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# Boundary controllability of a simplified stabilized Kuramoto-Sivashinsky system\*\*

Víctor Hernández-Santamaría\*    Alberto Mercado†    Piero Visconti‡

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## Abstract

In this paper, we study the controllability of a nonlinear system of coupled second- and fourth-order parabolic equations. This system can be regarded as a simplification of the well-known stabilized Kuramoto-Sivashinsky system. Using only one control applied on the boundary of the second-order equation, we prove that the local-null controllability of the system holds if the square root of the diffusion coefficient of the second-order equation is an irrational number with finite Liouville-Roth constant.

**Keywords:** parabolic system, boundary controllability, source term method.

**AMS subject classifications:** 93B05, 93C20, 35K41.

## 1 Introduction

The study of controllability of systems of coupled partial differential equations has deserved a lot of attention in the literature of the past recent years. For systems of second-order equations, we refer the reader to the classical works [1, 2, 3], and the references within. For systems involving fourth- and second-order differential equations, we refer to [8], [9], [6], and [12]. Particularly, in [7], the authors study controllability properties of a linear cascade system coupling a bilaplacian operator to a heat equation with a single boundary control, obtaining positive and negative controllability results depending of the diffusion coefficient.

The interest in the study of systems involving fourth-order parabolic equations comes from models of front propagation in reaction-diffusion phenomena. Indeed, in [17] a system was proposed as a model for such phenomena with both dissipative and dispersive features and allowing a

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stable solitary-pulse. The system consists of a one-dimensional Kuramoto-Sivashinsky-KdV (KS-KdV) equation, linearly coupled to an extra dissipative equation, under the name of the stabilized Kuramoto-Sivashinsky system, and given by

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + au_{xx} + uu_x = v_x, \\ v_t - dv_{xx} + cv_x = u_x, \end{cases} \quad (1.1)$$

where  $\gamma > 0$  accounts for the long-wave instability,  $a$  is the short-wave dissipation,  $d > 0$  is the dissipative parameter and  $c \in \mathbb{R}$  is the group-velocity mismatch between wave modes. Notice that the coupling is through first-order terms, which is harder to deal with than zero-order couplings.

The objective of this work is to perform one step in the direction of studying the boundary controllability of system (1.1), by considering the controllability of the nonlinear version of the system studied in [7]. That is, we consider the problem given by

$$\begin{cases} u_t + u_{xxxx} + uu_x = v, & t \in (0, T), x \in (0, \pi), \\ v_t - dv_{xx} = 0, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T), \end{cases} \quad (1.2)$$

where  $u = u(t, x)$  and  $v = v(t, x)$  are the state variables and  $h = h(t)$  is the control. Observe that the control acts only on the heat equation and influences indirectly the first equation by means of the coupling. The parameter  $d > 0$  is the diffusion of the heat equation and will play a crucial role in the analysis of controllability properties for the system.

The controllability problem we are interested in can be formulated as follows.

**Definition 1.1.** *We say system (1.2) is locally null-controllable at time  $T$  if there exists  $R > 0$  such that for any state  $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$  satisfying*

$$\|(u_0, v_0)\|_{L^2(0, \pi) \times H^{-1}(0, \pi)} \leq R,$$

*there exists a control  $h \in L^2(0, T)$  such that the solution of (1.2) with initial condition*

$$u(0, \cdot) = u_0 \quad \text{and} \quad v(0, \cdot) = v_0$$

*satisfies*

$$u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0.$$

In [7], the global null-controllability (*i.e.*, without imposing any constraint on the size of the initial data) of the linear version of system (1.2) (see (3.1) below) is proved, depending on the diffusion coefficient  $d > 0$ . To be more precise, we recall the following.

