefore we can write the following:

$$\sum_{n=1}^{N+1} n(U_{n+1} - U_n) = I(N+1, \sigma, \tau) - \sum_{n=1}^N \left(\frac{\delta_n}{n^{1+\sigma}} - \frac{\varepsilon_n}{n^2} \right) \quad (62)$$

$$= I(N+1, \sigma, \tau) - \alpha I(N+1, \sigma+1, \tau) - \beta J(N+1, \sigma+1, \tau) + \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (63)$$

Therefore

$$U(N, \sigma) = I(N+1, \sigma, \tau) - \sum_{n=1}^N n(U_{n+1} - U_n) \quad (64)$$

$$= \alpha I(N+1, \sigma+1, \tau) + \beta J(N+1, \sigma+1, \tau) - \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (65)$$

$$= a + b \frac{\cos(\tau \ln(N+1))}{(N+1)^\sigma} + c \frac{\sin(\tau \ln(N+1))}{(N+1)^\sigma} - \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (66)$$

Where

$$a(\sigma) = K(\sigma+1, \tau) \left(\alpha + \frac{\sigma}{\tau} \beta \right) \quad (67)$$

$$b(\sigma) = -K(\sigma+1, \tau) \left(\alpha + \frac{\sigma}{\tau} \beta \right) \quad (68)$$

$$c(\sigma) = K(\sigma+1, \tau) \left(\beta - \frac{\sigma}{\tau} \alpha \right) \quad (69)$$

$$\alpha + \frac{\sigma}{\tau} \beta = \frac{1}{2} \quad (70)$$

$$\beta - \frac{\sigma}{\tau} \alpha = -\frac{1}{2} \left(\tau + \frac{\sigma(1+\sigma)}{\tau} \right) \quad (71)$$

We have the series $\sum_{n \geq 1} \frac{\varepsilon_n}{n^2}$ converges since the sequences (ε_n) is bounded. Let's denote $\Delta(\sigma)$ such that:

$$\Delta(\sigma) = \lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (72)$$

We also have the following:

$$G(\sigma, \tau) = \lim_{N \rightarrow +\infty} U(N, \sigma) \quad (73)$$

$$= \lim_{N \rightarrow +\infty} a(\sigma) + b(\sigma) \frac{\cos(\tau \ln(N+1))}{(N+1)^\sigma} + c(\sigma) \frac{\sin(\tau \ln(N+1))}{(N+1)^\sigma} - \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (74)$$

$$= a(\sigma) - \Delta(\sigma) \quad (75)$$

Therefore:

$$a(\sigma) = \frac{K(\sigma+1, \tau)}{2} = \frac{G(1-\sigma, \tau)}{2} \quad (76)$$

$$\Delta(\sigma) = a(\sigma) - G(\sigma, \tau) \quad (77)$$

Now from the equation (52) and (61) we can write the following:

$$\frac{\varepsilon_n}{n^2} = n(U_{n+1} - U_n) - \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\delta_n}{n^{1+\sigma}} \quad (78)$$

$$= -K(\sigma + 1, \tau) \left\{ \left(\frac{n \cos(\tau \ln(n+1))}{(n+1)^\sigma} - \frac{\cos(\tau \ln(n))}{(n)^{\sigma-1}} \right) + \frac{\sigma}{\tau} \left(\frac{n \sin(\tau \ln(n+1))}{(n+1)^\sigma} - \frac{\sin(\tau \ln(n))}{(n)^{\sigma-1}} \right) \right\} \quad (79)$$

$$- \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\delta_n}{n^{1+\sigma}} \quad (80)$$

$$= -K(\sigma + 1, \tau) \left\{ \left(\frac{\cos(\tau \ln(n+1))}{(n+1)^{\sigma-1}} - \frac{\cos(\tau \ln(n))}{(n)^{\sigma-1}} - \frac{\cos(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right. \quad (81)$$

