



On the justification of the time-dependent Koiter's model for thermoelastic shells

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► To cite this version:

Paolo Piersanti. On the justification of the time-dependent Koiter's model for thermoelastic shells. Journal of Differential Equations, 2021. ⟨hal-03091265v5⟩

HAL Id: hal-03091265

<https://hal.science/hal-03091265v5>

Submitted on 2 Dec 2021

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On the justification of the frictionless time-dependent Koiter's model for thermoelastic shells

Paolo Piersanti

Received: date / Accepted: date

Abstract Our first objective is to identify two-dimensional equations that model the displacement of a linearly elastic flexural shell subjected to the action of an external heat source. To this end, we embed the shell into a family of linearly elastic flexural shells, all sharing the same middle surface $\boldsymbol{\theta}(\bar{\omega})$, where ω is a domain in \mathbb{R}^2 and $\boldsymbol{\theta} : \bar{\omega} \rightarrow \mathbb{E}^3$ is a smooth enough immersion and whose thickness $2\varepsilon > 0$ is considered as a “small” parameter approaching zero. We then identify, and justify by means of a rigorous asymptotic analysis as ε approaches zero, the corresponding “limit” two-dimensional variational problem.

Our second objective is to identify and justify a set of two-dimensional equations that are meant to approximate the original three-dimensional model in the case where the shell under consideration is either an elliptic membrane shell or a flexural shell.

Keywords Flexural shell · Thermoelasticity · Semi-groups · Asymptotic analysis · time-dependent problems

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1 Introduction

A thin linearly elastic shell can be *either* modelled by the three-dimensional equations of linearised elasticity *or* by means of a set two-dimensional equations obtained by W.T. Koiter in the seminal works [26, 27].

Koiter's model provides a different approach from the classical one (i.e., the one based upon the three-dimensional equations of linearised elasticity) for modelling the same kind of situation by means of specific *two-dimensional equations*. A rigorous asymptotic analysis (cf. [10] and [14]) shows that both the three-dimensional equations and the two-dimensional equations of Koiter's model have the same asymptotic behaviour as the thickness of the shell approaches zero. Moreover, Koiter's model does not *a priori* depend on the choice of the shell and it indeed includes both the *elliptic membrane shells* bilinear form and the *flexural shells* bilinear form. This feature is very convenient if one aims to approximate a variety of situations that would otherwise be substantially more delicate to handle by means of the classical *three-dimensional* approach.

Koiter's model has been extensively studied in the literature by Ciarlet and his associates [13, 14, 15, 16, 17, 18, 19, 20]. For what concerns the literature related to time-dependent problems in elasticity, it is worth mentioning the papers [3, 4, 5, 31, 32]. It is also worth mentioning the papers [33, 34, 36].

In this paper, we propose a suitable model of Koiter's type for time-dependent thermoelastic shells. Differently from the classical model, the model we are going to propose does not only incorporate bilinear forms associated with the *elliptic component* and the *flexural component*, but also additional terms related to the interaction between the temperature variation and the displacement. As a consequence of the Duhamel-Neumann law (cf., e.g., Chapter I, Section X of [28]), the model takes the form of a coupled *hyperbolic-parabolic system*.

To our best knowledge, there is no record in the literature treating the justification of Koiter's model for thermoelastic shells. The purpose of this paper is exactly to remedy this situation.

This paper is divided into eight sections (including this one). In section 2 we present some background and notation. In section 3 we introduce the problem under examination, modelling the displacement of a general three-dimensional linearly elastic shells as well as the evolution of its internal temperature variation. In section 4 we scale the problem presented in the previous section and, following [9] and [10], we make suitable assumptions on the given data in the case where the linearly elastic shell under consideration is a flexural shell. In section 5 we show that a certain initial boundary value problem – that will turn out to be the *two-dimensional limit problem* of the asymptotic analysis performed in section 6 – has a more regular velocity field. In section 6, as announced, we perform a rigorous asymptotic analysis of the scaled three-dimensional model in the case where the linearly elastic shell under consideration is a flexural shell, and we identify a suitable two-dimensional limit model, which exhibits the peculiar feature of being governed by two sets of *uncoupled* evolutionary equations, i.e., the equations governing the mechanical displacement, and the equations governing the internal temperature evolution. In section 7 we state our *proposed* Koiter's model for a family of “general” thermoelastic shells, and we study its convergence in the case where the shell under consideration is a flexural shell. Lastly, in section 8 we study the convergence of our proposed Koiter's model for a family of thermoelastic elliptic membrane shells, and we show that the recovered two-dimensional limit model coincides with the one recovered in [7].

2 Geometrical preliminaries

For details about the classical notions of differential geometry recalled in this section, see, e.g. [10] or [11].

Greek indices, except ε , take their values in the set $\{1, 2\}$, while Latin indices, except when they are used for indexing sequences, take their values in the set $\{1, 2, 3\}$, and the summation convention with respect to repeated indices is systematically used in conjunction with these two rules. The notation \mathbb{E}^3 designates the three-dimensional Euclidean space; the Euclidean inner product and the vector product of $\mathbf{u}, \mathbf{v} \in \mathbb{E}^3$ are denoted $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \wedge \mathbf{v}$; the Euclidean norm of $\mathbf{u} \in \mathbb{E}^3$ is denoted $|\mathbf{u}|$. The notation δ_i^j designates the Kronecker symbol.

Given an open subset Ω of \mathbb{R}^n , notations such as $L^2(\Omega)$, $H^m(\Omega)$, or $H_0^m(\Omega)$, $m \geq 1$, designate the usual Lebesgue and Sobolev spaces, and the notation $\mathcal{D}(\Omega)$ designates the space of all functions that are infinitely differentiable over Ω and have compact support in Ω . The notation $\|\cdot\|_X$ designates the norm in a normed vector space X . The dual space of a vector space X is denoted by X^* and the duality pair between X^* and X is denoted by ${}_X\langle \cdot, \cdot \rangle_X$. Spaces of vector-valued functions are denoted with boldface letters. Lebesgue-Bochner spaces defined over a bounded open interval I (cf. [29]), are denoted $L^p(I; H)$, where H is a Banach space and $1 \leq p \leq \infty$. The notation $\|\cdot\|_{L^p(I; H)}$ designates the norm of the Lebesgue-Bochner space $L^p(I; H)$. Sobolev-Bochner spaces defined over a bounded open interval I (cf. [29]), are denoted $W^{m,p}(I; H)$, where H is a Banach space, $m \geq 1$ and $1 \leq p \leq \infty$. The notation $\|\cdot\|_{W^{m,p}(I; H)}$ designates the norm of the Sobolev-Bochner space $W^{m,p}(I; H)$. The notations $\dot{\eta}$ and $\ddot{\eta}$ denote the first weak derivative with respect to $t \in I$ and the second weak derivative with respect to $t \in I$ of a scalar function η defined over the interval I . The notations $\dot{\boldsymbol{\eta}}$ and $\ddot{\boldsymbol{\eta}}$ denote the first weak derivative with respect to $t \in I$ and the second weak derivative with respect to $t \in I$ of a vector-valued function $\boldsymbol{\eta}$ defined over the interval I .

The boundary Γ of an open subset Ω in \mathbb{R}^n is said to be Lipschitz-continuous if the following conditions are satisfied (cf., e.g., Section 1.18 of [12]): Given an integer $s \geq 1$, there exist constants $\alpha_1 > 0$ and $L > 0$, and a finite number of local coordinate systems, with coordinates $\boldsymbol{\phi}'_r = (\phi'_1, \dots, \phi'_{n-1}) \in \mathbb{R}^{n-1}$ and $\phi_r = \phi_n$, and corresponding functions

$$\tilde{\theta}_r : \tilde{\omega}_r := \{\phi_r \in \mathbb{R}^{n-1}; |\phi_r| < \alpha_1\} \rightarrow \mathbb{R}, \quad 1 \leq r \leq s,$$

such that

$$\Gamma = \bigcup_{r=1}^s \{(\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \text{ and } \phi_r = \tilde{\theta}_r(\phi'_r)\},$$

and

$$|\tilde{\theta}_r(\phi'_r) - \tilde{\theta}_r(\mathbf{v}'_r)| \leq L|\phi'_r - \mathbf{v}'_r|, \text{ for all } \phi'_r, \mathbf{v}'_r \in \tilde{\omega}_r, \text{ and all } 1 \leq r \leq s.$$

We observe that the second last formula takes into account overlapping local charts, while the last set of inequalities express the Lipschitz continuity of the mappings $\tilde{\theta}_r$.

An open set Ω is said to be locally on the same side of its boundary Γ if, in addition, there exists a constant $\alpha_2 > 0$ such that

$$\begin{aligned} \{(\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \text{ and } \tilde{\theta}_r(\phi'_r) < \phi_r < \tilde{\theta}_r(\phi'_r) + \alpha_2\} &\subset \Omega, \quad \text{for all } 1 \leq r \leq s, \\ \{(\phi'_r, \phi_r); \phi'_r \in \tilde{\omega}_r \text{ and } \tilde{\theta}_r(\phi'_r) - \alpha_2 < \phi_r < \tilde{\theta}_r(\phi'_r)\} &\subset \mathbb{R}^n \setminus \overline{\Omega}, \quad \text{for all } 1 \leq r \leq s. \end{aligned}$$

A domain in \mathbb{R}^n is a bounded and connected open subset Ω of \mathbb{R}^n , whose boundary $\partial\Omega$ is Lipschitz-continuous, the set Ω being locally on a single side of $\partial\Omega$.

Let ω be a domain in \mathbb{R}^2 , let $y = (y_\alpha)$ denote a generic point in $\bar{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/(\partial y_\alpha \partial y_\beta)$. A mapping $\theta \in \mathcal{C}^1(\bar{\omega}; \mathbb{E}^3)$ is an *immersion* if the two vectors

$$\mathbf{a}_\alpha(y) := \partial_\alpha \theta(y)$$

are linearly independent at each point $y \in \bar{\omega}$. Then the image $\theta(\bar{\omega})$ of the set $\bar{\omega}$ under the mapping θ is a surface in \mathbb{E}^3 , equipped with y_1, y_2 as its *curvilinear coordinates*. Given any point $y \in \bar{\omega}$, the vectors $\mathbf{a}_\alpha(y)$ span the *tangent plane* to the surface $\theta(\bar{\omega})$ at the point $\theta(y)$, the unit vector

$$\mathbf{a}_3(y) := \frac{\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)|}$$

is normal to $\theta(\bar{\omega})$ at $\theta(y)$, the three vectors $\mathbf{a}_i(y)$ form the *covariant* basis at $\theta(y)$, and the three vectors $\mathbf{a}^j(y)$ defined by the relations

$$\mathbf{a}^j(y) \cdot \mathbf{a}_i(y) = \delta_i^j$$

form the *contravariant* basis at $\theta(y)$; note that the vectors $\mathbf{a}^\beta(y)$ also span the tangent plane to $\theta(\bar{\omega})$ at $\theta(y)$ and that $\mathbf{a}^3(y) = \mathbf{a}_3(y)$.

The *first fundamental form* of the surface $\theta(\bar{\omega})$ is defined by means of its *covariant components*

$$a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\beta\alpha} \in \mathcal{C}^0(\bar{\omega}),$$

or by means of its *contravariant components*

$$a^{\alpha\beta} := \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\beta\alpha} \in \mathcal{C}^0(\bar{\omega}).$$

Note that the symmetric matrix field $(a^{\alpha\beta})$ is the inverse of the matrix field $(a_{\alpha\beta})$, that $\mathbf{a}^\beta = a^{\alpha\beta} \mathbf{a}_\alpha$ and $\mathbf{a}_\alpha = a_{\alpha\beta} \mathbf{a}^\beta$, and that the *area element* along $\theta(\bar{\omega})$ is given at each point $\theta(y)$, $y \in \bar{\omega}$, by $\sqrt{a(y)} dy$, where

$$a := \det(a_{\alpha\beta}) \in \mathcal{C}^0(\bar{\omega}).$$

Given an immersion $\theta \in \mathcal{C}^2(\bar{\omega}; \mathbb{E}^3)$, the *second fundamental form* of the surface $\theta(\bar{\omega})$ is defined by means of its *covariant components*

$$b_{\alpha\beta} := \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 = -\mathbf{a}_\beta \cdot \partial_\alpha \mathbf{a}_3 = b_{\beta\alpha} \in \mathcal{C}^0(\bar{\omega}),$$

or by means of its *mixed components*

$$b_\alpha^\beta := a^{\beta\sigma} b_{\alpha\sigma} \in \mathcal{C}^0(\bar{\omega}),$$

and the *Christoffel symbols* associated with the immersion θ are defined by

$$\Gamma_{\alpha\beta}^\sigma := \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^\sigma = \Gamma_{\beta\alpha}^\sigma \in \mathcal{C}^0(\bar{\omega}).$$

The *Gaussian curvature* at each point $\theta(y)$, $y \in \bar{\omega}$, of the surface $\theta(\bar{\omega})$ is defined by

$$\kappa(y) := \frac{\det(b_{\alpha\beta}(y))}{\det(a_{\alpha\beta}(y))} = \det(b_\alpha^\beta(y)),$$

and we observe that the denominator in the above relation never vanishes as the mapping θ is assumed to be an immersion. Note that the Gaussian curvature $\kappa(y)$ at the point $\theta(y)$ is also equal to the product of the two principal curvatures at this point.

A surface $\theta(\bar{\omega})$ defined by means of an immersion $\theta \in \mathcal{C}^2(\bar{\omega}; \mathbb{E}^3)$ is said to be *elliptic* if its Gaussian curvature is everywhere > 0 in $\bar{\omega}$, or equivalently, if there exists a constant κ_0 such that

$$0 < \kappa_0 \leq \kappa(y) \quad \text{for all } y \in \bar{\omega}.$$

Given an immersion $\theta \in \mathcal{C}^2(\bar{\omega}; \mathbb{E}^3)$ and a vector field $\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^1(\bar{\omega}; \mathbb{R}^3)$, the vector field

$$\tilde{\boldsymbol{\eta}} := \eta_i \mathbf{a}^i$$

can be viewed as a *displacement field* of the surface $\theta(\bar{\omega})$, thus defined by means of its *covariant components* η_i over the vectors \mathbf{a}^i of the contravariant bases along the surface. If the norms $\|\eta_i\|_{\mathcal{C}^1(\bar{\omega})}$ are small enough, the mapping $(\theta + \eta_i \mathbf{a}^i) \in \mathcal{C}^1(\bar{\omega}; \mathbb{E}^3)$ is also an immersion, so that the set $(\theta + \eta_i \mathbf{a}^i)(\bar{\omega})$ is also a surface

in \mathbb{E}^3 , equipped with the same curvilinear coordinates as those of the surface $\boldsymbol{\theta}(\bar{\omega})$, called the *deformed surface* corresponding to the displacement field $\tilde{\boldsymbol{\eta}} = \eta_i \mathbf{a}^i$. One can then define the first fundamental form of the deformed surface by means of its covariant components

$$a_{\alpha\beta}(\boldsymbol{\eta}) := (\mathbf{a}_\alpha + \partial_\alpha \tilde{\boldsymbol{\eta}}) \cdot (\mathbf{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}}),$$

and the second fundamental form of the deformed surface by means of its covariant components

$$b_{\alpha\beta}(\boldsymbol{\eta}) := \partial_\alpha (\mathbf{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}}) \cdot \frac{(\mathbf{a}_1 + \partial_1 \tilde{\boldsymbol{\eta}}) \wedge (\mathbf{a}_2 + \partial_2 \tilde{\boldsymbol{\eta}})}{|(\mathbf{a}_1 + \partial_1 \tilde{\boldsymbol{\eta}}) \wedge (\mathbf{a}_2 + \partial_2 \tilde{\boldsymbol{\eta}})|}.$$

The *linear part with respect to $\tilde{\boldsymbol{\eta}}$* in the difference $\frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta})$ is called the *linearised change of metric tensor* associated with the displacement field $\eta_i \mathbf{a}^i$, the covariant components of which are then given by

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{\eta}) &= \frac{1}{2} (\mathbf{a}_\alpha \cdot \partial_\beta \tilde{\boldsymbol{\eta}} + \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \mathbf{a}_\beta) \\ &= \frac{1}{2} (\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta} \eta_3 = \gamma_{\beta\alpha}(\boldsymbol{\eta}). \end{aligned}$$

The *linear part with respect to $\tilde{\boldsymbol{\eta}}$* in the difference $\frac{1}{2}(b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta})$ is called the *linearised change of curvature tensor* associated with the displacement field $\eta_i \mathbf{a}^i$, the covariant components of which are then given by

$$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\eta}) &= (\partial_\alpha \tilde{\boldsymbol{\eta}} - \Gamma_{\alpha\beta}^\sigma \partial_\sigma \tilde{\boldsymbol{\eta}}) \cdot \mathbf{a}_3 \\ &= \partial_\alpha \eta_\beta - \Gamma_{\alpha\beta}^\sigma \partial_\sigma \eta_\beta - b_{\alpha\beta}^\sigma \eta_\sigma + b_\alpha^\sigma (\partial_\beta \eta_\sigma - \Gamma_{\beta\sigma}^\tau \eta_\tau) + b_\beta^\tau (\partial_\alpha \eta_\tau - \Gamma_{\alpha\tau}^\sigma \eta_\sigma) \\ &\quad + (\partial_\alpha b_\beta^\tau + \Gamma_{\alpha\sigma}^\tau b_\beta^\sigma - \Gamma_{\alpha\beta}^\sigma b_\sigma^\tau) \eta_\tau = \rho_{\beta\alpha}(\boldsymbol{\eta}). \end{aligned}$$

Let us now define the time-dependent “analogue” of the linearised change of metric tensor $\gamma_{\alpha\beta}$. Consider the operator

$$\tilde{\gamma}_{\alpha\beta} : L^2(0, T; H^1(\omega) \times H^1(\omega) \times L^2(\omega)) \rightarrow L^2(0, T; L^2(\omega)),$$

defined by

$$\tilde{\gamma}_{\alpha\beta}(\boldsymbol{\eta})(t) := \gamma_{\alpha\beta}(\boldsymbol{\eta}(t)) \text{ for all } \boldsymbol{\eta} \in L^2(0, T; H^1(\omega) \times H^1(\omega) \times L^2(\omega)),$$

for almost all (a.a. in what follows) $t \in (0, T)$. It has been shown in [31] that this operator is well-defined, linear and continuous.

Let us now define the time-dependent “analogue” of the linearised change of curvature tensor $\rho_{\alpha\beta}$. Consider the operator

$$\tilde{\rho}_{\alpha\beta} : L^2(0, T; H^1(\omega) \times H^1(\omega) \times H^2(\omega)) \rightarrow L^2(0, T; L^2(\omega)),$$

defined by

$$\tilde{\rho}_{\alpha\beta}(\boldsymbol{\eta})(t) := \rho_{\alpha\beta}(\boldsymbol{\eta}(t)) \text{ for all } \boldsymbol{\eta} \in L^2(0, T; H^1(\omega) \times H^1(\omega) \times H^2(\omega)),$$

for a.a. $t \in (0, T)$. It has been shown in [31] that this operator is well-defined, linear and continuous.

3 The three-dimensional problem for a “general” linearly elastic shell

Let ω be a domain in \mathbb{R}^2 , let $\gamma := \partial\omega$, and let γ_0 be a non-empty relatively open subset of γ . For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[\text{ and } \Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon] \text{ and } \Gamma_\pm^\varepsilon := \omega \times \{\pm\varepsilon\},$$

we let $x^\varepsilon = (x_i^\varepsilon)$ designate a generic point in the set $\overline{\Omega^\varepsilon}$, and we let $\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon$. Hence we also have $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$.

Given an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$ and $\varepsilon > 0$, consider a *shell* with *middle surface* $\boldsymbol{\theta}(\bar{\omega})$ and with *constant thickness* 2ε . This means that the *reference configuration* of the shell is the set $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$, where the mapping $\boldsymbol{\Theta} : \overline{\Omega^\varepsilon} \rightarrow \mathbb{E}^3$ is defined by

$$\boldsymbol{\Theta}(x^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \mathbf{a}^3(y) \text{ at each point } x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega^\varepsilon}.$$

One can then show (cf. Theorem 3.1-1 of [10]) that, if $\varepsilon > 0$ is small enough, such a mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega^\varepsilon}; \mathbb{E}^3)$ is an *immersion*, in the sense that the three vectors

$$\mathbf{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(x^\varepsilon)$$

are linearly independent at each point $x^\varepsilon \in \overline{\Omega^\varepsilon}$; these vectors then constitute the *covariant basis* at the point $\Theta(x^\varepsilon)$, while the three vectors $\mathbf{g}^{j,\varepsilon}(x^\varepsilon)$ defined by the relations

$$\mathbf{g}^{j,\varepsilon}(x^\varepsilon) \cdot \mathbf{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j$$

constitute the *contravariant basis* at the same point. It will be implicitly assumed in the sequel that $\varepsilon > 0$ is small enough so that $\Theta : \overline{\Omega^\varepsilon} \rightarrow \mathbb{R}^3$ is an immersion.

One then defines the *metric tensor associated with the immersion* Θ by means of its *covariant components*

$$g_{ij}^\varepsilon := \mathbf{g}_i^\varepsilon \cdot \mathbf{g}_j^\varepsilon \in C^1(\overline{\Omega^\varepsilon}),$$

or by means of its *contravariant components*

$$g^{ij,\varepsilon} := \mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}^{j,\varepsilon} \in C^1(\overline{\Omega^\varepsilon}).$$

Note that the symmetric matrix field $(g^{ij,\varepsilon})$ is then the inverse of the matrix field (g_{ij}^ε) , that $\mathbf{g}^{j,\varepsilon} = g^{ij,\varepsilon} \mathbf{g}_i^\varepsilon$ and $\mathbf{g}_i^\varepsilon = g_{ij}^\varepsilon \mathbf{g}^{j,\varepsilon}$, and that the *volume element* in $\Theta(\overline{\Omega^\varepsilon})$ is given at each point $\Theta(x^\varepsilon)$, $x^\varepsilon \in \overline{\Omega^\varepsilon}$, by $\sqrt{g^\varepsilon(x^\varepsilon)} dx^\varepsilon$, where

$$g^\varepsilon := \det(g_{ij}^\varepsilon) \in C^1(\overline{\Omega^\varepsilon}).$$

One also defines the *Christoffel symbols* associated with the immersion Θ by

$$\Gamma_{ij}^{p,\varepsilon} := \partial_i \mathbf{g}_j^\varepsilon \cdot \mathbf{g}^{p,\varepsilon} = \Gamma_{ji}^{p,\varepsilon} \in C^0(\overline{\Omega^\varepsilon}).$$

Note that $\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0$. Given a vector field $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in C^1(\overline{\Omega^\varepsilon}; \mathbb{R}^3)$, the associated vector field

$$\tilde{\mathbf{v}}^\varepsilon := v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$$

can be viewed as a *displacement field* of the reference configuration $\Theta(\overline{\Omega^\varepsilon})$ of the shell, thus defined by means of its *covariant components* v_i^ε over the vectors $\mathbf{g}^{i,\varepsilon}$ of the contravariant bases in the reference configuration.

If the norms $\|v_i^\varepsilon\|_{C^1(\overline{\Omega^\varepsilon})}$ are small enough, the mapping $(\Theta + v_i^\varepsilon \mathbf{g}^{i,\varepsilon})$ is also an immersion, so that one can also define the metric tensor of the *deformed configuration* $(\Theta + v_i^\varepsilon \mathbf{g}^{i,\varepsilon})(\overline{\Omega^\varepsilon})$ by means of its covariant components

$$g_{ij}^\varepsilon(\mathbf{v}^\varepsilon) := (\mathbf{g}_i^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon) \cdot (\mathbf{g}_j^\varepsilon + \partial_j^\varepsilon \tilde{\mathbf{v}}^\varepsilon) = g_{ij}^\varepsilon + \mathbf{g}_i^\varepsilon \cdot \partial_j^\varepsilon \tilde{\mathbf{v}}^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon \cdot \mathbf{g}_j^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon \cdot \partial_j^\varepsilon \tilde{\mathbf{v}}^\varepsilon.$$

The linear part with respect to $\tilde{\mathbf{v}}^\varepsilon$ in the difference $\frac{1}{2}(g_{ij}^\varepsilon(\mathbf{v}^\varepsilon) - g_{ij}^\varepsilon)$ is then called the *linearised strain tensor* associated with the displacement field $v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$, the covariant components of which are thus defined by

$$e_{ij\parallel j}^\varepsilon(\mathbf{v}^\varepsilon) := \frac{1}{2} (\mathbf{g}_i^\varepsilon \cdot \partial_j^\varepsilon \tilde{\mathbf{v}}^\varepsilon + \partial_i^\varepsilon \tilde{\mathbf{v}}^\varepsilon \cdot \mathbf{g}_j^\varepsilon) = \frac{1}{2} (\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon = e_{j\parallel i}^\varepsilon(\mathbf{v}^\varepsilon).$$

The functions $e_{i\parallel j}^\varepsilon(\mathbf{v}^\varepsilon)$ are called the *linearised strains in curvilinear coordinates* associated with the displacement field $v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$.

