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A Short and Simple Proof of the Riemann's Hypothesis

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Abstract

We present a short and simple proof of the Riemann's Hypothesis (RH) where only undergraduate mathematics is needed.

Keywords: Riemann Hypothesis; Zeta function; Prime Numbers; Millennium Problems.

MSC2020 Classification: 11Mxx, 11-XX, 26-XX, 30-xx.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part $1/2$.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (1)$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at $s = 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (2)$$

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Where $\{x\} = x - [x]$ is the fractional part and $[x]$ is the integer part of x . There is another way [2] to analytically continue $\zeta(s)$ to the region $\Re(s) > 0$. The idea is to observe that for $\Re(s) > 1$:

$$\left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} - \sum_{n=1}^{+\infty} \frac{2}{(2n)^s} \quad (3)$$

Thus,

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \quad (4)$$

$$= \left(1 - \frac{2}{2^s}\right)^{-1} \eta(s) \quad (5)$$

The Dirichlet eta function $\eta(s)$ converges conditionally when $\Re(s) > 0$ and $s \neq 1 + i\frac{2k\pi}{\ln(2)}$. $\eta(s)$ is used as analytical continuation of the Zeta function on the domain where $\Re(s) > 0$. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation¹

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (7)$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[2,3,6] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard and De la Vallée Poussin[2,3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[2,3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0, \forall \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k, k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s, \bar{s}$ and $1 - \bar{s}$ are also zeros of ζ : i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the ‘‘Riemann Hypothesis’’ (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

¹This is slightly different from the functional equation presented in Riemann’s paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (6)$$

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (8)$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \quad (9)$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}} dx \quad (10)$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma - 1)^2 + \tau^2} = \int_1^{+\infty} \frac{(\cos(\tau \ln(x)) - i \sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (11)$$

The integral is absolutely convergent. We take the real part and the imaginary part of both sides of the above equation and define the functions F and G as following:

$$F(\sigma, \tau) = \frac{\sigma - 1}{(\sigma - 1)^2 + \tau^2} \quad (12)$$

$$= \int_1^{+\infty} \frac{(\cos(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (13)$$

And

$$G(\sigma, \tau) = \frac{\tau}{(\sigma - 1)^2 + \tau^2} \quad (14)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{\sigma+1}} dx \quad (15)$$

We also have $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$F(1 - \sigma, \tau) = \frac{\sigma_1 - 1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (16)$$

$$= \frac{-\sigma}{\sigma^2 + \tau^2} \quad (17)$$

$$= \int_1^{+\infty} \frac{(\cos(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (18)$$

And

$$G(1 - \sigma, \tau) = \frac{\tau_1}{(\sigma_1 - 1)^2 + \tau_1^2} \quad (19)$$

$$= \frac{\tau}{\sigma^2 + \tau^2} \quad (20)$$

$$= \int_1^{+\infty} \frac{(\sin(\tau \ln(x)))\{x\}}{x^{2-\sigma}} dx \quad (21)$$

Before we move forward, we need to define the following function $I(a, \sigma, \tau)$ for $a > 0$:

$$I(a, \sigma, \tau) = \int_1^a \frac{\sin(\tau \ln(x))}{x^\sigma} dx \quad (22)$$

$$= \int_0^{\ln(a)} \sin(\tau x) e^{(1-\sigma)x} dx \quad (23)$$

$$= K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} - \frac{(\sigma-1)}{\tau} \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (24)$$

Where

$$K(\sigma, \tau) = \frac{\tau}{(\sigma-1)^2 + \tau^2} \quad (25)$$

Let's define the function $J(a, \sigma, \tau)$ for $a > 0$:

$$J(a, \sigma, \tau) = \int_1^a \frac{\cos(\tau \ln(x))}{x^\sigma} dx \quad (26)$$

$$= \int_0^{\ln(a)} \cos(\tau x) e^{(1-\sigma)x} dx \quad (27)$$

$$= K(\sigma, \tau) \left(\frac{(\sigma-1)}{\tau} - \frac{(\sigma-1)}{\tau} \frac{\cos(\tau \ln(a))}{a^{\sigma-1}} + \frac{\sin(\tau \ln(a))}{a^{\sigma-1}} \right) \quad (28)$$

Where $K(\sigma, \tau)$ is defined above in the equation (25).

