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► **To cite this version:**

Alin Bostan, Lucia Di Vizio, Kilian Raschel. Differential transcendence of Bell numbers and relatives: a Galois theoretic approach. American Journal of Mathematics, In press. hal-03091272v3

**HAL Id: hal-03091272**

**<https://hal.science/hal-03091272v3>**

Submitted on 27 Apr 2024

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# Differential transcendence of Bell numbers and relatives: a Galois theoretic approach

Alin Bostan

Inria, Université Paris-Saclay,  
1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France

Lucia Di Vizio

CNRS, Université Paris-Saclay, UVSQ, Laboratoire de mathématiques de Versailles,  
78000, Versailles, France

Kilian Raschel

CNRS, Université Angers, Laboratoire Angevin de Recherche en Mathématiques,  
2 Boulevard Lavoisier, 49000 Angers, France

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## Abstract

In 2003 Klazar proved that the ordinary generating function of the sequence of Bell numbers is differentially transcendental over the field  $\mathbb{C}(\{t\})$  of meromorphic functions at 0. We show that Klazar's result is an instance of a general phenomenon that can be proven in a compact way using difference Galois theory. We present the main principles of this theory in order to prove a general result about differential transcendence over  $\mathbb{C}(\{t\})$ , that we apply to many other (infinite classes of) examples of generating functions, including as very special cases the ones considered by Klazar. Most of our examples belong to Sheffer's class, well studied notably in umbral calculus. They all bring concrete evidence in support to the Pak-Yeliussizov conjecture, according to which a sequence whose both ordinary and exponential generating functions satisfy nonlinear differential equations with polynomial coefficients necessarily satisfies a *linear* recurrence with polynomial coefficients.

*Keywords:* Combinatorial power series, differential transcendence, Galois theory, Sheffer sequences, umbral calculus, Bell numbers, Bernoulli numbers, Euler numbers, Genocchi numbers.

2010 MSC: 11B68, 05A40, 12H05, 33B15, 33C45, 39A10, 30D30

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## 1. Introduction

In this paper we deal with what we will call the *strong differential transcendence* of some power series in  $\mathbb{C}[[t]]$ , i.e., with their differential transcendence over the field  $\mathbb{C}(\{t\})$  of germs of meromorphic functions at 0. More precisely, we prove that the solutions  $f \in \mathbb{C}[[t]]$  of some first-order linear functional equations must be either rational, i.e., in  $\mathbb{C}(t)$ , or differentially transcendental over  $\mathbb{C}(\{t\})$ , i.e., for any non-negative integer  $n$  and any polynomial  $P \in \mathbb{C}(\{t\})[X_0, X_1, \dots, X_n]$  we must have  $P(f, f', \dots, f^{(n)}) \neq 0$ , where  $f^{(k)}$  denotes the  $k$ -th derivative of  $f$  with respect to  $t$ . We conclude in this way that many well-known power series in  $\mathbb{C}[[t]]$  with a combinatorial origin are strongly differentially transcendental.

### State of the art

We have come to consider this problem, influenced by three earlier works, by Klazar [Kla03], by Pak [Pak18] and by Adamczewski, Dreyfus and Hardouin [ADH21].

First of all, in [Kla03], Klazar considers the ordinary generating function (OGF) of the Bell numbers  $\phi(t) := 1 + \sum_{n \geq 1} \phi_n t^n$ , where  $\phi_n$  is the number of partitions of a set of cardinality  $n \geq 1$ , and proves that  $\phi(t)$

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*Email addresses:* alin.bostan@inria.fr (Alin Bostan), lucia.di-vizio@uvsq.fr (Lucia Di Vizio), raschel@math.cnrs.fr (Kilian Raschel)

*URL:* <http://specfun.inria.fr/bostan> (Alin Bostan), <https://divizio.joomla.com> (Lucia Di Vizio), <https://raschel.perso.math.cnrs.fr/> (Kilian Raschel)

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No 759702, and from the ANR project [DeRerumNatura](#), ANR-19-CE40-0018.

is differentially transcendental over  $\mathbb{C}(\{t\})$ . To do so, he uses a functional equation satisfied by  $\phi(t)$ , namely:

$$\phi\left(\frac{t}{1+t}\right) = t\phi(t) + 1. \quad (1.1)$$

A classical and important property of the Bell numbers is that their *exponential generating function* (EGF)

$$\hat{\phi}(t) := 1 + \sum_{n \geq 1} \frac{\phi_n}{n!} t^n$$

satisfies

$$\hat{\phi}(t) = \exp(\exp t - 1).$$

As a consequence,  $\hat{\phi}(t)$  is D-algebraic, meaning that it satisfies an algebraic differential equation over  $\mathbb{Q}(t)$  (or equivalently over  $\mathbb{C}(t)$ ). However,  $\hat{\phi}(t)$  is not D-finite, that is, it does not satisfy any *linear* differential equation with coefficients in  $\mathbb{Q}(t)$ . This can be seen either analytically, using the asymptotics of  $\phi_n$  [Odl95, Eq. (5.47)], or algebraically, using [Sin86] and the fact that the power series  $\exp(t)$  is not algebraic.

Secondly, starting from the example of the Bell numbers, Pak and Yeliussizov formulated the following ambitious conjecture as an “advanced generalization of Klazar’s theorem”:

**Conjecture 1** ([Pak18, Open Problem 2.4]). *If for a sequence of rational numbers  $(a_n)_{n \geq 0}$  both ordinary and exponential generating functions  $\sum_{n \geq 0} a_n t^n$  and  $\sum_{n \geq 0} a_n \frac{t^n}{n!}$  are D-algebraic, then both are D-finite (equivalently,  $(a_n)_{n \geq 0}$  satisfies a linear recurrence with polynomial coefficients in  $\mathbb{Q}[n]$ ).*

Thirdly, a very recent work by Adamczewski, Dreyfus and Hardouin in difference Galois theory establishes the following general statement:

**Theorem 2** ([ADH21, Thm. 1.2]). *Let  $f \in \mathbb{C}((t))$  be a Laurent series satisfying a linear functional equation of the form*

$$\alpha_0 y + \alpha_1 \tau(y) + \cdots + \alpha_n \tau^n(y) = 0,$$

where  $\alpha_i \in \mathbb{C}(t)$ , not all zero, and  $\tau$  is one of the following operators:

- $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$ ;
- $\tau(f(t)) = f(qt)$  for some  $q \in \mathbb{C}^*$ , not a root of unity;
- $\tau(f(t)) = f(t^m)$  for some positive integer  $m$ .

Then either  $f \in \mathbb{C}(t^{1/r})$  for some positive integer  $r$ , or  $f$  is D-transcendental over  $\mathbb{C}(t)$ . Moreover, in the case of the first operator,  $r$  is necessarily equal to 1.

The juxtaposition of the three works above raises immediately three remarks. The first one is that Klazar is concerned with the *strong* differential transcendence, i.e. over  $\mathbb{C}(\{t\})$ , while Pak and Yeliussizov’s conjecture and the main theorem of [ADH21] are concerned with the differential transcendence over the field of rational functions. The second one is that Theorem 2 would prove Conjecture 1 if we were able to rephrase the differential properties of the series  $\sum_{n \geq 0} a_n \frac{t^n}{n!}$  in terms of a difference operator acting on  $\sum_{n \geq 0} a_n t^n$ . The third one is that we could not find in the literature any another functional equation of the form (1.1), satisfied by other generating functions of combinatorial interest, with the exception of the generating function of the Bernoulli numbers, considered in [Zag98], and of Fubini numbers, as in [Kla16]. In this paper, we address these three issues.

### Combinatorial examples for the Pak-Yeliussizov conjecture

We consider families of polynomials  $(P_n(x))_{n \geq 0}$  in  $\mathbb{C}[x]$ , with  $\deg(P_n) = n$ , and whose exponential generating functions (EGFs) are defined in terms of power series  $u, v, f, g, h \in \mathbb{C}[[t]]$  by

$$\sum_{n \geq 0} P_n(x) \frac{t^n}{n!} = u(x) v(t) f(g(x)h(t)). \quad (1.2)$$

A surprisingly large number of classical families of polynomials fit into the framework of Eq. (1.2). An important special case is provided by *Sheffer sequences*, very classical in umbral calculus [Rom84, Ch. 2], whose EGFs have the form

$$\sum_{n \geq 0} P_n(x) \frac{t^n}{n!} = v(t) e^{x h(t)}. \quad (1.3)$$

	polynomial $P_n(x)$	$v(t)$	$h(t)$
D-finite examples	Laguerre $L_n^\alpha(x)$	$(1-t)^{-1-\alpha}$	$-t(1-t)^{-1}$
	Hermite $H_n(x)$	$\exp(-t^2)$	$2t$
	Mott $M_n(x)$	1	$(\sqrt{1-t^2}-1)/t$
	Bessel $p_n(x)$	1	$1-\sqrt{1-2t}$
	Falling factorial $(x)_n$	1	$\log(1+t)$
exponential functions	Euler $E_n^{(\alpha)}(x)$	$2^\alpha(e^t+1)^{-\alpha}$	$t$
	Bernoulli $B_n^{(\alpha)}(x)$	$t^\alpha(e^t-1)^{-\alpha}$	$t$
	Bell-Touchard $\phi_n(x)$	1	$\exp(t)-1$
	Mahler $s_n(x)$	1	$1+t-\exp(t)$
	Actuarial $a_n^{(\beta)}(x)$	$\exp(\beta t)$	$1-\exp(t)$
logarithmic functions	Bernoulli, 2nd kind $b_n(x)$	$t/\log(1+t)$	$\log(1+t)$
	Poisson-Charlier $c_n(x; a)$	$\exp(-t)$	$\log(1+t/a)$
	Narumi $N_n^{(a)}(x)$	$t^a \log(1+t)^{-a}$	$\log(1+t)$
	Peters $P_n^{(\lambda, \mu)}(x)$	$(1+(1+t)^\lambda)^{-\mu}$	$\log(1+t)$
	Meixner-Pollaczek $P_n^{(\lambda)}(x; \phi)$	$(1+t^2-2t \cos \phi)^{-\lambda}$	$i \log\left(\frac{1-te^{i\phi}}{1-te^{-i\phi}}\right)$
	Meixner $m_n(x; \beta, c)$	$(1-t)^{-\beta}$	$\log\left(\frac{1-t/c}{1-t}\right)$
Krawtchouk $K_n(x; p, N)$	$(1+t)^N$	$\log\left(\frac{p-(1-p)t}{p(1+t)}\right)$	

Table 1: Examples of various families of polynomials  $P_n(x)$  of the Sheffer type [Rom84, Ch. 2], with the corresponding  $v, h$  as in (1.3). The first set of entries corresponds to D-finite examples. We focus on the second set of entries, also called *exponential functions* [EMOT81, §19.7]. The last entries are sometimes called *logarithmic functions*, and are not covered by our methods.

An important subclass is that of *Appell polynomials* [App80], for which  $h(t) = t$ . Other interesting examples (see Table 1) include some families of classical orthogonal polynomials (Hermite, Laguerre, Bessel, ...), for which both ordinary and exponential generating functions are D-algebraic, since they are even D-finite (always over  $\mathbb{C}(t)$  unless we clearly state otherwise). Even more interesting examples, for our purpose, are those for which D-finiteness does not hold: for instance the Bell-Touchard and the Bernoulli polynomials. This is a consequence of the fact that their EGFs possess an infinite number of complex singularities, which is incompatible with D-finiteness. In these cases, a natural question is whether the corresponding OGF

$$F(x, t) = \sum_{n \geq 0} P_n(x) t^n$$

can still be D-algebraic, at least when evaluated at special values  $x_0 \in \mathbb{C}$  of  $x$ .

As Klazar in his D-transcendence proof for the OGF of the Bell numbers [Kla03] (which correspond to the evaluation at  $x = 1$  of the Bell-Touchard polynomials  $\phi_n(x)$  in Table 1), we focus on the case where  $F(x, t)$  satisfies a functional equation of the form

$$F\left(x, \frac{t}{1+t}\right) = R(x, t) \cdot F(x, t) + S(x, t), \quad (1.4)$$

where  $R$  and  $S$  are non-zero rational functions in  $\mathbb{C}(x, t)$ . In §2.1, we explain a *recipe* to deduce a functional equation à la Klazar from the closed form of the EGFs in Table 2 (more precisely, from the differential equations with exponential coefficients that they satisfy). Therefore, all our examples bring further evidence and reinforce Conjecture 1, thanks to Theorem 2. This is the first contribution of our paper. To our knowledge, Klazar's examples of the Bell and of the (related) Uppuluri-Carpenter numbers were the only known combinatorial examples on which Conjecture 1 was proved prior to our work.

### Main Galois theoretic result

Our aim in this article is to demonstrate, using difference Galois theory, that Klazar's result is a very particular instance of a general phenomenon. To do so, we equip  $\mathbb{C}$  with the usual absolute value so that it makes sense to consider the field  $\mathbb{C}(\{t\})$ , which coincides with the field of fractions of the ring of convergent series at 0 with coefficients in  $\mathbb{C}$ . We can finally state the second contribution of this paper (see §4 below), which generalizes (from D-transcendence to strong D-transcendence) the first instance of Theorem 2, in the case of first-order inhomogeneous difference equations:

polynomial $P_n(x)$	EGF $\sum_{n \geq 0} P_n(x) \frac{t^n}{n!}$	$R$	$S$	ref.
Bernoulli $B_n(x)$	$\frac{t}{e^t - 1} \cdot \exp(xt)$	$t + 1$	$-\frac{t(t+1)}{(t+1-xt)^2}$	[Apo08]
Glaisher $U_n(x)$	$\frac{t}{e^t + 1} \cdot \exp(xt)$	$-(t + 1)$	$\frac{t(t+1)}{(t+1-xt)^2}$	[Gla98, §230, §234]
Apostol-Bernoulli $A_n^{(\lambda)}(x)$	$\frac{t}{\lambda e^t - 1} \cdot \exp(xt)$	$\lambda(t + 1)$	$-\frac{t(t+1)}{(t+1-xt)^2}$	[Apo51]
Imschenetzky $S_n(x)$	$\frac{t}{e^t - 1} \cdot (\exp(xt) - 1)$	$t + 1$	$\frac{xt^2(xt-2t-2)}{(t+1)(t+1-xt)^2}$	[Ims83], [EMOT81, p. 254, (38)]
Euler $E_n(x)$	$\frac{2}{e^t + 1} \cdot \exp(xt)$	$-(t + 1)$	$\frac{2(t+1)}{t+1-xt}$	[Car59]
Genocchi $G_n(x)$	$\frac{2t}{e^t + 1} \cdot \exp(xt)$	$-(t + 1)$	$\frac{2t(t+1)}{(t+1-xt)^2}$	[Hor91]
Carlitz $C_n^{(\lambda)}(x)$	$\frac{1-\lambda}{1-\lambda e^t} \cdot \exp(xt)$	$\lambda(t + 1)$	$\frac{(1-\lambda)(t+1)}{t+1-xt}$	[Car53, Car62]
Fubini $F_n(x)$	$1/(1 - x(e^t - 1))$	$\frac{x}{x+1} \cdot (t + 1)$	$\frac{1}{x+1}$	[Tan75]
Bell-Touchard $\phi_n(x)$	$\exp(x(e^t - 1))$	$xt$	1	[Bel34, Tou56]
Mahler $s_n(x)$	$\exp(x(t + 1 - e^t))$	$\frac{xt(t+1)}{xt-t-1}$	$\frac{t+1}{t+1-xt}$	[Mah32], [EMOT81, p. 254, (40)]
Toscano's actuarial $a_n^{(\lambda)}(x)$	$\exp(-xe^t + \lambda t + x)$	$\frac{xt(t+1)}{\lambda t - t - 1}$	$\frac{t+1}{t+1-\lambda t}$	[Tos50], [BB64, p. 42]

Table 2: Examples of power series  $F(x, t) = \sum_{n \geq 0} P_n(x)t^n \in \mathbb{C}[x][[t]]$  with D-algebraic exponential generating functions (EGF) and satisfying first-order difference equations of the form  $F(x, \frac{t}{t+1}) = R(x, t) \cdot F(x, t) + S(x, t)$ , for some rational functions  $R, S \in \mathbb{C}(x, t)$ . In the cases of Apostol-Bernoulli, Carlitz and Toscano,  $\lambda$  is assumed to be a fixed parameter in  $\mathbb{C}$ .