Let us now define the time-dependent “analogue” of the linearised strain tensor $e_{i\parallel j}^\varepsilon$. Consider the operator

$$\tilde{e}_{i\parallel j}^\varepsilon : L^2(0, T; \mathbf{H}^1(\Omega^\varepsilon)) \rightarrow L^2(0, T; L^2(\Omega^\varepsilon)),$$

defined by

$$\tilde{e}_{i\parallel j}^\varepsilon(\mathbf{v}^\varepsilon)(t) := e_{i\parallel j}^\varepsilon(\mathbf{v}^\varepsilon(t)) \text{ for all } \boldsymbol{\eta} \in L^2(0, T; \mathbf{H}^1(\Omega^\varepsilon)),$$

for a.a. $t \in (0, T)$. It has been shown in [32] that this operator is well-defined, linear and continuous.

We assume throughout this paper that, for each $\varepsilon > 0$, the reference configuration $\Theta(\overline{\Omega^\varepsilon})$ of the shell is a *natural state* (i.e., stress-free) and that the material constituting the shell is *homogeneous*, *isotropic*, and *linearly elastic*.

The behaviour of such an elastic material is thus entirely governed by its two *Lamé constants* $\lambda^\varepsilon \geq 0$ and $\mu^\varepsilon > 0$ (for details, see, e.g., Section 3.8 of [8]). The positive function $\rho^\varepsilon \in L^\infty(\Omega^\varepsilon)$ denotes the *mass density* of the shell per unit volume.

We will also assume that the shell is subjected to *applied body forces* whose density per unit volume is defined by means of its covariant components $f^{i,\varepsilon} \in H^1(0, T; L^2(\Omega^\varepsilon))$, to *applied surface forces* whose surface per unit area is defined by means of its covariant components $h^{i,\varepsilon} \in H^1(0, T; L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon))$ and to a *homogeneous boundary condition of place* along the portion Γ_0^ε of its lateral face (i.e., the displacement vanishes on Γ_0^ε).

Apart from the applied body forces and the applied surface forces mentioned beforehand, the shell is also subjected to the action of *heat* (or *cool*), defined by the function $Q^\varepsilon \in L^\infty(0, T; L^\infty(\Omega^\varepsilon))$. The *positive* function $A^\varepsilon \in L^\infty(\Omega^\varepsilon)$ denotes the *thermal dilatation* coefficient, the *positive* function $K^\varepsilon \in L^\infty(\Omega^\varepsilon)$

denotes the *conductivity* coefficient, and the *positive* function $S^\varepsilon \in L^\infty(\Omega^\varepsilon)$ denotes the *specific heat* coefficient.

Let

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu \left(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon} \right) = A^{jikl,\varepsilon} = A^{klij,\varepsilon}$$

denote the contravariant components of the *fourth order three-dimensional elasticity tensor* of the linearly elastic material constituting the shell. The unknown associated with the “*mechanical*” part of the problem is the vector field $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ where the functions $u_i^\varepsilon : [0, T] \times \overline{\Omega^\varepsilon} \rightarrow \mathbb{R}$ are the three covariant components of the unknown “three-dimensional” displacement vector field $u_i^\varepsilon \mathbf{g}^{i,\varepsilon}$ of the reference configuration of the shell. The space of admissible displacements is defined by:

$$\mathbf{V}(\Omega^\varepsilon) := \{\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon); \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\}.$$

The unknown associated with the “*thermal*” part of the problem is the function $\Theta^\varepsilon : [0, T] \times \overline{\Omega^\varepsilon} \rightarrow \mathbb{R}$, which denotes the (internal) temperature variation of the shell. The space of admissible temperature variations is defined by:

$$W(\Omega^\varepsilon) := \{\Xi^\varepsilon \in H^1(\Omega^\varepsilon); \Xi^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\}.$$

This gives rise to a system of coupled equations defined in terms of the displacement and the temperature variation. Such a coupling is achieved thanks to the Duhamel-Neumann law (cf., e.g., Chapter I, Section X of [28]), and gives rise to a *hyperbolic-parabolic system*.

Following [28], we assume that the temperature variation is very small if compared with the temperature of the surrounding environment. Besides, we assume that the surrounding environment is sufficiently big so that the heat irradiated from the linearly elastic shell induces a negligible change in the temperature of the surrounding environment.

In order to infer the well-posedness of the solution of the problem we are going to define next, we need to introduce the space

$$\mathbf{X}(\Omega^\varepsilon) := \mathbf{V}(\Omega^\varepsilon) \times \mathbf{L}^2(\Omega^\varepsilon) \times W(\Omega^\varepsilon),$$

and the space

$$\mathbf{H}(\Omega^\varepsilon) := \mathbf{V}(\Omega^\varepsilon) \times \mathbf{L}^2(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon),$$

and we equip them with the norms

$$\|\mathbf{U}^\varepsilon\|_{\mathbf{X}(\Omega^\varepsilon)} := \|\mathbf{v}^\varepsilon\|_{\mathbf{V}(\Omega^\varepsilon)} + \|\mathbf{w}^\varepsilon\|_{\mathbf{L}^2(\Omega^\varepsilon)} + \|\Xi^\varepsilon\|_{W(\Omega^\varepsilon)}, \quad \text{for all } \mathbf{U}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{w}^\varepsilon, \Xi^\varepsilon) \in \mathbf{X}(\Omega^\varepsilon),$$

and

$$\|\mathbf{U}^\varepsilon\|_{\mathbf{H}(\Omega^\varepsilon)} := \|\mathbf{v}^\varepsilon\|_{\mathbf{V}(\Omega^\varepsilon)} + \|\mathbf{w}^\varepsilon\|_{\mathbf{L}^2(\Omega^\varepsilon)} + \|\Xi^\varepsilon\|_{L^2(\Omega^\varepsilon)}, \quad \text{for all } \mathbf{U}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{w}^\varepsilon, \Xi^\varepsilon) \in \mathbf{H}(\Omega^\varepsilon),$$

respectively.

The three-dimensional problem modelling the displacement of a “general” linearly elastic shell takes the following form:

Problem $\mathcal{P}(\Omega^\varepsilon)$ Find a vector field $\mathbf{u}^\varepsilon = (u_i^\varepsilon) : (0, T) \rightarrow \mathbf{V}(\Omega^\varepsilon)$ and a function $\Theta^\varepsilon : (0, T) \rightarrow W(\Omega^\varepsilon)$ such that

$$\begin{aligned} \mathbf{u}^\varepsilon &\in L^\infty(0, T; \mathbf{V}(\Omega^\varepsilon)), \\ \dot{\mathbf{u}}^\varepsilon &\in L^\infty(0, T; \mathbf{L}^2(\Omega^\varepsilon)), \\ \ddot{\mathbf{u}}^\varepsilon &\in L^\infty(0, T; \mathbf{V}^*(\Omega^\varepsilon)), \\ \Theta^\varepsilon &\in L^\infty(0, T; W(\Omega^\varepsilon)), \\ \dot{\Theta}^\varepsilon &\in L^\infty(0, T; W^*(\Omega^\varepsilon)), \end{aligned}$$

that satisfy the following variational equations

$$\begin{aligned} &\frac{d^2}{dt^2} \int_{\Omega^\varepsilon} \rho^\varepsilon u_i^\varepsilon(t) g^{ij,\varepsilon} v_j^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k||l}^\varepsilon(\mathbf{u}^\varepsilon(t)) e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon \\ &\quad - \int_{\Omega^\varepsilon} A^\varepsilon (3\lambda^\varepsilon + 2\mu^\varepsilon) \Theta^\varepsilon(t) e_{i||j}^\varepsilon(\mathbf{v}^\varepsilon) g^{ij,\varepsilon} \sqrt{g^\varepsilon} dx^\varepsilon \\ &= \int_{\Omega^\varepsilon} f^{i,\varepsilon}(t) v_i^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^{i,\varepsilon}(t) v_i^\varepsilon \sqrt{g^\varepsilon} d\Gamma^\varepsilon, \end{aligned} \tag{1}$$

for all $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{V}(\Omega^\varepsilon)$, in the sense of distributions in $(0, T)$,

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega^\varepsilon} K^\varepsilon \Theta^\varepsilon(t) \Xi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} S^\varepsilon \partial_i^\varepsilon \Theta^\varepsilon(t) g^{ij,\varepsilon} \partial_j^\varepsilon \Xi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\
& + \frac{d}{dt} \int_{\Omega^\varepsilon} A^\varepsilon (3\lambda^\varepsilon + 2\mu^\varepsilon) \Xi^\varepsilon e_{i||j}^\varepsilon(\mathbf{u}^\varepsilon(t)) g^{ij,\varepsilon} \sqrt{g^\varepsilon} dx^\varepsilon = \int_{\Omega^\varepsilon} Q^\varepsilon(t) \Xi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon, \\
& \text{for all } \Xi^\varepsilon \in W(\Omega^\varepsilon), \text{ in the sense of distributions in } (0, T),
\end{aligned} \tag{2}$$

and that satisfy the following initial conditions

$$\begin{cases} \mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(0) = \mathbf{u}_1^\varepsilon, \\ \Theta^\varepsilon(0) = \Theta_0^\varepsilon, \end{cases} \tag{3}$$

where the initial data $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \Theta_0^\varepsilon) \in \mathbf{X}(\Omega^\varepsilon)$ is prescribed. \blacksquare

The existence and uniqueness of solutions of Problem $\mathcal{P}(\Omega^\varepsilon)$ immediately follows by the properties of semi-groups [22], which can be verified for the context under consideration by resorting, since $\mathbf{X}(\Omega^\varepsilon)$ is dense in $\mathbf{H}(\Omega^\varepsilon)$, to the same proof technique utilised in [23]. As a consequence of Proposition 6.2 of [22], we obtain that $\dot{\mathbf{u}}^\varepsilon \in L^\infty(0, T; \mathbf{V}(\Omega^\varepsilon))$.

Another proof, not resorting to semi-groups theory, can be given by mimicking the argument of Theorems 5 and 6 in section 5.

4 The scaled three-dimensional problem for a family of flexural shells

In Section 3, we considered a variational problem for “general” linearly elastic shells. From now on, we will restrict ourselves to a specific class of shells, according to the following definition (originally proposed in [16]; see also [10]).

Consider a linearly elastic shell, subjected to the various assumptions set forth in Section 3. Such a shell is said to be a *linearly elastic flexural shell* if the following two additional assumptions are satisfied: *first*, $\emptyset \neq \gamma_0 \subset \gamma$, i.e., the homogeneous boundary condition of place is imposed over a *non-zero area portion of the entire lateral face* $\gamma_0 \times [-\varepsilon, \varepsilon]$ of the shell, and *second*, the space

$$\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \text{ and } \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

contains non-zero functions, i.e., we have $\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}$. The symbol ∂_ν denotes the *outer unit normal derivative operator along* γ .

The corresponding space of admissible temperature variations takes the form:

$$W(\omega) := \{\xi \in H^1(\omega); \xi = 0 \text{ on } \gamma_0\}.$$

Similarly, we define the space

$$\mathbf{X}_F(\omega) := \mathbf{V}_F(\omega) \times \mathbf{L}^2(\omega) \times W(\omega),$$

and the space

$$\mathbf{H}_F(\omega) := \mathbf{V}_F(\omega) \times \mathbf{L}^2(\omega) \times L^2(\omega),$$

and we equip them with the norms

$$\|\mathbf{U}\|_{\mathbf{X}_F(\omega)} := \|\boldsymbol{\eta}\|_{\mathbf{V}_F(\omega)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(\omega)} + \|\tau\|_{W(\omega)}, \quad \text{for all } \mathbf{U} = (\boldsymbol{\eta}, \boldsymbol{\xi}, \tau) \in \mathbf{X}_F(\omega),$$

and

$$\|\mathbf{U}\|_{\mathbf{H}_F(\omega)} := \|\boldsymbol{\eta}\|_{\mathbf{V}_F(\omega)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(\omega)} + \|\tau\|_{L^2(\omega)}, \quad \text{for all } \mathbf{U} = (\boldsymbol{\eta}, \boldsymbol{\xi}, \tau) \in \mathbf{H}_F(\omega),$$

respectively.

If the initial data of Problem $\mathcal{P}(\Omega^\varepsilon)$ is *also* in $\mathbf{X}_F(\omega)$, then such an initial data can be identified with a uniquely determined vector field in $\mathbf{X}(\Omega^\varepsilon)$ independent of the transverse variable (cf. Section 4.2 of [10]). Such an assumption on the data will be used to correctly recover the sought two-dimensional limit problem in the forthcoming asymptotic analysis. Therefore, in the rest of the analysis, we assume that the prescribed initial data $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \Theta_0^\varepsilon)$ satisfies the following hypothesis: *There exists* $(\zeta_0, \zeta_1, \vartheta_0) \in \mathbf{V}_F(\omega) \times \mathbf{V}_F(\omega) \times L^2(\omega)$ *such that*:

$$(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \Theta_0^\varepsilon) = (\zeta_0, \zeta_1, \vartheta_0). \tag{4}$$

Our basic objective then consists in performing an *asymptotic analysis as* $\varepsilon \rightarrow 0$, so as to seek whether we can identify a *two-dimensional limit problem*. To this end, we shall resort to a (by now standard) methodology first proposed by Ciarlet, Lods and Miara (cf. Theorem 5.1 of [16] and Theorem 6.2-1 of [10]): To begin with, we “scale” each problem $\mathcal{P}(\Omega^\varepsilon)$, $\varepsilon > 0$, over a *fixed domain* Ω , using appropriate *scalings*

on the unknowns and assumptions on the data. Note that these scalings and assumptions definitely depend on the type of shells that are considered; for instance, those used for the *linearly elastic elliptic membrane shells* considered elsewhere (cf., e.g., [15], [17] and also [18]) are different.

Let us define the sets

$$\Omega := \omega \times]-1, 1[\text{ and } \Gamma_0 := \gamma_0 \times [-1, 1] \text{ and } \Gamma_{\pm} := \omega \times \{\pm 1\},$$

let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let $\partial_i := \partial/\partial x_i$. With each point $x = (x_i) \in \overline{\Omega}$, we associate the point $x^\varepsilon = (x_i^\varepsilon)$ defined by

$$x_\alpha^\varepsilon := x_\alpha = y_\alpha \text{ and } x_3^\varepsilon := \varepsilon x_3,$$

so that $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \varepsilon^{-1} \partial_3$. To the unknown $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ and to the vector fields $\mathbf{v}^\varepsilon = (v_i^\varepsilon)$ appearing in the equations (1) of Problem $\mathcal{P}(\Omega^\varepsilon)$ corresponding to a linearly elastic flexural shell, we then associate the *scaled unknown* $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ and the *scaled vector fields* $\mathbf{v} = (v_i)$ by letting

$$u_i(\varepsilon)(t)(x) := u_i^\varepsilon(t)(x^\varepsilon) \text{ and } v_i(x) := v_i^\varepsilon(x^\varepsilon),$$

at each $x \in \overline{\Omega}$ and at each $t \in [0, T]$.

For the temperature variations, we can proceed analogously: To the unknown Θ^ε and to the functions Ξ^ε appearing in the equations (2) of Problem $\mathcal{P}(\Omega^\varepsilon)$, we associate the *scaled unknown* $\Theta(\varepsilon)$ and the *scaled functions* Ξ by letting

$$\Theta(\varepsilon)(t)(x) := \Theta^\varepsilon(t)(x^\varepsilon) \text{ and } \Xi(x) := \Xi^\varepsilon(x^\varepsilon),$$

at each $x \in \overline{\Omega}$ and at each $t \in [0, T]$.

To begin, with, we *assume* that there exist functions $f^i \in H^1(0, T; L^2(\Omega))$, $h^i \in H^1(0, T; L^2(\Gamma_+ \cup \Gamma_-))$, $Q \in L^\infty(0, T; L^2(\Omega))$ independent on ε such that the following *assumptions on the data* hold:

$$\begin{aligned} f^{i,\varepsilon}(t)(x^\varepsilon) &= \varepsilon^2 f^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega, \\ h^{i,\varepsilon}(t)(x^\varepsilon) &= \varepsilon^3 h^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Gamma_+ \cup \Gamma_-, \\ Q^\varepsilon(t)(x^\varepsilon) &= \varepsilon^2 Q(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega. \end{aligned} \quad (5)$$

Finally, we *assume* that there exist positive constants ρ , A , K , S , λ and μ such that:

$$\begin{aligned} \rho^\varepsilon(x^\varepsilon) &= \varepsilon^2 \rho \text{ at a.a. } x \in \Omega, \\ A^\varepsilon(x^\varepsilon) &= \varepsilon A \text{ at a.a. } x \in \Omega, \\ K^\varepsilon(x^\varepsilon) &= \varepsilon^2 K \text{ at a.a. } x \in \Omega, \\ S^\varepsilon(x^\varepsilon) &= \varepsilon^2 S \text{ at a.a. } x \in \Omega, \\ \lambda^\varepsilon &= \lambda \quad \text{and} \quad \mu^\varepsilon = \mu. \end{aligned} \quad (6)$$

Note that the independence on ε of the Lamé constants assumed in Section 3 in the formulation of Problem $\mathcal{P}(\Omega^\varepsilon)$ implicitly constituted another *assumption on the data*. The scalings of the mass density per unit volume, of the applied body force, and the applied surface forces are made in accordance with [9] and [10].

The next lemma (cf., e.g., Theorems 3.3-1 and 3.3-2 in [10]) assembles various asymptotic properties as $\varepsilon \rightarrow 0$ of functions and vector fields appearing in the formulation of problem $\mathcal{P}(\varepsilon; \Omega)$; these properties will be repeatedly used in the proof of the convergence theorem (Theorem 7):

In the next statement, the notation “ $O(\varepsilon)$ ”, or “ $O(\varepsilon^2)$ ”, stands for a remainder that is of order ε , or ε^2 , with respect to the sup-norm over the set $\overline{\Omega}$, and any function, or vector-valued function, of the variable $y \in \overline{\omega}$, such as $a^{\alpha\beta}$, $b_{\alpha\beta}$, \mathbf{a}^i , etc. (all these are defined in section 2) is identified with the function, or vector-valued function, of $x = (y, x_3) \in \overline{\Omega} = \overline{\omega} \times [-1, 1]$ that takes the same value at $x_3 = 0$ and is independent of $x_3 \in [-1, 1]$; for brevity, this extension from $\overline{\omega}$ to $\overline{\Omega}$ is designated with the same notation.

Recall that $\varepsilon > 0$ is implicitly assumed to be *small enough* so that $\Theta : \overline{\Omega}^\varepsilon \rightarrow \mathbb{E}^3$ is an immersion.

For later purposes (like in Lemma 1 below), we also let

$$\mathbf{g}_i(\varepsilon)(x) := g_i^\varepsilon(x^\varepsilon) \text{ at each } x \in \overline{\Omega}.$$

Lemma 1 *Let ε_0 be defined as in Theorem 3.1-1 of [10]. The functions $A^{ijkl}(\varepsilon) = A^{jikl}(\varepsilon) = A^{klij}(\varepsilon)$ have the following properties:*

$$A^{ijkl}(\varepsilon) = A^{ijkl}(0) + O(\varepsilon), \quad A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0,$$

for all $0 < \varepsilon \leq \varepsilon_0$, where

$$\begin{aligned} A^{\alpha\beta\sigma\tau}(0) &= \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), \\ A^{\alpha\beta 33}(0) &= \lambda a^{\alpha\beta}, \quad A^{\alpha 333}(0) = \mu a^{\alpha\sigma}, \quad A^{3333}(0) = \lambda + 2\mu, \end{aligned}$$

and there exists a constant $C_0 > 0$ such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_0 A^{ijk\ell}(\varepsilon)(x) t_{k\ell} t_{ij}$$

for all $0 < \varepsilon \leq \varepsilon_0$, all $x \in \overline{\Omega}$, and all symmetric matrices (t_{ij}) .

The functions $\Gamma_{ij}^p(\varepsilon)$ and $g(\varepsilon)$ have the following properties:

$$\begin{aligned} \Gamma_{\alpha\beta}^\sigma(\varepsilon) &= \Gamma_{\alpha\beta}^\sigma - \varepsilon x_3 (\partial_\alpha b_\beta^\sigma + \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\alpha\beta}^\tau b_\tau^\sigma) + O(\varepsilon^2), \\ \Gamma_{\alpha\beta}^3(\varepsilon) &= b_{\alpha\beta} - \varepsilon x_3 b_\alpha^\sigma b_{\sigma\beta}, \quad \partial_3 \Gamma_{\alpha\beta}^p(\varepsilon) = O(\varepsilon), \\ \Gamma_{\alpha 3}^\sigma(\varepsilon) &= -b_\alpha^\sigma - \varepsilon x_3 b_\alpha^\tau b_\tau^\sigma + O(\varepsilon^2), \quad \Gamma_{\alpha 3}^3(\varepsilon) = \Gamma_{33}^p(\varepsilon) = 0, \\ g(\varepsilon) &= a + O(\varepsilon), \end{aligned}$$

for all $0 < \varepsilon \leq \varepsilon_0$ and all $x \in \overline{\Omega}$. In particular then, there exist constants g_0 and g_1 such that

$$0 < g_0 \leq g(\varepsilon)(x) \leq g_1 \text{ for all } 0 < \varepsilon \leq \varepsilon_0 \text{ and all } x \in \overline{\Omega}.$$

The vector fields $\mathbf{g}_i(\varepsilon)$ and $\mathbf{g}^j(\varepsilon)$ have the following properties:

$$\begin{aligned} \mathbf{g}_\alpha(\varepsilon) &= \mathbf{a}_\alpha - \varepsilon x_3 b_\alpha^\sigma \mathbf{a}_\sigma, \quad \mathbf{g}_3(\varepsilon) = \mathbf{a}_3, \\ \mathbf{g}^\alpha(\varepsilon) &= \mathbf{a}^\alpha + \varepsilon x_3 b_\sigma^\alpha \mathbf{a}^\sigma + O(\varepsilon^2), \quad \mathbf{g}^3(\varepsilon) = \mathbf{a}^3. \end{aligned}$$

□

Remark 1 The various relations and estimates in Lemma 1 hold in fact for *any* family of linearly elastic shells, i.e., irrespective of whether these shells are flexural ones or not.

The scaled variational problem $\mathcal{P}(\varepsilon; \Omega)$ defined in the next theorem will constitute the *point of departure* of our asymptotic analysis.