$$\left. + \frac{\sigma}{\tau} \left(\frac{\sin(\tau \ln(n+1))}{(n+1)^{\sigma-1}} - \frac{\sin(\tau \ln(n))}{(n)^{\sigma-1}} - \frac{\sin(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right\} - \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\delta_n}{n^{1+\sigma}} \quad (82)$$

$$(83)$$

Therefore

$$\sum_{n=1}^N \frac{\varepsilon_n}{n^2} = -K(\sigma + 1, \tau) \left\{ \sum_{n=1}^N \left(\frac{\cos(\tau \ln(n+1))}{(n+1)^{\sigma-1}} - \frac{\cos(\tau \ln(n))}{(n)^{\sigma-1}} - \frac{\cos(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right. \quad (84)$$

$$\left. + \frac{\sigma}{\tau} \sum_{n=1}^N \left(\frac{\sin(\tau \ln(n+1))}{(n+1)^{\sigma-1}} - \frac{\sin(\tau \ln(n))}{(n)^{\sigma-1}} - \frac{\sin(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right\} - \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \sum_{n=1}^N \frac{\delta_n}{n^{1+\sigma}} \quad (85)$$

$$= -K(\sigma + 1, \tau) \left\{ \left(\frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} - 1 - \sum_{n=1}^N \frac{\cos(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right. \quad (86)$$

$$\left. + \frac{\sigma}{\tau} \left(\frac{\sin(\tau \ln(N+1))}{(N+1)^{\sigma-1}} - 0 - \sum_{n=1}^N \frac{\sin(\tau \ln(n+1))}{(n+1)^\sigma} \right) \right\} - \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \sum_{n=1}^N \frac{\delta_n}{n^{1+\sigma}} \quad (87)$$

$$= -K(\sigma + 1, \tau) \left\{ \left(\frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} - J(N+1, \sigma, \tau) \right) + \frac{\sigma}{\tau} \left(\frac{\sin(\tau \ln(N+1))}{(N+1)^{\sigma-1}} - I(N+1, \sigma, \tau) \right) \right\} \quad (88)$$

$$- I(N+1, \sigma, \tau) + \alpha I(N+1, \sigma+1, \tau) + \beta J(N+1, \sigma+1, \tau) \quad (89)$$

$$= -K(\sigma + 1, \tau) \left\{ \frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} + \frac{\sigma \sin(\tau \ln(N+1))}{\tau (N+1)^{\sigma-1}} \right\} \quad (90)$$

$$+ K(\sigma + 1, \tau) \left(J(N+1, \sigma, \tau) + \frac{\sigma}{\tau} I(N+1, \sigma, \tau) \right) - I(N+1, \sigma, \tau) \quad (91)$$

$$+ \alpha I(N+1, \sigma+1, \tau) + \beta J(N+1, \sigma+1, \tau) \quad (92)$$

Also we have:

$$J(N+1, \sigma, \tau) + \frac{\sigma}{\tau} I(N+1, \sigma, \tau) = K(\sigma, \tau) \left(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau} \frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} + \frac{\sin(\tau \ln(N+1))}{(N+1)^{\sigma-1}} \right) \quad (93)$$

$$+ \frac{\sigma}{\tau} K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} - \frac{(\sigma-1)}{\tau} \frac{\sin(\tau \ln(N+1))}{(N+1)^{\sigma-1}} \right) \quad (94)$$

$$= K(\sigma, \tau) \left(\frac{(2\sigma-1)}{\tau} - \frac{(2\sigma-1)}{\tau} \frac{\cos(\tau \ln(N+1))}{(N+1)^{\sigma-1}} + \left(1 + \frac{\sigma(1-\sigma)}{\tau^2} \right) \frac{\sin(\tau \ln(N+1))}{(N+1)^{\sigma-1}} \right) \quad (95)$$