**Theorem 3.** Let  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  such that  $\alpha\delta - \beta\gamma \neq 0$  and  $(\delta - \alpha)^2 + 4\gamma\beta = 0$ . Let  $f \in \mathbb{C}((t)) \setminus \mathbb{C}(t)$  be any solution of

$$f\left(\frac{\beta + \alpha t}{\delta + \gamma t}\right) = a(t)f(t) + r(t),$$

where  $a, r \in \mathbb{C}(t) \setminus \{0\}$ . Then  $f$  is differentially transcendental over  $\mathbb{C}(\{t\})$ .

This result will be proven in §4 in the particular case  $\alpha = \gamma = \delta = 1$  and  $\beta = 0$ , which corresponds to the map  $\tau$  in Theorem 2. Note that the hypotheses on  $\alpha, \beta, \gamma, \delta$  are equivalent to assume that the homography  $t \mapsto \frac{\beta + \alpha t}{\delta + \gamma t}$  has only one fixed point, therefore Theorem 3 is an easy consequence of this particular case, by a rational change of variable, as any homography with only one fixed point is conjugated to  $\tau$ . See Corollary 40 below.

The theorem above applied to the OGFs of Table 2, including all the families obtained after appropriately specializing  $x$  and  $\lambda$ , allows to immediately obtain their strong differential transcendence:

**Corollary 4.** Let  $f(t)$  be any  $F(x, t)$  in Table 2 evaluated at some  $x \in \mathbb{C} \setminus \{0\}$  and  $\lambda \in \mathbb{C}$ . Then  $f(t)$  is differentially transcendental over  $\mathbb{C}(\{t\})$ .

Note that the assumption  $x \in \mathbb{C} \setminus \{0\}$  is only necessary for the last four entries of Table 2. Indeed, in the other cases the EGF is non D-finite for any  $x \in \mathbb{C}$ , hence the OGF as well, and therefore the OGF is strongly D-transcendental by Theorem 3.

In particular, we deduce the strong D-transcendence of interesting combinatorial OGFs, among which the two main examples in Klazar's paper (see §2.2 for a few other combinatorial examples):

- the OGF of the *Bell numbers* [Kla03, Prop. 3.3] (A000110)

$$\sum_{n \geq 0} \phi_n(1)t^n = 1 + t + 2t^2 + 5t^3 + 15t^4 + 52t^5 + 203t^6 + \dots$$

- the OGF of the *Uppuluri-Carpenter numbers* [Kla03, Thm. 3.5] (A000587)

$$\sum_{n \geq 0} \phi_n(-1)t^n = 1 - t + t^3 + t^4 - 2t^5 - 9t^6 - 9t^7 + 50t^8 + \dots$$

- the OGF of the *bicolored partitions* [BBMD<sup>+</sup>02, Tab. 2] (A001861)

$$\sum_{n \geq 0} \phi_n(2)t^n = 1 + 2t + 6t^2 + 22t^3 + 94t^4 + 454t^5 + \dots$$

- the OGF of the number of *set partitions without singletons* (A000296)

$$\sum_{n \geq 0} s_n(-1)t^n = 1 + t^2 + t^3 + 4t^4 + 11t^5 + 41t^6 + 162t^7 + 715t^8 + \dots$$

- the OGF of the *Genocchi numbers* [Dum74] (A001469)

$$\sum_{n \geq 0} G_n(1)t^n = t + t^2 - t^4 + 3t^6 - 17t^8 + 155t^{10} - \dots$$

- the OGF of the *surjection numbers* (also, *preferential arrangements*) [FS09, p. 109] (A000670)

$$\sum_{n \geq 0} F_n(1)t^n = 1 + t + 3t^2 + 13t^3 + 75t^4 + 541t^5 + \dots$$

While Theorem 3 and Corollary 4 demonstrate that Klazar’s theorem is an instance of a general phenomenon, they raise more questions than they answer. First, it is striking that so many concrete combinatorial objects are enumerated by strongly differentially transcendental functions. Secondly, although there is a huge gap between the D-transcendental and the strongly differentially transcendental classes, Theorem 3 and Corollary 4 show that their intersections with solutions of difference equations of order 1 coincide. Thirdly, it is natural to inquire whether an extension of Conjecture 1 might hold with *differentially algebraic over  $\mathbb{Q}(t)$*  replaced by *differentially algebraic over  $\mathbb{C}(\{t\})$* . We feel that this paper should provide a motivation to look further into these questions.

We should also point out that the (strong) D-transcendence of very natural combinatorial examples such as the OGF of labeled rooted trees  $\sum_{n \geq 1} n^{n-1}t^n$ , or the OGF of the logarithmic functions in Table 1, does not fit into our framework, and actually escapes for the moment any other attempt of proof. Three other challenging examples are presented in §2.2.6.

Finally, let us mention that there are several interesting examples satisfying higher order linear  $\tau$ -equations or linear  $q$ -difference equations, some of which arise from combinatorics of lattice walks [BBMR16, BBMR21, DHRS20, DHRS18, HS21]. We plan to consider them in subsequent publications under the viewpoint of the Pak-Yeliussizov conjecture, together with the differential transcendence with respect to the parameters  $x$  and  $\lambda$  of the examples in Table 2. We expect that similar techniques will allow us to obtain more general results, such as the algebraic-differential independence of the families of power series in Table 2.

### Content of the paper

In Section 2, we explain how to deduce a functional equation from a closed form of an exponential type as the EGFs in Table 2. As a corollary, we find functional equations satisfied by several combinatorial examples, on which Theorem 3 will be applied.

Section 3, and in particular §3.1, may be considered as a quick and gentle introduction to the parameterized Galois theory of difference equations, starting from the point of view of the usual Galois theory of difference equations. In §3.2 we have included some proofs, since we have stated some results in the exact form needed to prove Theorem 3. They are quite similar to some statements in [HS08], but under a weaker assumption on the field of constants, necessary in our applications. Besides, we have tried to provide a user-friendly exposition, avoiding as much as possible the use of the more sophisticated parameterized Galois theory, since we do not want to restrict the audience of the paper to specialists.

Theorem 3 is our main technical result. Its proof is given in Section 4, and it relies on results from Section 3.

### Acknowledgments

We thank Marni Mishna and Michael Singer for having suggested some additional bibliographic pointers. We are grateful to Michael Singer for the enthusiasm that he showed for the questions tackled in this paper: this is encouraging for us to continue further in this direction. We also thank Christopher Chiu, Martin Klazar and Sergey Yurkevich for interesting comments on the first version of the manuscript. We warmly thank the two anonymous referees for their detailed reports, their very interesting comments and their generous suggestions.

## 2. From exponential generating functions to difference equations: setting and examples

### 2.1. From exponential generating functions to $\tau$ -equations

Let  $C$  be a field of characteristic zero. As before, we consider the substitution map  $\tau : f(t) \mapsto f\left(\frac{t}{1+t}\right)$ , with compositional inverse  $\tau^{-1} : f(t) \mapsto f\left(\frac{t}{1-t}\right)$ , and the derivation  $D := \frac{d}{dt} : f(t) \mapsto f'(t)$ . It is not difficult to see that  $\tau$  defines an automorphism of  $C((t))$ ,  $C(\{t\})$  and  $C(t)$ . We will informally call *difference  $\tau$ -equation*, or simply  *$\tau$ -equation*, a linear functional equation with respect to  $\tau$ .

We want to tackle the relation between the closed forms of the EGFs of Table 2 and the existence of a difference  $\tau$ -equation for the corresponding OGFs. First of all, we recall the definition of the formal Borel transform  $\mathfrak{B} : C[[t]] \rightarrow C[[t]]$ :

$$g := \sum_{n \geq 0} g_n t^n \mapsto \hat{g} := \mathfrak{B}(g) = \sum_{n \geq 0} g_n \frac{t^n}{n!}.$$

Moreover, we consider the Euler-type transform  $\Phi : C[[t]] \rightarrow C[[t]]$  defined by:

$$f(t) \mapsto (\Phi f)(t) := \frac{1}{1-t} \cdot f\left(\frac{t}{1-t}\right).$$

Note for later use that, for all  $j \in \mathbb{Z}$ ,

$$\Phi^j(f(t)) = \frac{1}{1-jt} \cdot f\left(\frac{t}{1-jt}\right) \quad \text{and} \quad \tau^j(f(t)) = f\left(\frac{t}{1+jt}\right). \quad (2.1)$$

The maps  $\mathfrak{B}$  and  $\Phi$  intertwine in an interesting way, described in Lemma 5 below, reminiscent of the formal Fourier transform. Although not stated in this generality, (variants of) this result can be found in various papers from various areas of mathematics and computer science, such as [Gou90, FR92, Sch92, Pro94, FS95]. For instance, part (a) is a consequence of Theorems 1 and 2 in [Gou90], which relate the coefficients of  $f$  and those of  $\Phi(f)$  via a binomial (Euler-type) transform. Section 1 in [FS95] gives a slight variation of (a), while [Pro94] gives a generalization of (a) to transforms of the form  $f(t) \mapsto \frac{1}{1-bt} \cdot f\left(\frac{ct}{1-bt}\right)$ . The (inverse) Borel transform is used in [FR92, Sch92] for the analysis of some “digital trees” and “tree algorithms”, in order to solve different types of functional equations similar to ours (see Lemmas 1 and 2 in [FR92] and Eqs. (2.5) and (2.8) in [Sch92]).

**Lemma 5.** *For any  $f, g \in C[[t]]$ , we have:*

- (a)  $\Phi(f) = g$  if and only if  $\mathfrak{B}(g) = \mathfrak{B}(f) \cdot e^t$ ;
- (b)  $\frac{d}{dt}(\mathfrak{B}(f)) = \mathfrak{B}\left(\frac{f(t)-f(0)}{t}\right)$ ;
- (c) for any  $j, k \in \mathbb{N}$ , the following commutation rule holds

$$e^{jt} \circ D^k \circ \mathfrak{B} = \mathfrak{B} \circ \Phi^j \circ \Delta^k,$$

where  $\Delta : C[[t]] \rightarrow C[[t]]$  is the divided difference operator  $f(t) \mapsto (\Delta f)(t) := \frac{f(t)-f(0)}{t}$ .

*Proof.* Note that (a) is equivalent to (c) for the particular choice  $(j, k) = (1, 0)$  and that (b) is equivalent to (c) for the particular choice  $(j, k) = (0, 1)$ . Moreover, an easy induction on  $j$  and  $k$  shows that (c) follows from an iteration of (a) and (b). It is therefore sufficient to prove (a) and (b). By linearity, it is enough to show that, for any  $m \geq 0$ , we have

$$e^t \cdot (\mathfrak{B}(t^m)) = \mathfrak{B}(\Phi(t^m)) \quad \text{and} \quad D(\mathfrak{B}(t^m)) = \mathfrak{B}(\Delta(t^m)). \quad (2.2)$$

The second part of Eq. (2.2) is straightforward, since both terms are clearly equal to  $t^{m-1}/(m-1)!$ . It remains to prove the first part of Eq. (2.2). Using the binomial theorem, its right-hand side is equal to

$$\mathfrak{B}\left(\frac{1}{1-t} \cdot \left(\frac{t}{1-t}\right)^m\right) = \mathfrak{B}\left(t^m \cdot \sum_{n \geq 0} \binom{m+n}{n} t^n\right) = \sum_{n \geq 0} \frac{t^{m+n}}{n!m!} = e^t \cdot \frac{t^m}{m!},$$

and it is thus equal to its left-hand side. This finishes the proof of the lemma.  $\square$



Note that Lemma 5(c) also holds for non-integer values of  $j$ , by taking Eq. (2.1) as definition of  $\Phi^j$ . This simple remark will be used in some examples of Section 2.2.

Lemma 5 will be our main tool to transform any linear differential equation with polynomial coefficients in  $\exp(t)$  satisfied by  $\hat{f} = \mathfrak{B}(f)$  into a linear  $\Phi$ -difference equation satisfied by  $f$ . Then, using the rule (consequence of (2.1))

$$\tau^d \circ \Phi^j = \frac{1 + dt}{1 + (d-j)t} \circ \tau^{d-j}, \quad \text{for all } j, d \in \mathbb{Z} \quad (2.3)$$

will allow us to transform the latter  $\Phi$ -difference equation into a  $\tau$ -difference equation satisfied by  $f$ . Let us first illustrate this process on an example.

**Example 6.** The case of Bell numbers, considered by Klazar and mentioned in (1.1), coincides with the Bell-Touchard polynomials  $\phi_n(x)$  in Table 2 evaluated at  $x = 1$ . The associated EGF is  $\hat{\phi}(t) = \exp(e^t - 1)$ , which satisfies

$$D(\hat{\phi}) - e^t \cdot \hat{\phi} = 0,$$

thus  $\mathfrak{B}\left(\frac{\hat{\phi}-1}{t}\right) = \mathfrak{B}(\Phi(\phi))$ , by Lemma 5(a) and Lemma 5(b). Therefore,  $\phi$  satisfies the first-order  $\Phi$ -equation

$$\Phi(\phi) = \frac{\phi - 1}{t}.$$

Now, applying  $\tau$  to both sides of this equation, and using Eq. (2.3) with  $d = j = 1$ , we find

$$(t+1)\phi(t) = \tau\left(\frac{\phi-1}{t}\right).$$

Since on the other hand  $\tau\left(\frac{\phi-1}{t}\right) = \tau\left(\frac{1}{t}\right) \cdot \tau(\phi-1) = \frac{t+1}{t} \cdot (\tau(\phi) - 1)$ , we finally conclude that  $\phi$  satisfies the first-order  $\tau$ -equation  $\tau(\phi) = t\phi + 1$ , as expected. This provides an alternative proof to [Kla03, Prop. 2.1].

In general, an iteration of the same argument allows to show:

**Proposition 7.** Let  $f \in C[[t]]$ . If  $\hat{f} = \mathfrak{B}(f)$  satisfies a linear differential equation of order  $r$ , of the form

$$a_0(e^t)\hat{f} + a_1(e^t)D(\hat{f}) + \cdots + a_r(e^t)D^r(\hat{f}) = 0, \quad (2.4)$$

with  $a_0, \dots, a_r \in C[t]$  of degree at most  $d$ , then  $f$  satisfies a linear inhomogeneous difference  $\tau$ -equation of order at most  $d$ , with coefficients in  $C[t]$  of degree at most  $d+r$ .