**Definition 1.2.** *The Liouville-Roth constant of a real number  $x$  is the least upper bound of the set of positive real numbers  $\mu$  such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

*is satisfied by infinitely many integer pairs  $(p, q)$  with  $q > 0$ . A real number is said to be a Liouville number if it has infinite Liouville-Roth constant.*

**Remark 1.3** ([5, Theorem E.3]). *The set of Liouville numbers has null Lebesgue measure in  $\mathbb{R}$ .*

In this regard, if  $\sqrt{d}$  is not a Liouville number, then corresponding linear version of (1.2) (see (3.1)) is null-controllable at time  $T$  for any  $T > 0$  and any initial datum  $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ .

In this same spirit, the main result of this work is the following.

**Theorem 1.4.** *If  $\sqrt{d}$  is an irrational number with finite Liouville-Roth constant, then system (1.2) is locally null controllable.*

The proof of Theorem 1.4 follows from a precise estimate of the cost of the control of the linear system (3.1), which is based on bounds for the norm of biorthogonal families to exponential functions associated to sequences with condensation index, which have been recently found in [11], and an application of the so-called source term method, introduced by Y. Liu, T. Takahashi and M. Tucsnak [16]. It is important to mention that we are able to prove such a precise estimate only in the case where  $\sqrt{d}$  has finite Liouville-Roth constant (see Definition 1.2). As we have mentioned, in [7, Theorem 1.5], the global controllability of the linear system (3.1) is established also when  $\sqrt{d}$  satisfies the same condition. Nonetheless, such result does not provide enough information on the control cost which is crucial to perform the source term method and extend the result to the nonlinear case. Therefore, one of our main contributions is Theorem 3.4 which improves the result in [7].

**Remark 1.5.** *In Definition 1.1 there is an implicit well-posedness assumption, as we refer to “the solution of (1.2)”. Like in other works concerning the source term method (see e.g., [14, 10]), the a priori well-posedness of the nonlinear system (1.2) is not a prerequisite to study the controllability problem, since the existence and local uniqueness of the controlled solution can be obtained as a byproduct of the proof of Theorem 1.4. However, in our case, (1.2) enjoys uniqueness and, for initial data and control with small enough norms, existence. These features are made precise in Proposition 4.1 and Proposition 4.2. See also Remark 4.3.*

The rest of the paper is organized as follows. In Section 2 we establish some basic results of well-posedness and we apply the source term method to our system. In Section 3 we obtain adequate bounds for the cost of the control of the linear system under suitable hypothesis of the diffusion coefficient. Finally, in Section 4 we prove Theorem 1.4.

## 2 The source term method

In [16], the authors introduced the so-called source term method, a quite general methodology to deduce local controllability results for semilinear problems based on the controllability properties of the underlying linear system perturbed by an external force with some decay properties.

By adapting this methodology to our case, we are reduced to study the controllability of the linear system

$$\begin{cases} u_t + u_{xxxx} = v + f, & t \in (0, T), x \in (0, \pi), \\ v_t - dv_{xx} = 0, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T), \end{cases} \quad (2.1)$$

where  $f$  is a suitable source term acting only in the first equation, since this is where the nonlinearity appears.

We begin by stating a general well-posedness result for system (2.1).

**Proposition 2.1.** *Let  $t_1 < t_2$  be any arbitrary positive times. Assume*

$$f \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))'), \quad h \in L^2(t_1, t_2), \quad (u_1, v_1) \in L^2(0, \pi) \times H^{-1}(0, \pi).$$

*Then (2.1) admits a unique solution  $(u, v) \in C([t_1, t_2], L^2(0, \pi) \times H^{-1}(0, \pi)) \cap L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi)) \times L^2(0, \pi))$  satisfying  $(u, v)(t_1, \cdot) = (u_1, v_1)$ . Moreover, we have the following bound*

$$\begin{aligned} & \| (u, v) \|_{L^2(t_1, t_2; (H^2 \cap H_0^1) \times L^2)}^2 + \| (u, v) \|_{C([t_1, t_2], L^2 \times H^{-1})}^2 \\ & \leq C_1 \left( \| h \|_{L^2(t_1, t_2)}^2 + \| (u_1, v_1) \|_{L^2 \times H^{-1}}^2 + \| f \|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right), \end{aligned}$$

where  $C_1$  does not depend on  $t_1, t_2$ .