Therefore

$$K(\sigma + 1, \tau) \left(J(N + 1, \sigma, \tau) + \frac{\sigma}{\tau} I(N + 1, \sigma, \tau) \right) - I(N + 1, \sigma, \tau) \quad (96)$$

$$= K(\sigma + 1, \tau) K(\sigma, \tau) \left(\frac{(2\sigma - 1)}{\tau} - \frac{(2\sigma - 1)}{\tau} \frac{\cos(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} + \left(1 + \frac{\sigma(1 - \sigma)}{\tau^2} \right) \frac{\sin(\tau \ln(N + 1))}{a^{\sigma - 1}} \right) \quad (97)$$

$$- K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} - \frac{(\sigma - 1)}{\tau} \frac{\sin(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} \right) \quad (98)$$

$$= a_1 + b_1 \frac{\cos(\tau \ln(N + 1))}{(N + 1)^\sigma} + c_1 \frac{\sin(\tau \ln(N + 1))}{(N + 1)^\sigma} \quad (99)$$

Where

$$a_1(\sigma) = K(\sigma, \tau) \left(K(\sigma + 1, \tau) \frac{(2\sigma - 1)}{\tau} - 1 \right) \quad (100)$$

$$b_1(\sigma) = -K(\sigma, \tau) \left(K(\sigma + 1, \tau) \frac{(2\sigma - 1)}{\tau} - 1 \right) \quad (101)$$

$$c_1(\sigma) = K(\sigma, \tau) \left(K(\sigma + 1, \tau) \left(1 + \frac{\sigma(1 - \sigma)}{\tau^2} \right) - \frac{(1 - \sigma)}{\tau} \right) \quad (102)$$

Therefore

$$\sum_{n=1}^N \frac{\varepsilon_n}{n^2} = a_1 + b_1 \frac{\cos(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} + c_1 \frac{\sin(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} \quad (103)$$

$$- K(\sigma + 1, \tau) \left\{ \frac{\cos(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} + \frac{\sigma}{\tau} \frac{\sin(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} \right\} \quad (104)$$

$$+ \alpha I(N + 1, \sigma + 1, \tau) + \beta J(N + 1, \sigma + 1, \tau) \quad (105)$$

$$= b_2 \frac{\cos(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} + c_2 \frac{\sin(\tau \ln(N + 1))}{(N + 1)^{\sigma - 1}} \quad (106)$$

$$+ a_1 + a + b \frac{\cos(\tau \ln(N + 1))}{(N + 1)^\sigma} + c \frac{\sin(\tau \ln(N + 1))}{(N + 1)^\sigma} \quad (107)$$

Where

$$b_2(\sigma) = b_1(\sigma) - K(\sigma + 1, \tau) \quad (108)$$

$$c_2(\sigma) = c_1(\sigma) - \frac{\sigma}{\tau} K(\sigma + 1, \tau) \quad (109)$$

And a_1 and a defined previously. We have $0 < \sigma < 1$ and the series $\sum_{n \geq 1} \frac{\varepsilon_n}{n^2}$ converge. Therefore b_2 and c_2 must be equal zero. In fact, we have $b_2(\sigma) = c_2(\sigma) = 0$.

Therefore

$$\Delta(\sigma) = \lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\varepsilon_n}{n^2} \quad (110)$$

$$= a(\sigma) + a_1(\sigma) = a(\sigma) - b_1(\sigma) \quad (111)$$

$$= a(\sigma) - K(\sigma + 1, \tau) = a(\sigma) - G(1 - \sigma, \tau) \quad (112)$$

By combining the equations (77) and (112) we get that:

$$\Delta(\sigma) = a(\sigma) - G(1 - \sigma, \tau) = a(\sigma) - G(\sigma, \tau) \quad (113)$$

Therefore

$$G(\sigma, \tau) = G(1 - \sigma, \tau) \quad (114)$$

Therefore:

$$\sigma = \frac{1}{2} \quad (115)$$

Hence, the Riemann's Hypothesis is true. ■

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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