*Proof.* Setting  $a_i(t) = \sum_{j=0}^d a_{i,j}t^j$  with  $a_{i,j} \in C$ , Eq. (2.4) writes  $\sum_{i=0}^r \sum_{j=0}^d a_{i,j}e^{jt}D^i(\mathfrak{B}(f)) = 0$ . By Lemma 5, this implies  $\sum_{i=0}^r \sum_{j=0}^d a_{i,j}\Phi^j(\Delta^i(f)) = 0$ . Applying  $\tau^d$  to this last equation, and using (2.3), yields

$$\sum_{j=0}^d \sum_{i=0}^r a_{i,d-j} \cdot \frac{\tau^j(\Delta^i(f))}{1+jt} = 0. \quad (2.5)$$

Now, for  $i > 0$ , each term  $\tau^j(\Delta^i(f))$  in the above sum is equal to

$$\Delta^i(f)\left(\frac{t}{1+jt}\right) = \frac{f\left(\frac{t}{1+jt}\right) - R_{i,j}\left(\frac{t}{1+jt}\right)}{\left(\frac{t}{1+jt}\right)^i} = \frac{(1+jt)^i}{t^i} \cdot \left(\tau^j(f) - R_{i,j}\left(\frac{t}{1+jt}\right)\right),$$

where  $R_{i,j} \in C[t]$  is a polynomial of degree less than  $i$ . Therefore, multiplying (2.5) by  $t^r \prod_{j=1}^d (1+jt)$  and arranging terms yields an inhomogeneous  $\tau$ -equation of order at most  $d$  with polynomial coefficients of degree at most  $d+r$ .  $\square$

**Remark 8.** Proposition 7 generalizes without difficulty when Eq. (2.4) has a non-zero right-hand side, e.g. an exponential polynomial  $P \in C[t, e^t]$ . In this case, the (equivalent)  $\tau$ -equation (2.5) has a right-hand side equal to  $\frac{\tau^d(\mathfrak{B}^{-1}(P))}{1+dt}$ . For instance if  $P \in C[t]$ , then instead of multiplying (2.5) by just  $t^r \prod_{j=1}^d (1+jt)$ , we multiply this time by  $t^r(1+dt)^{d_P} \prod_{j=1}^d (1+jt)$ , where  $d_P := \deg P$ ; this produces a  $\tau$ -equation of order at most  $d$  and coefficients in  $C[t]$  of degree at most  $d_P + d + r$ . In the general case  $P = \sum_{i=0}^{d_P} \sum_{j=0}^{e_P} p_{i,j}t^i e^{jt}$  with  $p_{i,j} \in C$ , we use that  $\mathfrak{B}^{-1}(t^i e^{jt}) = i!t^i / (1-jt)^{i+1}$  to deduce, after multiplying this time the  $\tau$ -equation (2.5) by  $t^r(1+dt)^{\deg P} \prod_{\ell=d-e_P}^{d-1} (1+\ell t)^{d_P+1} \prod_{j=1}^d (1+jt)$ , a  $\tau$ -equation of order at most  $d$  and with coefficients in  $C[t]$  of degree at most  $d_P + r + d + (d_P + 1)e_P$ .



Before deducing from the proposition above the functional equations satisfied by the OGF in Table 2, let us just illustrate that one can obtain higher order difference equations in some other interesting cases:

**Example 9.** We consider the EGF  $\hat{f} = \exp((e^t - 1)^2/2)$ . The associated OGF is  $f = 1 + t^2 + 3t^3 + 10t^4 + 45t^5 + \dots$ , the generating function of the numbers of simple labeled graphs on  $n$  nodes in which each component is a complete bipartite graph, see [A060311](#). Then  $D(\hat{f}) = \hat{f} \cdot e^t(e^t - 1)$ , thus  $\mathfrak{B}(\frac{\hat{f}-1}{t}) = \mathfrak{B}(\Phi^2(f) - \Phi(f))$  by Lemma 5. Therefore,  $f$  satisfies the second-order  $\Phi$ -equation  $\Phi^2(f) - \Phi(f) = \frac{f-1}{t}$ . Applying  $\tau^2$  to both sides of this equation yields, using (2.3), the second-order  $\tau$ -equation  $(2t + 1)f - \frac{2t+1}{t+1}\tau(f) = \frac{2t+1}{t} \cdot (\tau^2(f) - 1)$ . Rearranging terms, we deduce that  $f$  satisfies the second-order  $\tau$ -equation  $(t + 1)\tau^2(f) + t\tau(f) - (t^2 + t)f = t + 1$ .

## 2.2. Other combinatorial examples

As promised in the introduction, we finish this section by considering the series in Table 2 plus a few more examples of  $\tau$ -equations arising from combinatorial generating functions.

As far as Table 2 is concerned, we will only detail the case of the family of Bernoulli polynomials  $B_n(x) = B_n^{(0)}(x)$  (see [\[Apo08\]](#)), defined by their exponential generating function as follows:

$$\frac{t}{e^t - 1} \cdot \exp(xt) = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}.$$

We apply our recipe to find a linear  $\tau$ -equation for  $B(x, t)$ .

**Lemma 10.** The OGF of the Bernoulli polynomials  $B(x, t) := \sum_{n \geq 0} B_n(x)t^n$  satisfies the functional equation

$$\tau(B) = (t + 1) \cdot B - \frac{t(t + 1)}{(t + 1 - tx)^2}. \quad (2.6)$$

*Proof.* We start with the inhomogeneous differential equation (of order 0)  $\hat{B} \cdot \exp(t) - \hat{B} = t \cdot \exp(xt)$ , and follow the procedure described in the proof of Proposition 7 and in Remark 8. We deduce  $\mathfrak{B}(\phi(B) - B) = \mathfrak{B}(t/(xt - 1)^2)$ , and by applying  $\tau \circ \mathfrak{B}^{-1}$  we get  $(t + 1)B - \tau(B) = t(t + 1)/(t + 1 - xt)^2$ .  $\square$

As a consequence of Lemma 10, Theorem 3 implies Corollary 4 for the first entry of Table 2; in particular, this proves that the OGF  $B(0, t)$  of the sequence of Bernoulli numbers  $B_n = B_n(0)$ , classical in number theory, is strongly D-transcendental. All the other functional equations for the OGFs in Table 2 are calculated in the same way, by using the procedure described in the proof of Proposition 7. In each case, Theorem 3 implies Corollary 4 for the corresponding entry of Table 2.

We complete our list with a few more examples from combinatorics.

### 2.2.1. Tangent numbers

The integer sequence of the so-called ‘‘tangent numbers’’  $(f_n)_{n \geq 0} = (1, 2, 16, 272, 7936, 353792, \dots)$  ([A000182](#)) appears in the expansion of the (D-algebraic) tangent function as an EGF:

$$\tan(t) = 1 \frac{t}{1!} + 2 \frac{t^3}{3!} + 16 \frac{t^5}{5!} + 272 \frac{t^7}{7!} + 7936 \frac{t^9}{9!} + 353792 \frac{t^{11}}{11!} + \dots$$

A slight variation of Lemma 5, based on the exponential form of the tangent function, proves that the corresponding OGF  $F(t) = t + 2t^3 + 16t^5 + 272t^7 + 7936t^9 + \dots$  satisfies the difference equation

$$F\left(\frac{t}{1 - 2it}\right) + (1 - 2it)F(t) = 2t, \quad (2.7)$$

and Theorem 3 implies that  $F(t)$  is strongly D-transcendental.

Indeed, as pointed out earlier, Lemma 5(c) also holds for non-integer values of  $j$ , by taking Eq. (2.1) as definition of  $\Phi^j$ . Thus, from the equality

$$\tan(t) = -i \frac{e^{2it} - 1}{e^{2it} + 1}$$

we deduce that  $(e^{2it} + 1)\mathfrak{B}(F) = i(1 - e^{2it})$ , and hence

$$\mathfrak{B}(\Phi^{2i}(F) + F) = \mathfrak{B}\left(i - \frac{i}{1 - 2it}\right),$$

which in turn implies Eq. (2.7).

As a side remark, note that one can deduce from the D-transcendence of  $F(t)$  an alternative proof for the D-transcendence of the OGF of the Bernoulli numbers via the classical relation  $f_n = 2^{2n+1}(2^{2n+2} - 1) \frac{(-1)^n}{n+1} B_{2n+2}$ .

### 2.2.2. Alternating permutations

A permutation  $\sigma = \sigma_1 \cdots \sigma_n$  of the set  $\{1, 2, \dots, n\}$  is called *alternating* if  $\sigma_1 < \sigma_2 > \sigma_3 < \dots$ . Alternating permutations are counted by the sequence  $(a_n)_{n \geq 0} = (1, 1, 1, 2, 5, 16, 61, 272, \dots)$  (A000111). By a famous result due to André [And81], its EGF is known to be  $\tan(t) + \sec(t)$ , which is clearly D-algebraic. It is natural to inquire about the nature of the corresponding OGF,  $A(t) := \sum_{n \geq 0} a_n t^n$ . Using the identity

$$\tan(t) + \sec(t) = \frac{1 - ie^{it}}{e^{it} - i}$$

we deduce as in §2.2.1 that

$$\mathfrak{B}(\Phi^i(A) - A) = (e^{it} - i)\mathfrak{B}(A) = 1 - ie^{it} = \mathfrak{B}(1 - i/(1 - it)),$$

and therefore  $A(t)$  satisfies the difference equation

$$A\left(\frac{t}{1 - it}\right) = (t + i)A(t) + 1 - i - it.$$

Theorem 3 then implies that  $A(t)$  is strongly D-transcendental.

### 2.2.3. Springer numbers

The sequence  $(s_n)_{n \geq 0} = (1, 1, 3, 11, 57, 361, 2763, \dots)$  (A001586) of the Springer numbers [Spr71] bears several combinatorial interpretations; for instance, it counts the topological types of odd functions with  $2n$  critical values [Arn92b, Arn92a]. By a result due to Glaisher [Gla98], its EGF is  $1/(\cos(t) - \sin(t))$ , which is D-algebraic. As in §2.2.1 one can prove that the corresponding OGF,  $S(t) = \sum_{n \geq 0} s_n t^n$ , satisfies the difference equation

$$S\left(\frac{t}{1 - 2it}\right) = (2t + i)S(t) + \frac{(1 - i)(2t + i)}{t + i},$$

and Theorem 3 implies that  $S(t)$  is strongly D-transcendental.

An alternative way to deduce this fact is to use the relations [Gla98, §253]

$$s_{2n} = (-1)^n \frac{4^{2n+1}}{4n + 2} \cdot U_{2n+1}(1/4), \quad s_{2n-1} = (-1)^n \frac{4^{2n}}{4n} \cdot U_{2n}(1/4)$$

between the Springer numbers and the values at  $x = 1/4$  of the Glaisher polynomials  $U_n(x)$  in Table 2.

### 2.2.4. Barred preferential arrangements

For any  $m \in \mathbb{N}$ , the sequence  $(r_{m,n})_{n \geq 0} = (1, m + 1, (m + 1)(m + 3), (m + 1)(m^2 + 8m + 13), (m + 1)(m^3 + 15m^2 + 63m + 75), \dots)$  (A226513) counting the number of “barred preferential arrangements” of  $n$  elements with  $m$  bars was studied in [AUP13]. When  $m = 0$ , this coincides with the sequence of surjection numbers mentioned on page 5. It was shown in [AUP13, Thm. 4] that the EGF of  $(r_{m,n})_{n \geq 0}$  is equal to  $1/(2 - e^t)^{m+1}$ . As in §2.2.1 one can prove that the corresponding OGF,  $R_m(t) = \sum_{n \geq 0} r_{m,n} t^n$ , satisfies the difference equation

$$\tau(R_m) = \frac{(m + 1)t + 1}{2} R_m + \frac{1}{2}$$

and Theorem 3 allows to conclude that  $R_m(t)$  is strongly D-transcendental.

### 2.2.5. Various other sequences

Many other examples of D-algebraic EGFs, whose corresponding OGFs can be shown to satisfy  $\tau$ -equations, may be found in various references such as the book [EMOT81] (e.g. on page 252 for generalized Bernoulli polynomials; on page 253 for generalized Euler polynomials; on page 254 for generalized Imschenetzky polynomials), and the articles [Gla98, Gla99, Fro10, Car59, BB64, KT75, Luo06, OSS10, AO11], that we will comment in detail in the sequel. Note however that these references never mention explicitly the corresponding  $\tau$ -equations for the OGFs. For all these examples, the OGF is D-transcendental by Theorem 2, and even strongly D-transcendental by Theorem 3 when the order of the  $\tau$ -equation is 1.

*Karande–Thakare polynomials.* In [KT75] Karande and Thakare introduced the family of polynomials  $D_n(x; a, k)$ , with  $a \neq 0$  and  $k \in \mathbb{N}$ , defined by

$$\frac{2 \left(\frac{t}{2}\right)^k}{e^t - a} \cdot \exp(xt) = \sum_{n \geq 0} D_n(x; a, k) \frac{t^n}{n!}. \quad (2.8)$$

This is a simultaneous generalization of several families from Table 2, since  $B_n(x) = D_n(x; 1, 1)$ ,  $E_n(x) = D_n(x; -1, 0)$ ,  $G_n(x) = D_n(x; -1, 1)$  and  $C_n^{(\lambda)}(x) = \frac{\lambda-1}{2\lambda} D_n(x; 1/\lambda, 0)$  for  $\lambda \neq 0$ . Using Proposition 7 one can prove that the OGF  $F(t) := \sum_{n \geq 0} D_n(x; a, k) t^n$  satisfies the  $\tau$ -equation

$$\tau(F) - \frac{t+1}{a} F = -\frac{k! t^k}{2^{k-1} a} \cdot \frac{t+1}{(t+1-tx)^{k+1}}.$$

The OGF  $F(t)$  cannot be D-finite, since otherwise  $\sum_{n \geq 0} D_n(x; a, k) \frac{t^n}{n!}$  would be D-finite too, hence  $1/(e^t - a)$  would be D-finite as well by (2.8), a contradiction with the fact that  $1/(e^t - a)$  has infinitely many complex singularities. By Theorem 3,  $F$  is therefore strongly D-transcendental.

*Frobenius–Carlitz rational functions.* Carlitz considered in [Car59] the “Eulerian” rational functions  $R_n(x)$  defined by

$$\frac{1-x}{e^t - x} = \sum_{n \geq 0} R_n(x) \frac{t^n}{n!},$$

and already studied by Frobenius [Fro10]. Using Proposition 7 one can prove that the OGF  $F(t, x) := \sum_{n \geq 0} R_n(x) t^n$  satisfies the  $\tau$ -equation

$$\tau(F) - \frac{t+1}{x} F = \frac{x-1}{x}.$$

Again, by Theorem 3,  $F(t, x_0)$  is strongly D-transcendental for any  $x_0 \neq 0$ .

*Generalized Bernoulli, Euler and Carlitz polynomials.* Extensions of the classical Bernoulli, Euler and Carlitz polynomials are defined by ([Nør22, §32, p. 185], [Nør24, §77, p. 145], [EMOT53, p. 252], [BB64, p. 30])

$$\begin{aligned} \sum_{n \geq 0} B_n^{(\alpha)}(x) \frac{t^n}{n!} &:= \left( \frac{t}{e^t - 1} \right)^\alpha \cdot e^{xt}, \\ \sum_{n \geq 0} E_n^{(\alpha)}(x) \frac{t^n}{n!} &:= \left( \frac{2t}{e^t + 1} \right)^\alpha \cdot e^{xt}, \\ \sum_{n \geq 0} C_n^{(\lambda, \alpha)}(x) \frac{t^n}{n!} &:= \left( \frac{1-\lambda}{1-\lambda e^t} \right)^\alpha \cdot e^{xt}. \end{aligned}$$

For  $\alpha \in \mathbb{N}$ , the corresponding OGFs satisfy  $\tau$ -equations of order  $\alpha$ , hence they are D-transcendental. We conjecture that they are even strongly D-transcendental. For instance,  $F(t) = \sum_{n \geq 0} B_n^{(2)}(x) t^n$  satisfies

$$\tau^2(F) = \frac{4t+2}{t+1} \cdot \tau(F) - (2t+1)F - \frac{2(2t+1)t^2}{(tx-2t-1)^3},$$

and since  $F$  is not D-finite, it is D-transcendental by Theorem 2. Similar conclusions can be drawn for other families of generalized polynomials, such as the generalized Apostol–Euler polynomials [Luo06] and the further unification of the Apostol–Bernoulli, Euler and Genocchi polynomials [OSS10], and its generalization to higher orders [AO11].