Theorem 1 Define the space

$$\mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

and the space

$$W(\Omega) := \{\Xi \in H^1(\Omega); \Xi = 0 \text{ on } \Gamma_0\}.$$

Let $\varepsilon_0 > 0$ be defined as in Lemma 1. For each $0 < \varepsilon \leq \varepsilon_0$, define the scaled functions

$$\begin{aligned} \mathbf{g}^i(\varepsilon)(x) &:= \mathbf{g}^{i,\varepsilon}(x^\varepsilon) \text{ at each } x \in \overline{\Omega}, \\ g(\varepsilon)(x) &:= g^\varepsilon(x^\varepsilon) \text{ and } A^{ijk\ell}(\varepsilon)(x) := A^{ijk\ell,\varepsilon}(x^\varepsilon) \text{ at each } x \in \overline{\Omega}, \\ e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) &:= \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma_{\alpha\beta}^k(\varepsilon) v_k = e_{\beta\|\alpha}(\varepsilon; \mathbf{v}), \\ e_{\alpha\|3}(\varepsilon; \mathbf{v}) = e_{3\|\alpha}(\varepsilon; \mathbf{v}) &:= \frac{1}{2} \left(\frac{1}{\varepsilon} \partial_3 v_\alpha + \partial_\alpha v_3 \right) - \Gamma_{\alpha 3}^\sigma(\varepsilon) v_\sigma, \\ e_{3\|3}(\varepsilon; \mathbf{v}) &:= \frac{1}{\varepsilon} \partial_3 v_3, \end{aligned}$$

where

$$\Gamma_{ij}^p(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon) \text{ at each } x \in \overline{\Omega}.$$

Define the scaled version of Problem $\mathcal{P}(\Omega^\varepsilon)$ as follows:

Problem $\mathcal{P}(\varepsilon; \Omega)$ Find a vector field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon)) : (0, T) \rightarrow \mathbf{V}(\Omega)$ and a function $\Theta(\varepsilon) : (0, T) \rightarrow W(\Omega)$ such that

$$\begin{aligned} \mathbf{u}(\varepsilon) &\in L^\infty(0, T; \mathbf{V}(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\in L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \ddot{\mathbf{u}}(\varepsilon) &\in L^\infty(0, T; \mathbf{V}^*(\Omega)), \\ \Theta(\varepsilon) &\in L^\infty(0, T; W(\Omega)), \\ \dot{\Theta}(\varepsilon) &\in L^\infty(0, T; W^*(\Omega)), \end{aligned}$$

that satisfy the following variational equations

$$\begin{aligned}
& \varepsilon^2 \rho \frac{d^2}{dt^2} \int_{\Omega} u_i(\varepsilon)(t) g^{ij}(\varepsilon) v_j \sqrt{g(\varepsilon)} dx + \int_{\Omega} A^{ijk\ell}(\varepsilon) e_{k\parallel\ell}(\varepsilon; \mathbf{u}(\varepsilon)(t)) e_{i\parallel j}(\varepsilon; \mathbf{v}) \sqrt{g(\varepsilon)} dx \\
& - A\varepsilon(3\lambda + 2\mu) \int_{\Omega} \Theta(\varepsilon)(t) e_{i\parallel j}(\varepsilon; \mathbf{v}) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} dx \\
& = \varepsilon^2 \int_{\Omega} f^i(t) v_i \sqrt{g(\varepsilon)} dx + \varepsilon^2 \int_{\Gamma_+ \cup \Gamma_-} h^i(t) v_i \sqrt{g(\varepsilon)} d\Gamma, \\
& \text{for all } \mathbf{v} = (v_i) \in \mathbf{V}(\Omega), \text{ in the sense of distributions in } (0, T),
\end{aligned} \tag{7}$$

$$\begin{aligned}
& K\varepsilon^2 \frac{d}{dt} \int_{\Omega} \Theta(\varepsilon)(t) \Xi \sqrt{g(\varepsilon)} dx + S\varepsilon^2 \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \Xi \sqrt{g(\varepsilon)} dx + S \int_{\Omega} \partial_3 \Theta(\varepsilon)(t) \partial_3 \Xi \sqrt{g(\varepsilon)} dx \\
& + A\varepsilon(3\lambda + 2\mu) \frac{d}{dt} \int_{\Omega} \Xi e_{i\parallel j}(\varepsilon; \mathbf{u}(\varepsilon)(t)) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} dx = \varepsilon^2 \int_{\Omega} Q(t) \Xi \sqrt{g(\varepsilon)} dx,
\end{aligned} \tag{8}$$

for all $\Xi \in W(\Omega)$, in the sense of distributions in $(0, T)$,

and that satisfy the initial conditions (3), which are assumed to appear in the form (4). \blacksquare

Then the pair of scaled unknowns $(\mathbf{u}(\varepsilon), \Theta(\varepsilon))$ is the unique solution of Problem $\mathcal{P}(\varepsilon; \Omega)$.

Proof The variational problem $\mathcal{P}(\varepsilon, \Omega)$ simply constitutes a re-writing of the variational problem $\mathcal{P}(\Omega^\varepsilon)$, this time in terms of the scaled unknowns $\mathbf{u}(\varepsilon)$ and $\Theta(\varepsilon)$, of the vector fields \mathbf{v} , and of the functions f^i , h^i and Q , as well as the constants ρ , A , K , S , λ and μ which are now all defined over the domain Ω . Then all the assertions follow from this observation. \square

The functions $e_{i\parallel j}(\varepsilon; \mathbf{v})$ appearing in problem $\mathcal{P}(\varepsilon; \Omega)$ are called the *scaled linearised strains in curvilinear coordinates* associated with the scaled displacement vector field $v_i \mathbf{g}^i(\varepsilon)$.

When one considers a family of linearly elastic *flexural shells* whose thickness 2ε approaches zero, a specific *Korn's inequality in curvilinear coordinates* holds over the fixed domain $\Omega = \omega \times]-1, 1[$, according to the following theorem (cf., e.g., Theorem 4.1 of [16] or Theorem 5.3-1 of [10]). That the constant C_1 that appears in this inequality is *independent of* $\varepsilon > 0$ plays a key role in the forthcoming asymptotic analysis.

Theorem 2 *Let there be given a family of linearly elastic flexural shells with the same middle surface $\boldsymbol{\theta}(\bar{\omega})$ and thickness $2\varepsilon > 0$. Recall the definition of the space*

$$\mathbf{V}(\Omega) = \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}.$$

Then there exist constants $0 < \varepsilon_1 \leq \varepsilon_0$ and $C_1 > 0$ such that

$$\left\{ \sum_i \|v_i\|_{H^1(\Omega)}^2 \right\}^{1/2} \leq \frac{C_1}{\varepsilon} \left\{ \sum_{i,j} \|e_{i\parallel j}(\varepsilon; \mathbf{v})\|_{L^2(\Omega)}^2 \right\}^{1/2},$$

for all $0 < \varepsilon \leq \varepsilon_1$ and all $\mathbf{v} \in \mathbf{V}(\Omega)$. \square

We recall a very important inequality which is used to study evolutionary problems: Gronwall's inequality (see the seminal paper [24] and Theorem 1.1 in Chapter III of [25]).

Theorem 3 *Let $T > 0$ and suppose that the function $y : [0, T] \rightarrow \mathbb{R}$ is absolutely continuous and such that*

$$\frac{dy}{dt}(t) \leq a(t)y(t) + b(t), \quad \text{for a.a. } t \in (0, T),$$

where $a, b \in L^1(0, T)$ and $a, b \geq 0$ almost everywhere in $(0, T)$. Then, it results

$$y(t) \leq \left[y(0) + \int_0^t b(s) ds \right] e^{\int_0^t a(s) ds}, \quad \text{for all } t \in [0, T].$$

\square

5 A preparatory result for the asymptotic analysis

In this section we consider the following problem $\mathcal{P}_F(\omega)$, that will be recovered by means of a rigorous asymptotic analysis in the next section. Before rigorously stating the problem, we define the space

$$\mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Define the norm $\|\cdot\|_{\mathbf{V}_K(\omega)}$ by

$$\|\boldsymbol{\eta}\|_{\mathbf{V}_K(\omega)} := \left\{ \sum_{\alpha} \|\eta_{\alpha}\|_{H^1(\omega)}^2 + \|\eta_3\|_{H^2(\omega)}^2 \right\}^{1/2} \quad \text{for each } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega).$$

Define the fourth order two-dimensional elasticity tensor $(a^{\alpha\beta\sigma\tau})$, the vector $\mathbf{p} = (p^i) \in H^1(0, T; \mathbf{L}^2(\omega))$, and the function $q \in L^\infty(0, T; L^2(\omega))$ respectively by:

$$\begin{aligned} a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu \left(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma} \right), \\ p^i(t) &:= \int_{-1}^1 f^i(t)(\cdot, x_3) dx_3 + h^i(t)(\cdot, 1) + h^i(t)(\cdot, -1) \in L^2(\omega), \quad \text{for a.a. } t \in (0, T), \\ q(t) &:= \int_{-1}^1 Q(t)(\cdot, x_3) dx_3 \in L^2(\omega), \quad \text{for a.a. } t \in (0, T). \end{aligned}$$

Problem $\mathcal{P}_F(\omega)$ Find a vector field $\boldsymbol{\zeta} = (\zeta_i) : (0, T) \rightarrow \mathbf{V}_F(\omega)$ and a function $\vartheta : (0, T) \rightarrow W(\omega)$ such that

$$\begin{aligned} \boldsymbol{\zeta} &\in L^\infty(0, T; \mathbf{V}_F(\omega)), \\ \dot{\boldsymbol{\zeta}} &\in L^\infty(0, T; \mathbf{L}^2(\omega)), \\ \ddot{\boldsymbol{\zeta}} &\in L^\infty(0, T; \mathbf{V}_F^*(\omega)), \\ \vartheta &\in L^\infty(0, T; W(\omega)), \\ \dot{\vartheta} &\in L^\infty(0, T; W^*(\omega)), \end{aligned}$$

that satisfy the following variational equations

$$2\rho \frac{d^2}{dt^2} \int_{\omega} (\zeta_{\alpha}(t) a^{\alpha\beta} \eta_{\beta} + \zeta_3(t) \eta_3) \sqrt{a} dy + \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}(t)) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy = \int_{\omega} p^i(t) \eta_i \sqrt{a} dy, \quad (9)$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$, in the sense of distributions in $(0, T)$,

$$2 \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \vartheta(t) \xi \sqrt{a} dy + 2S \int_{\omega} \partial_{\alpha} \vartheta(t) a^{\alpha\beta} \partial_{\beta} \xi \sqrt{a} dy = \int_{\omega} q(t) \xi \sqrt{a} dy, \quad (10)$$

for all $\xi \in W(\omega)$, in the sense of distributions in $(0, T)$,

and that satisfy the initial conditions

$$\begin{cases} \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0, \\ \dot{\boldsymbol{\zeta}}(0) = \boldsymbol{\zeta}_1, \\ \vartheta(0) = \vartheta_0, \end{cases} \quad (11)$$

where the right hand side of (11) is as in (4). ■

Following [30] and [31], it can be shown that this problem admits a unique solution. In this fashion, the “mechanical” part of Problem $\mathcal{P}_F(\omega)$ is treated via a penalty scheme, amenable for the realisation of numerical simulations (cf., e.g., [38]). In order to establish this existence and uniqueness result, we resort to the following inequality of Korn's type on a general surface (cf., e.g., Theorem 2.6-4 of [10] for a proof).

Theorem 4 Let ω be a domain in \mathbb{R}^2 and let $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbb{E}^3)$ be an immersion. Let γ_0 be a non-empty relatively open subset of $\gamma = \partial\omega$. Recall the definition of the space

$$\mathbf{V}_K(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Then there exists a constant $c = c(\omega, \gamma_0, \boldsymbol{\theta}) > 0$ such that

$$\left\{ \sum_{\alpha} \|\eta_{\alpha}\|_{H^1(\omega)}^2 + \|\eta_3\|_{H^2(\omega)}^2 \right\}^{1/2} \leq c \left\{ \sum_{\alpha, \beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2(\omega)}^2 + \sum_{\alpha, \beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2(\omega)}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$. □

The above inequality, which is due to [1] and was later given a simpler proof in [2], constitutes an example of a *Korn's inequality on a general surface*; it constitutes a “Korn inequality” in the sense that it provides a basic estimate of an appropriate norm of a displacement field defined on a surface in terms of an appropriate norm of a specific “measure of strain” (here, the linearised change of metric tensor and the linearised change of curvature tensor) corresponding to the displacement field considered.

The purpose of the next theorem, which constitutes the *first new result* in this paper, is to show that if the applied body forces and applied surface forces are as regular as in the assumption (5), we have that the velocity field $\dot{\zeta}$ is of class $L^\infty(0, T; \mathbf{V}_F(\omega))$.

The technique we propose is *independent of semi-groups theory*, and the proof is worth working out, as the space $\mathbf{V}_F(\omega)$ incorporates a linear constraint which renders the analysis more complicated and prevents many of the standard results from being applied.

Let $\kappa > 0$ denote a *penalty parameter* and let us introduce the “penalised” problem $\mathcal{P}_{F,\kappa}(\omega)$ corresponding to equations (9) of Problem $\mathcal{P}_F(\omega)$.

Problem $\mathcal{P}_{F,\kappa}(\omega)$ Find a vector field $\zeta_\kappa = (\zeta_{i,\kappa}) : [0, T] \rightarrow \mathbf{V}_K(\omega)$ such that

$$\zeta_\kappa \in \mathcal{C}^0([0, T]; \mathbf{V}_K(\omega)) \cap \mathcal{C}^1([0, T]; \mathbf{L}^2(\omega)),$$

that satisfies the following variational equations

$$\begin{aligned} 2\rho \frac{d^2}{dt^2} \int_{\omega} (\zeta_{\alpha,\kappa}(t) a^{\alpha\beta} \eta_\beta + \zeta_{3,\kappa}(t) \eta_3) \sqrt{a} \, dy + \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_\kappa(t)) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \\ + \frac{1}{\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_\kappa(t)) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy = \int_{\omega} p^i(t) \eta_i \sqrt{a} \, dy, \end{aligned}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$, in the sense of distributions in $(0, T)$, and which satisfies the initial conditions

$$\begin{cases} \zeta_\kappa(0) &= \zeta_0, \\ \dot{\zeta}_\kappa(0) &= \dot{\zeta}_1. \end{cases}$$

■

Problem $\mathcal{P}_{F,\kappa}(\omega)$ admits a unique solution by virtue of the results contained in [31]. We show next that the velocity field $\dot{\zeta}_\kappa$ is more regular with respect to the spatial variable.

Theorem 5 Let $\kappa > 0$, and let ζ_κ be the unique solution of Problem $\mathcal{P}_{F,\kappa}(\omega)$. If $\mathbf{p} = (p^i) \in H^1(0, T; \mathbf{L}^2(\omega))$, then

$$\dot{\zeta}_\kappa \in L^\infty(0, T; \mathbf{V}_K(\omega)) \text{ and } \ddot{\zeta}_\kappa \in L^\infty(0, T; \mathbf{L}^2(\omega)).$$

Proof Strong, weak and weak-star convergences are respectively denoted \rightarrow , \rightharpoonup , and $\overset{*}{\rightharpoonup}$.

The *outline* of the proof is to a large extent inspired by the proof of Theorem 5.1 of [31], where no thermal action was considered. For sake of clarity, the proof is broken into three parts, numbered (i)–(iii).

(i) *Construction of Galerkin approximation.* Observe that the space $\mathbf{V}_K(\omega)$ is an infinite-dimensional and separable Hilbert space. Therefore, by Theorem 6.2-1 of [35], there exists an orthonormal Hilbert basis $(\mathbf{w}^k)_{k=1}^\infty$ of the space $\mathbf{L}^2(\omega)$ which also constitutes an orthogonal Hilbert basis of the space $\mathbf{V}_K(\omega)$.

For each positive integer $m \geq 1$, let us denote by \mathbf{E}^m the following m -dimensional linear hull

$$\mathbf{E}^m := \text{Span}(\mathbf{w}^k)_{k=1}^m \subset \mathbf{V}_K(\omega) \subset \mathbf{L}^2(\omega).$$

Since each element of the Hilbert basis $(\mathbf{w}^k)_{k=1}^\infty$ is independent of the time variable t , we have $\mathbf{w}^k \in L^\infty(0, T; \mathbf{V}_K(\omega))$, for each integer $1 \leq k \leq m$. We now discretize Problem $\mathcal{P}_{F,\kappa}(\omega)$ and, in order to keep the notation simple, we drop the dependence of the vector fields entering the variational equations on the parameters κ and ε . Let us observe that, the duality pair between \mathbf{E}^m and its dual coincides with the inner product of $\mathbf{L}^2(\omega)$.

For each positive integer $m \geq 1$, the “penalised” discrete problem corresponding to Problem $\mathcal{P}_{F,\kappa}(\omega)$, that we denote by $\mathcal{P}_{F,\kappa}^m(\omega)$, amounts to:

Problem $\mathcal{P}_{F,\kappa}^m(\omega)$ Find functions $c_k : [0, T] \rightarrow \mathbb{R}$, $1 \leq k \leq m$, such that

$$\zeta^m(t) := \sum_{k=1}^m c_k(t) \mathbf{w}^k, \text{ for a.a. } t \in (0, T),$$

which satisfies the following variational equations in the sense of distributions in $(0, T)$, for each integer $1 \leq p \leq m$

$$2\rho \int_{\omega} \ddot{\zeta}_\alpha^m(t) a^{\alpha\beta} w_\beta^p \sqrt{a} \, dy + 2\rho \int_{\omega} \ddot{\zeta}_3^m(t) w_3^p \sqrt{a} \, dy + \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta^m(t)) \rho_{\alpha\beta}(\mathbf{w}^p) \sqrt{a} \, dy$$

$$+ \frac{1}{\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta^m(t)) \gamma_{\alpha\beta}(\mathbf{w}^p) \sqrt{a} \, dy = \int_{\omega} p^i(t) w_i^p \sqrt{a} \, dy,$$

and for which the following initial conditions hold:

$$\begin{cases} \zeta^m(0) &= \zeta_0^m, \\ \dot{\zeta}^m(0) &= \zeta_1^m, \end{cases} \quad (12)$$

where initial data ζ_0^m and ζ_1^m are, respectively, the projections of ζ_0 and ζ_1 onto the finite dimensional space \mathbf{E}^m . \blacksquare

For each $\kappa > 0$, we define the following continuous and $\mathbf{V}_K(\omega)$ -elliptic bilinear form:

$$a_{\kappa}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\xi}) \sqrt{a} \, dy + \frac{1}{\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\xi}) \sqrt{a} \, dy,$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$ and all $\boldsymbol{\xi} = (\xi_i) \in \mathbf{V}_K(\omega)$.

We immediately observe that the projections of $\zeta_0 = (\zeta_{0,i}) \in \mathbf{V}_F(\omega)$ and $\zeta_1 = (\zeta_{1,i}) \in \mathbf{V}_F(\omega)$ onto \mathbf{E}^m can be expanded as follows (cf. Theorem 4.9-1 of [12])

$$\begin{aligned} \zeta_0^m &= \sum_{k=1}^m \left(\int_{\omega} \zeta_{0,i} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha} \zeta_{0,i} \partial_{\alpha} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha\beta} \zeta_{0,3} \partial_{\alpha\beta} w_3^k \sqrt{a} \, dy \right) \mathbf{w}^k, \\ \zeta_1^m &= \sum_{k=1}^m \left(\int_{\omega} \zeta_{1,i} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha} \zeta_{1,i} \partial_{\alpha} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha\beta} \zeta_{1,3} \partial_{\alpha\beta} w_3^k \sqrt{a} \, dy \right) \mathbf{w}^k, \end{aligned}$$

so that $\zeta_0^m \rightarrow \zeta_0$ in $\mathbf{V}_K(\omega)$ and $\zeta_1^m \rightarrow \zeta_1$ in $\mathbf{V}_K(\omega)$.

Since the elements of the Hilbert basis do not depend on the time variable we can take the coefficients c_k as well as their derivatives \dot{c}_k and \ddot{c}_k outside the integral sign. This gives, for each $1 \leq k \leq m$, the following second order linear ordinary differential equation with respect to the variable t :

$$2\rho \ddot{c}_k(t) + a_{\kappa}(\mathbf{w}^k, \mathbf{w}^k) c_k(t) = \int_{\omega} p^i(t) w_i^k \sqrt{a} \, dy, \quad (13)$$

and satisfying the following *initial conditions*:

$$\begin{aligned} c_k(0) &= \int_{\omega} \zeta_{0,i} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha} \zeta_{0,i} \partial_{\alpha} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha\beta} \zeta_{0,3} \partial_{\alpha\beta} w_3^k \sqrt{a} \, dy, \\ \dot{c}_k(0) &= \int_{\omega} \zeta_{1,i} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha} \zeta_{1,i} \partial_{\alpha} w_i^k \sqrt{a} \, dy + \int_{\omega} \partial_{\alpha\beta} \zeta_{1,3} \partial_{\alpha\beta} w_3^k \sqrt{a} \, dy. \end{aligned}$$

By the Cauchy-Lipschitz theorem (cf., e.g., Theorem 3.8-1 of [12]), such an ordinary differential equation admits a unique solution, which clearly depends on the parameter κ .

Since, by Theorem 8.60 of [29], we have that $p^i \in H^1(0, T; L^2(\omega)) \hookrightarrow C^0([0, T]; L^2(\omega))$, and since the left hand side of the variational equations in Problem $\mathcal{P}_{F,\kappa}(\omega)$ is an inner product in $\mathbf{L}^2(\omega)$ (recall that the matrix $(a^{\alpha\beta})$ is uniformly positive-definite in $\bar{\omega}$), we are in a position to infer that the vector field $\ddot{\zeta}_{\kappa}(0)$ is well-defined in $\mathbf{L}^2(\omega)$, and is independent of κ as well. We are thus in a position to infer the existence of a vector field $\zeta_2 = (\zeta_{2,i}) \in \mathbf{L}^2(\omega)$ in a way such that

$$\zeta_{2,i} := \ddot{\zeta}_{\kappa,i}(0),$$

and this is the unique vector field for which the following variational equations are satisfied

$$\begin{aligned} 2\rho \int_{\omega} \zeta_{2,\alpha} a^{\alpha\beta} \eta_{\beta} \sqrt{a} \, dy + 2\rho \int_{\omega} \zeta_{2,3} \eta_3 \sqrt{a} \, dy + \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_0) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \\ + \frac{1}{\kappa} \underbrace{\int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_0) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy}_{=0 \text{ as } \zeta_0 \in \mathbf{V}_F(\omega)} = \int_{\omega} p^i(0) \eta_i \sqrt{a} \, dy, \end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$. Letting ζ_2^m be the projection of ζ_2 onto \mathbf{E}^m gives

$$\zeta_2^m \rightarrow \zeta_2, \quad \text{in } \mathbf{L}^2(\omega).$$

(ii) *Energy estimates.* For a.a. $t \in (0, T)$, let $h > 0$ be such that $0 < t+h < T$. Multiplying the variational equations in Problem $\mathcal{P}_{F,\kappa}^m(\omega)$ at times t and $(t+h)$ by $\{\dot{c}_k(t+h) - \dot{c}_k(t)\}$ gives:

$$2\rho \int_{\omega} (\ddot{\zeta}_{\alpha}^m(t+h) - \ddot{\zeta}_{\alpha}^m(t)) a^{\alpha\beta} (\dot{\zeta}_{\alpha}^m(t+h) - \dot{\zeta}_{\alpha}^m(t)) \sqrt{a} \, dy$$

$$\begin{aligned}
& + 2\rho \int_{\omega} (\ddot{\zeta}_3^m(t+h) - \ddot{\zeta}_3^m(t))(\dot{\zeta}_3^m(t+h) - \dot{\zeta}_3^m(t))\sqrt{a} \, dy \\
& + \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta^m(t+h) - \zeta^m(t))\rho_{\alpha\beta}(\dot{\zeta}^m(t+h) - \dot{\zeta}^m(t))\sqrt{a} \, dy \\
& + \frac{1}{\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta^m(t+h) - \zeta^m(t))\gamma_{\alpha\beta}(\dot{\zeta}^m(t+h) - \dot{\zeta}^m(t))\sqrt{a} \, dy \\
& = \int_{\omega} (p^i(t+h) - p^i(t))(\dot{\zeta}_i^m(t+h) - \dot{\zeta}_i^m(t))\sqrt{a} \, dy.
\end{aligned}$$

An integration over $(0, t)$ gives:

$$\begin{aligned}
& \rho \int_{\omega} (\dot{\zeta}_{\alpha}^m(t+h) - \dot{\zeta}_{\alpha}^m(t))a^{\alpha\beta}(\dot{\zeta}_{\alpha}^m(t+h) - \dot{\zeta}_{\alpha}^m(t))\sqrt{a} \, dy + \rho \int_{\omega} |\dot{\zeta}_3^m(t+h) - \dot{\zeta}_3^m(t)|^2 \sqrt{a} \, dy \\
& + \frac{1}{6} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta^m(t+h) - \zeta^m(t))\rho_{\alpha\beta}(\zeta^m(t+h) - \zeta^m(t))\sqrt{a} \, dy \\
& + \frac{1}{2\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta^m(t+h) - \zeta^m(t))\gamma_{\alpha\beta}(\zeta^m(t+h) - \zeta^m(t))\sqrt{a} \, dy \\
& = \int_0^t \int_{\omega} (p^i(\tau+h) - p^i(\tau))(\dot{\zeta}_i^m(\tau+h) - \dot{\zeta}_i^m(\tau))\sqrt{a} \, dy \, d\tau \\
& + \rho \int_{\omega} (\dot{\zeta}_{\alpha}^m(h) - \dot{\zeta}_{\alpha}^m(0))a^{\alpha\beta}(\dot{\zeta}_{\alpha}^m(h) - \dot{\zeta}_{\alpha}^m(0))\sqrt{a} \, dy + \rho \int_{\omega} |\dot{\zeta}_3^m(h) - \dot{\zeta}_3^m(0)|^2 \sqrt{a} \, dy \\
& + \frac{1}{6} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta^m(h) - \zeta^m(0))\rho_{\alpha\beta}(\zeta^m(h) - \zeta^m(0))\sqrt{a} \, dy \\
& + \frac{1}{2\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta^m(h) - \zeta^m(0))\gamma_{\alpha\beta}(\zeta^m(h) - \zeta^m(0))\sqrt{a} \, dy.
\end{aligned}$$

