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A Short and Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

1 The Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

Where $\{x\} = x - [x]$ and $[x]$ is the integer part of x . Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

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Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (4)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[3] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard[3] and De la Vallée Poussin[3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

3 Proof of the Riemann Hypothesis

Let’s take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let’s suppose that $0 < \sigma < 1, \tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann’s integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (5)$$

We have $s \neq 1, s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (6)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (7)$$

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the literature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (8)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{\left(\cos(\tau \ln(x)) - i \sin(\tau \ln(x))\right)\{x\}}{x^{\sigma+1}} dx \quad (8)$$

The integral is absolutely convergent. We take the real part of both sides of the above equation and define the functions F and G as following:

$$F(\sigma, \tau) = \frac{\sigma - 1}{(\sigma - 1)^2 + \tau^2} \quad (9)$$

$$= \int_1^{+\infty} \frac{\left(\cos(\tau \ln(x))\right)\{x\}}{x^{\sigma+1}} dx \quad (10)$$

And

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (11)$$

$$= \int_1^{+\infty} \frac{\left(\sin(\tau \ln(x))\right)\{x\}}{x^{\sigma+1}} dx \quad (12)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$F(1 - \sigma, \tau) = \frac{\sigma_1 - 1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (13)$$

$$= \frac{-\sigma}{\sigma^2 + \tau^2} \quad (14)$$

$$= \int_1^{+\infty} \frac{\left(\cos(\tau \ln(x))\right)\{x\}}{x^{2-\sigma}} dx \quad (15)$$

And

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (16)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (17)$$

$$= \int_1^{+\infty} \frac{\left(\sin(\tau \ln(x))\right)\{x\}}{x^{2-\sigma}} dx \quad (18)$$

Therefore:

$$\frac{F(\sigma, \tau)}{F(1 - \sigma, \tau)} = \frac{1 - \sigma}{\sigma} \frac{\sigma^2 + \tau^2}{(1 - \sigma)^2 + \tau^2} \quad (19)$$

And

$$\frac{G(\sigma, \tau)}{G(1 - \sigma, \tau)} = \frac{\sigma^2 + \tau^2}{(1 - \sigma)^2 + \tau^2} \quad (20)$$

By combining the equations (19) and (20) we get:

$$\boxed{\frac{1-\sigma}{\sigma} = \frac{F(\sigma, \tau) G(1-\sigma, \tau)}{G(\sigma, \tau) F(1-\sigma, \tau)}} \quad (21)$$

We apply Fubini theorem to rewrite $F(\sigma, \tau)G(1-\sigma, \tau)$. The conditions for Fubini theorem are present since the functions involved are peicewise continuous and absolutely integrable ($1 + \sigma > 1$ and $2 - \sigma > 1$):

$$\int_1^{+\infty} \int_1^{+\infty} dx dy \left| \frac{(\cos(\tau \ln(x)))\{x\}}{x^{1+\sigma}} \frac{(\sin(\tau \ln(y)))\{y\}}{y^{2-\sigma}} \right| \leq \int_1^{+\infty} \int_1^{+\infty} \frac{dx}{x^{1+\sigma}} \frac{dy}{y^{2-\sigma}} \quad (22)$$

$$= \frac{1}{\sigma(1-\sigma)} \quad (23)$$

$$< +\infty \quad (24)$$

So we have:

$$F(\sigma, \tau)G(1-\sigma, \tau) = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)))\{x\}}{x^{1+\sigma}} dx \int_1^{+\infty} \frac{(\sin(\tau \ln(y)))\{y\}}{y^{2-\sigma}} dy \quad (25)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{(\cos(\tau \ln(x)))\{x\}}{x^{1+\sigma}} \frac{(\sin(\tau \ln(y)))\{y\}}{y^{2-\sigma}} \quad (26)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\cos(\tau \ln(x)) \sin(\tau \ln(y))}{x^{1+\sigma} y^{2-\sigma}} \{x\}\{y\} \quad (27)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy)) - \sin(\tau \ln(x/y))}{2 x^{1+\sigma} y^{2-\sigma}} \{x\}\{y\} \quad (28)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy)) - \sin(\tau \ln(x/y))}{2 (xy)^{1+\sigma} y^{1-2\sigma}} \{x\}\{y\} \quad (29)$$