*Glaisher’s sequences.* Glaisher introduced in [Gla98, Gla99] several interesting sequences, that are nowadays called Glaisher’s  $I, J, H, P, Q, R$  and  $T$  numbers. For instance, the  $R$ -numbers  $1, 7, 305, 33367, \dots$  (A002437) are the coefficients  $R_n$  in the expansion  $\cosh(t)/(2 \cosh(2t) - 1) = (1 + \cosh(2t))/(2 \cosh(3t)) = \sum_{n \geq 0} (-1)^n R_n t^{2n} / (2n)!$  [Gla98, §132, p. 70]. The corresponding OGF  $R(t) := \sum_{n \geq 0} (-1)^n R_n t^{2n}$  satisfies the difference equation

$$R\left(\frac{t}{1+6t}\right) + (6t+1)R(t) - \frac{2(6t+1)(7t^2+6t+1)}{(5t+1)(3t+1)(t+1)} = 0.$$

Similarly, the  $T$ -numbers  $1, 23, 1681, 257543, \dots$  (A002439) are the coefficients  $T_n$  in the expansion  $\sinh(t)/(2 \cosh(2t) - 1) = \sinh(2t)/(2 \cosh(3t)) = \sum_{n \geq 0} (-1)^n T_n t^{2n+1}/(2n+1)!$  [Gla98, §143, p. 75]. The corresponding OGF  $T(t) := \sum_{n \geq 0} (-1)^n T_n t^{2n+1}$  satisfies the difference equation

$$T\left(\frac{t}{1+6t}\right) + (6t+1)T(t) - \frac{2t(6t+1)}{(t+1)(5t+1)} = 0.$$

More generally, one can prove using Proposition 7 that, for any complex numbers  $a, b, c, u, v$ , the OGF  $F(t)$  associated to the EGF of  $(u \sinh(at) + v \cosh(at))/(c \cosh(bt))$  satisfies the difference equation

$$F\left(\frac{t}{1+2bt}\right) + (2bt+1)F(t) + \frac{2(2bt+1)((au+bv)t+v)}{c((a-b)t-1)((a+b)t+1)} = 0.$$

By Theorem 3, we conclude that  $T(t)$ , and hence also  $\sum_{n \geq 0} T_n t^n$ , are (strongly) D-transcendental.

The following is a generalization of Glaisher's  $I$ -numbers and  $J$ -numbers: one starts with

$$\sum_{n \geq 0} f_n \frac{t^n}{n!} := \frac{3}{2} \cdot \frac{e^{at} + e^{-at} + c}{e^{bt} + e^{-bt} + 1} = 1 + \frac{c}{2} + \left(a^2 - \frac{2}{3}b^2 - \frac{1}{3}b^2c\right)t^2 + (b^4c + a^4 - 4b^2a^2 + 2b^4)t^4 + \dots$$

Then, the corresponding OGF  $F(t) = \sum_{n \geq 0} f_n t^n$  satisfies the difference equation

$$F\left(\frac{t}{1+3bt}\right) - (3bt+1)F(t) + r(t) = 0,$$

where  $r(t)$  is the rational function

$$\frac{3bt(3bt+1)((at-bt-1)(at+bt+1)(at-2bt-1)(at+2bt+1)c + 2(bt+1)(2bt+1)(a^2t^2 + 2b^2t^2 + 3bt+1))}{2(bt+1)(2bt+1)(at-bt-1)(at+bt+1)(at-2bt-1)(at+2bt+1)}.$$

Glaisher's  $I$ -numbers [Gla98, §57, p. 35] correspond to the particular choice  $(a, b, c) = (0, 1, 0)$  and the  $J$ -numbers [Gla98, §75, p. 44] correspond to the particular choice  $(a, b, c) = (1, 2, 0)$ . For any  $b \neq 0$ , the EGF admits infinitely many singularities, hence it cannot be D-finite; thus  $F(t)$  cannot be D-finite either, and by Theorem 3  $F(t)$  is necessarily strongly D-transcendental.

### 2.2.6. Three more challenging examples

Here we describe three interesting functional equations from the literature where we cannot conclude differential transcendence using the methods of this article.

*Fishburn-Stoimenow numbers.* The sequence  $(p_n)_{n \geq 0} = (1, 1, 2, 5, 15, 53, 217, 1014, \dots)$  (A022493) with OGF

$$P(t) = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i)$$

is known to count several interesting combinatorial objects. For instance, Zagier proved in [Zag01, Thm. 1] that  $p_n$  is equal to the number of certain involutions on  $2n$  points, called *regular linearized chord diagrams*: these are involutions  $\pi$  in  $S_{2n}$  with no fixed points and such that if  $\pi_i > \pi_{i+1}$ , then  $\pi_i > i \geq \pi_{i+1}$ . On the other hand, Bousquet-Mélou et al. [BMCDK10, Thm. 13] proved that  $p_n$  is the number of unlabeled posets of size  $n$  that do not contain any induced subposet isomorphic to the union of two disjoint 2-element chains (these are called  $(2+2)$ -free posets). Zagier proved [Zag01, Thm. 4] that  $p_n/n! \sim \kappa \cdot (6/\pi^2)^n \cdot n^{1/2}$  with  $\kappa \approx 2.7$ , which implies that  $P(t)$  is not D-finite. He also proved [Zag01, Thm. 3] that

$$P(1 - e^{-24t}) = e^t \cdot \sum_{n \geq 0} \frac{T_n}{n!} t^n,$$

where  $(T_n)_{n \geq 0} = (1, 23, 1681, 257543, \dots)$  is the sequence of Glaisher's  $T$ -numbers. This implies that  $P(t)$  is D-transcendental if and only if the EGF of Glaisher's  $T$ -numbers is D-transcendental. Note that  $\sum_{n \geq 0} T_n t^{2n+1}/(2n+1)!$  is equal to  $\sin(2t)/(2 \cos(3t))$ , hence it is D-algebraic. Recall that we proved that the OGF  $\sum_{n \geq 0} T_n t^n$  is (strongly) D-transcendental. However, none of these results allows to conclude whether the EGF  $\sum_{n \geq 0} \frac{T_n}{n!} t^n$  (equivalently,  $P(t)$ ) is D-transcendental or not. We leave this as an open question. Note that it was proved in [BMCDK10, §6.2] that  $P(t)$  is equal to  $F(t, 1)$  where  $F(t, u) \in \mathbb{Q}[u][[t]] \cap \mathbb{Q}(t)[[u]]$  starts

$$1 + t + (u+1)t^2 + (u^2 + 3u+1)t^3 + (u^3 + 7u^2 + 6u+1)t^4 + \dots = \frac{1}{1-t} + \frac{t^2u}{(1-t)^3} + \frac{t^3(1+t-t^2)u^2}{(1-t)^6} + \dots$$

and satisfies the functional equation

$$F(t, u) = \frac{(1-u)(1-t)}{(tu-t+1)^2} + \frac{u}{(tu-t+1)^2} \cdot F\left(t, \frac{u}{tu-t+1}\right).$$

From the second part of Theorem 2, we can deduce that for any  $t_0 \in \mathbb{C} \setminus \{0\}$  with  $|t_0| < 1$ , we have that  $F(t_0, u) \in \mathbb{C}[[u]]$  is D-transcendental in  $u$  (this is because the homography  $u/(t_0u - t_0 + 1)$  has two fixed points, hence it is conjugated to a dilation). But, once again, this does not give any information about the nature of  $F(t, 1) = P(t)$ . Another example related to the previous one is the following. An unlabeled poset is called  $(3+1)$ -free if it does not contain the disjoint union of chains of lengths 3 and 1 as an induced subposet. The sequence  $(q_n)_{n \geq 0} = (1, 2, 5, 15, 49, 173, 639, 2469, \dots)$  (A079146), counting  $(3+1)$ -free posets with  $n$  unlabelled vertices, was studied by Guay-Paquet, Morales and Rowland in [GPMR14]. They showed that  $q_n \sim 2^{n^2/4 - \kappa n \log n + O(n)}$  for some  $\kappa > 0$ , see [GPMR14, Thm. 4.4 and Table 1], a result which implies that the OGF  $\sum_{n \geq 0} q_n t^n$  is not D-finite. As for  $(2+2)$ -free posets, it is natural to ask whether  $\sum_{n \geq 0} q_n t^n$  is D-transcendental, see Question 1 in Morales' compilation of combinatorial conjectures.

*Schmid's equation.* The following functional equation, with  $q \in \mathbb{N}$  and  $a, c \in (0, 1)$ , was considered in [Sch92, Eq. (2.8)]:

$$f(t) = qf(t/q) - \frac{aq}{1+t} \cdot f\left(\frac{ct/q}{1+(1-c)t}\right) + \frac{qt^2}{1+t^2}.$$

His analysis in Section 3 shows that  $f$  is not a rational function. It would be interesting to prove that  $f$  is (strongly) D-transcendental, by using an extension of the methods in this paper.

*Generalized digital trees.* Flajolet and Richmond [FR92, Lem. 1] solved the following difference-differential equation, with  $b \in \mathbb{N}$ :

$$f^{(b)}(t) = 2e^{t/2} \cdot f\left(\frac{t}{2}\right) + e^t.$$

Here  $f$  is the EGF of the expected number  $f_n$  of nonempty nodes in a random tree built from  $n$  elements, in a tree partitioning process depending on some fixed parameter  $b$  (see the introduction of [FR92] for more details). They showed ([FR92, Lem. 2]) that if  $G(t)$  denotes the inverse Borel transform  $\mathfrak{B}^{-1}(e^{-t} \cdot f(t))$ , then

$$(1+t)^b \cdot G(t) = 2t^b \cdot G\left(\frac{t}{2}\right) + t(1+t)^{b-1}.$$

They proved that  $G$  has infinitely many singularities, and that  $f_n$  has asymptotic behavior incompatible with D-finiteness. It would be nice to be able to study the D-algebraic nature of  $f$  and  $G$ .

### 2.3. An elementary treatment of one example: Bernoulli polynomials

We give here an elementary proof of the D-transcendence of  $B(x, t)$ , based on two main steps: first, a strong link between the expansion of  $B(x, t)$  at  $t = \infty$  and the Euler gamma function  $\Gamma$ , see (2.9); second, Hölder's theorem on the D-transcendence of the  $\Gamma$ -function [Hö86].

Beyond the case of the OGF  $B(x, t)$  of Bernoulli polynomials, thinking of the two-step approach mentioned above, a result of Praagman [Pra86, Thm. 1] ensures that solutions to general  $\tau$ -equations are meromorphic at  $\infty$ ; however, there is no hope in general to express such expansions at  $\infty$  using special functions, nor to prove that these expansions are D-transcendental. As we shall see in Section 2.4, Galois theory comes into play to solve this problem, focusing on the equation itself rather than on its solutions. (Praagman's result applies to the meromorphic behavior on  $\mathbb{C}$  of solutions to difference equations with the shift  $t \mapsto t+1$ , which translates into a similar result on  $\mathbb{C} \cup \{\infty\} \setminus \{0\}$  after our change of variable  $t \mapsto \frac{1}{t}$ .)

**Proposition 11.** *The ordinary generating function  $B(x, t)$  is D-transcendental over  $\mathbb{C}(t)$  for any  $x \in \mathbb{C}$ .*

Notice that the conclusion of Proposition 11 (which can alternatively be deduced by combining Theorem 2 and Lemma 10) is weaker than the one in Theorem 3.

*Proof.* We first prove the result for  $x = 0$ , that is for the OGF of the sequence of Bernoulli numbers  $(B_n)_{n \geq 0}$ :

$$B_0(t) = B(0, t) = 1 - \frac{1}{2}t + \frac{1}{6}t^2 - \frac{1}{30}t^4 + \frac{1}{42}t^6 - \frac{1}{30}t^8 + \dots$$

The idea is to use the "logarithmic Stirling formula": as  $t \rightarrow \infty$ ,

$$\log \Gamma(t) \sim \left(t - \frac{1}{2}\right) \log t - t + \frac{\log(2\pi)}{2} + \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)} t^{2n-1}.$$

More precisely, we will use its consequence on the asymptotic expansion as  $t \rightarrow \infty$  of the derivative of the digamma function  $\Psi(t) := \Gamma'(t)/\Gamma(t)$ , see [EMOT53, Sec. 1.18]:

$$\Psi'(t) \sim_{t \rightarrow \infty} \frac{1}{t} + \frac{1}{2t^2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{t^{2n+1}}, \quad (2.9)$$

and the fact that  $\Gamma(t)$  is D-transcendental. More formally, let us introduce the power series in  $\mathbb{C}[[t]]$  defined by

$$S(t) := t + \frac{t^2}{2} + \sum_{n=1}^{\infty} B_{2n} t^{2n+1}.$$

As  $\Gamma(t)$  is D-transcendental, it follows that  $\Psi(t)$  is D-transcendental, hence the asymptotic expansion (2.9) of  $\Psi'(t)$  at infinity is also D-transcendental (if a meromorphic function on  $\mathbb{C}$  is D-algebraic, then its potential asymptotic expansion at infinity satisfies the same differential equations), and finally  $S(t)$  is D-transcendental. Since  $S(t) = tB_0(t) + t^2$ , it also follows that  $B_0(t)$  is D-transcendental.

Now, let us treat the case of a general  $x \in \mathbb{C}$ . The key is the following equality, which holds in  $\mathbb{C}[x][[t]]$ :

$$B(x, t) = \frac{1}{t} \cdot S\left(\frac{t}{1+t-tx}\right),$$

and which readily implies that  $B(x, t)$  is D-transcendental. To prove this last equality, it is sufficient to check that both sides are power series in  $\mathbb{C}[x][[t]]$  that satisfy the same  $\tau$ -equation from Lemma 10 (since (2.6) admits at most one power series solution). This is an easy consequence of the fact that  $B_0$  satisfies the equation  $\tau(B_0) = (t+1) \cdot B_0 - \frac{t}{1+t}$ , hence  $S$  satisfies  $\tau(S) = S - t^2$ .  $\square$

#### 2.4. Why the situation is not so easy in general

We know from [Pra86, Thm. 1] that any linear functional equation of the form

$$a_0(t)y + a_1(t)\tau(y) + \dots + a_n(t)\tau^n(y) = 0,$$

with  $a_0(t), \dots, a_n(t) \in \mathbb{C}(t)$ , has a basis of meromorphic solutions at  $\infty$ , i.e., in  $\mathbb{C}\left(\left\{\frac{1}{t}\right\}\right)$ . This means that the functional equation (2.6)

$$\tau(B) = (1+t) \cdot B - \frac{t(1+t)}{(1+t-tx)^2}$$

satisfied by the OGF of the Bernoulli polynomials, has a meromorphic solution at  $\infty$ . As it turns out, we can find explicitly this solution at  $\infty$  with some lucky and elementary manipulations, which might give the impression that the proof of Proposition 11 is a bit magical and pulled out of a hat.

These manipulations are alternatively described in the next lemma. Once the solution is found, checking its correctness simply amounts to using the classical identity  $\Psi'(t) - \Psi'(1+t) = \frac{1}{t^2}$ .

**Lemma 12.** *The meromorphic function  $F(x, t) = \frac{1}{t}\Psi'\left(\frac{1+t-tx}{t}\right)$  at  $\infty$  is a solution of (2.6), for any  $x \in \mathbb{C}$ .*

*Proof.* We first remark that  $z(t) = 1/t$  is solution to the homogeneous equation  $\tau(z) = (1+t) \cdot z$ . Setting  $G(x, t) := F(x, t)/z(t) = tF(x, t)$  yields a simpler functional equation, amenable to telescopic summation:

$$\tau(G) = G - \left(\frac{t}{1+t-tx}\right)^2. \quad (2.10)$$

Iterating identity (2.10), we obtain that  $G$  satisfies the following equation for any  $n \geq 1$

$$G - \tau^n(G) = G(x, t) - G\left(x, \frac{t}{1+nt}\right) = \sum_{k=0}^{n-1} \tau^k\left(\frac{t}{1+t-tx}\right)^2 = \sum_{k=1}^n \left(\frac{t}{1+kt-tx}\right)^2.$$

We remind that  $\Psi'(t) = \sum_{k \geq 0} \frac{1}{(t+k)^2}$ . Now, letting  $n \rightarrow \infty$ , we conclude that  $G(x, t) = G(x, 0) + \Psi'\left(\frac{1+t-tx}{t}\right)$ . Since  $\lim_{t \rightarrow 0^+} \Psi'(1/t) = 0$  we can choose  $G(x, 0) = 0$ , hence we have found a solution of the functional equation in Lemma 10 satisfied by  $B(x, t)$ , namely

$$F(x, t) = \frac{1}{t}\Psi'\left(\frac{1+t-tx}{t}\right).$$

This ends the proof.  $\square$



The solution  $F(x, t)$  is a “nice” solution, due to its link with the gamma function. Hölder’s theorem implies immediately that it is D-transcendental over  $\mathbb{C}(t)$ , for any  $x \in \mathbb{C}$ . One could try to use (2.9) to deduce that  $B(x, t)$  is the expansion of  $F(x, t)$ , but this may be a delicate procedure. Galois theory comes into the picture to solve this problem. One can even say that the solution of such a problem is the core of Galois theory, namely recognizing the properties of solutions that depend on the equation, and therefore being able to transfer a property from a solution to another, in spite of the fact that they live in very different algebras of functions. For Eq. (2.10), the question is treated in Example 31, using Proposition 30, where the differential transcendence of  $G(x, t)$ , or of  $tB(x, t)$ , is proven to be equivalent to the differential properties of the rational function  $(t/1 + t - tx)^2$ , that is the inhomogeneous right-hand side of (2.10).