Thanks to the uniform positive-definiteness of the matrix $(a^{\alpha\beta})$, the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$, Korn's inequality on a general surface (Theorem 4), Hölder's inequality and Young's inequality [39], we obtain that there exists a constant $C > 0$ independent of m and κ such that:

$$\begin{aligned}
& \|\dot{\zeta}^m(t+h) - \dot{\zeta}^m(t)\|_{L^2(\omega)}^2 + \|\zeta^m(t+h) - \zeta^m(t)\|_{\mathbf{V}_{\kappa}(\omega)}^2 + \frac{1}{2\kappa} \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\zeta^m(t+h) - \zeta^m(t))\|_{L^2(\omega)}^2 \\
& \leq C \left\{ \sum_i \int_0^T \|p^i(t+h) - p^i(t)\|_{L^2(\omega)}^2 \, dt + \sum_i \int_0^T \|\dot{\zeta}_i^m(t+h) - \dot{\zeta}_i^m(t)\|_{L^2(\omega)}^2 \, dt \right. \\
& \quad + \|\dot{\zeta}^m(h) - \dot{\zeta}^m(0)\|_{L^2(\omega)}^2 + \|\zeta^m(h) - \zeta^m(0)\|_{\mathbf{V}_{\kappa}(\omega)}^2 \\
& \quad \left. + \frac{1}{2\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta^m(h) - \zeta^m(0))\gamma_{\alpha\beta}(\zeta^m(h) - \zeta^m(0))\sqrt{a} \, dy \right\}.
\end{aligned}$$

Divide both sides by h^2 and let $h \rightarrow 0$. An application of Theorem 8.60 of [29] together with the convergence of the family $(\zeta_2^m)_{m=1}^{\infty}$ and the convergence of the family $(\zeta_1^m)_{m=1}^{\infty}$, gives:

$$\begin{aligned}
& \|\dot{\zeta}^m(t)\|_{L^2(\omega)}^2 + \|\dot{\zeta}^m(t)\|_{\mathbf{V}_{\kappa}(\omega)}^2 + \frac{1}{\kappa} \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\dot{\zeta}^m(t))\|_{L^2(\omega)}^2 \\
& \leq C \left\{ \int_0^T \|\dot{p}(t)\|_{L^2(\omega)}^2 \, dt + \int_0^T \|\dot{\zeta}^m(t)\|_{L^2(\omega)}^2 \, dt + \|\zeta_2\|_{L^2(\omega)}^2 + \|\zeta_1\|_{\mathbf{V}_{\kappa}(\omega)}^2 + \frac{1}{2\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_1^m)\gamma_{\alpha\beta}(\zeta_1^m)\sqrt{a} \, dy \right\}.
\end{aligned}$$

We also observe that, since $\zeta_1^m \rightarrow \zeta_1 \in \mathbf{V}_F(\omega)$ in $\mathbf{V}_{\kappa}(\omega)$, there exists an integer $\tilde{m} = \tilde{m}(\kappa)$ such that for all $m \geq \tilde{m}$ it results

$$\frac{1}{2\kappa} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_1^m)\gamma_{\alpha\beta}(\zeta_1^m)\sqrt{a} \, dy \leq \frac{1}{2}.$$

Proceeding like in [31], we apply Gronwall's inequality and we infer that

$$\begin{aligned}
& (\dot{\zeta}^m)_{m=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T; \mathbf{V}_{\kappa}(\omega)), \\
& (\ddot{\zeta}^m)_{m=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T; L^2(\omega)), \\
& \left(\frac{1}{\kappa} \tilde{\gamma}_{\alpha\beta}(\dot{\zeta}^m) \right)_{m=1}^{\infty} \text{ is bounded in } L^2(0, T; L^2(\omega)).
\end{aligned}$$

It is also possible to observe, by Fatou's lemma, that there exists a constant $L > 0$ independent of κ such that

$$\|\tilde{\gamma}_{\alpha\beta}(\dot{\zeta}_{\kappa})\|_{L^2(0,T;L^2(\omega))} \leq \sqrt{\kappa L}. \quad (14)$$

(iii) *Passage to the limit.* An application of the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.16 of [6]) gives that, up to passing to suitable subsequences,

$$\begin{aligned}\dot{\zeta}^m &\overset{*}{\rightharpoonup} \dot{\zeta}_\kappa && \text{in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ \ddot{\zeta}^m &\overset{*}{\rightharpoonup} \ddot{\zeta}_\kappa && \text{in } L^\infty(0, T; \mathbf{L}^2(\omega)),\end{aligned}$$

and the proof is complete

Theorem 6 *Let $\zeta \in L^\infty(0, T; \mathbf{V}_F(\omega))$ be the unique solution of Problem $\mathcal{P}_F(\omega)$. If $\mathbf{p} = (p^i) \in H^1(0, T; \mathbf{L}^2(\omega))$ we have that*

$$\dot{\zeta} \in L^\infty(0, T; \mathbf{V}_F(\omega)) \text{ and } \ddot{\zeta} \in L^\infty(0, T; \mathbf{L}^2(\omega)).$$

Proof By virtue of the conclusion obtained in Theorem 5 can thus re-perform the same passages as in the energy estimates. Thanks to the uniform positive-definiteness of the matrix $(a^{\alpha\beta})$, the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$, Korn's inequality on a general surface (Theorem 4), Hölder's inequality and Young's inequality [39], we obtain that there exists a constant $C > 0$ independent of κ such that:

$$\begin{aligned}& \|\dot{\zeta}_\kappa(t+h) - \dot{\zeta}_\kappa(t)\|_{L^2(\omega)}^2 + \|\zeta_\kappa(t+h) - \zeta_\kappa(t)\|_{\mathbf{V}_K(\omega)}^2 + \frac{1}{2\kappa} \sum_{\alpha, \beta} \|\gamma_{\alpha\beta}(\zeta_\kappa(t+h) - \zeta_\kappa(t))\|_{L^2(\omega)}^2 \\ & \leq C \left\{ \sum_i \int_0^T \|p^i(t+h) - p^i(t)\|_{L^2(\omega)}^2 dt + \sum_i \int_0^T \|\dot{\zeta}_{\kappa,i}(t+h) - \dot{\zeta}_{\kappa,i}(t)\|_{L^2(\omega)}^2 dt \right. \\ & \quad \left. + \|\dot{\zeta}_\kappa(h) - \dot{\zeta}_\kappa(0)\|_{L^2(\omega)}^2 + \|\zeta_\kappa(h) - \zeta_\kappa(0)\|_{\mathbf{V}_K(\omega)}^2 + \frac{1}{2\kappa} \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_\kappa(h) - \zeta_\kappa(0)) \gamma_{\alpha\beta}(\zeta_\kappa(h) - \zeta_\kappa(0)) \sqrt{a} dy \right\}.\end{aligned}$$

Divide both sides by h^2 and let $h \rightarrow 0$. An application of Theorem 8.60 of [29] gives:

$$\begin{aligned}& \|\ddot{\zeta}_\kappa(t)\|_{L^2(\omega)}^2 + \|\dot{\zeta}_\kappa(t)\|_{\mathbf{V}_K(\omega)}^2 + \frac{1}{\kappa} \sum_{\alpha, \beta} \|\gamma_{\alpha\beta}(\dot{\zeta}_\kappa(t))\|_{L^2(\omega)}^2 \\ & \leq C \left\{ \int_0^T \|\dot{\mathbf{p}}(t)\|_{L^2(\omega)}^2 dt + \int_0^T \|\dot{\zeta}_\kappa(t)\|_{L^2(\omega)}^2 dt + \|\zeta_2\|_{L^2(\omega)}^2 + \|\zeta_1\|_{\mathbf{V}_K(\omega)}^2 \right\}.\end{aligned}$$

Proceeding like in [31], we apply Gronwall's inequality and we infer that

$$\begin{aligned}(\dot{\zeta}_\kappa)_{\kappa>0} &\text{ is bounded in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ (\ddot{\zeta}_\kappa)_{\kappa>0} &\text{ is bounded in } L^\infty(0, T; \mathbf{L}^2(\omega)).\end{aligned}$$

We apply the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [6]) and we observe that the weak-star limits obtained are the weak time-derivatives of the unique solution ζ obtained in Theorem 6.1 of [31]. Combining the convergence process above with (14) and applying Fatou's lemma gives that

$$\|\gamma_{\alpha\beta}(\dot{\zeta})\|_{L^2(0, T; L^2(\omega))} = 0,$$

which means that $\dot{\zeta} \in L^\infty(0, T; \mathbf{V}_F(\omega))$ as it was to be proved.

6 Asymptotic analysis

The following theorem, that resorts to an argument very similar to the one of [16] (see also Theorem 6.2-1 of [10]), constitutes the *second new result* in this paper. It shows that, in a specific sense, *the solutions $\mathbf{u}(\varepsilon)$ of the (scaled) three-dimensional problems $\mathcal{P}(\varepsilon; \Omega)$ converge as ε approaches zero to the solution of a *ad hoc* two-dimensional problem*. It will turn out that such a two-dimensional limit problem exactly coincides with Problem $\mathcal{P}_F(\omega)$ introduced in section 5.

The vector fields \mathbf{a}^i and the functions $\Gamma_{\alpha\beta}^\sigma, b_{\alpha\beta}, a^{\alpha\beta}, a$, and $\gamma_{\alpha\beta}(\boldsymbol{\eta})$, have already been defined in section 2.

Theorem 7 *Let ω be a domain in \mathbb{R}^2 , let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$ be the middle surface of a flexural shell, let γ_0 be a non-zero length portion of the boundary γ (cf. Section 4).*

Let us consider the non-trivial space of admissible linearised inextensional displacements introduced in section 4

$$\mathbf{V}_F(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \text{ and } \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

as well as the space of admissible temperature variations

$$W(\omega) := \{\xi \in H^1(\omega); \xi = 0 \text{ on } \gamma_0\}.$$

Let there be given a family of linearly elastic flexural shells with each having the same middle surface $\boldsymbol{\theta}(\bar{\omega})$ and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, and let the assumptions on the data be as in (5) and (6).

For $0 < \varepsilon \leq \varepsilon_1$, where ε_1 was defined in Korn's inequality (Theorem 2), let the pair $(\mathbf{u}(\varepsilon), \Theta(\varepsilon))$ denote the solution of the associated scaled three-dimensional problem $\mathcal{P}(\varepsilon; \Omega)$ introduced in Theorem 1. Then there exists a vector field $\mathbf{u} = (u_i) \in L^\infty(0, T; \mathbf{H}^1(\Omega))$ satisfying $\mathbf{u}(t) = \mathbf{0}$ on Γ_0 for a.a. $t \in (0, T)$ and there exists a function $\Theta \in L^\infty(0, T; H^1(\Omega))$ satisfying $\Theta(t) = 0$ on Γ_0 for a.a. $t \in (0, T)$ such that

$$\begin{aligned} \mathbf{u}(\varepsilon) &\rightarrow \mathbf{u}, & \text{in } L^\infty(0, T; \mathbf{V}(\Omega)), \\ \mathbf{u} = (u_i) &\text{ is independent of the transverse variable } x_3, \\ \dot{\mathbf{u}}(\varepsilon) &\rightarrow \dot{\mathbf{u}}, & \text{in } L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \Theta(\varepsilon) &\rightarrow \Theta, & \text{in } L^\infty(0, T; W(\Omega)), \\ \Theta &\text{ is independent of the transverse variable } x_3. \end{aligned}$$

For a.a. $t \in (0, T)$, define the averages

$$\bar{\mathbf{u}}(t) = (\bar{u}_i(t)) := \frac{1}{2} \int_{-1}^1 \mathbf{u}(t)(\cdot, x_3) dx_3, \quad \text{and} \quad \bar{\Theta}(t) := \frac{1}{2} \int_{-1}^1 \Theta(t) dx_3.$$

Then

$$\bar{\mathbf{u}} = \boldsymbol{\zeta} \quad \text{and} \quad \bar{\Theta} = \vartheta,$$

where the pair $(\boldsymbol{\zeta}, \vartheta)$ is the unique solution of the two-dimensional limit problem $\mathcal{P}_F(\omega)$ introduced in section 5.

Proof Strong, weak and weak-star convergences as $\varepsilon \rightarrow 0$ are respectively denoted \rightarrow , \rightharpoonup , and $\overset{*}{\rightharpoonup}$. For sake of brevity, we let

$$\tilde{e}_{i\|j}(\varepsilon)(t) := e_{i\|j}(\varepsilon; \mathbf{u}(\varepsilon)(t)) \quad \text{for a.a. } t \in (0, T).$$

The outline of the proof is to a large extent inspired by the proof of Theorem 6.2-1 in [10] (itself adapted from Theorem 5.1 in [16]), where no thermal action was considered. This is why some parts of the proof are reminiscent of those in [10]; otherwise, considering the time dependence requires substantial extra care. For sake of clarity, the proof is broken into six parts, numbered (i)–(vi).

(i) There exist subsequences, still denoted $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ and $(\Theta(\varepsilon))_{\varepsilon>0}$, and there exist $\mathbf{u} \in L^\infty(0, T; \mathbf{V}(\Omega))$, $\Theta \in L^\infty(0, T; W(\Omega))$ and there exist $e_{i\|j}^1 \in L^\infty(0, T; L^2(\Omega))$ satisfying

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{0} \text{ on } \Gamma_0 = \gamma_0 \times [-1, 1] \text{ for a.a. } t \in (0, T), \\ \Theta(t) &= 0 \text{ on } \Gamma_0 \text{ for a.a. } t \in (0, T), \end{aligned}$$

and such that the following convergence process holds:

$$\begin{aligned} \mathbf{u}(\varepsilon) &\overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0, T; \mathbf{V}(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\overset{*}{\rightharpoonup} \dot{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \ddot{\mathbf{u}}(\varepsilon) &\overset{*}{\rightharpoonup} \ddot{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{V}^*(\Omega)), \\ \frac{1}{\varepsilon} \tilde{e}_{i\|j}(\varepsilon) &\overset{*}{\rightharpoonup} e_{i\|j}^1 \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \Theta(\varepsilon) &\overset{*}{\rightharpoonup} \Theta \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \partial_\alpha \Theta(\varepsilon) &\rightharpoonup \partial_\alpha \Theta \text{ in } L^2(0, T; L^2(\Omega)), \\ \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon) &\rightharpoonup \partial_3^1 \Theta \text{ in } L^2(0, T; L^2(\Omega)), \\ \dot{\Theta}(\varepsilon) &\overset{*}{\rightharpoonup} \dot{\Theta} \text{ in } L^\infty(0, T; W^*(\Omega)). \end{aligned} \tag{15}$$

Let us fix a time instant $t \in (0, T)$. By virtue of the consequences drawn in Proposition 1 of [23] (see also Theorems 5 and 6), we can plug $(\mathbf{v}, \Xi) = (\dot{\mathbf{u}}(\varepsilon)(t), \Theta(\varepsilon)(t))$ in the equations of Problem $\mathcal{P}(\varepsilon; \Omega)$, getting:

$$\begin{aligned} \rho \int_\Omega \ddot{u}_i(\varepsilon)(t) g^{ij}(\varepsilon) \dot{u}_j(\varepsilon)(t) \sqrt{g(\varepsilon)} dx + \int_\Omega A^{ijk\ell}(\varepsilon) \tilde{e}_{k\|l}(\varepsilon)(t) e_{i\|j}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) \sqrt{g(\varepsilon)} dx \\ + K \int_\Omega \dot{\Theta}(\varepsilon)(t) \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} dx + S \int_\Omega \{ \partial_\alpha \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_\beta \Theta(\varepsilon)(t) + \varepsilon^{-2} |\partial_3 \Theta(\varepsilon)(t)|^2 \} \sqrt{g(\varepsilon)} dx \end{aligned}$$

$$= \int_{\Omega} f^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} dx + \int_{\Gamma_+ \cup \Gamma_-} h^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} d\Gamma + \int_{\Omega} Q(t) \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} dx.$$

Let us now perform an integration over the interval $(0, t)$, getting:

$$\begin{aligned} & \rho \int_{\Omega} \dot{u}_i(\varepsilon)(t) g^{ij}(\varepsilon) \dot{u}_j(\varepsilon)(t) \sqrt{g(\varepsilon)} dx \\ & + \frac{1}{2} \int_{\Omega} A^{ijk\ell}(\varepsilon) \tilde{e}_{k\parallel\ell}(\varepsilon)(t) \tilde{e}_{i\parallel j}(\varepsilon)(t) \sqrt{g(\varepsilon)} dx + \frac{K}{2} \int_{\Omega} |\Theta(\varepsilon)(t)|^2 \sqrt{g(\varepsilon)} dx \\ & + S \int_0^t \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(\tau) g^{ij}(\varepsilon) \partial_{\beta} \Theta(\varepsilon)(\tau) \sqrt{g(\varepsilon)} dx d\tau + \frac{S}{\varepsilon^2} \int_0^t \int_{\Omega} |\partial_3 \Theta(\varepsilon)(\tau)|^2 dx d\tau \\ & = \int_0^t \int_{\Omega} f^i(\tau) \dot{u}_i(\varepsilon)(\tau) \sqrt{g(\varepsilon)} dx d\tau + \int_0^t \int_{\Gamma_+ \cup \Gamma_-} h^i(\tau) \dot{u}_i(\varepsilon)(\tau) \sqrt{g(\varepsilon)} d\Gamma d\tau \\ & + \int_0^t \int_{\Omega} Q(\tau) \Theta(\varepsilon)(\tau) \sqrt{g(\varepsilon)} dx d\tau + \rho \int_{\Omega} u_{1,i}^{\varepsilon} g^{ij}(\varepsilon) u_{1,j}^{\varepsilon} \sqrt{g(\varepsilon)} dx \\ & + \frac{1}{2} \int_{\Omega} A^{ijk\ell}(\varepsilon) e_{k\parallel\ell}(\varepsilon; \mathbf{u}_0^{\varepsilon}) e_{i\parallel j}(\varepsilon; \mathbf{u}_0^{\varepsilon}) \sqrt{g(\varepsilon)} dx + \frac{K}{2} \int_{\Omega} |\Theta_0^{\varepsilon}|^2 \sqrt{g(\varepsilon)} dx. \end{aligned} \tag{16}$$

Combining (16), the uniform positive-definiteness of the matrix $(g^{ij}(\varepsilon))$, the positive-definiteness of the tensor $(A^{ijk\ell}(\varepsilon))$, Korn's inequality (Theorem 2), Hölder's inequality, Young's inequality [39], and the asymptotic behaviour of the function $g(\varepsilon)$ (Lemma 1), we obtain that for $\varepsilon > 0$ sufficiently small we can find a constant $C > 0$ independent of ε such that:

$$\begin{aligned} & \frac{1}{C} \left\{ \|\dot{\mathbf{u}}(\varepsilon)(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}(\varepsilon)(t)\|_{\mathbf{V}(\Omega)}^2 + \|\Theta(\varepsilon)(t)\|_{L^2(\Omega)}^2 \right\} \\ & \leq \rho \|\dot{\mathbf{u}}(\varepsilon)(t)\|_{L^2(\Omega)}^2 + \sum_{i,j} \left\| \frac{1}{\varepsilon} e_{i\parallel j}(\varepsilon)(t) \right\|_{L^2(\Omega)}^2 + \|\Theta(\varepsilon)(t)\|_{L^2(\Omega)}^2 \\ & \leq C \left\{ \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{h}\|_{L^2(0,T;L^2(\Gamma_+ \cup \Gamma_-))}^2 + \|Q\|_{L^2(0,T;L^2(\Omega))}^2 + \|\Theta(\varepsilon)\|_{L^2(0,T;L^2(\Omega))}^2 \right. \\ & \quad + \rho \int_0^T \int_{\Omega} \dot{u}_i(\varepsilon)(\tau) g^{ij}(\varepsilon) \dot{u}_j(\varepsilon)(\tau) \sqrt{g(\varepsilon)} dx d\tau + \frac{K}{2} \int_0^T \|\Theta(\varepsilon)(t)\|_{L^2(\Omega)}^2 dt \\ & \quad \left. + \frac{1}{2} \int_0^T \int_{\Omega} A^{ijk\ell}(\varepsilon) \tilde{e}_{k\parallel\ell}(\varepsilon)(t) \tilde{e}_{i\parallel j}(\varepsilon)(t) \sqrt{g(\varepsilon)} dx dt \right\}. \end{aligned}$$

By Gronwall's inequality (Theorem 3) we infer that:

$$\begin{aligned} & (\mathbf{u}(\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; \mathbf{V}(\Omega)), \\ & (\dot{\mathbf{u}}(\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; \mathbf{L}^2(\Omega)), \\ & \left(\frac{1}{\varepsilon} \tilde{e}_{i\parallel j}(\varepsilon) \right)_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)), \\ & (\Theta(\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)). \end{aligned}$$

Similarly, starting again from (16), we can prove that there exists a constant $C > 0$ independent of ε such that:

$$S \sum_{\alpha} \|\partial_{\alpha} \Theta(\varepsilon)\|_{L^2(0,T;L^2(\Omega))}^2 + S \left\| \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon) \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C.$$

In view of the boundedness of the families above, we have that equation (7) gives:

$$(\ddot{\mathbf{u}}(\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; \mathbf{V}^*(\Omega)).$$

Likewise, from equation (8) we derive that:

$$(\dot{\Theta}(\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^{\infty}(0, T; W^*(\Omega)).$$

By the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [6]), we have that the convergence process (15) holds.

Besides, the fact that $\mathbf{u}(t) = \mathbf{0}$ on $\gamma_0 \times [-1, 1]$ for a.a. $t \in (0, T)$ follows from the continuity of the trace operator $\text{tr} : H^1(\Omega) \rightarrow L^2(\gamma_0 \times [-1, 1])$. Likewise, we can infer that $\Theta(t) = 0$ on $\gamma_0 \times [-1, 1]$ for a.a. $t \in (0, T)$.

(ii) *The weak limits $u_i \in L^\infty((0, T); H^1(\Omega))$ found in (i) are independent of the transverse variable $x_3 \in]-1, 1[$, in the sense that they satisfy*

$$\partial_3 u_i = 0 \text{ in } \mathcal{D}'((0, T) \times \Omega).$$

Also the weak limit $\Theta \in L^\infty(0, T; L^2(\Omega))$ found in (i) is independent of the transverse variable $x_3 \in]-1, 1[$, in the sense that

$$\partial_3 \Theta = 0 \text{ in } \mathcal{D}'((0, T) \times \Omega).$$

Besides, the average $\bar{\mathbf{u}}$ satisfies $\bar{\mathbf{u}} \in L^\infty(0, T; \mathbf{V}_F(\omega))$, namely,

$$\begin{aligned} \bar{\mathbf{u}} &= (\bar{u}_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \\ \bar{u}_i(t) &= \partial_\nu \bar{u}_3(t) = 0 \text{ on } \gamma_0 \text{ for a.a. } t \in (0, T), \\ \tilde{\gamma}_{\alpha\beta}(\bar{\mathbf{u}}) &= 0 \text{ in } (0, T) \times \omega. \end{aligned}$$

Apart from the part concerning the *limit temperature* Θ , the proof is identical to that of part (ii) of the proof of Theorem 6.2-1 in [10]. Let us thus prove that $\partial_3 \Theta = 0$ in $\mathcal{D}'((0, T) \times \Omega)$. Thanks to the seventh convergence in (15), we obtain that there exists a constant $C > 0$ such that:

$$\frac{1}{\varepsilon} \|\partial_3 \Theta(\varepsilon)\|_{L^2(0, T; L^2(\Omega))} \leq C.$$

This means that

$$\limsup_{\varepsilon \rightarrow 0} \|\partial_3 \Theta(\varepsilon)\|_{L^2(0, T; L^2(\Omega))} = 0,$$

and thanks to Theorem 8.28 of [29], we have that

$$\int_{(0, T) \times \Omega} \partial_3 \Theta(\varepsilon) \varphi \, dx \, dt = - \int_{(0, T) \times \Omega} \Theta(\varepsilon) \partial_3 \varphi \, dx \, dt \rightarrow 0, \quad \text{for all } \varphi \in \mathcal{D}((0, T) \times \Omega),$$

which in turn implies that $\partial_3 \Theta = 0$ in $\mathcal{D}'((0, T) \times \Omega)$.