Now, let's write $G(\sigma, \tau)$ as the limit of a sequence as follows:

$$G(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_1^N \frac{\{x\} \sin(\tau \ln(x))}{x^{1+\sigma}} dx \quad (29)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx \sin(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (30)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (31)$$

$$= \lim_{N \rightarrow +\infty} U(N, \sigma) \quad (32)$$

Where

$$U(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (33)$$

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau} x} \quad (34)$$

$$\epsilon(x) = \{x\} \quad (35)$$

And the same for $F(\sigma, \tau)$:

$$F(\sigma, \tau) = \lim_{N \rightarrow +\infty} \int_1^N \frac{\{x\} \cos(\tau \ln(x))}{x^{1+\sigma}} dx \quad (36)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx \cos(x) \frac{\epsilon(e^{\frac{x}{\tau}}) e^{-\frac{\sigma}{\tau} x}}{\tau} \quad (37)$$

$$= \lim_{N \rightarrow +\infty} \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (38)$$

$$= \lim_{N \rightarrow +\infty} V(N, \sigma) \quad (39)$$

Where

$$V(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (40)$$

Let's study the function g over \mathbb{R} . We have the function g piecewise continuous and its derivatives are also piecewise continuous and:

$$g(x) = \frac{\epsilon(e^{\frac{x}{\tau}})}{\tau} e^{-\frac{\sigma}{\tau} x} \quad (41)$$

$$g'(x) = \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) - \frac{\sigma}{\tau} g(x) \quad (42)$$

The interval $(0, \tau \ln(N))$ is the union of the intervals $(\tau \ln(n), \tau \ln(n+1))$ where $1 \leq n < N$. We have $\epsilon'(x) = 1$ for each $x \in (\tau \ln(n), \tau \ln(n+1))$ for each $1 \leq n < N$. Therefore, thanks to the integration by parts we can write:

$$U(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (43)$$

$$= g(0^+) \cos(0) + \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{N-1} [g(\tau \ln(n) + \epsilon) \cos(\tau \ln(n) + \epsilon) - g(\tau \ln(n) - \epsilon) \cos(\tau \ln(n) - \epsilon)] \quad (44)$$

$$-g(\tau \ln(N)^-) \cos(\tau \ln(N)) + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (45)$$

$$= \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{N-1} [g(\tau \ln(n) + \epsilon) - g(\tau \ln(n) - \epsilon)] \cos(\tau \ln(n)) \quad (46)$$

$$- \frac{\cos(\tau \ln(N))}{\tau N^\sigma} + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (47)$$

$$= -\frac{1}{\tau} \sum_{n=1}^{N-1} \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\cos(\tau \ln(N))}{\tau N^\sigma} + \int_0^{\tau \ln(N)} dx g'(x) \cos(x) \quad (48)$$

$$= -\frac{1}{\tau} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\sigma}{\tau} \int_0^{\tau \ln(N)} dx g(x) \cos(x) + \int_0^{\tau \ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \epsilon'(e^{\frac{x}{\tau}}) \right) \cos(x) \quad (49)$$

$$= -\frac{1}{\tau} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - \frac{\sigma}{\tau} V(N, \sigma) + \int_0^{\tau \ln(N)} dx \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \cos(x) \quad (50)$$

Therefore

$$U(N, \sigma) + \frac{\sigma}{\tau} V(N, \sigma) = -\frac{1}{\tau} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} + \frac{1}{\tau} J(N, \sigma, \tau) \quad (51)$$

Therefore

$$\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) = -\tau U(N, \sigma) - \sigma V(N, \sigma) \quad (52)$$

Therefore, the sequence $\left(\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - J(N, \sigma, \tau) = -\frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} \quad (53)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} = J(N, \sigma, \tau) - \frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} + \alpha_N \quad (54)$$

Where (α_N) is a bounded sequence with limit zero.