### 3. Differential transcendence via Galois theory

#### 3.1. A survey of difference Galois theory

We quickly review the basic concepts of Galois theory for difference equations. We follow the classical book [vdPS97], in order to state the two main theorems of the theory: the Galois correspondence (see Theorem 23) and the theorem on the dimension of the Galois group (see Theorem 19). All the criteria for the applications we have in mind are their consequences; they form the object of the subsequent sections.

Let us consider a field  $\mathbb{K}$  of characteristic zero equipped with an automorphism  $\tau : \mathbb{K} \rightarrow \mathbb{K}$ . We call  $C$  the subfield of  $\mathbb{K}$  of  $\tau$ -invariant elements of  $\mathbb{K}$ , i.e.,  $C := \mathbb{K}^\tau := \{f \in \mathbb{K} : \tau(f) = f\}$ . The elements of  $C$  are the “constants of the theory”, therefore they are also called  $\tau$ -constants or simply constants, when the meaning is clear from the context.

**Example 13.** We can take for instance  $\mathbb{K} = \mathbb{C}((t))$  and set  $\tau(f(t)) := f\left(\frac{t}{t+1}\right)$ , for any  $f \in \mathbb{C}((t))$ . We claim that the field of constants  $\mathbb{C}((t))^\tau$  of  $\mathbb{C}((t))$  coincides with  $\mathbb{C}$ . Let us assume that there exists  $\sum_{n \geq -N} a_n t^n \in \mathbb{C}((t))$ , for some positive integer  $N$ , with  $a_{-N} \neq 0$ , which is invariant by  $\tau$ . We have

$$\sum_{n=1}^N a_{-n} \left(1 + \frac{1}{t}\right)^n + \sum_{n \geq 0} a_n \frac{t^n}{(t+1)^n} = \sum_{n=1}^N \frac{a_{-n}}{t^n} + \sum_{n \geq 0} a_n t^n.$$

By identifying the coefficients of  $t^{1-N}$ , one sees that  $N a_{-N} + a_{-N+1} = a_{-N+1}$ , hence  $a_{-N} = 0$ . This is in contradiction with our assumptions and therefore we conclude that we must have  $N \leq 0$ . A similar argument allows to exclude the case of a formal power series with  $N > 0$ , and to conclude that the only  $\tau$ -constants are the actual constants. Notice that  $\tau$  induces an automorphism of  $\mathbb{C}(t)$  and of  $\mathbb{C}(\{t\})$  as well. Therefore we also have  $\mathbb{C}(t)^\tau = \mathbb{C}(\{t\})^\tau = \mathbb{C}$ .

We consider a linear functional system  $\tau(\vec{y}) = A\vec{y}$ , where  $A$  is an invertible square matrix of order  $\nu$  with coefficients in  $\mathbb{K}$ ,  $\vec{y}$  is a vector of unknowns and  $\tau$  acts on vectors (and later also on matrices) componentwisely.

*Picard-Vessiot rings.* The Galois theory of difference equations follows the general structure of classical Galois theory. Picard-Vessiot rings play the role of the splitting fields, where we can find abstract solutions that can be manipulated in the proofs.

**Definition 14** ([vdPS97, Def. 1.5]). A Picard-Vessiot ring for  $\tau(\vec{y}) = A\vec{y}$  over  $\mathbb{K}$  is a  $\mathbb{K}$ -algebra  $R$  equipped with an automorphism extending the action of  $\tau$ , that we still call  $\tau$ , and such that:

1.  $R$  does not have any non-trivial proper ideal invariant under  $\tau$ , i.e.,  $R$  is  $\tau$ -simple;
2. there exists  $Y \in \text{GL}_\nu(R)$  such that  $\tau(Y) = AY$  and  $R$  is generated by the entries of  $Y$  and the inverse of  $\det Y$ , that is  $R = \mathbb{K}[Y, \det Y^{-1}]$ .

**Proposition 15** ([vdPS97, §1.1]). A Picard-Vessiot ring always exists. If  $C$  is algebraically closed, then  $R^\tau = C$  and  $R$  is unique up to an isomorphism of  $\mathbb{K}$ -algebras commuting with  $\tau$ .

**Remark 16.** We will not need the explicit construction of  $R$ , but it is quite simple and it may be helpful to have it in mind: one considers the ring of polynomials in the  $\nu^2$  variables  $X = (x_{i,j})$  with coefficients in  $\mathbb{K}$ . Inverting  $\det X$  and setting  $\tau(X) = AX$ , we obtain a ring  $\mathbb{K}[X, \det X^{-1}]$  with an automorphism  $\tau$ . Any of its quotients by a maximal  $\tau$ -invariant ideal is a Picard-Vessiot ring of  $\tau(\vec{y}) = A\vec{y}$  over  $\mathbb{K}$ .

It is important to notice that the ring  $R$  does not need to be a domain. If  $C$  is algebraically closed, we can say more about its structure, namely that it can be written as a direct sum  $R_1 \oplus \cdots \oplus R_r$ , such that:  $R_i = e_i R$ , for some  $e_i \in R$  with  $e_i^2 = e_i$ ;  $R_i$  is a domain;  $\tau$  acts transitively on the  $R_i$ ’s, i.e., changing the order of the  $R_i$ ’s and identifying  $\{1, \dots, r\}$  with the elements of  $\mathbb{Z}/r\mathbb{Z}$ , we have  $\tau(R_i) \subset R_{i+1}$ . See [vdPS97, Cor. 1.16].



**Example 17.** Let  $a \in \mathbb{K}$ ,  $a \neq 0$  and  $R$  be the Picard-Vessiot ring of  $\tau(y) = ay$ . Then there exists  $z \in R$  such that  $\tau(z) = az$  and  $R = \mathbb{K}[z, z^{-1}]$ .

**Example 18.** Let  $a, f$  be non-zero elements of  $\mathbb{K}$  and let us consider the functional equation  $\tau(y) = ay + f$ . It can be rewritten as  $\tau(\vec{y}) = \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix} \vec{y}$ . This system has two linearly independent solution vectors, so that an invertible matrix of solutions has the form  $Y = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix}$ , where  $w, z$  are elements of a Picard-Vessiot ring  $R$ , with  $\tau(z) = az$  and  $\tau(w) = aw + f$ . Then  $R = \mathbb{K}[z, z^{-1}, w]$ .

*The Galois group.* We suppose that the field of constants  $C$  of  $\mathbb{K}$  is algebraically closed. The Galois group  $G$  of  $\tau(\vec{y}) = A\vec{y}$  over  $\mathbb{K}$  is defined to be the group  $\text{Aut}^\tau(R/\mathbb{K})$  of automorphisms of rings  $\varphi : R \rightarrow R$  that commute with  $\tau$  and such that  $\varphi|_{\mathbb{K}}$  is the identity.

Since  $\varphi \in G$  leaves  $\mathbb{K}$  invariant, the matrix  $\varphi(Y)$  is another invertible matrix of solutions of  $\tau(\vec{y}) = A\vec{y}$ . It follows that  $\tau(Y^{-1}\varphi(Y)) = Y^{-1}\varphi(Y)$  and hence that  $Y^{-1}\varphi(Y) \in \text{GL}_\nu(C)$ . In other words, we have a natural group morphism  $G \rightarrow \text{GL}_\nu(C)$ . It depends on the choice of  $Y$  and a different choice gives a conjugated map. The theorem below contains two crucial pieces of information: First of all,  $G$  is a geometric object, more precisely it can be identified with the  $C$ -points of a linear algebraic group. Roughly, this is another way of saying that  $G$  can be identified with a subgroup of matrices of  $\text{GL}_\nu(C)$ , whose entries and their determinant are exactly the points in an algebraic variety of the affine space  $\mathbb{A}_C^{\nu^2+1}$ . Secondly, an algebraic relation among the entries of  $Y$  exists if and only if the dimension of  $G$  is “smaller than expected”. These ideas can be formalized as follows:

**Theorem 19** ([vdPS97, Thm. 1.13 and Cor. 1.18]). *The morphism  $G \rightarrow \text{GL}_\nu(C)$ ,  $\varphi \mapsto Y^{-1}\varphi(Y)$ , represents  $G$  as the group of the  $C$ -points of a linear algebraic subgroup of  $\text{GL}_\nu(C)$ . Moreover, the dimension of  $G$  over  $C$  as an algebraic variety is equal to the transcendence degree of  $R$  over  $\mathbb{K}$ , i.e.,  $\text{tr. deg}_{\mathbb{K}} R = \dim_C G$ .*

**Example 20.** We consider an equation of the form  $\tau(y) = ay$ , as in Example 17. Its Galois group  $G$  is represented through its action on a solution  $z \in R$ . Since any element  $\varphi$  of  $G$  must send  $z$  to another solution of  $\tau(y) = ay$ , we have  $\varphi(z) = c_\varphi z$ , for some  $c_\varphi \in C$ . As noticed in Example 17,  $R = \mathbb{K}[z, z^{-1}]$ , hence  $z$  is invertible in  $R$ , which forces  $c_\varphi$  to be non-zero. Therefore  $G$  is an algebraic subgroup of the multiplicative group  $C^*$  of the field  $C$ . The solution  $z$  is transcendental over  $\mathbb{K}$  if and only if  $G = C^*$ . Since the only proper algebraic subgroups of  $C^*$  are the groups of roots of unity,  $z$  is algebraic over  $\mathbb{K}$  if and only if  $G$  is a group of roots of unity, i.e., if and only if there exists a positive integer  $N$  such that  $c_\varphi^N = 1$ , for any  $\varphi \in G$ . Therefore, for any  $\varphi \in G$ , we have  $\varphi(z^N) = c_\varphi^N z^N = z^N$ . As we will see in Theorem 23, the Galois correspondence implies that  $z^N \in \mathbb{K}$ .

For further reference, we point out the following fact. As explained in Remark 16, the Picard-Vessiot ring  $R$  is a quotient of a polynomial ring  $\mathbb{K}[X, X^{-1}]$  by an ideal invariant under the action of  $\tau$ . If  $z$  is transcendental over  $\mathbb{K}$ , such ideal is necessarily 0 and  $R$  is a domain.

**Example 21.** Let us go back to Example 18. We consider the associated Galois group  $G$  over  $C$ . Any  $\varphi \in G$  must map a matrix of solutions of the functional equation into another matrix of solutions, therefore there exist  $c_\varphi, d_\varphi \in C$  such that  $\varphi(z) = c_\varphi z$  and  $\varphi(w) = w + d_\varphi z$ . In other words, we must have:  $\varphi \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\varphi & d_\varphi \\ 0 & 1 \end{pmatrix}$ . Therefore,  $G$  is a subgroup of  $\tilde{G} := \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} : c, d \in C, c \neq 0 \right\} \subset \text{GL}_2(C)$ . According to whether  $z$  and  $w$  are algebraically dependent or not, either  $G$  will be a proper linear algebraic subgroup of  $\tilde{G}$ , or  $G = \tilde{G}$ .

*The total Picard-Vessiot ring.* We have seen in Remark 16 that  $R$  is not a domain, therefore we cannot consider its field of fractions. However, it is a direct sum of domains  $R_i$  and we can consider its ring  $\mathbb{L}$  of total fractions, which is the direct sum of the fields of fractions  $\mathbb{L}_i$  of the  $R_i$ 's. This means that  $\mathbb{L} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_r$ , where, for any  $i = 1, \dots, r$ ,  $\mathbb{L}_i$  is a field and  $\tau^r$  induces an automorphism of  $\mathbb{L}_i$ . Moreover,  $\tau$  acts transitively on the  $\mathbb{L}_i$ 's, i.e.,  $\tau(\mathbb{L}_i) = \mathbb{L}_{i+1}$ , with the same notation as in Remark 16. We will call  $\mathbb{L}$  the total Picard-Vessiot ring of  $\tau(\vec{y}) = A\vec{y}$ .

The action of the Galois group  $\text{Aut}^\tau(R/\mathbb{K})$  naturally extends from  $R$  to  $\mathbb{L}$ . See [vdPS97, §1.3] for details. For further reference, we recall the following characterization of total Picard-Vessiot rings.

**Proposition 22** ([vdPS97, Prop. 1.23]). *In the notation above, the total Picard-Vessiot ring  $\mathbb{L}$  of  $\tau(\vec{y}) = A\vec{y}$  is uniquely determined, up to an isomorphism of  $\mathbb{K}$ -algebras commuting with  $\tau$ , by the following properties:*

1.  $\mathbb{L}$  has no nilpotent elements and any non-zero divisor of  $\mathbb{L}$  is invertible.

2.  $\mathbb{L}^\tau = C$ .
3. The system  $\tau(\vec{y}) = A\vec{y}$  has a matrix solution in  $\mathrm{GL}_\nu(\mathbb{L})$ .
4.  $\mathbb{L}$  is minimal with respect to the inclusion and the three previous properties.

In particular, any  $\tau$ -ring satisfying the first three properties above contains a copy of the Picard-Vessiot ring  $R$  of  $\tau(\vec{y}) = A\vec{y}$  over  $\mathbb{K}$ .

*The Galois correspondence.* We consider the set  $\mathcal{F}$  of all  $\tau$ -stable rings  $F \subset \mathbb{L}$ , such that  $\mathbb{K} \subset F$  and any element of  $F$  is either a zero divisor or a unit in  $F$ , and the set  $\mathcal{G}$  of all linear algebraic subgroups of  $G$ . For any  $H \in \mathcal{G}$ , we set  $\mathbb{L}^H = \{f \in \mathbb{L} : \varphi(f) = f \text{ for all } \varphi \in H\}$  and, for any  $F \in \mathcal{F}$ , we set  $H_F := \{\varphi \in \mathcal{G} : \varphi(f) = f \text{ for all } f \in F\}$ .

**Theorem 23** (Galois correspondence [vdPS97, Thm. 1.29]). *In the notation above, the following two maps are each other's inverses:*

$$\begin{array}{ccc} \mathcal{G} & \rightarrow & \mathcal{F} \\ H & \mapsto & \mathbb{L}^H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F} & \rightarrow & \mathcal{G} \\ F & \mapsto & H_F \end{array} .$$

In particular,  $\mathbb{L}^H = \mathbb{K}$  if and only if  $H = G$ .

**Remark 24.** *We would like to warn the reader that the normal subgroups of  $G$  do not satisfy the property that one would expect. See [vdPS97, Cor. 1.30]. If  $H$  is a normal subgroup of  $G$  then the group  $\mathrm{Aut}^\tau(\mathbb{L}^H/\mathbb{K})$  is isomorphic to  $G/H$ . Moreover,  $H$  is a normal subgroup if and only if for any element  $f \in \mathbb{L}^H$  there exists  $\varphi \in \mathrm{Aut}^\tau(\mathbb{L}^H/\mathbb{K})$  such that  $\varphi(f) \neq f$ .*

We start mentioning the following immediate consequence of the theorem above, which illustrates how the transcendental nature of a solution to the homogeneous equation transfers to solutions of the inhomogeneous system. This statement is well known to specialists.