(iii) *The weak limits $e_{i\|j}^1 \in L^\infty(0, T; L^2(\Omega))$, $\mathbf{u} \in L^\infty(0, T; \mathbf{V}(\Omega))$ and $\Theta \in L^\infty(0, T; W(\Omega))$ found in (i) satisfy*

$$\begin{aligned} -\partial_3 e_{\alpha\| \beta}^1 &= \tilde{\rho}_{\alpha\beta}(\mathbf{u}) \text{ in } L^\infty(0, T; L^2(\Omega)), \\ e_{\alpha\| 3}^1 &= 0 \text{ in } L^\infty(0, T; L^2(\Omega)), \\ e_{3\| 3}^1 &= \frac{A(3\lambda + 2\mu)}{\lambda + 2\mu} \Theta - \frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e_{\alpha\| \beta}^1 \text{ in } L^\infty(0, T; L^2(\Omega)). \end{aligned}$$

The equality $-\partial_3 e_{\alpha\| \beta}^1 = \tilde{\rho}_{\alpha\beta}(\mathbf{u})$ in $L^\infty(0, T; L^2(\Omega))$ follows from Theorem 8.28 of [29] and Theorem 5.2-2 of [10]. Besides, since $\dot{\bar{\mathbf{u}}} \in L^\infty(0, T; \mathbf{V}_F(\omega))$ (Theorem 6), we have that the interchangeability of the weak time derivative with the weak derivative with respect to the transverse variable (which holds as a consequence of Theorem 8.28 of [29]) gives:

$$-\frac{d}{dt}(\partial_3 e_{\alpha\| \beta}^1) = \tilde{\rho}_{\alpha\beta}(\dot{\bar{\mathbf{u}}}) = -\partial_3 \dot{e}_{\alpha\| \beta}^1 \text{ in } L^\infty(0, T; L^2(\omega)). \quad (17)$$

Given any vector field $\mathbf{v} = (v_i) \in \mathbf{V}(\Omega)$, the variational equations (7) of Problem $\mathcal{P}(\varepsilon; \Omega)$ read as follows (recall that $A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0$):

$$\begin{aligned}
& \rho \varepsilon^2 \int_0^T (H^1(\Omega))^* \langle \ddot{u}_i(\varepsilon)(t), g^{ij}(\varepsilon) v_j \sqrt{g(\varepsilon)} \rangle_{H^1(\Omega)} \psi(t) \, dt \\
& + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{\alpha\beta\sigma\tau}(\varepsilon) \tilde{e}_{\sigma\|\tau}(\varepsilon)(t) \{ \varepsilon e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) \} \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{\alpha\beta 33}(\varepsilon) \tilde{e}_{3\|3}(\varepsilon)(t) \{ \varepsilon e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) \} \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} 4A^{\alpha 3\sigma 3}(\varepsilon) \tilde{e}_{\sigma\|3}(\varepsilon)(t) \{ \varepsilon e_{\alpha\|3}(\varepsilon; \mathbf{v}) \} \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{33\sigma\tau}(\varepsilon) \tilde{e}_{\sigma\|\tau}(\varepsilon)(t) \{ \varepsilon e_{3\|3}(\varepsilon; \mathbf{v}) \} \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{3333}(\varepsilon) \tilde{e}_{3\|3}(\varepsilon)(t) \{ \varepsilon e_{3\|3}(\varepsilon; \mathbf{v}) \} \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& - A\varepsilon(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(\varepsilon)(t) e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) g^{\alpha\beta}(\varepsilon) \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& - A\varepsilon(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(\varepsilon)(t) e_{3\|3}(\varepsilon; \mathbf{v}) \sqrt{g(\varepsilon)} \, dx \psi(t) \, dt \\
& = \varepsilon^2 \int_0^T \int_{\Omega} f^i(t) v_i \sqrt{g(\varepsilon)} \, dx \, dt + \varepsilon^2 \int_0^T \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^i(t) v_i \sqrt{g(\varepsilon)} \, d\Gamma \psi(t) \, dt,
\end{aligned} \tag{18}$$

for all $\psi \in \mathcal{D}(0, T)$.

Observe, in the same spirit as part (iii) of Theorem 6.2-1 of [10], that the following asymptotic behaviours as $\varepsilon \rightarrow 0$ hold for each $\mathbf{v} \in \mathbf{V}(\Omega)$:

$$\begin{aligned}
& \varepsilon e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) \rightarrow 0 \text{ in } L^2(\Omega), \\
& \varepsilon e_{\alpha\|3}(\varepsilon; \mathbf{v}) \rightarrow \frac{1}{2} \partial_3 v_\alpha \text{ in } L^2(\Omega), \\
& \varepsilon e_{3\|3}(\varepsilon; \mathbf{v}) = \partial_3 v_3 \text{ in } L^2(\Omega) \text{ for all } \varepsilon > 0.
\end{aligned} \tag{19}$$

The relations (19), combined with the convergence process (15), and the asymptotic behaviour of the functions $A^{ijk\ell}(\varepsilon)$ and $\sqrt{g(\varepsilon)}$ as $\varepsilon \rightarrow 0$ (Lemma 1), give:

$$\begin{aligned}
& \rho\varepsilon^2 \int_0^T (H^1(\Omega))^* \langle \ddot{u}_i(\varepsilon)(t), g^{ij}(\varepsilon) v_j \sqrt{g(\varepsilon)} \rangle_{H^1(\Omega)} \psi(t) dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0; \\
& \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{\alpha\beta\sigma\tau}(\varepsilon) \tilde{e}_{\sigma\|\tau}(\varepsilon)(t) \{\varepsilon e_{\alpha\|\beta}(\varepsilon; \mathbf{v})\} \sqrt{g(\varepsilon)} dx \psi(t) dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0; \\
& \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{\alpha\beta 33}(\varepsilon) \tilde{e}_{3\|3}(\varepsilon)(t) \{\varepsilon e_{\alpha\|\beta}(\varepsilon; \mathbf{v})\} \sqrt{g(\varepsilon)} dx \psi(t) dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0; \\
& \frac{1}{\varepsilon} \int_0^T \int_{\Omega} 4A^{\alpha 3\sigma 3}(\varepsilon) \tilde{e}_{\sigma\|3}(\varepsilon)(t) \{\varepsilon e_{\alpha\|3}(\varepsilon; \mathbf{v})\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\
& \quad \rightarrow \int_0^T \int_{\Omega} 2\mu a^{\alpha\sigma} e_{\sigma\|3}^1(t) \partial_3 v_{\alpha} \sqrt{a} dx \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\
& \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{33\sigma\tau}(\varepsilon) \tilde{e}_{\sigma\|\tau}(\varepsilon)(t) \{\varepsilon e_{3\|3}(\varepsilon; \mathbf{v})\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\
& \quad \rightarrow \int_0^T \int_{\Omega} \lambda a^{\sigma\tau} e_{\sigma\|\tau}^1(t) \partial_3 v_3 \sqrt{a} dx \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\
& \frac{1}{\varepsilon} \int_0^T \int_{\Omega} A^{3333}(\varepsilon) \tilde{e}_{3\|3}(\varepsilon)(t) \{\varepsilon e_{3\|3}(\varepsilon; \mathbf{v})\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\
& \quad \rightarrow \int_0^T \int_{\Omega} (\lambda + 2\mu) e_{3\|3}^1(t) \partial_3 v_3 \sqrt{a} dx \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\
& A\varepsilon(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(\varepsilon)(t) e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) g^{\alpha\beta}(\varepsilon) \sqrt{g(\varepsilon)} dx \psi(t) dt \\
& \quad + A\varepsilon(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(\varepsilon)(t) e_{3\|3}(\varepsilon; \mathbf{v}) \sqrt{g(\varepsilon)} dx \psi(t) dt \\
& \quad \rightarrow A(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(t) \partial_3 v_3 \sqrt{a} dx \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\
& \varepsilon^2 \int_0^T \int_{\Omega} f^i(t) v_i \sqrt{g(\varepsilon)} dx \psi(t) dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0; \\
& \varepsilon^2 \int_0^T \int_{\Gamma_+ \cup \Gamma_-} h^i(t) v_i \sqrt{g(\varepsilon)} d\Gamma \psi(t) dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned} \tag{20}$$

Consequently, we are able to draw the following conclusion from (18) and (20):

$$\begin{aligned}
& \int_{\Omega} \left((2\mu a^{\alpha\sigma} e_{\sigma\|3}^1(t)) \partial_3 v_{\alpha} + (\lambda a^{\sigma\tau} e_{\sigma\|\tau}^1(t) + (\lambda + 2\mu) e_{3\|3}^1(t)) \partial_3 v_3 \right) \sqrt{a} dx \\
& = A(3\lambda + 2\mu) \int_{\Omega} \Theta(t) \partial_3 v_3 \sqrt{a} dx, \quad \text{for a.a. } t \in (0, T).
\end{aligned}$$

Since this inequality holds for any vector field $\mathbf{v} = (v_i) \in \mathbf{V}(\Omega)$ and for a.a. $t \in (0, T)$, by Theorem 3.4-1 of [10] it follows that:

$$\begin{aligned}
& e_{\sigma\|3}^1 = 0 \text{ in } L^{\infty}(0, T; L^2(\Omega)), \\
& \lambda a^{\sigma\tau} e_{\sigma\|\tau}^1 + (\lambda + 2\mu) e_{3\|3}^1 = A(3\lambda + 2\mu) \Theta \text{ in } L^{\infty}(0, T; L^2(\Omega)).
\end{aligned}$$

In particular, the latter gives:

$$e_{3\|3}^1 = A \frac{3\lambda + 2\mu}{\lambda + 2\mu} \Theta - \frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e_{\alpha\|\beta}^1 \text{ in } L^{\infty}(0, T; L^2(\Omega)),$$

and the conclusion is thus reached.

(iv) The vector field $\bar{\mathbf{u}} = (\bar{u}_i)$ satisfies the variational equations (9) of Problem $\mathcal{P}_F(\omega)$. Fix $\boldsymbol{\eta} \in \mathbf{V}_F(\omega)$ and, following the same steps as in part (iv) of Theorem 6.2-1 of [10], define the vector field $\mathbf{w}(\varepsilon; \boldsymbol{\eta})$ by

$$w_{\alpha}(\varepsilon; \boldsymbol{\eta}) := \eta_{\alpha} - \varepsilon x_3 (\partial_{\alpha} \eta_3 + b_{\alpha}^{\sigma} \eta_{\sigma}), \quad \text{and} \quad w_3(\varepsilon; \boldsymbol{\eta}) := \eta_3.$$

Observe that $\mathbf{w}(\varepsilon; \boldsymbol{\eta}) \in \mathbf{V}(\Omega)$ and that, as $\varepsilon \rightarrow 0$,

$$\begin{aligned}
& \mathbf{w}(\varepsilon; \boldsymbol{\eta}) \rightarrow \boldsymbol{\eta} \text{ in } \mathbf{V}(\Omega), \\
& \frac{1}{\varepsilon} e_{\alpha\|\beta}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \rightarrow \{-x_3 \rho_{\alpha\beta}(\boldsymbol{\eta})\} \text{ in } L^2(\Omega),
\end{aligned}$$

$$\left(\frac{1}{\varepsilon} e_{\alpha\|3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right)_{\varepsilon > 0} \text{ converges in } L^2(\Omega),$$

$$e_{3\|3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) = 0 \text{ for all } \varepsilon > 0.$$

Letting $\mathbf{v} = \mathbf{w}(\varepsilon; \boldsymbol{\eta})$ in the variational equations (7) of Problem $\mathcal{P}(\varepsilon; \Omega)$ gives:

$$\begin{aligned} & \rho \int_0^T \int_{(H^1(\Omega))^*} \langle \ddot{u}_i(\varepsilon)(t), g^{ij}(\varepsilon) w_j(\varepsilon; \boldsymbol{\eta}) \sqrt{g(\varepsilon)} \rangle_{H^1(\Omega)} \psi(t) dt \\ & + \int_0^T \int_{\Omega} A^{\alpha\beta\sigma\tau}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{\sigma\| \tau}(\varepsilon)(t) \right\} \left\{ \frac{1}{\varepsilon} e_{\alpha\| \beta}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & + \int_0^T \int_{\Omega} A^{\alpha\beta 33}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{3\| 3}(\varepsilon)(t) \right\} \left\{ \frac{1}{\varepsilon} e_{\alpha\| \beta}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & + 4 \int_0^T \int_{\Omega} A^{\alpha 3 \sigma 3}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{\sigma\| 3}(\varepsilon)(t) \right\} \left\{ \frac{1}{\varepsilon} e_{\alpha\| 3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & + \int_0^T \int_{\Omega} A^{33\sigma\tau}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{\sigma\| \tau}(\varepsilon)(t) \right\} \left\{ \frac{1}{\varepsilon} e_{3\| 3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & + \int_0^T \int_{\Omega} A^{3333}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{3\| 3}(\varepsilon)(t) \right\} \left\{ \frac{1}{\varepsilon} e_{3\| 3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & - A(3\lambda + 2\mu) \int_0^T \int_{\Omega} \Theta(\varepsilon)(t) \left\{ \frac{1}{\varepsilon} e_{\alpha\| \beta}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta})) g^{\alpha\beta}(\varepsilon) + \frac{1}{\varepsilon} \underbrace{e_{3\| 3}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta}))}_{=0} \right\} \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & = \int_0^T \int_{\Omega} f^i(t) w_i(\varepsilon; \boldsymbol{\eta}) \sqrt{g(\varepsilon)} dx \psi(t) dt + \int_0^T \int_{\Gamma_+ \cup \Gamma_-} h^i(t) w_i(\varepsilon; \boldsymbol{\eta}) \sqrt{g(\varepsilon)} d\Gamma \psi(t) dt, \quad \text{for all } \psi \in \mathcal{D}(0, T). \end{aligned}$$

The asymptotic behaviour of the functions $\mathbf{w}(\varepsilon; \boldsymbol{\eta})$ and $\frac{1}{\varepsilon} e_{i\|j}(\varepsilon; \mathbf{w}(\varepsilon; \boldsymbol{\eta}))$, the asymptotic behaviour of the three-dimensional elasticity tensor $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$ (Lemma 1), the weak-star convergences $\frac{1}{\varepsilon} \tilde{e}_{i\|j}(\varepsilon) \xrightarrow{*} e_{i\|j}^1$ in $L^\infty(0, T; L^2(\Omega))$ established in part (i), and the relations satisfied by $e_{i\|3}^1$ (part (iii)) together give:

$$2\rho \frac{d^2}{dt^2} \int_{\omega} \{ \bar{u}_\alpha(t) a^{\alpha\beta} \eta_\beta + \bar{u}_3(t) \eta_3 \} \sqrt{a} dy - \frac{1}{2} \int_{\Omega} a^{\alpha\beta\sigma\tau} e_{\sigma\| \tau}^1(t) \{ x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) \} \sqrt{a} dx = \int_{\omega} p^i(t) \eta_i \sqrt{a} dx, \quad (21)$$

for all $\boldsymbol{\eta} \in \mathbf{V}_F(\omega)$ and a.a. $t \in (0, T)$.

We have yet to take into account the relations $\tilde{\rho}_{\alpha\beta}(\mathbf{u}) = -\partial_3 e_{\alpha\| \beta}^1$ in $L^\infty(0, T; L^2(\Omega))$ established in part (iii). Since \mathbf{u} is independent of x_3 (part (ii)), these relations show that the functions $e_{\alpha\| \beta}^1$ are of the form

$$e_{\alpha\| \beta}^1 = \tilde{\Upsilon}_{\alpha\beta} - x_3 \tilde{\rho}_{\alpha\beta}(\bar{\mathbf{u}}), \quad \text{for a unique } \tilde{\Upsilon}_{\alpha\beta} \in L^\infty(0, T; L^2(\omega)).$$

Likewise, we infer from (17) that there exists a unique function $\tilde{v}_\alpha \in L^\infty(0, T; L^2(\omega))$ such that:

$$\dot{e}_{\alpha\| \beta}^1 = \tilde{v}_{\alpha\beta} - x_3 \tilde{\rho}_{\alpha\beta}(\dot{\bar{\mathbf{u}}}).$$

By the uniqueness of the time weak derivative, we infer that

$$\tilde{v}_{\alpha\beta} = \dot{\tilde{\Upsilon}}_{\alpha\beta}, \quad \text{in } L^\infty(0, T; L^2(\omega)), \quad (22)$$

which implies that $\tilde{\Upsilon}_{\alpha\beta} \in W^{1,\infty}(0, T; L^2(\omega))$.

Since, for a.a. $t \in (0, T)$, we have

$$\int_{\Omega} a^{\alpha\beta\sigma\tau} e_{\sigma\| \tau}^1(t) \{ -x_3 \rho_{\alpha\beta}(\bar{\mathbf{u}}(t)) \} \sqrt{a} dx = \frac{2}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\bar{\mathbf{u}}(t)) \rho_{\alpha\beta}(\bar{\mathbf{u}}(t)) dy - \underbrace{\int_{-1}^1 \int_{\omega} a^{\alpha\beta\sigma\tau} x_3 \rho_{\sigma\tau}(\bar{\mathbf{u}}(t)) \tilde{\Upsilon}_{\alpha\beta}(t) \sqrt{a} dy dx_3}_{=0},$$

we thus get

$$2\rho \frac{d^2}{dt^2} \int_{\omega} \{ \bar{u}_\alpha(t) a^{\alpha\beta} \eta_\beta + \bar{u}_3(t) \eta_3 \} \sqrt{a} dy - \frac{1}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\bar{\mathbf{u}}(t)) \rho_{\alpha\beta}(\bar{\mathbf{u}}(t)) dy = \int_{\omega} p^i(t) \eta_i \sqrt{a} dx, \quad (23)$$

for all $\boldsymbol{\eta} \in \mathbf{V}_F(\omega)$ in the sense of distributions in $(0, T)$.

Using the representation for linearised strains in Theorem 4.3-2 of [11], we infer that

$$\frac{1}{\varepsilon} \tilde{\Upsilon}_{\alpha\beta}(\mathbf{u}(\varepsilon)) \rightharpoonup \tilde{\Upsilon}_{\alpha\beta}, \quad \text{in } H^1(0, T; L^2(\omega)), \quad (24)$$

where here, without loss of generality, we denote by $\tilde{\gamma}_{\alpha\beta}$ the standard extension to the three-dimensional case of the time-dependent “analogue” of the change of metric tensor (cf., e.g., [11]).

Moreover, the fact that $\tilde{\Upsilon}_{\alpha\beta} \in W^{1,\infty}(0, T; L^2(\omega))$ (cf. (22)), (24), and the fact that $\mathbf{u}(\varepsilon)(0) = \zeta_0 \in \mathbf{V}_F(\omega)$ imply that

$$\tilde{\Upsilon}_{\alpha\beta}(0) = 0 \quad \text{in } L^2(\omega).$$

(v) *The function $\bar{\Theta}$ satisfies the variational equations (10) of Problem $\mathcal{P}_F(\omega)$. Since this problem has a unique solution as a consequence of Theorem 6.1 of [31], we infer that the convergences established in (15) actually holds for the whole families. Testing the variational equations (8) of Problem $\mathcal{P}(\varepsilon; \Omega)$ at any $\xi \in W(\Omega)$ independent of the transverse variable x_3 gives*

$$\begin{aligned} & K \int_0^T \int_{W^*(\Omega)} \langle \dot{\Theta}(\varepsilon)(t), \xi \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \psi(t) dt \\ & + S \int_0^T \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \xi \sqrt{g(\varepsilon)} dx \psi(t) dt + \frac{S}{\varepsilon} \int_0^T \int_{\Omega} \left(\frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) \right) \partial_3 \xi \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & + A(3\lambda + 2\mu) \int_0^T \int_{\Omega} \left\{ \frac{1}{\varepsilon} e_{\alpha\|\beta}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) g^{\alpha\beta}(\varepsilon) + \frac{1}{\varepsilon} e_{3\|3}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) \right\} \xi \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & = \int_0^T \int_{\Omega} Q(t) \xi \sqrt{g(\varepsilon)} dx \psi(t) dt, \end{aligned}$$

for all $\psi \in \mathcal{D}(0, T)$.

Since $\dot{\mathbf{u}}(t) \in \mathbf{V}_F(\omega)$ for a.a. $t \in (0, T)$, the relations established in part (iii) imply that the following time derivatives are well-defined:

$$\begin{aligned} \dot{e}_{i\|j}^1(t) &:= \frac{d}{dt} e_{i\|j}^1(t) \in L^2(\Omega) \text{ for a.a. } t \in (0, T), \\ \dot{\tilde{e}}_{i\|j}(\varepsilon)(t) &:= \frac{d}{dt} \tilde{e}_{i\|j}(\varepsilon)(t) = e_{i\|j}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) \in L^2(\Omega) \text{ for a.a. } t \in (0, T). \end{aligned}$$

The convergence process (15) and the definition of weak derivative in time give:

$$\begin{aligned} & K \int_0^T \int_{W^*(\Omega)} \langle \dot{\Theta}(\varepsilon)(t), \xi \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \psi(t) dt \rightarrow 2K \int_0^T \int_{W^*(\omega)} \langle \dot{\bar{\Theta}}(t), \xi \sqrt{a} \rangle_{W(\omega)} \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\ & S \int_0^T \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \xi \sqrt{g(\varepsilon)} dx \psi(t) dt \rightarrow 2S \int_0^T \int_{\omega} \partial_{\alpha} \bar{\Theta}(t) a^{\alpha\beta} \partial_{\beta} \xi \sqrt{a} dy \psi(t) dt \text{ as } \varepsilon \rightarrow 0; \\ & \frac{S}{\varepsilon} \int_0^T \int_{\Omega} \left(\frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) \right) \partial_3 \xi \sqrt{g(\varepsilon)} dx \psi(t) dt = 0 \text{ for all } \varepsilon > 0 \text{ being } \xi \text{ independent of } x_3; \\ & A(3\lambda + 2\mu) \int_0^T \int_{\Omega} \left\{ \frac{1}{\varepsilon} e_{\alpha\|\beta}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) g^{\alpha\beta}(\varepsilon) + \frac{1}{\varepsilon} e_{3\|3}(\varepsilon; \dot{\mathbf{u}}(\varepsilon)(t)) \right\} \xi \sqrt{g(\varepsilon)} dx \psi(t) dt \\ & \rightarrow \left(2A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_0^T \int_{W^*(\omega)} \langle \dot{\bar{\Theta}}(t), \xi \sqrt{a} \rangle_{W(\omega)} \psi(t) dt \\ & + 4A \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \int_{\mathcal{D}'(0, T; W^*(\omega))} \langle \dot{e}_{\alpha\|\beta}^1, a^{\alpha\beta} \xi \psi \sqrt{a} \rangle_{\mathcal{D}(0, T; W(\omega))} \\ & = \left(2A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_0^T \int_{W^*(\omega)} \langle \dot{\bar{\Theta}}(t), \xi \sqrt{a} \rangle_{W(\omega)} \psi(t) dt \\ & + 4A \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \int_{\mathcal{D}'(0, T; W^*(\omega))} \langle \dot{\tilde{\Upsilon}}_{\alpha\beta} a^{\alpha\beta}, \xi \psi \sqrt{a} \rangle_{\mathcal{D}(0, T; W(\omega))} \text{ as } \varepsilon \rightarrow 0; \\ & \int_0^T \int_{\Omega} Q(t) \xi \sqrt{g(\varepsilon)} dx \psi(t) dt \rightarrow \int_{\omega} q(t) \xi \sqrt{a} dy \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

By the Duhamel-Neumann law, it results that $\tilde{\Upsilon}_{\alpha\beta} = c_1 \gamma_{\alpha\beta}(\bar{\mathbf{u}}) + c_2 \rho_{\alpha\beta}(\bar{\mathbf{u}})$, for some $c_1, c_2 \in \mathbb{R}$. Moreover, up to changing the observation time instant t_0 to a point where $\bar{\mathbf{u}}(t_0) \in \mathbf{V}_F(\omega) \setminus \{\mathbf{0}\}$, we have that

$$0 = \tilde{\Upsilon}_{\alpha\beta}(0) = c_2 \rho_{\alpha\beta}(\zeta_0).$$

Thanks to Theorem 2.6-3 of [10], it must be $c_2 = 0$. In conclusion, it must occur that for a.a. $t \in (0, T)$ the term $\tilde{\Upsilon}_{\alpha\beta}(t) a^{\alpha\beta}$ is directly proportional to the change of metric tensor at $\bar{\mathbf{u}}$ and so it vanishes almost everywhere in ω . In conclusion, we have that $\tilde{\Upsilon}_{\alpha\beta} = 0$ in $\mathcal{C}^0([0, T]; L^2(\omega))$.