And the same for $V(N, \sigma)$ as follows:

$$V(N, \sigma) = \int_0^{\tau \ln(N)} dx g(x) \cos(x) \quad (55)$$

$$= -g(0^+) \sin(0) + \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{N-1} \left[-g(\tau \ln(n) + \epsilon) \sin(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \sin(\tau \ln(n) - \epsilon) \right] \quad (56)$$

$$+ g(\tau \ln(N)^-) \sin(\tau \ln(N)) - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (57)$$

$$= \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{N-1} \left[-g(\tau \ln(n) + \epsilon) + g(\tau \ln(n) - \epsilon) \right] \sin(\tau \ln(n)) \quad (58)$$

$$+ \frac{\sin(\tau \ln(N))}{\tau N^\sigma} - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (59)$$

$$= \frac{1}{\tau} \sum_{n=1}^{N-1} \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sin(\tau \ln(N))}{\tau N^\sigma} - \int_0^{\tau \ln(N)} dx g(x) \sin(x) \quad (60)$$

$$= \frac{1}{\tau} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sigma}{\tau} \int_0^{\tau \ln(N)} dx g(x) \sin(x) - \int_0^{\tau \ln(N)} dx \left(\frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} e^{\frac{x}{\tau}} \right) \sin(x) \quad (61)$$

$$= \frac{1}{\tau} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} + \frac{\sigma}{\tau} U(N, \sigma) - \int_0^{\tau \ln(N)} dx \frac{e^{\frac{1-\sigma}{\tau} x}}{\tau^2} \sin(x) \quad (62)$$

Therefore

$$V(N, \sigma) - \frac{\sigma}{\tau} U(N, \sigma) = \frac{1}{\tau} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - \frac{1}{\tau} I(N, \sigma, \tau) \quad (63)$$

Therefore

$$\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) = \tau V(N, \sigma) - \sigma U(N, \sigma) \quad (64)$$

Therefore, the sequence $\left(\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) \right)_{N \geq 1}$ converges and its limit is as follows:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} - I(N, \sigma, \tau) = -\frac{\tau}{(1-\sigma)^2 + \tau^2} \quad (65)$$

Therefore we can write the following:

$$\sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} = I(N, \sigma, \tau) - \frac{\tau}{(1-\sigma)^2 + \tau^2} + \beta_N \quad (66)$$

Where (β_N) is a bounded sequence with limit zero.

From equation (5), the $\zeta(s)$ function is also defined through the Dirichlet eta function $\eta(s)$ as follows:

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \eta(s) \quad (67)$$

$$= \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} \quad (68)$$

Let's define the sequences X_N and Y_N as follows:

$$X_N = \sum_{n=1}^N \frac{1}{n^s} \quad (69)$$

$$Y_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{n^s} \quad (70)$$

We can also write the following:

$$Y_{2N} = X_{2N} - 2^{1-s} X_N \quad (71)$$

In our case, s is a zero of $\zeta(s)$. Therefore

$$\lim_{N \rightarrow +\infty} Y_{2N} = \lim_{N \rightarrow +\infty} X_{2N} - 2^{1-s} X_N = 0 \quad (72)$$

Therefore, the limit of the real part of Y_{2N} is also zero:

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^{2N} \frac{\cos(\tau \ln(n))}{n^\sigma} - 2^{1-\sigma} \left(\cos(\tau \ln(2)) \sum_{n=1}^N \frac{\cos(\tau \ln(n))}{n^\sigma} - \sin(\tau \ln(2)) \sum_{n=1}^N \frac{\sin(\tau \ln(n))}{n^\sigma} \right) = 0 \quad (73)$$

We inject the equations (54) and (66) into the last equation to get the following:

$$\lim_{N \rightarrow +\infty} J(2N, \sigma, \tau) - \frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} + \alpha_{2N} - 2^{1-\sigma} \left(\cos(\tau \ln(2)) \left(J(N, \sigma, \tau) \right. \right. \quad (74)$$

$$\left. - \frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} + \alpha_N \right) - \sin(\tau \ln(2)) \left(I(N, \sigma, \tau) - \frac{\tau}{(1 - \sigma)^2 + \tau^2} + \beta_N \right) = 0 \quad (75)$$