**Corollary 25.** *In the notation of Example 21, if  $z$  is transcendental over  $\mathbb{K}$  and  $w \notin \mathbb{K}$ , then also  $w$  is transcendental over  $\mathbb{K}$ .*

*Proof.* We suppose that  $w$  is algebraic over  $\mathbb{K}$ . Since  $w \notin \mathbb{K}$ , there exists  $\varphi \in G$  such that  $\varphi(w) = d_\varphi z + w \neq w$ , i.e.,  $d_\varphi \neq 0$ . The element  $\varphi(w)$  is necessarily algebraic over  $\mathbb{K}$ , and therefore  $\varphi(w) - w = d_\varphi z$  is also algebraic over  $\mathbb{K}$ .  $\square$

The purpose of the following subsections is to give a proof of an analogous statement for differential transcendence.

### 3.2. Application to differential transcendence

*The  $\partial$ -Picard-Vessiot ring.* Now, we suppose that there exists a derivation  $\partial$  on  $\mathbb{K}$  commuting with  $\tau$ .

**Example 26.** *In the situation of Example 13, for  $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$ , we can take  $\partial := t^2 \frac{d}{dt}$ .*

**Proposition 27** ([Wib12] and [DVH12, Prop. 1.16, Rem. 1.18, Cor. 1.19]). *For any linear system of the form  $\tau(\vec{y}) = A\vec{y}$ , with  $A \in \mathrm{GL}_\nu(\mathbb{K})$ , there exists a  $\mathbb{K}$ -algebra  $\mathcal{R}$ , equipped with an extension of  $\tau$  and of  $\partial$ , preserving the commutation, such that:*

1. there exists  $Z \in \mathrm{GL}_\nu(\mathcal{R})$  such that  $\tau(Z) = AZ$ ;
2.  $\mathcal{R}$  is generated over  $\mathbb{K}$  by the entries of  $Z$ ,  $\frac{1}{\det(Z)}$  and all their derivatives;
3.  $\mathcal{R}$  is  $\tau$ -simple.

Moreover, the total ring of fractions of  $\mathcal{R}$  satisfies the first three properties of Proposition 22.

We will call the ring  $\mathcal{R}$  the  $\partial$ -Picard-Vessiot ring of  $\tau(\vec{y}) = A\vec{y}$  over  $\mathbb{K}$ , without giving a formal definition. Applying  $\partial^n$  to the system  $\tau(\vec{y}) = A\vec{y}$  for any positive integer  $n$ , we can consider the difference system:

$$\tau(\vec{y}) = \begin{pmatrix} A & \binom{n}{1}\partial(A) & \cdots & \binom{n}{n}\partial^n(A) \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \binom{n}{1}\partial(A) \\ 0 & \cdots & 0 & A \end{pmatrix} \vec{y}, \quad \text{with solution} \quad \begin{pmatrix} Z & \partial(Z) & \cdots & \partial^n(Z) \\ 0 & Z & \ddots & \vdots \\ \vdots & \ddots & \ddots & \partial(Z) \\ 0 & \cdots & 0 & Z \end{pmatrix}. \quad (3.1)$$

The following corollary is hidden in the proof of [DVH12, Prop. 1.16]. Since  $\mathcal{R}$  is generated by  $Z$  and its derivatives, it is a fairly natural consequence of Proposition 22. We do not prove it, however we can point out that one would mainly need to prove that the rings generated by  $Z$ , its derivatives up to a certain order and  $Z^{-1}$  are  $\tau$ -simple.

**Corollary 28.** *For any integer  $n \geq 0$ , the ring  $\mathcal{R}$  contains a copy of the Picard-Vessiot ring of (3.1).*

Before dealing with the general case of linear  $\tau$ -difference equations  $\tau(y) = ay + f$  of order 1, we study the particular cases  $f = 0$  and  $a = 1$ .

**Notation 29.** *From now on,  $F$  will be a  $\mathbb{K}$ -algebra with no nilpotent elements, and such that any element is either a zero divisor, or invertible. We suppose that  $F$  is equipped with an extension of  $\tau$  and of  $\partial$ , preserving the commutation, and that  $F^\tau = C$ . Of course, it does not need to be the same algebra in all the statements.*

Proposition 22 thus implies that  $F$  contains a copy of the (total) Picard-Vessiot ring of (3.1) for any  $n \geq 0$ . They form an ascending chain of subrings of  $F$  and their union coincides with  $\mathcal{R}$  (see Corollary 28).

We say that an element of  $F$  is differentially algebraic over  $\mathbb{K}$  if it satisfies an algebraic differential equation (with respect to the derivation  $\partial$ ) with coefficients in  $\mathbb{K}$ , and that it is differentially transcendental otherwise.

**Proposition 30.** *Let  $f$  be a non-zero element of  $\mathbb{K}$ , and let  $w \in F$  be such that  $\tau(w) = w + f$ . Then the following assertions are equivalent:*

1.  $w$  is differentially algebraic over  $\mathbb{K}$ .
2. There exist a non-negative integer  $n$ ,  $\alpha_0, \dots, \alpha_n \in C$  (not all zero) and  $g \in \mathbb{K}$  such that  $\alpha_0 f + \alpha_1 \partial(f) + \dots + \alpha_n \partial^n(f) = \tau(g) - g$ .
3. There exist a non-negative integer  $n$  and  $\alpha_0, \dots, \alpha_n \in C$  (not all zero) such that  $g := \sum_{i=0}^n \alpha_i \partial^i(w) \in \mathbb{K}$ .

*Proof.* It follows from Corollary 28 that  $F$  contains a copy of the Picard-Vessiot ring  $R_{f,n}$  of the system  $\{\tau(y_i) = y_i + \partial^i(f), i = 0, \dots, n\}$ . Notice that the latter system can be written in the form of a linear  $\tau$ -difference system as follows (where  $\text{diagonal}(A_1, A_2, \dots, A_n)$  indicates a block-diagonal matrix, having the blocks  $A_1, A_2, \dots, A_n$  on the diagonal):

$$\tau(\vec{y}) = \text{diagonal} \left( \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \partial(f) \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & \partial^n(f) \\ 0 & 1 \end{pmatrix} \right) \vec{y}. \quad (3.2)$$

For any  $i \geq 0$ ,  $\partial^i(w)$  is a solution of  $\tau(y_i) = y_i + \partial^i(f)$ , so that  $R_{f,n} = \mathbb{K}[w, \partial(w), \dots, \partial^n(w)] \subset F$ , as in Example 18 and Example 21.

By definition, the element  $w \in F$  is differentially algebraic over  $\mathbb{K}$  if and only if there exists  $n \geq 0$  such that  $w, \partial(w), \dots, \partial^n(w)$  are algebraically dependent over  $\mathbb{K}$ , therefore if and only if there exists  $n \geq 0$  such that  $\text{tr. deg}_{\mathbb{K}} R_{f,n} \leq n$ . Moreover, for any  $\varphi \in G_n := \text{Aut}^\tau(R_{f,n}/\mathbb{K})$ , we must have  $\varphi(\partial^i(w)) = \partial^i(w) + d_{\varphi,i}$ , for some  $d_{\varphi,i} \in C$ . The composition of two automorphisms  $\varphi$  and  $\psi$  in the Galois group is represented by the sum  $d_{\varphi,i} + d_{\psi,i}$ . This means that we can identify the Galois group to a subgroup of the vector space  $(C, +)^{n+1}$ . Theorem 19 implies that  $\text{tr. deg}_{\mathbb{K}} R_{f,n} \leq n$  if and only if  $\dim_C G_n \leq n$ , hence if and only if  $G_n$  is a proper linear subgroup of  $(C, +)^{n+1}$ , i.e.,  $G_n$  is contained in a hyperplane of  $C^{n+1}$ . Thus, the property  $\text{tr. deg}_{\mathbb{K}} R_{f,n} \leq n$  is equivalent to the existence of  $\alpha_0, \dots, \alpha_n \in C$  (not all zero) such that for any  $\varphi \in G$ , we have  $\sum_{i=0}^n \alpha_i d_{\varphi,i} = 0$ . The last linear relation is equivalent to the fact that  $g := \sum_{i=0}^n \alpha_i \partial^i(w)$  is  $G$ -invariant, that is  $\varphi(\sum_{i=0}^n \alpha_i \partial^i(w)) = \sum_{i=0}^n \alpha_i \partial^i(w)$  for any  $\varphi \in G$ . Theorem 23 implies that the latter condition is equivalent to the fact that  $g$  belongs to  $\mathbb{K}$  and we obtain:

$$\tau(g) - g = \tau \left( \sum_{i=0}^n \alpha_i \partial^i(w) \right) - \sum_{i=0}^n \alpha_i \partial^i(w) = \sum_{i=0}^n \alpha_i \partial^i(f).$$

The proof is completed.  $\square$

The equivalence between the first two assertions of the previous statement is an immediate corollary of [Har08, Prop. 2.1], applied to the system  $\{\tau(y_i) = y_i + \partial^i(f), i = 0, \dots, n\}$ . The equivalence with the third assertion is contained in the cited proof and explains how, under these assumptions, differential algebraicity (i.e., existence of an algebraic differential equation for  $w$ ) and D-finiteness (i.e., existence of a linear differential equation for  $w$ ) are equivalent. It also explains how to recover a linear differential equation for the solution  $w$  from the “telescoper” for  $f$  in the second assertion. One can find it in this form in [DV21, Cor. 6.7]. Let us notice that a similar criterion, but in a much more sophisticated setting, has been proved by Papanikolas in [Pap08] and applied to the algebraic independence of Carlitz logarithms (see Thm. 1.2.6 in *loc. cit.*). Even though they are defined in positive characteristic, they present some analogies with the ordinary and exponential generating series considered in §2.

**Example 31.** We consider the OGF of the family of Bernoulli polynomials, which satisfies the functional equation

$$\tau(B) = (1+t) \cdot B - \frac{t(1+t)}{(1+t-tx)^2},$$

as proved in Lemma 10. Setting  $G(x, t) := tB(x, t)$ , one obtains the functional equation:

$$\tau(G) = G - \left( \frac{t}{1+t-tx} \right)^2.$$

We need to prove that  $G$  is differentially transcendental over  $\mathbb{C}(t)$ , for any fixed value of  $x \in \mathbb{C}$ , to conclude the differential transcendence of  $B$ . We do so supposing by contradiction that  $G$  is differentially algebraic over  $\mathbb{C}(t)$ . The argument that follows will appear again in the proof of Theorem 39 (precisely in the proof of Lemma 46) and is actually used frequently in this kind of problems. One can apply Proposition 30 and show that for any non-negative integer  $n$ , there do not exist any constants  $\alpha_0, \dots, \alpha_n$  (not all zero) and any function  $g \in \mathbb{C}(t)$  such that

$$\alpha_0 f + \alpha_1 \partial(f) + \dots + \alpha_n \partial^n(f) = \tau(g) - g, \quad (3.3)$$

with  $\partial = t^2 \frac{d}{dt}$  and  $f = \left( \frac{t}{1+t-tx} \right)^2$ . We easily compute, for all  $k \geq 0$ ,

$$\partial^k(f) = (k+1)! \left( \frac{t}{1+t-tx} \right)^{k+2}. \quad (3.4)$$

Let us first deal with the case  $x \neq 1$ . By (3.4), the left-hand side of (3.3) has a unique pole, namely, at  $t_0$  such that  $1+t_0-t_0x=0$  (i.e.,  $t_0 = \frac{1}{x-1}$ ). This proves that the right-hand side of (3.3) has only one pole at  $t_0$ , and hence either  $g$  or  $\tau(g)$  has a pole at  $t_0$ . To fix ideas, let us assume that  $g$  has a pole at  $t_0$ . Then  $\tau^{-1}(t_0)$  is a pole of  $\tau(g)$ . If  $\tau(g) - g$  has a pole at  $\tau^{-1}(t_0)$ , we have proved that  $\tau(g) - g$  has at least two poles, a contradiction with (3.3). If  $\tau(g) - g$  has no pole at  $\tau^{-1}(t_0)$ , then we conclude that  $g$  has also a pole at  $\tau^{-1}(t_0)$ , that cancels the pole of  $\tau(g)$  in  $\tau(g) - g$ . It means that  $\tau(g)$  must have a pole at  $\tau^{-2}(t_0)$  and we can repeat the argument. Since  $g$  and  $\tau(g)$  are rational functions, they admit a finite number of poles, therefore eventually one shows that  $\tau(g) - g$  has two poles, contradicting the fact that the left-hand side of (3.3) has only one pole.

In the case  $x = 1$ , the left-hand side of (3.3) becomes a non-zero polynomial with no constant term. One first proves that  $g \in \mathbb{C}(t)$  must also be a polynomial and finally gets a contradiction, showing that it can only be a constant, in contradiction with the fact that the left-hand side is non-zero. The key point here is that  $\infty$  is the only pole of a polynomial and that  $\tau$  does not fix  $\infty$ .

Let  $a$  be a non-zero element of  $\mathbb{K}$ . We consider the equation  $\tau(y) = ay$  and the Picard-Vessiot ring  $R_{a,n}$  over  $\mathbb{K}$  of the linear difference system obtained as in (3.1) for  $A = (a)$ . The following proposition is proved by using the trick of taking logarithmic derivatives, as in [Har08, Prop. 2.2].

**Proposition 32.** Let  $a \in \mathbb{K}$  be non-zero,  $F$  be a  $\mathbb{K}$ -algebra as in Notation 29 and let  $z \in F$  be a non-zero solution of  $\tau(y) = ay$ . With the notation introduced above, the following assertions are equivalent:

1. The element  $z$  is differentially algebraic over  $\mathbb{K}$ .
2. There exists a non-negative integer  $n$  such that  $\text{tr. deg}_{\mathbb{K}} R_{a,n} \leq n$ .
3. There exists a non-negative integer  $n$  such that there exist  $\alpha_0, \dots, \alpha_n \in C$  (not all zero) and  $g \in \mathbb{K}$  satisfying

$$\alpha_0 \frac{\partial(a)}{a} + \alpha_1 \partial \left( \frac{\partial(a)}{a} \right) + \dots + \alpha_n \partial^n \left( \frac{\partial(a)}{a} \right) = \tau(g) - g. \quad (3.5)$$

*Proof.* We have  $R_{a,n} = \mathbb{K}[z, \partial(z), \dots, \partial^n(z), z^{-1}] \subset F$ , up to an automorphism commuting with  $\tau$ . The first equivalence follows immediately from the definition of differential algebraicity and Theorem 19. We notice that  $z$  is differentially algebraic over  $\mathbb{K}$  if and only if  $w := \frac{\partial(z)}{z} \in F$  is differentially algebraic over  $\mathbb{K}$ . Since we have  $\tau(w) = \frac{\tau(\partial(z))}{\tau(z)} = \frac{\partial(\tau(z))}{\tau(z)} = \frac{\partial(az)}{az} = w + \frac{\partial(a)}{a}$ , we conclude by applying Proposition 30.  $\square$

**Example 33.** Let us consider the setting of Examples 13 and 26, with  $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$  and  $\partial = t^2 \frac{d}{dt}$ . The function  $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$ , which lives in the algebra of analytic functions over  $\mathbb{C}^* \cup \{\infty\}$ , is a solution of the

equation  $\tau(y) = ty$ , that is the homogenous equation associated to Klazar's example (1.1). Since  $\frac{\partial(t)}{t} = t$  and  $\partial^n(t) = n!t^{n+1}$ , one deduces that for any  $\alpha_0, \dots, \alpha_n \in \mathbb{C}$  (not all zero), the linear combination

$$\alpha_0 t + \alpha_1 \partial(t) + \dots + \alpha_n \partial^n(t) = \alpha_0 t + \alpha_1 t^2 + \dots + \alpha_n n! t^{n+1}$$

cannot be written as  $g\left(\frac{t}{1+t}\right) - g(t)$ , for any  $g \in \mathbb{C}(t)$ . This is easily proved by reasoning on the poles of the potential rational functions  $g$ , as in Example 31. We deduce from Proposition 32 that  $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$  is  $D$ -transcendental over  $\mathbb{K} := \mathbb{C}(t)$ , which reproves Hölder's theorem [Hö86, BK78, HS08] on the  $D$ -transcendence of the gamma function. We also deduce from Proposition 32 that any non-zero solution of  $\tau(y) = ty$  in any algebra  $F$  as above is differentially transcendental<sup>1</sup> over  $\mathbb{C}(t)$ .