In conclusion, we obtain that

$$2 \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \bar{\Theta}(t) \xi \sqrt{a} dy + 2S \int_{\omega} \partial_{\alpha} \bar{\Theta}(t) a^{\alpha\beta} \partial_{\beta} \xi \sqrt{a} dy = \int_{\omega} q(t) \xi \sqrt{a} dy,$$

for all $\xi \in W(\omega)$ in the sense of distributions in $(0, T)$.

By virtue of (4), we have that $\bar{\mathbf{u}}$ and $\bar{\Theta}$ satisfy the initial conditions (11), and this puts us in a position to infer that the pair $(\bar{\mathbf{u}}, \bar{\Theta})$ is the unique solution of Problem $\mathcal{P}_F(\omega)$ (see Chapter 3 of [30] and Theorem 6.1 of [31]).

(vi) *The weak-star convergences*

$$\begin{aligned} \mathbf{u}(\varepsilon) &\overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0, T; \mathbf{V}(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\overset{*}{\rightharpoonup} \dot{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \frac{1}{\varepsilon} \tilde{e}_{i||j}(\varepsilon) &\overset{*}{\rightharpoonup} e_{i||j}^1 \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \Theta(\varepsilon) &\overset{*}{\rightharpoonup} \Theta \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \partial_\alpha \Theta(\varepsilon) &\overset{*}{\rightharpoonup} \partial_\alpha \Theta \text{ in } L^\infty(0, T; L^2(\Omega)), \end{aligned}$$

established in part (i) are actually stronger, in the sense that,

$$\begin{aligned} \mathbf{u}(\varepsilon) &\rightarrow \mathbf{u} \text{ in } L^\infty(0, T; \mathbf{V}(\Omega)), \\ \dot{\mathbf{u}}(\varepsilon) &\rightarrow \dot{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \frac{1}{\varepsilon} \tilde{e}_{i||j}(\varepsilon) &\rightarrow e_{i||j}^1 \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \Theta(\varepsilon) &\rightarrow \Theta \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \partial_\alpha \Theta(\varepsilon) &\rightarrow \partial_\alpha \Theta \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \partial_3 \Theta(\varepsilon) &\rightarrow \partial_3^1 \Theta \text{ in } L^\infty(0, T; L^2(\Omega)). \end{aligned}$$

Besides, since these limits are uniquely determined, these convergences hold for the whole families.

For each $\varepsilon \leq \varepsilon_1$ and for a.a. $t \in (0, T)$, let us consider the number

$$\begin{aligned} A(\varepsilon)(t) &:= \rho \mathbf{v}^*(\Omega) \langle (\ddot{u}_i(\varepsilon)(t) - \ddot{u}_i(t)), g^{ij}(\varepsilon)(\dot{u}_j(\varepsilon)(t) - \dot{u}_j(t)) \sqrt{g(\varepsilon)} \rangle_{\mathbf{V}(\Omega)} \\ &\quad + \int_{\Omega} A^{ijk\ell}(\varepsilon) \left\{ \frac{\tilde{e}_{k||\ell}(\varepsilon)(t)}{\varepsilon} - e_{k||\ell}^1(t) \right\} \left\{ \frac{\dot{\tilde{e}}_{k||\ell}(\varepsilon)(t)}{\varepsilon} - \dot{e}_{k||\ell}^1(t) \right\} \sqrt{g(\varepsilon)} \, dx \\ &\quad + K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(\varepsilon)(t) - \dot{\Theta}(t), (\Theta(\varepsilon)(t) - \Theta(t)) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \\ &\quad + S \int_{\Omega} \partial_\alpha (\Theta(\varepsilon)(t) - \Theta(t)) g^{\alpha\beta}(\varepsilon) \partial_\beta (\Theta(\varepsilon)(t) - \Theta(t)) \sqrt{g(\varepsilon)} \, dx \\ &\quad + S \int_{\Omega} \left| \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) - \partial_3^1 \Theta(t) \right|^2 \sqrt{g(\varepsilon)} \, dx. \end{aligned}$$

On the one hand, by virtue of equations (7) and (8) of Problem $\mathcal{P}(\varepsilon; \Omega)$, we get that

$$\begin{aligned} A(\varepsilon)(t) &= \int_{\Omega} f^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} \, dx + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} \, d\Gamma + A(3\lambda + 2\mu) \int_{\Omega} \Theta(\varepsilon)(t) \dot{\tilde{e}}_{i||j}(\varepsilon)(t) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \, dx \\ &\quad - 2\rho \mathbf{v}^*(\Omega) \langle \ddot{u}_i(\varepsilon)(t), \dot{u}_j(t) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{\mathbf{V}(\Omega)} + \rho \mathbf{v}^*(\Omega) \langle \ddot{u}_i(t), \dot{u}_j(t) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{\mathbf{V}(\Omega)} \\ &\quad - 2 \int_{\Omega} A^{ijk\ell}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{k||\ell}(\varepsilon)(t) \right\} \dot{e}_{i||j}^1(t) \sqrt{g(\varepsilon)} \, dx + \int_{\Omega} A^{ijk\ell}(\varepsilon) e_{k||\ell}^1(t) \dot{e}_{i||j}^1(t) \sqrt{g(\varepsilon)} \, dx \\ &\quad + K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(\varepsilon)(t), \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} - K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(t), \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \\ &\quad - K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(\varepsilon)(t), \Theta(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} + K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(t), \Theta(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \\ &\quad + S \int_{\Omega} \partial_\alpha \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_\beta \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} \, dx - 2S \int_{\Omega} \partial_\alpha \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_\beta \Theta(t) \sqrt{g(\varepsilon)} \, dx \\ &\quad + S \int_{\Omega} \partial_\alpha \Theta(t) g^{\alpha\beta}(\varepsilon) \partial_\beta \Theta(t) \sqrt{g(\varepsilon)} \, dx + \frac{S}{\varepsilon^2} \int_{\Omega} |\partial_3 \Theta(\varepsilon)(t)|^2 \sqrt{g(\varepsilon)} \, dx \\ &\quad - 2S \int_{\Omega} \left\{ \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) \right\} \partial_3^1 \Theta(t) \sqrt{g(\varepsilon)} \, dx + S \int_{\Omega} |\partial_3^1 \Theta(t)|^2 \sqrt{g(\varepsilon)} \, dx \\ &= \int_{\Omega} f^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} \, dx + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^i(t) \dot{u}_i(\varepsilon)(t) \sqrt{g(\varepsilon)} \, d\Gamma \\ &\quad - 2\rho \mathbf{v}^*(\Omega) \langle \ddot{u}_i(\varepsilon)(t), \dot{u}_j(t) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{\mathbf{V}(\Omega)} + \rho \mathbf{v}^*(\Omega) \langle \ddot{u}_i(t), \dot{u}_j(t) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{\mathbf{V}(\Omega)} \\ &\quad - 2 \int_{\Omega} A^{ijk\ell}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{k||\ell}(\varepsilon)(t) \right\} \dot{e}_{i||j}^1(t) \sqrt{g(\varepsilon)} \, dx + \int_{\Omega} A^{ijk\ell}(\varepsilon) e_{k||\ell}^1(t) \dot{e}_{i||j}^1(t) \sqrt{g(\varepsilon)} \, dx \\ &\quad - K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(t), \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} - K \mathbf{w}^*(\Omega) \langle \dot{\Theta}(\varepsilon)(t), \Theta(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \end{aligned}$$

$$\begin{aligned}
& + K \, {}_{W^*(\Omega)} \langle \dot{\Theta}(t), \Theta(t) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} - 2S \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(t) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \Theta(t) \sqrt{g(\varepsilon)} \, dx \\
& + S \int_{\Omega} \partial_{\alpha} \Theta(t) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \Theta(t) \sqrt{g(\varepsilon)} \, dx - 2S \int_{\Omega} \left\{ \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) \right\} \partial_3^1 \Theta(t) \sqrt{g(\varepsilon)} \, dx \\
& + S \int_{\Omega} |\partial_3^1 \Theta(t)|^2 \sqrt{g(\varepsilon)} \, dx + \int_{\Omega} Q(t) \Theta(\varepsilon)(t) \sqrt{g(\varepsilon)} \, dx.
\end{aligned}$$

An integration over the interval $(0, t)$ gives:

$$\begin{aligned}
\int_0^t \Lambda(\varepsilon)(\tau) \, d\tau &= \int_0^t \int_{\Omega} f^i(\tau) \dot{u}_i(\varepsilon)(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau + \int_0^t \int_{\Gamma_{\pm}^{\varepsilon} \cup \Gamma_{\pm}^{\varepsilon}} h^i(\tau) \dot{u}_i(\varepsilon)(\tau) \sqrt{g(\varepsilon)} \, d\Gamma \, d\tau \\
&- 2\rho \int_0^t (H^1(\Omega))^* \langle \ddot{u}_i(\varepsilon)(\tau), \dot{u}_j(\tau) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{H^1(\Omega)} \, d\tau + \rho \int_0^t (H^1(\Omega))^* \langle \ddot{u}_i(\tau), \dot{u}_j(\tau) g^{ij}(\varepsilon) \sqrt{g(\varepsilon)} \rangle_{H^1(\Omega)} \, d\tau \\
&- 2 \int_0^t \int_{\Omega} A^{ijkl}(\varepsilon) \left\{ \frac{1}{\varepsilon} \tilde{e}_{k\parallel\ell}(\varepsilon)(\tau) \right\} \dot{e}_{i\parallel j}^1(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau + \int_0^t \int_{\Omega} A^{ijkl}(\varepsilon) e_{k\parallel\ell}^1(\tau) \dot{e}_{i\parallel j}^1(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau \\
&- K \int_0^t {}_{W^*(\Omega)} \langle \dot{\Theta}(\tau), \Theta(\varepsilon)(\tau) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \, d\tau - K \int_0^t {}_{W^*(\Omega)} \langle \dot{\Theta}(\varepsilon)(\tau), \Theta(\tau) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \, d\tau \\
&+ K \int_0^t {}_{W^*(\Omega)} \langle \dot{\Theta}(\tau), \Theta(\tau) \sqrt{g(\varepsilon)} \rangle_{W(\Omega)} \, d\tau + \int_0^t \int_{\Omega} Q(\tau) \Theta(\varepsilon)(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau \\
&- 2S \int_0^t \int_{\Omega} \partial_{\alpha} \Theta(\varepsilon)(\tau) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \Theta(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau + S \int_0^t \int_{\Omega} \partial_{\alpha} \Theta(\tau) g^{\alpha\beta}(\varepsilon) \partial_{\beta} \Theta(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau \\
&- 2S \int_0^t \int_{\Omega} \left\{ \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(\tau) \right\} \partial_3^1 \Theta(\tau) \sqrt{g(\varepsilon)} \, dx \, d\tau + S \int_0^t \int_{\Omega} |\partial_3^1 \Theta(\tau)|^2 \sqrt{g(\varepsilon)} \, dx \, d\tau.
\end{aligned}$$

Let $\varepsilon \rightarrow 0$, and observe that the Lebesgue dominated convergence theorem gives:

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_0^t \Lambda(\varepsilon)(\tau) \, d\tau &= \int_0^t \int_{\omega} p^i(\tau) \dot{u}_i(\tau) \sqrt{a} \, dy \, d\tau - \frac{1}{3} \int_0^t \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\bar{\mathbf{u}}(\tau)) \rho_{\alpha\beta}(\dot{\bar{\mathbf{u}}}(\tau)) \sqrt{a} \, dy \, d\tau \\
&- 2\rho \int_0^t (H^1(\omega))^* \langle \ddot{u}_{\alpha}(\tau), \dot{u}_{\beta}(\tau) a^{\alpha\beta} \sqrt{a} \rangle_{H^1(\omega)} \, d\tau - 2\rho \int_0^t (H^2(\omega))^* \langle \ddot{u}_3(\tau), \dot{u}_3(\tau) \sqrt{a} \rangle_{H^2(\omega)} \, d\tau \\
&- 2 \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_0^t {}_{W^*(\omega)} \langle \dot{\bar{\Theta}}(\tau), \bar{\Theta}(\tau) \sqrt{a} \rangle_{W(\omega)} \, d\tau - 2S \int_0^t \int_{\omega} \partial_{\alpha} \bar{\Theta}(\tau) a^{\alpha\beta} \partial_{\beta} \bar{\Theta}(\tau) \sqrt{a} \, dy \, d\tau \\
&- S \int_0^t \int_{\omega} |\partial_3^1 \bar{\Theta}(\tau)|^2 \sqrt{a} \, dx \, d\tau + \int_0^t \int_{\omega} q(\tau) \bar{\Theta}(\tau) \sqrt{a} \, dy \, d\tau = -S \int_0^t \int_{\omega} |\partial_3^1 \bar{\Theta}(\tau)|^2 \sqrt{a} \, dx \, d\tau.
\end{aligned}$$

In conclusion, we have shown that:

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \Lambda(\varepsilon)(\tau) \, d\tau \leq 0. \quad (25)$$

On the other hand, the uniform positive-definiteness of the fourth order three-dimensional elasticity tensor $(A^{ijkl}(\varepsilon))$, Korn's inequality (Theorem 2), and the initial conditions, give that there exists a constant $C > 0$ such that:

$$\begin{aligned}
\int_0^t \Lambda(\varepsilon)(\tau) \, d\tau &\geq \|\dot{\mathbf{u}}(\varepsilon)(t) - \dot{\mathbf{u}}(t)\|_{L^2(\Omega)}^2 + \sum_{i,j} \left\| \frac{1}{\varepsilon} \tilde{e}_{i\parallel j}(\varepsilon)(t) - e_{i\parallel j}^1(t) \right\|_{L^2(\Omega)}^2 + \|\Theta(\varepsilon)(t) - \Theta(t)\|_{L^2(\Omega)}^2 \\
&+ \sum_{\alpha} \|\partial_{\alpha} \Theta(\varepsilon)(t) - \partial_{\alpha} \Theta(t)\|_{L^2(\Omega)}^2 + \left\| \frac{1}{\varepsilon} \partial_3 \Theta(\varepsilon)(t) - \partial_3^1 \Theta(t) \right\|_{L^2(\Omega)}^2, \quad (26)
\end{aligned}$$

for a.a. $t \in (0, T)$. Combining (25), and (26) thus gives that

$$\partial_3^1 \Theta = 0 \text{ in } L^{\infty}(0, T; L^2(\Omega)).$$

and that the claimed strong convergences hold. The proof is complete.

In the asymptotic analysis conducted in Theorem 7 we noticed that, as the thickness ε to zero, the mutual interaction between the temperature and the displacement tends to zero. Indeed, the variational equations appearing in Problem $\mathcal{P}_F(\omega)$ are *uncoupled*. This does not mean that there is no interaction between the displacement and the temperature variation. The fact that the mechanical variational equations and the thermal variational equations in Problem $\mathcal{P}_F(\omega)$ are uncoupled is *solely* owing to the fact that the heat induces a deformation expressed in terms of the change of metric tensor $\gamma_{\alpha\beta}$ only. The action of the change of metric tensor $\gamma_{\alpha\beta}$ at the elements of the space $\mathbf{V}_F(\omega)$ is null by definition.

It remains to “*de-scale*” the results of Theorem 2, which apply to the solutions $\mathbf{u}(\varepsilon)$ of the *scaled* problem $\mathcal{P}(\varepsilon; \Omega)$. This means that we need to convert these results into ones about the *unknown* $u_i^\varepsilon \mathbf{g}^{i,\varepsilon} : [0, T] \times \overline{\Omega^\varepsilon} \rightarrow \mathbb{E}^3$, which represents the physical *three-dimensional vector field* of the actual reference configuration of the shell. As shown in the next theorem, this conversion is most conveniently achieved through the introduction of the *averages* $\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \mathbf{g}^{i,\varepsilon} dx_3^\varepsilon$ across the thickness of the shell, a procedure which also clearly highlights the striking *difference* (in terms of function spaces) between the asymptotic behaviors of the *tangential* and *normal* components of the displacement field of the middle surface of the shell.

Theorem 8 *Let the assumptions on the data be as in Section 4 and let the assumptions on the immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$ be as in Theorem 7. Let $\mathbf{u}^\varepsilon = (u_i^\varepsilon) \in L^\infty(0, T; \mathbf{V}(\Omega^\varepsilon))$ and $\Theta^\varepsilon \in L^\infty(0, T; L^2(\Omega^\varepsilon))$ denote for each $\varepsilon > 0$ the unique solution of Problem $\mathcal{P}(\Omega^\varepsilon)$.*

Let $\boldsymbol{\zeta} \in L^\infty(0, T; \mathbf{V}_F(\omega))$ and $\vartheta \in L^\infty(0, T; L^2(\omega))$ denote the unique solution of Problem $\mathcal{P}_F(\omega)$. Then

$$\begin{aligned} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \mathbf{g}^{\alpha,\varepsilon} dx_3^\varepsilon &\rightarrow \zeta_\alpha \mathbf{a}^\alpha \text{ in } L^\infty(0, T; \mathbf{H}^1(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \mathbf{g}^{3,\varepsilon} dx_3^\varepsilon &\rightarrow \zeta_3 \mathbf{a}^3 \text{ in } L^\infty(0, T; \mathbf{H}^1(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \Theta^\varepsilon dx_3^\varepsilon &\rightarrow \vartheta \text{ in } L^\infty(0, T; L^2(\omega)) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Proof The proof strategy, apart from a few slight changes concerning the time-dependent part, is analogous to that of Theorem 4.6-1 in [10] and for this reason is omitted.

7 Justification of Koiter's model for thermoelastic flexural shells

A commonly used *two-dimensional* set of equations for modelling such a shell (“two-dimensional” in the sense that it is posed over $\bar{\omega}$ instead of $\bar{\Omega^\varepsilon}$) was proposed in 1970 by Koiter [26, 27]. We now recall, following Chapter 7 of [10], the modern formulation of this model in the static (i.e., *time-independent*) case. Koiter's model has also been recently studied in the context of obstacle problems by Ciarlet, Mardare and Piersanti [17, 18, 19, 20].

First, we recall the definition of the space

$$\mathbf{V}_K(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

and we recall that the norm $\|\cdot\|_{\mathbf{V}_K(\omega)}$ is defined by:

$$\|\boldsymbol{\eta}\|_{\mathbf{V}_K(\omega)} = \left\{ \sum_\alpha \|\eta_\alpha\|_{H^1(\omega)}^2 + \|\eta_3\|_{H^2(\omega)}^2 \right\}^{1/2} \quad \text{for each } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega).$$

Next, we recall the definition the *fourth-order two-dimensional elasticity tensor of the shell*, viewed here as a two-dimensional linearly elastic body, by means of its contravariant components (at this point, we have not made any assumptions on the data yet):

$$a^{\alpha\beta\sigma\tau,\varepsilon} = \frac{4\lambda^\varepsilon \mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^\varepsilon \left(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma} \right).$$

Finally, define the bilinear forms $B_M(\cdot, \cdot)$ and $B_F(\cdot, \cdot)$ by

$$B_M(\boldsymbol{\xi}, \boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\xi}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy, \quad \text{and} \quad B_F(\boldsymbol{\xi}, \boldsymbol{\eta}) := \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\xi}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy,$$

for each $\boldsymbol{\xi} = (\xi_i) \in \mathbf{V}_K(\omega)$ and each $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$.

Define the linear form ℓ^ε by

$$\ell^\varepsilon(\boldsymbol{\eta}) := \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a} dy, \quad \text{for each } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega),$$

where $p^{i,\varepsilon}(y) := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(y, x_3^\varepsilon) dx_3^\varepsilon + h^{i,\varepsilon}(y, \varepsilon) + h^{i,\varepsilon}(y, -\varepsilon)$ for a.a. $y \in \omega$.

Then the *total energy* of the shell is the *quadratic functional* $J : \mathbf{V}_K(\omega) \rightarrow \mathbb{R}$ defined by:

$$J(\boldsymbol{\eta}) := \frac{\varepsilon}{2} B_M(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{\varepsilon^3}{2} B_F(\boldsymbol{\eta}, \boldsymbol{\eta}) - \ell^\varepsilon(\boldsymbol{\eta}) \quad \text{for each } \boldsymbol{\eta} \in \mathbf{V}_K(\omega).$$

The term $\frac{\varepsilon}{2}B_M(\cdot, \cdot)$ and $\frac{\varepsilon^3}{2}B_F(\cdot, \cdot)$ respectively represent the *membrane part* and the *flexural part* of the total energy, as aptly recalled by the subscripts “ M ” and “ F ”.

In Koiter’s model, the unknown “two-dimensional” displacement field $\zeta_{i,K}^\varepsilon \mathbf{a}^i : \bar{\omega} \rightarrow \mathbb{E}^3$ of the middle surface $\boldsymbol{\theta}(\bar{\omega})$ of the shell is such that the vector field $\boldsymbol{\zeta}_K^\varepsilon := (\zeta_{i,K}^\varepsilon)$ should be the solution of the following problem: Find a vector field $\boldsymbol{\zeta}_K^\varepsilon : \bar{\omega} \rightarrow \mathbb{R}^3$ that satisfies

$$\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}_K(\omega) \quad \text{and} \quad J(\boldsymbol{\zeta}_K^\varepsilon) = \inf_{\boldsymbol{\eta} \in \mathbf{V}_K(\omega)} J(\boldsymbol{\eta}),$$

or equivalently, find $\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}_K(\omega)$ that satisfies the following variational equations:

$$\varepsilon B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^3 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) = \ell^\varepsilon(\boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}_K(\omega).$$

As first shown in [1] (see also [2]), this problem has one and only one solution. To begin with, define the space

$$\mathbf{X}_K(\omega) := \mathbf{V}_K(\omega) \times \mathbf{L}^2(\omega) \times W(\omega),$$

and the space

$$\mathbf{H}_K(\omega) := \mathbf{V}_K(\omega) \times \mathbf{L}^2(\omega) \times \mathbf{L}^2(\omega),$$

and we equip them with the norms

$$\|\mathbf{U}\|_{\mathbf{X}_K(\omega)} := \|\boldsymbol{\eta}\|_{\mathbf{V}_K(\omega)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(\omega)} + \|\tau\|_{W(\omega)}, \quad \text{for all } \mathbf{U} = (\boldsymbol{\eta}, \boldsymbol{\xi}, \tau) \in \mathbf{X}_K(\omega),$$

and

$$\|\mathbf{U}\|_{\mathbf{H}_K(\omega)} := \|\boldsymbol{\eta}\|_{\mathbf{V}_K(\omega)} + \|\boldsymbol{\xi}\|_{\mathbf{L}^2(\omega)} + \|\tau\|_{\mathbf{L}^2(\omega)}, \quad \text{for all } \mathbf{U} = (\boldsymbol{\eta}, \boldsymbol{\xi}, \tau) \in \mathbf{H}_K(\omega),$$

respectively. For a.a. $t \in (0, T)$, we also define

$$q^\varepsilon(t) := \int_{-\varepsilon}^{\varepsilon} Q^\varepsilon(t)(\cdot, x_3^\varepsilon) dx_3^\varepsilon \in L^2(\omega),$$

and, in the same fashion as the static case, we define for a.a. $t \in (0, T)$ the functions

$$p^{i,\varepsilon}(t) := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(t)(\cdot, x_3^\varepsilon) dx_3^\varepsilon + h^{i,\varepsilon}(t)(\cdot, \varepsilon) + h^{i,\varepsilon}(t)(\cdot, -\varepsilon) \in L^2(\omega).$$

Our aim is to now *formulate* and *justify* the analogue of Koiter’s model in the case where the shell under consideration is also subjected to the action of a heat source, apart from the usual applied body forces and applied surface forces.