The proof of this result is classical and relies on well-known arguments. For the sake of completeness, we present it in Appendix A.

Now, we recall the method introduced in [16] that will allow us to deal with the controllability of the nonlinear system (1.2).

We introduce the following weighted spaces: consider a measurable function  $\rho : [0, T] \mapsto \mathbb{R}$  which is nonzero a.e. on  $[0, T]$ . We denote by  $L_\rho^2(0, T; X)$  the weighted  $L^2$  space with values on  $X$  and with measure  $m_T/\rho^2$ , where  $m_T$  denotes the Lebesgue measure of the interval  $(0, T)$ . Notice that when  $\rho$  is bounded from above,  $L_\rho^2(0, T; X)$  injects continuously into  $L^2(0, T; X)$ , this happens, for example if  $\rho$  is continuous.

In the spirit of [16], we have the following controllability result for (2.1) in weighted spaces.

**Proposition 2.2.** *Assume that system (2.1) with  $f \equiv 0$  is null controllable in any time  $\tau$ , with control cost  $\mathcal{C}(\tau)$  bounded from above by a continuous and nonincreasing function  $K(\tau)$ . Suppose that  $q > 1$  and  $\rho_0, \rho_1 : [0, T] \rightarrow \mathbb{R}$  are continuous, nonincreasing functions which are positive on  $[0, T)$ , satisfy  $\rho_0(T) = \rho_1(T) = 0$  and verify the identity*

$$\rho_0(t) = \rho_1(q^2 t + (1 - q^2)T)K((q - 1)(T - t)), \quad \left( t \in \left[ \left(1 - \frac{1}{q^2}\right)T, T \right] \right).$$

*Then, for each  $f \in L_{\rho_1}^2(0, T; (H^2 \cap H_0^1)')$  and any initial condition  $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ , there exists a control  $h \in L_{\rho_0}^2(0, T)$  driving the state to zero at time  $T$  such that the solution  $(u, v)$  to (2.1) satisfies*

$$\begin{aligned} & \| (u, v) \|_{L_{\rho_0}^2(0, T; (H^2 \cap H_0^1) \times L^2)}^2 + \left\| \frac{(u, v)}{\rho_0} \right\|_{C([0, T], L^2 \times H^{-1})}^2 + \| h \|_{L_{\rho_0}^2(0, T)}^2 \\ & \leq C_2 \left( \| (u_0, v_0) \|_{L^2 \times H^{-1}}^2 + \| f \|_{L_{\rho_1}^2(0, T; (H^2 \cap H_0^1)')}^2 \right) \end{aligned} \tag{2.2}$$

where  $C_2 > 0$  is uniform with respect to  $f, y_0$  and  $h$ .

The proof of this result is very close to [16, Proposition 2.3], where it is assumed that the underlying operator is self-adjoint, which is not fulfilled for our system. Therefore we only give a brief sketch of the proof in Appendix B.

### 3 Bounds on control cost for the linear system

As we anticipated in Section 1, one of the key ingredients of the proof of Theorem 1.4 is a precise bound of the cost of the control of a linear version of (1.2), which allow us to prove the hypotheses

of Proposition 2.2. In this section, we give sufficient conditions on  $d$  such that the control cost has bounds that allow us to deal with the controllability of the nonlinear problem.

We begin by recalling the known results on the linear control problem, which is given by the following system.

$$\begin{cases} u_t + u_{xxxx} = v, & t \in (0, T), x \in (0, \pi), \\ v_t - dv_{xx} = 0, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T). \end{cases} \quad (3.1)$$

**Definition 3.1.** Let  $T > 0$ . System (3.1) is said to be globally null-controllable in time  $T$  if for any state  $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ , there exists a control  $h \in L^2(0, T)$  such that the solution of (3.1) with initial condition

$$u(0, \cdot) = u_0 \quad \text{and} \quad v(0, \cdot) = v_0$$

satisfies

$$u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0.$$

We have the following.