Therefore

$$\lim_{N \rightarrow +\infty} J(2N, \sigma, \tau) - 2^{1-\sigma} \left(\cos(\tau \ln(2)) J(N, \sigma, \tau) - \sin(\tau \ln(2)) I(N, \sigma, \tau) \right) = \quad (76)$$

$$\frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} - 2^{1-\sigma} \left(\cos(\tau \ln(2)) \frac{\tau^2 + \sigma(\sigma - 1)}{(1 - \sigma)^2 + \tau^2} - \sin(\tau \ln(2)) \frac{\tau}{(1 - \sigma)^2 + \tau^2} \right) = 0 \quad (77)$$

We have from the equations (24) and (28) the following:

$$I(N, \sigma, \tau) = K(\sigma, \tau) \left(1 - \frac{\cos(\tau \ln(N))}{N^{\sigma-1}} - \frac{(\sigma - 1) \sin(\tau \ln(N))}{\tau N^{\sigma-1}} \right) \quad (78)$$

$$J(N, \sigma, \tau) = K(\sigma, \tau) \left(\frac{(\sigma - 1)}{\tau} - \frac{(\sigma - 1) \cos(\tau \ln(N))}{\tau N^{\sigma-1}} + \frac{\sin(\tau \ln(N))}{N^{\sigma-1}} \right) \quad (79)$$

$$J(2N, \sigma, \tau) = K(\sigma, \tau) \left(\frac{(\sigma - 1)}{\tau} - \frac{(\sigma - 1) \cos(\tau \ln(2N))}{\tau (2N)^{\sigma-1}} + \frac{\sin(\tau \ln(2N))}{(2N)^{\sigma-1}} \right) \quad (80)$$

We inject the equations (78-80) into the equation (76-77) to get the following:

$$K(\sigma, \tau) \left(\frac{(\sigma - 1)}{\tau} - \frac{(\sigma - 1)}{\tau} 2^{1-\sigma} \cos(\tau \ln(2)) + 2^{1-\sigma} \sin(\tau \ln(2)) \right) \quad (81)$$

$$= K(\sigma, \tau) \left(\frac{\tau^2 + \sigma(\sigma - 1)}{\tau} - 2^{1-\sigma} \cos(\tau \ln(2)) \frac{\tau^2 + \sigma(\sigma - 1)}{\tau} + 2^{1-\sigma} \sin(\tau \ln(2)) \right) \quad (82)$$

Therefore

$$\frac{(\sigma - 1)}{\tau} \left(1 - 2^{1-\sigma} \cos(\tau \ln(2)) \right) = \frac{\tau^2 + \sigma(\sigma - 1)}{\tau} \left(1 - 2^{1-\sigma} \cos(\tau \ln(2)) \right) \quad (83)$$

Therefore

$$\left(1 - 2^{1-\sigma} \cos(\tau \ln(2)) \right) \frac{\tau^2 + \sigma(\sigma - 1) - (\sigma - 1)}{\tau} = \left(1 - 2^{1-\sigma} \cos(\tau \ln(2)) \right) \frac{\tau^2 + (\sigma - 1)^2}{\tau} = 0 \quad (84)$$

Since $\tau^2 + (\sigma - 1)^2 > 0$, we conclude that:

$$\boxed{\cos(\tau \ln(2)) = \frac{1}{2^{1-\sigma}}} \quad (85)$$

We saw above that $1 - \bar{s} = 1 - \sigma + i\tau = \sigma_1 + i\tau_1$ is also a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$. We apply the same technique above, we will get:

$$\cos(\tau \ln(2)) = \frac{1}{2^{1-\sigma_1}} = \frac{1}{2^\sigma} \quad (86)$$

Therefore

$$\cos(\tau \ln(2)) = \frac{1}{2^{1-\sigma}} = \frac{1}{2^\sigma} \quad (87)$$

Therefore

$$\sigma = 1 - \sigma \quad (88)$$

Therefore

$$\sigma = \frac{1}{2} \quad (89)$$

Hence, the Riemann's Hypothesis is true. ■

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.* In the next article, we will try to apply the same method to prove the Generalized Riemann Hypothesis (GRH).

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