To begin with, we *propose* a Koiter’s model for thermoelastic shells, and we will proceed to its justification in the case where the shell under consideration is a *flexural shell*.

Problem $\mathcal{P}_K^\varepsilon(\omega)$ Find a vector field $\boldsymbol{\zeta}_K^\varepsilon = (\zeta_{K,i}^\varepsilon) : (0, T) \rightarrow \mathbf{V}_K(\omega)$ and a function $\vartheta_K^\varepsilon : (0, T) \rightarrow W(\omega)$ such that

$$\begin{aligned} \boldsymbol{\zeta}_K^\varepsilon &\in L^\infty(0, T; \mathbf{V}_K(\omega)), \\ \dot{\boldsymbol{\zeta}}_K^\varepsilon &\in L^\infty(0, T; \mathbf{L}^2(\omega)), \\ \ddot{\boldsymbol{\zeta}}_K^\varepsilon &\in L^\infty(0, T; \mathbf{V}_K^*(\omega)), \\ \vartheta_K^\varepsilon &\in L^\infty(0, T; W(\omega)), \\ \dot{\vartheta}_K^\varepsilon &\in L^\infty(0, T; W^*(\omega)), \end{aligned}$$

that satisfies the following variational equations

$$\begin{aligned} 2\varepsilon \frac{d^2}{dt^2} \int_{\omega} \rho^\varepsilon (\zeta_{K,\alpha}^\varepsilon(t) a^{\alpha\beta} \eta_\beta + \zeta_{K,3}^\varepsilon \eta_3) \sqrt{a} dy + \frac{\varepsilon^3}{3} \int_{\omega} a^{\alpha\beta\sigma\tau, \varepsilon} \rho_{\sigma\tau}(\zeta_K^\varepsilon(t)) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy \\ - 4\varepsilon \int_{\omega} A^\varepsilon \mu^\varepsilon \frac{3\lambda^\varepsilon + 2\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} \vartheta_K^\varepsilon(t) \gamma_{\alpha\beta}(\boldsymbol{\eta}) a^{\alpha\beta} \sqrt{a} dy = \int_{\omega} p^{i,\varepsilon}(t) \eta_i \sqrt{a} dy, \end{aligned} \quad (27)$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$, in the sense of distributions in $(0, T)$,

$$\begin{aligned} 2\varepsilon \frac{d}{dt} \int_{\omega} \left(K^\varepsilon + (A^\varepsilon)^2 \frac{(3\lambda^\varepsilon + 2\mu^\varepsilon)^2}{\lambda^\varepsilon + 2\mu^\varepsilon} \right) \vartheta_K^\varepsilon(t) \xi \sqrt{a} dy + 2\varepsilon S^\varepsilon \int_{\omega} \partial_\alpha \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_\beta \xi \sqrt{a} dy \\ + 4\varepsilon \frac{d}{dt} \int_{\omega} A^\varepsilon \mu^\varepsilon \frac{3\lambda^\varepsilon + 2\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} \xi \gamma_{\alpha\beta}(\zeta_K^\varepsilon(t)) a^{\alpha\beta} \sqrt{a} dy = \int_{\omega} q^\varepsilon(t) \xi \sqrt{a} dy, \end{aligned} \quad (28)$$

for all $\xi \in W(\omega)$, in the sense of distributions in $(0, T)$,

and that satisfy the initial conditions

$$\begin{cases} \zeta_K^\varepsilon(0) = \zeta_0, \\ \dot{\zeta}_K^\varepsilon(0) = \dot{\zeta}_1, \\ \Theta_K^\varepsilon(0) = \vartheta_0, \end{cases} \quad (29)$$

where the initial data satisfies (4). ■

Once again, mimicking the proof of the results contained in [23], noticing that $\mathbf{X}_K(\omega)$ is dense in $\mathbf{H}_K(\omega)$, and applying Proposition 6.2 of [22] gives that Problem $\mathcal{P}_K^\varepsilon(\omega)$ admits a unique solution and, moreover, the velocity field $\dot{\zeta}_K^\varepsilon(t)$ is in the space $\mathbf{V}_K(\omega)$ for a.a. $t \in (0, T)$.

We are now ready to state Koiter's model for a general linearly elastic shell. The forthcoming analysis resorts to an argument similar to the one used in Theorem 7.2-3 of [10] (itself based on [37] and, especially, on [16]), and constitutes the *third new result* in this paper.

Theorem 9 *Let ω be a domain in \mathbb{R}^2 and let $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of flexural shells with thickness 2ε approaching zero and with each having the same middle surface $\boldsymbol{\theta}(\bar{\omega})$. Let γ_0 be a non-empty relatively open subset of γ , and assume that each shell is subject to a boundary condition of place along a portion of its lateral face, whose middle curve is the set $\boldsymbol{\theta}(\gamma_0)$.*

Finally, assume that there exist functions $f^i \in H^1(0, T; L^2(\Omega))$, $h^i \in H^1(0, T; L^2(\Gamma_+ \cup \Gamma_-))$ and $Q \in L^\infty(0, T; L^2(\Omega))$ independent of ε such that the assumptions (5) hold, i.e.,

$$\begin{aligned} f^{i,\varepsilon}(t)(x^\varepsilon) &= \varepsilon^2 f^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega, \\ h^{i,\varepsilon}(t)(x^\varepsilon) &= \varepsilon^3 h^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Gamma_+ \cup \Gamma_-, \\ Q^\varepsilon(t)(x^\varepsilon) &= \varepsilon^2 Q(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega, \end{aligned}$$

as well as the assumptions on the coefficients (6) hold, i.e., there exist positive constants ρ , A , K , S , λ and μ such that:

$$\begin{aligned} \rho^\varepsilon(x^\varepsilon) &= \varepsilon^2 \rho \text{ at a.a. } x \in \Omega, \\ A^\varepsilon(x^\varepsilon) &= \varepsilon A \text{ at a.a. } x \in \Omega, \\ K^\varepsilon(x^\varepsilon) &= \varepsilon^2 K \text{ at a.a. } x \in \Omega, \\ S^\varepsilon(x^\varepsilon) &= \varepsilon^2 S \text{ at a.a. } x \in \Omega, \\ \lambda^\varepsilon &= \lambda, \quad \text{and} \quad \mu^\varepsilon = \mu. \end{aligned}$$

For each $\varepsilon > 0$, let the pair $(\zeta_K^\varepsilon, \vartheta_K^\varepsilon)$ denote the solution of Problem $\mathcal{P}_K^\varepsilon(\omega)$. Then the following convergences hold:

$$\begin{aligned} \zeta_K^\varepsilon &\rightarrow \zeta \quad \text{in } L^\infty(0, T; \mathbf{V}_K(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \dot{\zeta}_K^\varepsilon &\rightarrow \dot{\zeta} \quad \text{in } L^\infty(0, T; \mathbf{L}^2(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \vartheta_K^\varepsilon &\rightarrow \vartheta \quad \text{in } L^\infty(0, T; L^2(\omega)) \text{ as } \varepsilon \rightarrow 0, \end{aligned} \quad (30)$$

where the pair (ζ, ϑ) denotes the unique solution of Problem $\mathcal{P}_F(\omega)$.

Proof The outline of the proof is to a large extent inspired by the proof of Theorem 7.2-3 in [10] (itself adapted from [14]), where only the static case is considered. This is why some parts of the proof are reminiscent of those in [10]; otherwise, considering the time-dependence and requires extra care. For sake of clarity, we break the proof into three parts, numbered (i)–(iii).

(i) *There exist subsequences, still denoted $(\zeta_K^\varepsilon)_{\varepsilon>0}$ and $(\vartheta_K^\varepsilon)_{\varepsilon>0}$, and there exist $\zeta \in L^\infty(0, T; \mathbf{V}_K(\omega))$ and $\vartheta \in L^\infty(0, T; W(\omega))$ such that the following convergence process holds:*

$$\begin{aligned} \zeta_K^\varepsilon &\overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ \dot{\zeta}_K^\varepsilon &\overset{*}{\rightharpoonup} \dot{\zeta} \text{ in } L^\infty(0, T; \mathbf{L}^2(\omega)), \\ \ddot{\zeta}_K^\varepsilon &\overset{*}{\rightharpoonup} \ddot{\zeta} \text{ in } L^\infty(0, T; \mathbf{V}_K^*(\omega)), \\ \vartheta_K^\varepsilon &\overset{*}{\rightharpoonup} \vartheta \text{ in } L^\infty(0, T; L^2(\omega)), \\ \partial_\alpha \vartheta_K^\varepsilon &\rightarrow \partial_\alpha \vartheta \text{ in } L^2(0, T; L^2(\omega)), \\ \dot{\vartheta}_K^\varepsilon &\overset{*}{\rightharpoonup} \dot{\vartheta} \text{ in } L^\infty(0, T; W^*(\omega)). \end{aligned} \quad (31)$$

By virtue of the results contained in [23] and Proposition 6.2 of [22], we can specialise $\boldsymbol{\eta} = \dot{\zeta}_K^\varepsilon(t)$ and $\xi = \vartheta_K^\varepsilon(t)$ in the variational equations (27) and (28) of Problem $\mathcal{P}_K^\varepsilon(\omega)$ at a certain time instant $t \in (0, T)$.

After such a specialisation, we sum equations (27) and (28) of Problem $\mathcal{P}_K^\varepsilon(\omega)$, getting:

$$\begin{aligned} & 2\rho\varepsilon^3 (H^1(\omega))^* \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t), a^{\alpha\beta} \dot{\zeta}_{K,\beta}^\varepsilon(t) \sqrt{a} \rangle_{(H^1(\omega))^*} + 2\rho\varepsilon^3 (H^2(\omega))^* \langle \ddot{\zeta}_{K,3}^\varepsilon(t), \dot{\zeta}_{K,3}^\varepsilon(t) \sqrt{a} \rangle_{H^2(\omega)} \\ & + \varepsilon \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^\varepsilon(t)) \gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) \sqrt{a} \, dy + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_K^\varepsilon(t)) \rho_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) \sqrt{a} \, dy \\ & + 2\varepsilon^3 \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) W^*(\omega) \langle \dot{\vartheta}_K^\varepsilon(t), \vartheta_K^\varepsilon(t) \sqrt{a} \rangle_{W(\omega)} \\ & + 2S\varepsilon^3 \int_\omega \partial_\alpha \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_\beta \vartheta_K^\varepsilon(t) \sqrt{a} \, dy = \int_\omega p^{i,\varepsilon}(t) \dot{\zeta}_{K,i}^\varepsilon(t) \sqrt{a} \, dy + \int_\omega q^\varepsilon(t) \vartheta_K^\varepsilon(t) \sqrt{a} \, dy. \end{aligned} \quad (32)$$

Carrying an integration in the interval $(0, t)$, dividing (32) by ε^3 , and resorting to the uniform positive-definiteness of the fourth order two-dimensional elasticity tensor, Korn's inequality on general surface (Theorem 4), Hölder's inequality and Young's inequality [39] gives:

$$\begin{aligned} & \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \frac{1}{c} \|\zeta_K^\varepsilon(t)\|_{\mathbf{V}_K(\omega)}^2 + \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_\alpha \int_0^T \|\partial_\alpha \vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \\ & \leq \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_{\alpha,\beta} \left\| \frac{1}{\varepsilon} \gamma_{\alpha\beta}(\zeta_K^\varepsilon(t)) \right\|_{L^2(\omega)}^2 + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\zeta_K^\varepsilon(t))\|_{L^2(\omega)}^2 + \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_\alpha \int_0^T \|\partial_\alpha \vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \\ & \leq C \left\{ \|\zeta_0\|_{\mathbf{V}_F(\omega)}^2 + \|\zeta_1\|_{L^2(\omega)}^2 + \|\vartheta_0\|_{W(\omega)}^2 + \int_0^T \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt + \int_0^T \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \right\}, \end{aligned}$$

for some $C = C(p^i, q) > 0$ and some $c = c(\omega, \gamma_0, \theta) > 0$. Hence, by Gronwall's inequality, we can conclude that

$$\begin{aligned} & (\zeta_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ & (\dot{\zeta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\ddot{\zeta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; \mathbf{V}_K^*(\omega)), \\ & \left(\frac{1}{\varepsilon} \gamma_{\alpha\beta}(\zeta_K^\varepsilon) \right)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\vartheta_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\partial_\alpha \vartheta_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2(0, T; L^2(\omega)), \\ & (\dot{\vartheta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; W^*(\omega)), \end{aligned} \quad (33)$$

and the conclusion is thus reached by the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [6]).

(ii) *The found weak-star pair (ζ, ϑ) is the unique solution of Problem $\mathcal{P}_F(\omega)$.* Let us test the variational equations (27) at $\eta \in \mathbf{V}_F(\omega)$, and at $\psi \in \mathcal{D}(0, T)$, and divide these equations by ε^3 . We obtain:

$$\begin{aligned} & 2\rho \int_0^T (H^1(\omega))^* \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t), a^{\alpha\beta} \eta_\beta \sqrt{a} \rangle_{H^1(\omega)} \psi(t) \, dt + 2\rho \int_0^T (H^2(\omega))^* \langle \ddot{\zeta}_{K,3}^\varepsilon(t), \eta_3 \sqrt{a} \rangle_{H^2(\omega)} \psi(t) \, dt \\ & + \frac{1}{\varepsilon^2} \int_0^T \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^\varepsilon(t)) \gamma_{\alpha\beta}(\eta) \sqrt{a} \, dy \psi(t) \, dt + \frac{1}{3} \int_0^T \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_K^\varepsilon(t)) \rho_{\alpha\beta}(\eta) \sqrt{a} \, dy \psi(t) \, dt \\ & - \frac{4}{\varepsilon^2} A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_0^T \int_\omega \vartheta_K^\varepsilon(t) a^{\alpha\beta} \gamma_{\alpha\beta}(\eta) \sqrt{a} \, dy \psi(t) \, dt = \int_0^T \int_\omega p^i(t) \eta_i \sqrt{a} \, dy \psi(t) \, dt. \end{aligned}$$

By virtue of the convergence process (33), letting $\varepsilon \rightarrow 0$ in the equations above gives:

$$2\rho \frac{d^2}{dt^2} \int_\omega (\zeta_\alpha(t) a^{\alpha\beta} \eta_\beta + \zeta_3(t) \eta_3) \sqrt{a} \, dy + \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta(t)) \rho_{\alpha\beta}(\eta) \sqrt{a} \, dy = \int_\omega p^i(t) \eta_i \sqrt{a} \, dy,$$

for all $\eta = (\eta_i) \in \mathbf{V}_F(\omega)$ in the sense of distributions in $(0, T)$.

Let us test the variational equations (28) at $\xi \in W(\omega)$, and at $\psi \in \mathcal{D}(0, T)$, and divide these equations by ε^3 . We obtain:

$$\begin{aligned} & 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_0^T W^*(\omega) \langle \dot{\vartheta}_K^\varepsilon(t), \xi \sqrt{a} \rangle_{W(\omega)} \psi(t) \, dt + 2S \int_0^T \int_\omega \partial_\alpha \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_\beta \xi \sqrt{a} \, dy \psi(t) \, dt \\ & + \frac{4A}{\varepsilon} \mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_0^T \int_\omega \gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) a^{\alpha\beta} \xi \sqrt{a} \, dy \psi(t) \, dt = \int_0^T \int_\omega q(t) \xi \sqrt{a} \, dy \psi(t) \, dt. \end{aligned}$$

By virtue of the convergence process (31), the Duhamel-Neumann law, and the fact that $\dot{\zeta}(t) \in \mathbf{V}_F(\omega)$ for a.a. $t \in (0, T)$ (Theorem 6), letting $\varepsilon \rightarrow 0$ in the equations above gives:

$$2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \frac{d}{dt} \int_\omega \vartheta(t) \xi \sqrt{a} \, dy + 2S \int_\omega \partial_\alpha \vartheta(t) a^{\alpha\beta} \partial_\beta \xi \sqrt{a} \, dy = \int_\omega q(t) \xi \sqrt{a} \, dy,$$

for all $\xi \in W(\omega)$ in the sense of distributions in $(0, T)$, proving the claim.

(iii) *The weak-star convergences established in (i) are actually stronger, in the sense of the process (30), i.e.,*

$$\begin{aligned}\zeta_K^\varepsilon &\rightarrow \zeta && \text{in } L^\infty(0, T; \mathbf{V}_K(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \dot{\zeta}_K^\varepsilon &\rightarrow \dot{\zeta} && \text{in } L^\infty(0, T; \mathbf{L}^2(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \vartheta_K^\varepsilon &\rightarrow \vartheta && \text{in } L^\infty(0, T; L^2(\omega)) \text{ as } \varepsilon \rightarrow 0.\end{aligned}$$

Let us define, for a.a. $t \in (0, T)$ and for all $0 < \varepsilon \leq \varepsilon_1$,

$$\begin{aligned}\lambda(\varepsilon)(t) &:= 2\rho_{(H^1(\omega))^*} \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t) - \ddot{\zeta}_\alpha(t), a^{\alpha\beta}(\dot{\zeta}_{K,\beta}^\varepsilon(t) - \dot{\zeta}_\beta(t))\sqrt{a} \rangle_{H^1(\omega)} \\ &\quad + 2\rho_{(H^2(\omega))^*} \langle \ddot{\zeta}_{K,3}^\varepsilon(t) - \ddot{\zeta}_3(t), (\dot{\zeta}_{K,3}^\varepsilon(t) - \dot{\zeta}_3(t))\sqrt{a} \rangle_{H^2(\omega)} \\ &\quad + \frac{1}{\varepsilon^2} \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^\varepsilon(t) - \zeta(t)) \gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t) - \dot{\zeta}(t)) \sqrt{a} \, dy \\ &\quad + \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_K^\varepsilon(t) - \zeta(t)) \rho_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t) - \dot{\zeta}(t)) \sqrt{a} \, dy \\ &\quad + 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_\omega (\vartheta_K^\varepsilon(t) - \vartheta(t)) (\dot{\vartheta}_K^\varepsilon(t) - \dot{\vartheta}(t)) \sqrt{a} \, dy \\ &\quad + 2S \int_\omega \partial_\alpha(\vartheta_K^\varepsilon(t) - \vartheta(t)) a^{\alpha\beta} \partial_\beta(\vartheta_K^\varepsilon(t) - \vartheta(t)) \sqrt{a} \, dy.\end{aligned}$$

By virtue of equations (27) and (28) of Problem $\mathcal{P}_K^\varepsilon(\omega)$ and the fact that $\zeta(t)$ and $\dot{\zeta}(t)$ belong to $\mathbf{V}_F(\omega)$ for a.a. $t \in (0, T)$, we get that:

$$\begin{aligned}\lambda(\varepsilon)(t) &= \int_\omega p^i(t) \dot{\zeta}_{K,i}^\varepsilon(t) \sqrt{a} \, dy + \int_\omega q(t) \vartheta_K^\varepsilon(t) \sqrt{a} \, dy \\ &\quad - 2\rho_{(H^1(\omega))^*} \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t), a^{\alpha\beta} \dot{\zeta}_\beta(t) \rangle_{H^1(\omega)} - 2\rho_{(H^1(\omega))^*} \langle \ddot{\zeta}_\alpha(t), a^{\alpha\beta} \dot{\zeta}_{K,\beta}^\varepsilon(t) \rangle_{H^1(\omega)} \\ &\quad - 4\rho \int_\omega \ddot{\zeta}_3(t) \dot{\zeta}_{K,3}^\varepsilon(t) \sqrt{a} \, dy + 2\rho \int_\omega \ddot{\zeta}_3(t) \dot{\zeta}_3(t) \sqrt{a} \, dy + 2\rho_{(H^1(\omega))^*} \langle \ddot{\zeta}_\alpha(t), a^{\alpha\beta} \dot{\zeta}_\beta(t) \rangle_{H^1(\omega)} \\ &\quad - \frac{2}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_K^\varepsilon(t)) \rho_{\alpha\beta}(\dot{\zeta}(t)) \sqrt{a} \, dy + \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta(t)) \rho_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) \sqrt{a} \, dy \\ &\quad - 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_\omega \dot{\vartheta}_K^\varepsilon(t) \vartheta(t) \sqrt{a} \, dy - 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_\omega \vartheta_K^\varepsilon(t) \dot{\vartheta}(t) \sqrt{a} \, dy \\ &\quad + 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_\omega \dot{\vartheta}(t) \vartheta(t) \sqrt{a} \, dy - 4S \int_\omega \partial_\alpha \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_\beta \vartheta(t) \sqrt{a} \, dy + 2S \int_\omega \partial_\alpha \vartheta(t) a^{\alpha\beta} \partial_\beta \vartheta_K^\varepsilon(t) \sqrt{a} \, dy.\end{aligned}$$

On the one hand, letting $\varepsilon \rightarrow 0$ gives

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \lambda(\varepsilon)(\tau) \, d\tau = 0, \quad (34)$$

as the limit satisfies the equations (9) and (10) of Problem $\mathcal{P}_F(\omega)$ for a.a. $t \in (0, T)$.

On the other hand, by virtue of the uniform positive-definiteness of the fourth order two-dimensional elasticity tensor $(a^{\alpha\beta\sigma\tau})$, Korn's inequality on general surfaces (Theorem 4), the uniform positive-definiteness of the matrix $(a^{\alpha\beta})$, and the initial conditions, we obtain that there exists a constant $C > 0$ independent of ε for which:

$$\begin{aligned}\frac{1}{C} \int_0^t \lambda(\varepsilon)(\tau) \, d\tau &\geq \|\dot{\zeta}_K^\varepsilon - \dot{\zeta}(t)\|_{L^2(\omega)}^2 + \|\zeta_K^\varepsilon - \zeta(t)\|_{\mathbf{V}_K(\omega)}^2 \\ &\quad + \|\vartheta_K^\varepsilon(t) - \vartheta(t)\|_{L^2(\omega)}^2 + \sum_\alpha \|\partial_\alpha \vartheta_K^\varepsilon(t) - \partial_\alpha \vartheta(t)\|_{L^2(\omega)}^2,\end{aligned} \quad (35)$$

for a.a. $t \in (0, T)$. Putting together (35) and (34), the conclusion immediately follows and the proof is complete.

8 Justification of Koiter's model for thermoelastic elliptic membranes

The last part of the paper is devoted to the justification of Koiter's model in the case where the linearly elastic shells under consideration is a *linearly elastic elliptic membrane shell* (from now on simply *elliptic membrane*).

In section 3, we considered a variational problem for “general” linearly elastic shells. From now on, we will restrict ourselves to a specific class of shells, according to the following definition (proposed in [13]; see also [10]).

Consider a linearly elastic shell, subjected to the various assumptions set forth in section 3. Such a shell is said to be a *linearly elastic elliptic membrane shell* if the following two additional assumptions are satisfied: *first*, $\gamma_0 = \gamma$, i.e., the homogeneous boundary condition of place is imposed over the *entire lateral face* $\gamma \times [-\varepsilon, \varepsilon]$ of the shell, and *second*, its middle surface $\theta(\bar{\omega})$ is *elliptic*, in the sense of section 2.

The following result is of paramount importance when studying elliptic membrane shells.

Theorem 10 *Let ω be a domain in \mathbb{R}^2 and let an immersion $\theta \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$ be given such that the surface $\theta(\bar{\omega})$ is elliptic. Define the space*

$$\mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega).$$

Then there exists a constant c_0 such that

$$\left\{ \sum_{\alpha} \|\eta_{\alpha}\|_{H^1(\omega)}^2 + \|\eta_3\|_{L^2(\omega)}^2 \right\}^{1/2} \leq c_0 \left\{ \sum_{\alpha, \beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{L^2(\omega)}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$. □

The above inequality, which is due to [15] and [21] (see also Theorem 2.7-3 of [10]), constitutes an example of a *Korn's inequality on a surface*, in the sense that it provides an estimate of an appropriate norm of a displacement field defined on a surface in terms of an appropriate norm of a specific “measure of strain” (here, the linearised change of metric tensor) corresponding to the displacement field considered.