**Theorem 3.2** (Theorem 1.5 of [7]). *If  $\sqrt{d}$  is an irrational number with finite Liouville-Roth constant, then system (3.1) is null-controllable in time  $T$  for any  $T > 0$ .*

Theorem 3.2 indeed tell us that system (3.1) is null-controllable at time  $T$  for a wide class of diffusion coefficients  $d$ . In particular, this means that for every  $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$  the set of admissible controls

$$\mathcal{H}(T, u_0, v_0) := \{h \in L^2(0, T) : (u, v)(T, \cdot) = 0\} \quad (3.2)$$

is nonempty. Thus the control cost in time  $T$  is defined as

$$\mathcal{C}(T) := \sup \left\{ \inf_{h \in \mathcal{H}(T, u_0, v_0)} \|h\|_{L^2(0, T)} : \|(u_0, v_0)\|_{L^2 \times H^{-1}} \leq 1 \right\}. \quad (3.3)$$

If (3.1) is not null controllable in time  $T$ , we set  $\mathcal{C}(T) = \infty$ .

**Remark 3.3.** For fixed  $(u_0, v_0)$ , it follows from the linearity and well-posedness of (3.1), that  $\mathcal{H}(T, u_0, v_0)$  is a closed affine subspace of  $L^2(0, T)$ , which is a translation of  $\mathcal{H}(T, 0, 0)$  whenever the former is nonempty. If system (3.1) is null controllable, since  $L^2(0, T)$  is Hilbert,  $\mathcal{H}(T, u_0, v_0)$  contains a unique element with minimal norm. Denoting  $N$  as the map which sends the initial data  $(u_0, v_0)$  to this null control, due to the linearity of (3.1), it follows that  $N$  is a linear operator with norm  $\|N\| = \mathcal{C}(T)$ , making clear the use of the term control cost.

The techniques used in [7] to prove Theorem 3.2 do not allow to directly obtain an estimation of (3.3) in all the cases where the system (3.1) is null-controllable. We will see that using the bounds recently found in [11], we can obtain a suitable estimation of the control cost for the existence of a control  $h$  for system (3.1). The result is the following.

**Theorem 3.4.** *Let  $0 < T \leq 1$ . If  $\sqrt{d}$  is an irrational number with finite Liouville-Roth constant, then there exist positive constants  $C$  and  $M$ , independent of  $T$ , such that the control cost for system (3.1) satisfies*

$$\mathcal{C}(T) \leq C \exp(M/T).$$

In order to prove Theorem 3.4, we begin by rewriting system (3.1) in a more compact way, that is,

$$y_t = Ay + Bh, \quad (3.4)$$

where  $y = (u, v)$ ,  $D(A) := \{u \in H^4(0, \pi) : u(0) = u(\pi) = u''(0) = u''(\pi) = 0\} \times H_0^1(0, \pi)$  and  $A : D(A) \subset L^2(0, \pi) \times H^{-1}(0, \pi) \rightarrow L^2(0, \pi) \times H^{-1}(0, \pi)$  is given by

$$A = - \begin{pmatrix} \partial_{xxxx} & -I \\ 0 & -d\partial_{xx} \end{pmatrix}, \quad (3.5)$$

and  $B \in \mathcal{L}(\mathbb{C}, ((H^2(0, \pi) \cap H_0^1(0, \pi))^2)')$  is given by

$$(Bh)(\phi_1, \phi_2) = dh\phi_{2x}(0). \quad (3.6)$$

Notice that with this we have for the adjoint operator,

$$\begin{aligned} D(A^*) &= \{u \in H^4(0, \pi) : u(0) = u(\pi) = u''(0) = u''(\pi) = 0\} \\ &\quad \times \{u \in H^3(0, \pi) : u(0) = u(\pi) = u''(0) = u''(\pi) = 0\} \end{aligned}$$

and  $A^* : D(A^*) \subset L^2(0, \pi) \times H_0^1(0, \pi) \rightarrow L^2(0, \pi) \times H_0^1(0, \pi)$  is given by

$$A^* = - \begin{pmatrix} \partial_{xxxx} & 0 \\ -I & -d\partial_{xx} \end{pmatrix}. \quad (3.7)$$