*Differential transcendence:* the equation  $\tau(y) = ay + f$ . We want to prove a generalization of Corollary 25 to differential transcendence (see Theorem 34 below). Let  $f$  and  $a$  be non-zero elements of  $\mathbb{K}$ , and let us consider the difference equation  $\tau(y) = ay + f$ . It can be transformed in a linear system as in the examples above:  $\tau(\vec{y}) = A\vec{y}$ , with  $A := \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix}$ . The following statement generalizes [HS08, Item 1 of Prop. 3.8] to the case of an algebraically closed field of constants. We remind that in [HS08] the authors assume that the field of constants is differentially closed (although it is not difficult to prove a descent to any algebraically closed field, for the readers familiar with this kind of reasoning). The advantage of the statement below, compared to [HS08], is that one can look for the solutions of the inhomogeneous equation and the associated homogeneous equation in two different algebras. As we have already observed in §2.4, this is particularly useful in our setting. We will illustrate further the situation in Remark 36 below.

**Theorem 34.** *Let us consider an equation of the form  $\tau(y) = ay + f$ , with  $a, f \in \mathbb{K}$ , such that  $a \neq 0, 1$  and  $f \neq 0$ . Let  $F/\mathbb{K}$  be a field extension, with an extension of  $\tau$  and  $\partial$  to  $F$  preserving the commutativity, such that there exists  $w \in F \setminus \mathbb{K}$  satisfying the equation  $\tau(w) = aw + f$ . Moreover, let  $F_a$  be a  $\mathbb{K}$ -algebra as in Notation 29, such that there exists  $z \in F_a$  satisfying the equation  $\tau(z) = az$ . If  $z$  is differentially transcendental over  $\mathbb{K}$ , then  $w$  is differentially transcendental over  $\mathbb{K}$ .*

Proposition 32 and Theorem 34 imply directly the following corollary:

**Corollary 35.** *In the notation of the theorem above, if for any  $n \geq 0$ , any  $\alpha_0, \dots, \alpha_n \in \mathbb{C}$  (not all zero) and any  $g \in \mathbb{K}$ , we have*

$$\alpha_0 \frac{\partial(a)}{a} + \alpha_1 \partial\left(\frac{\partial(a)}{a}\right) + \dots + \alpha_n \partial^n\left(\frac{\partial(a)}{a}\right) \neq \tau(g) - g, \quad (3.6)$$

then  $w$  is differentially transcendental over  $\mathbb{K}$ .

**Remark 36.** *The corollary above allows us to conclude immediately that the generating function of the Bell numbers is differentially transcendental over  $\mathbb{C}(t)$ , as a consequence of Hölder's theorem. See Example 33. If we want to apply Theorem 34, it is enough to take  $F = \mathbb{C}((t))$  and  $F_a$  the field of meromorphic functions at  $\infty$ . Notice that there is no common natural extension of the fields  $F$  and  $F_a$ .*

We will rather prove the following statement, which is obviously equivalent to Theorem 34:

**Proposition 37.** *In the notation of Theorem 34, if  $w \in F \setminus \mathbb{K}$  is differentially algebraic over  $\mathbb{K}$ , then  $z \in F_a$  is differentially algebraic over  $\mathbb{K}$ .*

*Proof.* Let  $R_n$  be the Picard-Vessiot ring over  $\mathbb{K}$  of the system obtained from  $\tau(\vec{y}) = \begin{pmatrix} a & f \\ 0 & 1 \end{pmatrix} \vec{y}$  by  $n$ -fold derivation as in (3.1), and  $\mathcal{R}$  be its  $\partial$ -Picard-Vessiot ring over  $F$ . It follows from Proposition 27 and Proposition 22 that we can suppose without loss of generality that  $R_n \subset \mathcal{R}$ , because  $\mathbb{K} \subset F$ , by assumption. Moreover,

$$R_n = \mathbb{K}[\tilde{z}, \partial(\tilde{z}), \dots, \partial^n(\tilde{z}), w, \partial(w), \dots, \partial^n(w), \tilde{z}^{-1}],$$

for some  $\tilde{z}$  satisfying  $\tau(\tilde{z}) = a\tilde{z}$ .

Let us assume that  $w$  is differentially algebraic over  $\mathbb{K}$ . This means that  $w$  satisfies an algebraic differential equation of order  $\kappa$  with coefficients in  $\mathbb{K}$ . Let  $\varphi \in \text{Aut}^\tau(R_\kappa/\mathbb{K})$  and let us consider  $\varphi(w) \in R_\kappa$  such that

<sup>1</sup>After we submitted to the arXiv a first version of the present paper, Michael F. Singer has brought to our attention the fact that in 2010, in a private letter to Charlotte Hardouin, he had given a Galoisian proof of Klazar's theorem on the OGF of Bell numbers, based on some calculations very similar to this example and [HS08, Prop. 3.9].



$\tau(\varphi(w)) = a\varphi(w) + f$ . The derivatives of  $\varphi(w)$  may not belong to  $R_\kappa$ , since  $\varphi$  commutes only with  $\tau$  and not necessarily with  $\partial$ , however the  $\mathbb{K}$ -algebra

$$\tilde{R}_\kappa = \mathbb{K}[\tilde{z}, \partial(\tilde{z}), \dots, \partial^\kappa(\tilde{z}), \varphi(w), \partial(\varphi(w)), \dots, \partial^\kappa(\varphi(w)), z^{-1}] \subset \mathcal{R}$$

is another Picard-Vessiot ring of the difference system obtained by  $\kappa$ -fold iteration. Hence  $R_\kappa$  and  $\tilde{R}_\kappa$  are isomorphic as difference  $\mathbb{K}$ -algebras (meaning that the isomorphism of  $\mathbb{K}$ -algebras commutes with  $\tau$ ) and, in particular, they have the same transcendence degree over  $\mathbb{K}$ . We deduce that  $\varphi(w) \in \mathcal{R}$  is solution of an algebraic differential equation of order  $\kappa$ , with coefficients in  $\mathbb{K}$ . Finally,  $w - \varphi(w)$  satisfies an algebraic differential equation of order at most  $2(\kappa + 1)$  (see for instance [BR86, Thm. 2.2]). Since  $\tilde{z} := w - \varphi(w) \in \mathcal{R}$  is a solution of  $\tau(y) = ay$ , we deduce by Proposition 32, applied to  $\tilde{z} \in \mathcal{R}$ , that there exist  $\alpha_0, \dots, \alpha_n \in C$  (not all zero) and  $g \in \mathbb{K}$  such that  $\sum_{i=0}^n \alpha_i \partial^i \left( \frac{\partial(a)}{a} \right) = \tau(g) - g$ . We conclude, by applying again Proposition 32 to  $z \in F_a$ , that  $z \in F_a$  is itself differentially algebraic over  $\mathbb{K}$ .  $\square$

**Remark 38.** *The previous proof would be easier if we were able to find an automorphism  $\varphi$  which commutes with both  $\tau$  and  $\partial$ . This is the starting point of the paper [HS08], where a more sophisticated Galois theory to deal with differential transcendence is developed. The idea is to consider as Galois group the group of automorphisms of the  $\partial$ -Picard-Vessiot ring introduced in Proposition 27, which commutes with  $\tau$  and  $\partial$ . Unfortunately, under the assumption that  $C$  is algebraically closed, such a group can be reduced to the identity. To avoid this problem one has either to enlarge drastically  $C$ , as in [HS08], or to adopt a group scheme point of view, as for instance in [DVH12]. Here, we have decided to avoid more abstract theories, since we only deal with order 1, which can be handled more easily.*

#### 4. Main result: Strong differential transcendence for first-order difference equations

We consider a field  $\mathbb{K}_0 := C(t)$ , where  $C$  is an algebraically closed field of characteristic zero, equipped with an Archimedean norm  $|\cdot|$ . Typically, we will choose  $C$  to be the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$  or  $\mathbb{C}$ , with the usual norm.

We will look for solutions in  $\mathbb{F} := C((t))$  and we will establish their differential transcendence over the field  $\mathbb{K} := C(\{t\})$ , where the convergence is considered with respect to  $|\cdot|$ . We recall (Example 26) that the derivation  $\partial := t^2 \frac{d}{dt}$  commutes with  $\tau : f(t) \mapsto f(t/(1+t))$  and that establishing the differential transcendence with respect to  $\partial$  is equivalent to establishing it with respect to  $\frac{d}{dt}$ .

The next theorem generalizes [Nis84, Thm. 2], where the author proves a similar statement for the differential transcendence over  $\mathbb{K}_0$ . It should be the first step of a generalization of [ADH21, Thm. 1.2] to higher-order difference equations (in the case called  $S_\infty$  in *loc. cit.*).

**Theorem 39.** *Let  $a, f \in \mathbb{K}_0$ , with  $a \neq 0$ , and let  $w \in \mathbb{F} \setminus \mathbb{K}_0$  satisfy the difference equation  $\tau(w) = aw + f$ . Then  $w$  is differentially transcendental over  $\mathbb{K}$ .*

A simple rational change of variable shows the following corollary:

**Corollary 40.** *The theorem above holds if we replace  $\tau$  with the endomorphism associated to any homography with only one fixed point  $t_0$ ,  $\mathbb{K}$  with the field of germs of meromorphic functions at  $t_0$  and  $F$  with the field of formal Laurent series based at  $t_0$ .*

Theorem 39 above implicitly says that any solution  $w$  of  $\tau(w) = aw + f$ , meromorphic in a punctured neighborhood of 0, is the Taylor expansion of a rational function. We start by proving this fact:

**Lemma 41.** *Let  $a, f \in \mathbb{K}_0$  with  $a \neq 0$ , and let  $w \in \mathbb{K}$  satisfy the difference equation  $\tau(w) = aw + f$ . Then  $w \in \mathbb{K}_0$ .*

*Proof.* Notice that if  $\tau(w) = aw + f$ , then  $\tau^2(w) = \tau(a)aw + (\tau(a)f + \tau(f))$ , with  $\tau(a)a, \tau(a)f + \tau(f) \in \mathbb{K}_0$ . By induction, one proves that for any  $n \geq 1$ ,  $w$  satisfies a functional equation of the form  $\tau^n(w) = a_n w + f_n$ , with  $a_n := \tau^{n-1}(a) \cdots \tau(a)a \in \mathbb{K}_0$  and  $f_n \in \mathbb{K}_0$  defined recursively by  $f_n = \tau(a_{n-1})f + \tau(f_{n-1})$ .

By assumption,  $w$  is analytic in an open punctured disk of radius  $r$  centered at 0. Let  $t_0 \in C \setminus \{-\frac{1}{k}, k \in \mathbb{Z}_{\geq 1}\}$  be outside such a disk, i.e.,  $|t_0| \geq r$ . Then  $\tau^n(t_0) = \frac{t_0}{1+nt_0}$ , for any  $n \geq 1$ . Therefore  $\tau^n(t_0)$  tends to 0 as  $n$  tends to  $+\infty$ . This means that there exists  $n$  such that  $|\tau^n(t_0)| < r$ , so that the functional equation  $\tau^n(w) = a_n w + f_n$  allows to continue  $w$  to a meromorphic function in a neighborhood of  $t_0$ . If  $t_0 = -\frac{1}{k}$ , for some positive integer  $k$ , then  $\tau^{-n}(-\frac{1}{k}) = -\frac{1}{n+k}$ , which also tends to 0. Applying  $\tau^{-1}$  recursively to  $\tau(w) = aw + f$ , we obtain a functional equation that allows to continue  $w$  in a neighborhood of  $t_0$ . Summarizing,  $w$  is actually meromorphic on the whole of  $C$ . In particular  $w$  is meromorphic in a neighborhood of  $-1$ , with  $\tau(-1) = \infty$ ,

therefore the initial functional equation  $\tau(w) = aw + f$  allows to continue  $w$  to a meromorphic function in a neighborhood of  $\infty$ . Finally  $w$  can be continued to a meromorphic function on  $C \cup \{\infty\}$ , hence it is actually a rational function.  $\square$

The following corollary explains the strategy of the proof:

**Corollary 42.** *Theorem 39 is equivalent to the following statement:*

(T) *Let  $a, f \in \mathbb{K}_0$  with  $a \neq 0$ , and let  $w \in \mathbb{F} \setminus \mathbb{K}$  satisfy the difference equation  $\tau(w) = aw + f$ . Then  $w$  is differentially transcendental over  $\mathbb{K}$ .*

*Proof.* If  $w$  satisfies the hypotheses of (T), then Theorem 39 implies that  $w$  is differentially transcendental over  $\mathbb{K}$ . Therefore (T) is a consequence of Theorem 39.

On the other hand, let  $w \in \mathbb{F} \setminus \mathbb{K}_0$  as in Theorem 39. If  $w \in \mathbb{K}$ , then  $w \in \mathbb{K}_0$ , thanks to Lemma 41, which contradicts the hypotheses of Theorem 39, hence  $w \in \mathbb{F} \setminus \mathbb{K}$ . Statement (T) implies that  $w$  is differentially transcendental over  $\mathbb{K}$ . This proves that (T) implies Theorem 39.  $\square$

Thanks to the statement (T), the strategy of the proof is becoming clear: if the equation  $\tau(y) = ay$  has a differentially transcendental solution, one applies Theorem 34; otherwise we need to look closer to the structure of the space of solutions of  $\tau(y) = ay$ . The following couple of lemmas are the main steps in the proof of Theorem 39.

**Lemma 43.** *Let  $a \in C \setminus \{0, 1, \text{roots of unity}\}$  and let  $F_a$  be a  $\mathbb{K}$ -algebra as in Notation 29 containing a solution  $z$  of  $\tau(y) = ay$ , minimal for the inclusion. Then  $z$  satisfies a differential equation of the form  $\partial(z) = cz$ , for some  $c \in C$ , and  $F_a$  is a field of the form  $F_a = \mathbb{K}(z)$ , with  $z$  transcendental over  $\mathbb{K}$ .*

**Remark 44.** *In Lemma 43 we may of course take  $F_a = \mathbb{F}$ , but such a result would not be enough to prove Theorem 39. Indeed, generically one cannot expect that both the functional equations  $\tau(w) = aw + b$  and  $\tau(z) = az$  have a solution in  $\mathbb{F}$ . Most examples, such as the generating series of Bell's numbers, actually do not.*

*We point out that if  $a = 1$ , then  $F_a = \mathbb{K}$ , since  $z \in C$ .*

*Proof of Lemma 43.* It follows from Proposition 22 that  $F_a$  is generated by  $z$  over  $\mathbb{K}$  and that  $F_a$  is unique up to a  $\mathbb{K}$ -algebra isomorphism commuting with  $\tau$ . Since  $\partial$  and  $\tau$  commute, we have  $\tau(\partial(z)) = a\partial(z)$ . Therefore  $z$  and  $\partial(z)$  are solutions to the same functional equation and  $\partial(z)/z = c \in C$ .