In the recent paper [7] the authors performed a formal asymptotic analysis of the model, the point of departure of which was Problem $\mathcal{P}(\Omega^\varepsilon)$. In order to conduct the asymptotic analyses in [7], the authors made the following assumptions on the data: there exist functions $f^i \in H^1(0, T; L^2(\Omega))$, $h^i \in H^1(0, T; L^2(\Gamma_+ \cup \Gamma_-))$, $Q \in L^\infty(0, T; L^2(\Omega))$ independent on ε such that the following *assumptions on the data* hold:

$$\begin{aligned} f^{i,\varepsilon}(t)(x^\varepsilon) &= f^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega, \\ h^{i,\varepsilon}(t)(x^\varepsilon) &= \varepsilon h^i(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Gamma_+ \cup \Gamma_-, \\ Q^\varepsilon(t)(x^\varepsilon) &= Q(t)(x) \text{ at a.a. } t \in (0, T) \text{ and a.a. } x \in \Omega. \end{aligned} \tag{36}$$

Finally, we *assume* that there exist positive constants ρ , A , K , S , λ and μ such that

$$\begin{aligned} \rho^\varepsilon(x^\varepsilon) &= \rho \text{ at a.a. } x \in \Omega, \\ A^\varepsilon(x^\varepsilon) &= A \text{ at a.a. } x \in \Omega, \\ K^\varepsilon(x^\varepsilon) &= K \text{ at a.a. } x \in \Omega, \\ S^\varepsilon(x^\varepsilon) &= S \text{ at a.a. } x \in \Omega, \\ \lambda^\varepsilon &= \lambda, \quad \text{and} \quad \mu^\varepsilon = \mu. \end{aligned} \tag{37}$$

The following convergence result was obtained (see Theorem 5 of [7]).

Theorem 11 *Assume that $\theta \in \mathcal{C}^3(\bar{\omega}; \mathbb{E}^3)$. Consider a family of elliptic membranes with thickness 2ε approaching zero and all sharing the same elliptic middle surface $\theta(\bar{\omega})$. For all $0 < \varepsilon \leq \varepsilon_0$ let the pair $(\mathbf{u}(\varepsilon), \Theta(\varepsilon))$ be the solution of the associated three-dimensional scaled Problem $\mathcal{P}(\varepsilon; \Omega)$.*

Then, there exist functions $\Theta, u_\alpha \in H^1(\Omega)$ satisfying $\Theta = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in L^2(\Omega)$ such that

$$\begin{aligned} u_\alpha(\varepsilon) &\rightarrow u_\alpha \text{ in } L^\infty(0, T; H^1(\Omega)), \\ u_3(\varepsilon) &\rightarrow u_3 \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \Theta(\varepsilon) &\rightarrow \Theta \text{ in } L^\infty(0, T; L^2(\Omega)). \end{aligned}$$

Besides, the functions u_i and ϑ are independent of the transverse variable x_3 , and the pair (\mathbf{u}, Θ) is the unique solution of the two-dimensional limit problem $\mathcal{P}_M(\omega)$:

Problem $\mathcal{P}_M(\omega)$ Find a vector field $\zeta = (\zeta_i) : (0, T) \rightarrow \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$ and a function $\vartheta : (0, T) \rightarrow H_0^1(\omega)$ such that

$$\begin{aligned}\zeta &\in L^\infty(0, T; \mathbf{V}_M(\omega)), \\ \dot{\zeta} &\in L^\infty(0, T; \mathbf{L}^2(\omega)), \\ \ddot{\zeta} &\in L^\infty(0, T; \mathbf{V}_M^*(\omega)), \\ \vartheta &\in L^\infty(0, T; H_0^1(\omega)), \\ \dot{\vartheta} &\in L^\infty(0, T; H^{-1}(\omega)),\end{aligned}$$

that satisfies the following variational equations

$$\begin{aligned}2\rho \frac{d^2}{dt^2} \int_{\omega} (\zeta_\alpha(t) a^{\alpha\beta} \eta_\beta + \zeta_3 \eta_3) \sqrt{a} \, dy + \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta(t)) \gamma_{\alpha\beta}(\eta) \sqrt{a} \, dy \\ - 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_{\omega} \vartheta(t) \gamma_{\alpha\beta}(\eta) a^{\alpha\beta} \sqrt{a} \, dy = \int_{\omega} p^i(t) \eta_i \sqrt{a} \, dy,\end{aligned}\tag{38}$$

for all $\eta = (\eta_i) \in \mathbf{V}_M(\omega)$, in the sense of distributions in $(0, T)$,

$$\begin{aligned}2 \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \vartheta(t) \xi \sqrt{a} \, dy + 2S \int_{\omega} \partial_\alpha \vartheta(t) a^{\alpha\beta} \partial_\beta \xi \sqrt{a} \, dy \\ + 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \xi \gamma_{\alpha\beta}(\zeta(t)) a^{\alpha\beta} \sqrt{a} \, dy = \int_{\omega} q(t) \xi \sqrt{a} \, dy,\end{aligned}\tag{39}$$

for all $\xi \in H_0^1(\omega)$, in the sense of distributions in $(0, T)$,

and that satisfy the initial conditions

$$\begin{cases} \zeta(0) = \zeta_0, \\ \dot{\zeta}(0) = \dot{\zeta}_1, \\ \vartheta(0) = \vartheta_0, \end{cases}\tag{40}$$

for a sufficiently smooth initial data $(\zeta_0, \dot{\zeta}_1, \vartheta_0)$. ■

Clearly, by semi-groups theory (cf., e.g., [23]), it results that $\dot{\zeta} \in L^\infty(0, T; \mathbf{V}_M(\omega))$. □

The purpose of the next theorem is to justify Koiter's model by means of a rigorous asymptotic analysis as the thickness of the shell approaches zero in the case where the shell under consideration is an elliptic membrane. This constitutes the *fourth new result* in this paper.

Theorem 12 Let ω be a domain in \mathbb{R}^2 and let $\theta \in C^3(\bar{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of elliptic membranes with thickness 2ε approaching zero and with each having the same middle surface $\theta(\bar{\omega})$.

Finally, assume that there exist functions $f^i \in H^1(0, T; L^2(\Omega))$, $h^i \in H^1(0, T; L^2(\Gamma_+ \cup \Gamma_-))$ and $Q \in L^\infty(0, T; L^2(\Omega))$ independent of ε such that the assumptions (36) hold.

For each $\varepsilon > 0$, let the pair $(\zeta_K^\varepsilon, \vartheta_K^\varepsilon)$ denote the solution of Problem $\mathcal{P}_K^\varepsilon(\omega)$. Then the following convergence holds:

$$\begin{aligned}\zeta_K^\varepsilon &\rightarrow \zeta \quad \text{in } L^2(0, T; \mathbf{V}_K(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \dot{\zeta}_K^\varepsilon &\rightarrow \dot{\zeta} \quad \text{in } L^2(0, T; \mathbf{L}^2(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \vartheta_K^\varepsilon &\rightarrow \vartheta \quad \text{in } L^2(0, T; L^2(\omega)) \text{ as } \varepsilon \rightarrow 0,\end{aligned}\tag{41}$$

where the pair (ζ, ϑ) denotes the unique solution of Problem $\mathcal{P}_M(\omega)$.

Proof The outline of the proof is to a large extent inspired by the proof of Theorem 7.2-3 in [10] (itself adapted from [14]), where only the static case is considered, and Theorem 9. This is why some parts of the proof are reminiscent of those in [10]; otherwise, considering the time dependence requires substantial extra care. For sake of clarity, we break the proof into three parts, numbered (i)–(iii).

(i) There exist subsequences, still denoted $(\zeta_K^\varepsilon)_{\varepsilon>0}$ and $(\vartheta_K^\varepsilon)_{\varepsilon>0}$, and there exist $\zeta \in L^\infty(0, T; \mathbf{V}_M(\omega))$ and $\vartheta \in L^\infty(0, T; H_0^1(\omega))$ such that the following convergence process holds:

$$\begin{aligned}\zeta_K^\varepsilon &\xrightarrow{*} \zeta \text{ in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ \dot{\zeta}_K^\varepsilon &\xrightarrow{*} \dot{\zeta} \text{ in } L^\infty(0, T; \mathbf{L}^2(\omega)), \\ \ddot{\zeta}_K^\varepsilon &\xrightarrow{*} \ddot{\zeta} \text{ in } L^\infty(0, T; \mathbf{V}_K^*(\omega)), \\ \vartheta_K^\varepsilon &\xrightarrow{*} \vartheta \text{ in } L^\infty(0, T; L^2(\omega)), \\ \partial_\alpha \vartheta_K^\varepsilon &\rightarrow \partial_\alpha \vartheta \text{ in } L^2(0, T; L^2(\omega)), \\ \dot{\vartheta}_K^\varepsilon &\xrightarrow{*} \dot{\vartheta} \text{ in } L^\infty(0, T; H^{-1}(\omega)).\end{aligned}\tag{42}$$

By virtue of the results contained in [23] and Proposition 6.2 of [22], we can specialise $\boldsymbol{\eta} = \dot{\boldsymbol{\zeta}}_K^\varepsilon(t)$ and $\xi = \vartheta_K^\varepsilon(t)$ in the variational equations (27) and (28) of Problem $\mathcal{P}_K^\varepsilon(\omega)$ at a certain time instant $t \in (0, T)$. After specialising in this fashion, we sum equations (27) and (28) of Problem $\mathcal{P}_K^\varepsilon(\omega)$, getting:

$$\begin{aligned} & 2\rho\varepsilon_{H^{-1}(\omega)} \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t), a^{\alpha\beta} \dot{\zeta}_{K,\beta}^\varepsilon(t) \sqrt{a} \rangle_{H_0^1(\omega)} + 2\rho\varepsilon_{H^{-2}(\omega)} \langle \ddot{\zeta}_{K,3}^\varepsilon(t), \dot{\zeta}_{K,3}^\varepsilon(t) \sqrt{a} \rangle_{H_0^2(\omega)} \\ & + \varepsilon \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\dot{\zeta}_K^\varepsilon(t)) \gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) \sqrt{a} \, dy + \frac{\varepsilon^3}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\dot{\zeta}_K^\varepsilon(t)) \rho_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) \sqrt{a} \, dy \\ & + 2\varepsilon \left(K + A^2 \frac{(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right)_{H^{-1}(\omega)} \langle \dot{\vartheta}_K^\varepsilon(t), \vartheta_K^\varepsilon(t) \sqrt{a} \rangle_{H_0^1(\omega)} + 2S\varepsilon \int_{\omega} \partial_{\alpha} \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_{\beta} \vartheta_K^\varepsilon(t) \sqrt{a} \, dy \\ & = \int_{\omega} p^{i,\varepsilon}(t) \dot{\zeta}_{K,i}^\varepsilon(t) \sqrt{a} \, dy + \int_{\omega} q^\varepsilon(t) \vartheta_K^\varepsilon(t) \sqrt{a} \, dy. \end{aligned} \quad (43)$$

Carrying an integration in the interval $(0, t)$, dividing by ε in (43), and resorting to the uniform positive-definiteness of the fourth order two-dimensional elasticity tensor, Korn's inequality on an elliptic surface (Theorem 10) and Hölder's inequality gives:

$$\begin{aligned} & \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \frac{1}{c} \|\dot{\zeta}_K^\varepsilon(t)\|_{\mathbf{V}_K(\omega)}^2 + \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_{\alpha} \int_0^T \|\partial_{\alpha} \vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \\ & \leq \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t))\|_{L^2(\omega)}^2 + \sum_{\alpha,\beta} \|\varepsilon \rho_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t))\|_{L^2(\omega)}^2 + \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 + \sum_{\alpha} \int_0^T \|\partial_{\alpha} \vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \\ & \leq C \left\{ \|\dot{\zeta}_0\|_{\mathbf{V}_M(\omega)}^2 + \|\dot{\zeta}_1\|_{L^2(\omega)}^2 + \|\vartheta_0\|_{H^1(\omega)}^2 + \int_0^T \|\dot{\zeta}_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt + \int_0^T \|\vartheta_K^\varepsilon(t)\|_{L^2(\omega)}^2 \, dt \right\}, \end{aligned}$$

for some $C = C(p^i, q) > 0$ and some $c = c(\omega, \gamma_0, \boldsymbol{\theta}) > 0$. Hence, by Gronwall's inequality, we can conclude that

$$\begin{aligned} & (\dot{\zeta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; \mathbf{V}_K(\omega)), \\ & (\dot{\zeta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; \mathbf{L}^2(\omega)), \\ & (\ddot{\zeta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; \mathbf{V}_K^*(\omega)), \\ & (\varepsilon \tilde{\rho}_{\alpha\beta}(\dot{\zeta}_K^\varepsilon))_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\vartheta_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\partial_{\alpha} \vartheta_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\omega)), \\ & (\dot{\vartheta}_K^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; H^{-1}(\omega)), \end{aligned}$$

and the conclusion is thus reached by the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [6]).

(ii) *The weak-star pair (ζ, ϑ) is the unique solution of Problem $\mathcal{P}_M(\omega)$.* Let us test the variational equations (27) at $\boldsymbol{\eta} \in \mathbf{V}_M(\omega)$, and at $\psi \in \mathcal{D}(0, T)$, and divide these equations by ε . We obtain:

$$\begin{aligned} & 2\rho \int_0^T \int_{\omega} \langle \ddot{\zeta}_{K,\alpha}^\varepsilon(t), a^{\alpha\beta} \eta_{\beta} \sqrt{a} \rangle_{H_0^1(\omega)} \psi(t) \, dt + \int_0^T \int_{\omega} \ddot{\zeta}_{K,3}^\varepsilon(t) \eta_3 \sqrt{a} \, dy \psi(t) \, dt \\ & + \int_0^T \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\dot{\zeta}_K^\varepsilon(t)) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \psi(t) \, dt + \frac{\varepsilon^2}{3} \int_0^T \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\dot{\zeta}_K^\varepsilon(t)) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \psi(t) \, dt \\ & - 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_0^T \int_{\omega} \vartheta_K^\varepsilon(t) a^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \psi(t) \, dt = \int_0^T \int_{\omega} p^i(t) \eta_i \sqrt{a} \, dy \psi(t) \, dt. \end{aligned}$$

By virtue of the convergence process (31), letting $\varepsilon \rightarrow 0$ in the equations above gives:

$$\begin{aligned} & 2\rho \frac{d^2}{dt^2} \int_{\omega} (\zeta_{\alpha}(t) a^{\alpha\beta} \eta_{\beta} + \zeta_3(t) \eta_3) \sqrt{a} \, dy + \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}(t)) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \\ & - 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_{\omega} \vartheta(t) a^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy = \int_{\omega} p^i(t) \eta_i \sqrt{a} \, dy, \end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$ in the sense of distributions in $(0, T)$.

Let us test the variational equations (28) at $\xi \in H_0^1(\omega)$, and at $\psi \in \mathcal{D}(0, T)$, and divide these equations by ε . We obtain

$$\begin{aligned} & 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_0^T \int_{\omega} \langle \dot{\vartheta}_K^\varepsilon(t), \xi \sqrt{a} \rangle_{H_0^1(\omega)} \psi(t) \, dt + 2S \int_0^T \int_{\omega} \partial_{\alpha} \vartheta_K^\varepsilon(t) a^{\alpha\beta} \partial_{\beta} \xi \sqrt{a} \, dy \psi(t) \, dt \\ & + 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \int_0^T \int_{\omega} \gamma_{\alpha\beta}(\dot{\zeta}_K^\varepsilon(t)) a^{\alpha\beta} \xi \sqrt{a} \, dy \psi(t) \, dt = \int_0^T \int_{\omega} q^\varepsilon(t) \xi \sqrt{a} \, dy \psi(t) \, dt. \end{aligned}$$

By virtue of the convergence process (42) and the fact that $\dot{\zeta}(t) \in \mathbf{V}_M(\omega)$ for a.a. $t \in (0, T)$ (cf., e.g., [23]), letting $\varepsilon \rightarrow 0$ in the equations above gives:

$$\begin{aligned} & 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \vartheta(t) \xi \sqrt{a} \, dy + 2S \int_{\omega} \partial_{\alpha} \vartheta(t) a^{\alpha\beta} \partial_{\beta} \xi \sqrt{a} \, dy \\ & + 4A\mu \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu} \right) \frac{d}{dt} \int_{\omega} \gamma_{\alpha\beta}(\zeta(t)) a^{\alpha\beta} \xi \sqrt{a} \, dy = \int_{\omega} q(t) \xi \sqrt{a} \, dy, \end{aligned}$$

for all $\xi \in H_0^1(\omega)$ in the sense of distributions in $(0, T)$.

(iii) *The weak-star convergences established in (i) are actually stronger, in the sense of the process (41), i.e.,*

$$\begin{aligned} \zeta_K^{\varepsilon} &\rightarrow \zeta && \text{in } L^{\infty}(0, T; \mathbf{V}_K(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \dot{\zeta}_K^{\varepsilon} &\rightarrow \dot{\zeta} && \text{in } L^{\infty}(0, T; \mathbf{L}^2(\omega)) \text{ as } \varepsilon \rightarrow 0, \\ \vartheta_K^{\varepsilon} &\rightarrow \vartheta && \text{in } L^{\infty}(0, T; L^2(\omega)) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Let us define, for a.a. $t \in (0, T)$ and for all $0 < \varepsilon \leq \varepsilon_1$,

$$\begin{aligned} \lambda(\varepsilon)(t) &:= 2\rho_{H^{-1}(\omega)} \langle \ddot{\zeta}_{K,\alpha}^{\varepsilon}(t) - \ddot{\zeta}_{\alpha}(t), a^{\alpha\beta} (\dot{\zeta}_{K,\beta}^{\varepsilon} - \dot{\zeta}_{\beta}(t)) \sqrt{a} \rangle_{H_0^1(\omega)} + 2\rho \int_{\omega} (\ddot{\zeta}_{K,3}^{\varepsilon}(t) - \ddot{\zeta}_3(t)) (\dot{\zeta}_{K,3}^{\varepsilon} - \dot{\zeta}_3(t)) \sqrt{a} \, dy \\ &+ \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^{\varepsilon}(t) - \zeta(t)) \gamma_{\alpha\beta}(\dot{\zeta}_K^{\varepsilon}(t) - \dot{\zeta}(t)) \sqrt{a} \, dy + \frac{\varepsilon^2}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^{\varepsilon}(t) - \zeta(t)) \gamma_{\alpha\beta}(\dot{\zeta}_K^{\varepsilon}(t) - \dot{\zeta}(t)) \sqrt{a} \, dy \\ &+ 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_{\omega} (\vartheta_K^{\varepsilon}(t) - \vartheta(t)) (\dot{\vartheta}_K^{\varepsilon}(t) - \dot{\vartheta}(t)) \sqrt{a} \, dy \\ &+ 2S \int_{\omega} \partial_{\alpha} (\vartheta_K^{\varepsilon}(t) - \vartheta(t)) a^{\alpha\beta} \partial_{\beta} (\vartheta_K^{\varepsilon}(t) - \vartheta(t)) \sqrt{a} \, dy. \end{aligned}$$

By virtue of equations (27) and (28) of Problem $\mathcal{P}_K^{\varepsilon}(\omega)$ and the fact that $\dot{\zeta}(t)$ and $\zeta(t)$ belong to $\mathbf{V}_M(\omega)$ for a.a. $t \in (0, T)$, we get that

$$\begin{aligned} \lambda(\varepsilon)(t) &= \int_{\omega} p^i(t) \dot{\zeta}_{K,i}^{\varepsilon}(t) \sqrt{a} \, dy + \int_{\omega} q(t) \vartheta_K^{\varepsilon}(t) \sqrt{a} \, dy \\ &- 2\rho_{H^{-1}(\omega)} \langle \ddot{\zeta}_{K,\alpha}^{\varepsilon}(t), a^{\alpha\beta} \dot{\zeta}_{\beta}^{\varepsilon}(t) \sqrt{a} \rangle_{H_0^1(\omega)} - 2\rho_{H^{-1}(\omega)} \langle \ddot{\zeta}_{\alpha}(t), a^{\alpha\beta} \dot{\zeta}_{K,\beta}^{\varepsilon}(t) \sqrt{a} \rangle_{H_0^1(\omega)} \\ &- 4\rho \int_{\omega} \ddot{\zeta}_3(t) \dot{\zeta}_{K,3}^{\varepsilon}(t) \sqrt{a} \, dy + 2\rho_{H^{-1}(\omega)} \langle \ddot{\zeta}_{\alpha}(t), a^{\alpha\beta} \dot{\zeta}_{\beta}^{\varepsilon}(t) \sqrt{a} \rangle_{H_0^1(\omega)} + 2\rho \int_{\omega} \ddot{\zeta}_3(t) \dot{\zeta}_3(t) \sqrt{a} \, dy \\ &- 2 \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta_K^{\varepsilon}(t)) \gamma_{\alpha\beta}(\dot{\zeta}(t)) \sqrt{a} \, dy + \int_{\omega} a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\zeta(t)) \gamma_{\alpha\beta}(\dot{\zeta}(t)) \sqrt{a} \, dy \\ &- \frac{2\varepsilon^2}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta_K^{\varepsilon}(t)) \rho_{\alpha\beta}(\dot{\zeta}(t)) \sqrt{a} \, dy + \frac{\varepsilon^2}{3} \int_{\omega} a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\zeta(t)) \rho_{\alpha\beta}(\dot{\zeta}(t)) \sqrt{a} \, dy \\ &- 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_{\omega} \dot{\vartheta}_K^{\varepsilon}(t) \vartheta(t) \sqrt{a} \, dy - 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_{\omega} \vartheta_K^{\varepsilon}(t) \dot{\vartheta}(t) \sqrt{a} \, dy \\ &+ 2 \left(K + \frac{A^2(3\lambda + 2\mu)^2}{\lambda + 2\mu} \right) \int_{\omega} \dot{\vartheta}(t) \vartheta(t) \sqrt{a} \, dy - 4S \int_{\omega} \partial_{\alpha} \vartheta_K^{\varepsilon}(t) a^{\alpha\beta} \partial_{\beta} \vartheta(t) \sqrt{a} \, dy + 2S \int_{\omega} \partial_{\alpha} \vartheta(t) a^{\alpha\beta} \partial_{\beta} \vartheta(t) \sqrt{a} \, dy. \end{aligned}$$

On the one hand, letting $\varepsilon \rightarrow 0$ gives

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \lambda(\varepsilon)(\tau) \, d\tau = 0, \quad (44)$$

as the limit satisfies the equations (38) and (39) of Problem $\mathcal{P}_M(\omega)$ for a.a. $t \in (0, T)$.

On the other hand, by virtue of the uniform positive-definiteness of the fourth order two-dimensional elasticity tensor $(a^{\alpha\beta\sigma\tau})$, Korn's inequality for elliptic surfaces (Theorem 10), as well as the uniform positive-definiteness of the matrix $(a^{\alpha\beta})$, and the initial conditions, we obtain that there exists a constant $C > 0$ independent of ε for which:

$$\begin{aligned} \frac{1}{C} \int_0^t \lambda(\varepsilon)(\tau) \, d\tau &\geq \|\dot{\zeta}_K^{\varepsilon} - \dot{\zeta}(t)\|_{L^2(\omega)}^2 + \|\zeta_K^{\varepsilon} - \zeta(t)\|_{\mathbf{V}_K(\omega)}^2 \\ &+ \|\vartheta_K^{\varepsilon}(t) - \vartheta(t)\|_{L^2(\omega)}^2 + \sum_{\alpha} \|\partial_{\alpha} \vartheta_K^{\varepsilon}(t) - \partial_{\alpha} \vartheta(t)\|_{L^2(\omega)}^2, \end{aligned} \quad (45)$$

for a.a. $t \in (0, T)$. Putting together (44) and (45), the conclusion immediately follows and the proof is complete.

Conclusions

In this paper we first conducted – after making a suitable assumption on the data – a rigorous asymptotic analysis on a time-dependent thermoelastic three-dimensional flexural shell model as the shell thickness approaches zero, and we recovered a time-dependent thermoelastic two-dimensional limit model, where the mechanical and thermal equations do not exhibit any couplings.

Secondly, we proposed and justified a suitable time-dependent thermoelastic model of Koiter's type for flexural shells. We showed that under the same assumption on the data made beforehand, the proposed model of Koiter's type converges to the same two-dimensional limit model recovered as outcome of the asymptotic analysis conducted on the original three-dimensional mode in the case where the shell under consideration is a flexural shell.

Finally, we showed that our proposed time-dependent thermoelastic Koiter's model converges to the time-dependent thermoelastic two-dimensional limit model for elliptic membrane shells recovered in [7] provided that the given data satisfies the same assumptions made in the aforementioned bibliographical reference.

Acknowledgements

The author is greatly indebted to Professor Philippe G. Ciarlet for his encouragement and guidance.

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