Therefore:

$$\boxed{F(\sigma, \tau)G(1-\sigma, \tau) = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy)) - \sin(\tau \ln(x/y))}{2 (xy)^{1+\sigma} y^{1-2\sigma}} \{x\}\{y\}} \quad (30)$$

And the same for $G(\sigma, \tau)F(1-\sigma, \tau)$. So we have:

$$G(\sigma, \tau)F(1-\sigma, \tau) = \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{1+\sigma}} dx \int_1^{+\infty} \frac{(\cos(\tau \ln(y)))\{y\}}{y^{2-\sigma}} dy \quad (31)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{(\sin(\tau \ln(x)))\{x\}}{x^{1+\sigma}} \frac{(\cos(\tau \ln(y)))\{y\}}{y^{2-\sigma}} \quad (32)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x)) \cos(\tau \ln(y))}{x^{1+\sigma} y^{2-\sigma}} \{x\}\{y\} \quad (33)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy)) - \sin(\tau \ln(y/x))}{2 x^{1+\sigma} y^{2-\sigma}} \{x\}\{y\} \quad (34)$$

We do a change of variables and we get:

$$\boxed{G(\sigma, \tau)F(1 - \sigma, \tau) = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy)) - \sin(\tau \ln(x/y))}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\}\{y\}} \quad (35)$$

By a change of variables we also have the following:

$$\boxed{\int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(xy))}{2(xy)^{1+\sigma} y^{1-2\sigma}} \{x\}\{y\} = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x/y))}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\}\{y\} = C} \quad (36)$$

And we denote this last term as C .

Let's now focus on the terms A and B where:

$$A = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x/y))}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\}\{y\} \quad (37)$$

$$B = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x/y))}{2(xy)^{1+\sigma} y^{1-2\sigma}} \{x\}\{y\} \quad (38)$$

By a change of variables we have:

$$\boxed{A = -B} \quad (39)$$

Let's now summarize. We have so far the following relations:

$$\boxed{F(\sigma, \tau)G(1 - \sigma, \tau) = C + A} \quad (40)$$

$$\boxed{G(\sigma, \tau)F(1 - \sigma, \tau) = C - A} \quad (41)$$

Remark. At this point, one hopes that $A = 0$. Because if such is the case, we will have $\frac{1-\sigma}{\sigma} = 1$ and then $\sigma = \frac{1}{2}$. Well it sounds too good to be true but it is indeed true. And it is the subject of the following lemma.

Lemma 3.1. Let's consider the function of two variables σ and τ that corresponds to the term $2 \times A$ defined above:

$$u(\sigma, \tau) = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x/y))}{(xy)^{1+\sigma} x^{1-2\sigma}} \{x\}\{y\} \quad (42)$$

Therefore, the function $u(\sigma, \tau)$ is a null function over $(0, 1) \times (0, +\infty)$.

i.e $u(\sigma, \tau) = 0, \quad \forall \sigma \in (0, 1), \forall \tau \in (0, +\infty)$.

In the context of this lemma, the variables σ and τ are arbitrary and not necessarily related to the zeros of the zeta function.

Proof. Without loss of generality, let's consider $0 < \sigma < 1, \tau > 0$. We use the Euler's formula for the sinus $\left(\sin(x) = \frac{e^{ix} - e^{-ix}}{2}\right)$ and Fubini, like in the case above of $F(\sigma, \tau)G(1 - \sigma, \tau)$, to write $u(\sigma, \tau)$ as following:

$$u(\sigma, \tau) = \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{\sin(\tau \ln(x/y))}{(xy)^{1+\sigma} x^{1-2\sigma}} \{x\} \{y\} \quad (43)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{e^{i(\tau \ln(x/y))} - e^{-i(\tau \ln(x/y))}}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\} \{y\} \quad (44)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{e^{i(\tau \ln(x))} e^{-i(\tau \ln(y))} - e^{-i(\tau \ln(x))} e^{i(\tau \ln(y))}}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\} \{y\} \quad (45)$$