We want to show that  $z$  is transcendental over  $\mathbb{K}$ , by contradiction. Following Example 20, if  $z$  is algebraic over  $\mathbb{K}$  then there exists a positive integer  $N$  such that  $z^N \in \mathbb{K}$ . It follows from Lemma 41 that we actually have  $z^N \in \mathbb{K}_0$ , with  $\tau(z^N) = a^N z^N$ . Since  $a \notin \{0, 1, \text{roots of unity}\}$ , iterating the functional equation  $\tau(z^N) = a^N z^N$  one shows that if  $z^N$  has a non-zero pole (resp. zero), then it has an infinite number of poles (resp. zeros). Therefore  $z^N$  has a pole or a zero only at 0 and it must be a monomial. If  $z^N$  is a monomial, then  $a = \tau(z^N)z^{-N}$  is an integer power of  $(1+t)$  and cannot be a constant, unless  $z^N$  is a constant itself, i.e.,  $N = 0$  and  $a = 1$ . We have found a contradiction, hence  $z$  is transcendental. Thanks to Example 20, we conclude that the Picard-Vessiot ring of  $\tau(z) = az$  is a domain and hence that  $F_a = \mathbb{K}(z)$  is a field.  $\square$

**Example 45.** *Let  $a \in C \setminus \{0, 1, \text{roots of unity}\}$ . Let us fix a branch of the complex logarithm, so that  $\exp\left(-\frac{\log a}{t}\right)$  is a solution to  $\tau(y) = ay$ . The field  $\mathbb{K}\left(\exp\left(-\frac{\log a}{t}\right)\right)$  is a total Picard-Vessiot ring for  $\tau(y) = ay$  over  $\mathbb{K}$ . Notice that  $\exp\left(-\frac{\log a}{t}\right)$  is not meromorphic at 0, hence its expansion does not belong to  $\mathbb{F}$ . One can show by a direct substitution that in this case  $\tau(y) = ay$  cannot have a solution in  $\mathbb{F}$ . In other words, the hypothesis of Theorem 39 with  $f = 0$  is never satisfied.*

The proof of the following lemma is an elaborated version of the argument of Example 31:

**Lemma 46.** *Let  $a \in C$ ,  $f \in \mathbb{K}_0$ ,  $a \neq 0$ , and let  $w \in \mathbb{F} \setminus \mathbb{K}_0$  satisfy  $\tau(w) = aw + f$ . Then  $w$  is differentially transcendental over  $\mathbb{K}$ .*

*Proof.* Before starting the actual proof we show that we can make some extra assumptions without loss of generality:

1. If  $a$  is a root of unity, we can assume that  $a = 1$ . In fact, if  $a$  is a root of unity of order  $N$ , we can proceed as in the proof of Lemma 41 and consider the iterated functional equation  $\tau^N(w) = a^N w + f_N = w + f_N$ . Recall that  $\tau^N(t) = \frac{t}{1+Nt}$ , by (2.1). If we set  $\tilde{t} := Nt$ , then  $\tau^N(\tilde{t}) = \frac{\tilde{t}}{1+\tilde{t}}$ , therefore we have an equation of the form  $\tau(w) = w + f_N$  in the variable  $\tilde{t}$ . Notice that neither the iteration of the initial functional equation nor the change of variable that we have performed change the nature of the solutions  $w$ .



2. We may assume that  $f$  does not have a pole at  $\infty$ . Indeed, we can replace  $w$  by  $\tau^n(w)$ , which satisfies the functional equation  $\tau(\tau^n(w)) = a\tau^n(w) + \tau^n(f)$ . If  $f$  has a pole at  $\infty$ , then  $\tau^n(f) = f\left(\frac{t}{1+nt}\right)$  has a pole at  $t = -1/n$ , and by conveniently choosing  $n$ , it does not have a pole at  $\infty$ .
3. Finally, we can assume that  $f$  has only one pole in each  $\tau$ -orbit, that does not contain 0. Indeed, notice that we can replace  $w$  by  $w + r$  for any  $r \in \mathbb{K}_0$ , and change the equation accordingly, since this will change neither the hypotheses nor the conclusion of the lemma. Indeed, we have:

$$\tau(w + r) = a(w + r) + (f + \tau(r) - ar).$$

We remark that the element  $w + r$  of  $\mathbb{F}$  cannot be in  $C$ , since  $w \notin \mathbb{K}_0$ , therefore  $f + \tau(r) - ar$  cannot be 0. With this in mind, one sees that for any  $\alpha \in C$ ,  $\alpha \neq 0, 1$ , and any  $m \in \mathbb{Z}$  we have:

$$\tau\left(\frac{1}{(t-\alpha)^m}\right) - \frac{a}{(t-\alpha)^m} = \frac{(1+t)^m}{((1-\alpha)t-\alpha)^m} - \frac{a}{(t-\alpha)^m}.$$

Notice that  $\tau\left(\frac{\alpha}{1-\alpha}\right) = \alpha$ , hence the poles of the right-hand side of the expression above are on the same  $\tau$ -orbit. Therefore, replacing  $w$  by  $w + r$ , where  $r$  is a rational function, we can “shift the poles  $\neq 0, 1$  of  $f$  along their  $\tau$ -orbit”. The poles of the form  $\frac{1}{n}$ , for any integer  $n \geq 2$ , can all be moved to the pole 1, since  $\tau\left(\frac{1}{n}\right) = \frac{1}{n-1}$ . We conclude that, adding a convenient rational function to  $w$ , we can suppose that  $f$  has only one pole in each  $\tau$ -orbit, that does not contain 0. We point out that the change of unknown function that we performed does not change the fact that  $f$  does not have a pole at  $\infty$ .

Finally, if  $f = 0$  and  $a \neq 1$ , the functional equation  $\tau(y) = ay$  cannot have a solution  $w \in \mathbb{F}$ , as noticed in Example 45. On the other hand, if  $a = 1$ , the solution  $w$  is a constant, hence  $w \in \mathbb{K}_0$ .

From now on we suppose that  $f \neq 0$ . Under the assumptions 1, 2 and 3 above, we now assume by contradiction that  $w \in \mathbb{F} \setminus \mathbb{K}_0$  is differentially algebraic over  $\mathbb{K}$ . Let us consider the total ring of fractions  $\mathbb{F}_a$  of the  $\partial$ -Picard-Vessiot ring of  $\tau(y) = ay$  over the field  $\mathbb{F}$  (see Proposition 27). Then  $\mathbb{F}_a$  contains a copy of the total Picard-Vessiot ring  $F_a$  of  $\tau(y) = ay$  over  $\mathbb{K}$ , which is a field, by Lemma 43 above. Since  $w$  is differentially algebraic over  $\mathbb{K}$ , it is differentially algebraic over  $F_a$  and we can consider the functional equation  $\tau\left(\frac{w}{z}\right) = \frac{w}{z} + \frac{f}{z}$ , where  $z \in F_a$  satisfies  $\tau(z) = az$ . It follows from Proposition 32, that there exist an integer  $n \geq 0$ , and elements  $\alpha_0, \dots, \alpha_n \in C$  with  $\alpha_n \neq 0$ , and  $g \in F_a$ , such that

$$\alpha_0 \frac{f}{z} + \alpha_1 \partial\left(\frac{f}{z}\right) + \dots + \alpha_n \partial^n\left(\frac{f}{z}\right) = \tau\left(\frac{g}{z}\right) - \frac{g}{z}.$$

Since  $\partial(z) = cz$  for some  $c \in C$ , we have  $\partial\left(\frac{f}{z}\right) = \frac{\partial(f)}{z} - c\frac{f}{z}$ , with  $c \neq 0$  if  $a \neq 1$  and  $c = 0$  otherwise. Calculating recursively  $\partial^i\left(\frac{f}{z}\right)$ , one proves that there exist  $\beta_0, \dots, \beta_n \in C$ , with  $\beta_n = \alpha_n \neq 0$ , such that

$$\beta_0 f + \beta_1 \partial(f) + \dots + \beta_n \partial^n(f) = \tau(g) - ag.$$

If  $a = 1$ , then  $g \in F_a = \mathbb{K}$ . If  $a \neq 1$ , then  $z$  is transcendental over  $\mathbb{K}$  and  $g \in F_a = \mathbb{K}(z) \subset \mathbb{K}((z))$ , therefore we can write  $g$  as a Laurent series  $g = \sum_i g_i z^i \in \mathbb{K}((z))$ . Plugging  $g$  in the identity above, we find:

$$\beta_0 f + \beta_1 \partial(f) + \dots + \beta_n \partial^n(f) = \sum_i (\tau(g_i) a^i - a g_i) z^i.$$

Since the left-hand side of such identity is in  $\mathbb{K}_0$ , the right-hand side must be in  $\mathbb{K}_0$  too, hence  $g = g_0 \in \mathbb{K}$ . Summarizing, both for  $a = 1$  and  $a \neq 1$ , we have  $\beta_0 f + \beta_1 \partial(f) + \dots + \beta_n \partial^n(f) = \tau(g) - ag$ , for some  $g \in \mathbb{K}$ . Notice that  $\beta_0 f + \beta_1 \partial(f) + \dots + \beta_n \partial^n(f)$  has the same non-zero poles as  $f$ , therefore it has at most one pole per non-zero orbit.

By assumption,  $g$  is analytic in a punctured disk around zero. Let us suppose by reductio ad absurdum that there exists a singularity  $t_0 \in C^*$  on the border of the domain of analyticity of  $g$ , i.e. that  $g$  is not analytic on  $C^*$ . We notice that  $\tau^{-m}(t_0) = \frac{t_0}{1-mt_0}$  for any  $m \in \mathbb{Z}$ ,  $m \geq 0$ , therefore the orbit of  $t_0$  has an accumulation point at 0 as  $m \rightarrow \infty$ . Since  $g$  is analytic in the open punctured disk of center 0 and radius  $|t_0|$ , possibly replacing  $t_0$  by another singularity in its orbit, we can suppose that, for any positive integer  $m$ , no  $\tau^{-m}(t_0)$  is a singularity of  $g$ . Then  $t_0$  is a singularity of  $g$  but not of  $\tau(g)$ , while  $\tau^{-1}(t_0)$  is a singularity of  $\tau(g)$  but not of  $g$ , therefore  $\tau(g) - g$  is forced to have a singularity both at  $t_0$  and at  $\tau^{-1}(t_0)$ , in contradiction with the fact that  $f$  has at most one singularity in each  $\tau$ -orbit. We conclude that the domain of analyticity of  $g$  is the

whole  $C^*$  and that  $f$  cannot have any pole other than 0 and  $\infty$ . In other words,  $f \in C[t, t^{-1}]$  and  $g$  is analytic over  $C^*$ , with a pole at 0.

We now notice that for any integer  $m \geq 1$ , we have

$$\tau\left(\frac{1}{t^m}\right) - \frac{a}{t^m} = \frac{1-a}{t^m} + \sum_{j=0}^{m-1} \binom{m}{j} \frac{1}{t^j}.$$

This means that replacing  $w$  by  $w + r$  for a convenient choice of  $r \in \mathbb{K}_0$ , we can replace  $f$  by  $f + \tau(r) - ar$  and assume that  $f$  is a non-zero polynomial in  $t$  without constant coefficient. We are finally reduced to an  $f \in tC[t]$ , which implies that  $f = 0$ , since we have supposed that  $f$  does not have any pole at  $\infty$ . As we have noticed at the beginning, we can never obtain a zero inhomogeneous term in the functional equation adding a rational function  $r$  to  $w$ , therefore we have found a contradiction. This means that  $w$  cannot be differentially algebraic over  $\mathbb{K}$ , unless it is rational.  $\square$

We have finished the core of the proof. It only remains to put the pieces together.

**Lemma 47.** ([HS08, Prop. 3.9]) *Let  $a \in \mathbb{K}_0$ ,  $a \neq 0$ , and let  $F_a$  be a  $\mathbb{K}$ -algebra as in Notation 29 containing a solution  $z$  of  $\tau(y) = ay$ . Then either  $z$  is differentially algebraic over  $\mathbb{K}_0$  and there exist  $a^* \in C$  and  $b \in \mathbb{K}_0$ , both non-zero, such that  $a = a^* \frac{\tau(b)}{b}$ , or the solution  $z$  (and hence any solution) of  $\tau(y) = ay$  is differentially transcendental over  $\mathbb{K}$ .*

Remark 45 above shows that, if  $z$  is differentially algebraic, it is essentially an exponential function. The following example illustrates the case when  $z$  is differentially transcendental:

**Example 48.** *We set  $C = \mathbb{C}$  and we go back to Example 33. Since we know that the gamma function is not rational, we immediately obtain that  $\Gamma\left(\frac{1}{t}\right)^{-1}$  is strongly differentially transcendental, being a solution of  $\tau(y) = ty$ . Notice that the statement makes sense since  $\Gamma\left(\frac{1}{t}\right)^{-1}$  is analytic in any punctured disk around 0, hence we can consider the  $\mathbb{K}$ -field of functions  $\mathbb{K}\left(\Gamma\left(\frac{1}{t}\right)^{-1}\right)$ .*

*Proof of Lemma 47.* If  $z$  is differentially algebraic over  $\mathbb{K}$ , so is  $\partial(z)/z$ , which is solution of  $\tau(y) = y + \partial(a)/a$ . It follows from Lemma 46 that  $\partial(z)/z$  is differentially algebraic over  $\mathbb{K}$  if and only if  $v := \partial(z)/z \in \mathbb{K}_0$ . We have  $\partial(a)/a = \tau(v) - v$ . Since both  $a$  and  $v$  are rational functions, we make the change of variable  $s = \frac{1}{t}$ , so that  $\tau(s) = s + 1$  and  $\partial = -\frac{d}{ds}$ . As we have remarked in Example 31, if  $v$ , as a function of the variable  $s$ , has a pole in  $C$ , then  $\tau(v) - v$  must have at least two poles in the same orbit. In particular, if  $v$  has a pole of order greater than 1, then  $\tau(v) - v$  has at least two poles of order greater than 1, while  $\frac{\partial(a)}{a}$  has none. Therefore  $v$  does not have any finite pole of order greater than 1. Since all the residues of  $\frac{\partial(a)}{a}$  are integers, the same argument shows that  $v$  cannot have any simple pole with a residue in  $C \setminus \mathbb{Z}$ . We conclude that  $v$  is the sum of a polynomial  $p \in C[t]$  and of some terms of the form  $\frac{m}{s-a}$ , with  $m \in \mathbb{Z}$  and  $a \in C$ . Then  $\tau(p) - p \in C[t]$ , while  $\frac{\partial(a)}{a}$  is the logarithmic derivative of a rational function. Therefore  $\tau(p) - p = 0$  and  $p \in C$ . Since  $v$  is determined up to a constant, we can suppose that  $p = 0$ . Hence there exists a rational function  $b$  such that  $v = \frac{\partial(b)}{b}$ . We conclude that  $\tau\left(\frac{\partial(z)}{z} - \frac{\partial(b)}{b}\right) = \frac{\partial(z)}{z} - \frac{\partial(b)}{b}$ , for some  $b \in \mathbb{K}_0$ , hence that there exists  $a^* \in C$  such that  $a = a^* \frac{\tau(b)}{b}$ .  $\square$

*Proof of Theorem 39.* Let  $F_a$  be the total ring of fractions of the  $\partial$ -Picard-Vessiot ring of  $\tau(y) = ay$  over  $\mathbb{K}$ , constructed in Proposition 27 and let  $z \in F_a$  be a solution of  $\tau(y) = ay$ . If  $z$  is differentially transcendental over  $\mathbb{K}$ , then  $w$  is differentially transcendental over  $\mathbb{K}$  because of Theorem 34, therefore there is nothing more to prove. If  $z \in F_a$  is differentially algebraic over  $\mathbb{K}$  then Lemma 47 implies that there exist  $a^* \in C$  and  $b \in \mathbb{K}_0$  such that  $a = a^* \frac{\tau(b)}{b}$ . Therefore  $w/b \in \mathbb{F}$  is differentially algebraic over  $\mathbb{K}$  and  $\tau(w/b) = a^*(w/b) + f/(ab)$ . Without loss of generality, we can write  $w$  for  $w/b$ ,  $a$  for  $a^*$  and  $f$  for  $f/(ab)$ , so that  $\tau(w) = aw + f$ , with  $a \in C \setminus \{0\}$  and  $f \in \mathbb{K}_0$ . This is the situation of Lemma 46, which allows to conclude that  $w \in \mathbb{K}_0$ .  $\square$

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