$$= \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{e^{i(\tau \ln(x))} e^{-i(\tau \ln(y))}}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\} \{y\} - \int_1^{+\infty} \int_1^{+\infty} dx dy \frac{e^{-i(\tau \ln(x))} e^{i(\tau \ln(y))}}{2(xy)^{1+\sigma} x^{1-2\sigma}} \{x\} \{y\} \quad (46)$$

$$= \frac{1}{2} \int_1^{+\infty} dx \frac{e^{i(\tau \ln(x))}}{x^{2-\sigma}} \{x\} \int_1^{+\infty} dy \frac{e^{-i(\tau \ln(y))}}{y^{1+\sigma}} \{y\} - \frac{1}{2} \int_1^{+\infty} dx \frac{e^{-i(\tau \ln(x))}}{x^{2-\sigma}} \{x\} \int_1^{+\infty} dy \frac{e^{i(\tau \ln(y))}}{y^{1+\sigma}} \{y\} \quad (47)$$

$$= \frac{1}{2} E(1-\sigma, \tau) E(\sigma, -\tau) - \frac{1}{2} E(1-\sigma, -\tau) E(\sigma, \tau) \quad (48)$$

Where the function $E(\sigma, \tau)$ is defined as the following:

$$E(\sigma, \tau) = \int_1^{+\infty} dx \frac{e^{i(\tau \ln(x))}}{x^{1+\sigma}} \{x\} \quad (49)$$

We note that the function $E(\sigma, \tau)$ is a complex function and the complex conjugate of $E(\sigma, \tau)$ is actually $E(\sigma, -\tau)$:

$$\overline{E(\sigma, \tau)} = E(\sigma, -\tau) \quad (50)$$

Therefore:

$$2u(\sigma, \tau) = E(1-\sigma, \tau) E(\sigma, -\tau) - E(1-\sigma, -\tau) E(\sigma, \tau) \quad (51)$$

$$= E(1-\sigma, \tau) \overline{E(\sigma, \tau)} - \overline{E(1-\sigma, \tau)} E(\sigma, \tau) \quad (52)$$

Let's now calculate the complex conjugate of $u(\sigma, \tau)$:

$$\overline{2u(\sigma, \tau)} = \overline{E(1-\sigma, \tau) \overline{E(\sigma, \tau)} - \overline{E(1-\sigma, \tau)} E(\sigma, \tau)} \quad (53)$$

$$= \overline{E(1-\sigma, \tau)} E(\sigma, \tau) - E(1-\sigma, \tau) \overline{E(\sigma, \tau)} \quad (54)$$

$$= -2u(\sigma, \tau) \quad (55)$$

Therefore

$$\Re(u(\sigma, \tau)) = \frac{u(\sigma, \tau) + \overline{u(\sigma, \tau)}}{2} = 0 \quad (56)$$

But the function $u(\sigma, \tau)$ is a real function by its definition.
Therefore

$$u(\sigma, \tau) = \Re(u(\sigma, \tau)) = 0 \quad (57)$$

■

Therefore $u(\sigma, \tau)$ is a null function.

□

Let's now go back to finish our proof of the Riemann's Hypothesis. The variables σ and τ are here again the real part and the imaginary part of a non-trivial zeta zero as stated in the beginning of the proof. i.e $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

From the lemma 3.1 above, we have that

$$\boxed{A = \frac{u(\sigma, \tau)}{2} = 0} \quad (58)$$

Therefore from the equation (40) and (41) we have

$$\boxed{F(\sigma, \tau)G(1 - \sigma, \tau) = C = G(\sigma, \tau)F(1 - \sigma, \tau)} \quad (59)$$

Since $C \neq 0$, from the equation (21) we conclude that:

$$\frac{1 - \sigma}{\sigma} = \frac{F(\sigma, \tau) G(1 - \sigma, \tau)}{G(\sigma, \tau) F(1 - \sigma, \tau)} \quad (60)$$

$$= 1 \quad (61)$$

Therefore

$$\sigma = \frac{1}{2} \quad (62)$$

Hence, the Riemann's Hypothesis is true. ■

4 Conclusion

We saw that if s is a non-trivial zero of the zeta function, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.*

We also applied in [5] the same method used here to prove the Generalized Riemann Hypothesis.

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