In [7, Section 3.2] it was found that there exists a Riesz basis of  $L^2(0, \pi) \times H^{-1}(0, \pi)$  made up of eigenfunctions of  $A$ , these are

$$\Phi_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k, \quad \Phi_{2,k} = k \begin{pmatrix} (k^4 - dk^2)^{-1} \\ 1 \end{pmatrix} \varphi_k, \quad k \in \mathbb{N}, \quad (3.8)$$

where  $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$  and  $\mathbb{N}$  denotes the set of positive integers. The eigenfunction  $\Phi_{1,k}$  is associated to the eigenvalue  $-k^4$  while  $\Phi_{2,k}$  is associated to the eigenvalue  $-dk^2$ .

The biorthogonal basis (of  $L^2(0, \pi) \times H_0^1(0, \pi)$ ) corresponding to this Riesz basis is given by

$$\Psi_{1,k} = \begin{pmatrix} 1 \\ (dk^2 - k^4)^{-1} \end{pmatrix} \varphi_k, \quad \Psi_{2,k} = k^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad k \in \mathbb{N}. \quad (3.9)$$

The vectors  $\Psi_{1,k}$  and  $\Psi_{2,k}$  are then the eigenvectors for the operator  $A^*$ , associated with  $-k^4$  and  $-dk^2$ , respectively.

We have the following auxiliary result.

**Lemma 3.5.** *If  $\sqrt{d}$  is an irrational number with finite Liouville-Roth constant, then there exist positive constants  $\gamma, \delta, \epsilon, \zeta$  (independent of  $T$ ) such that for each  $T > 0$  there exists a family  $\{q_{1,k}, q_{2,k}\}$  biorthogonal to  $\{\exp(-k^2t), \exp(-dk^2t) : k \in \mathbb{N}\}$  in  $L^2(0, T)$  such that*

$$\|q_{i,k}\|_{L^2(0,T)} \leq C \exp(\gamma/T) \exp((\delta\sqrt{T} + \epsilon)k)k^\zeta, \quad i = 1, 2, \quad k \in \mathbb{N}. \quad (3.10)$$

**Remark 3.6.** *Notice that since  $\{\exp(-k^4t), \exp(-dk^2t)\} \subset \{\exp(-k^2t), \exp(-dk^2t)\}$ , the subset  $\{q_{1,k^2}, q_{2,k}\}$  is biorthogonal to  $\{\exp(-k^4t), \exp(-dk^2t)\}$ .*

*Proof of Lemma 3.5.* Let us consider the sequence  $\Lambda = \{k^2\}_{k \geq 1} \cup \{dk^2\}_{k \geq 1}$ . Since  $\sqrt{d}$  is an irrational number, it is clear that  $k^2 \neq dn^2$  for any  $k, n \geq 1$ . Thus, the sequence  $\Lambda = \{k^2\}_{k \geq 1} \cup \{dk^2\}_{k \geq 1}$  can be rearranged as an increasing sequence  $\Lambda = \{\Lambda_j\}_{j \geq 1}$  which verifies all the hypothesis from [11, Definition 1.4] (see [11, Proposition 2.13]). In turn, [11, Theorem 1.7] yields the existence of a family of functions  $\{q_j\}_{j \geq 1} \subset L^2(0, T)$  which are biorthogonal to  $e^{-\Lambda_j t}$ ,  $t \in (0, T)$ , and positive constants  $\gamma$ ,  $\delta_0$  and  $\epsilon_0$  such that

$$\|q_j\|_{L^2(0, T)} \leq C \exp(\gamma/T) \exp\left((\delta_0 \sqrt{T} + \epsilon_0) \sqrt{\Lambda_j}\right) \mathcal{P}_j, \quad \forall j \geq 1$$

where

$$\mathcal{P}_j := \frac{1}{|\Lambda_j - \Lambda_{j-1}| |\Lambda_j - \Lambda_{j+1}|}, \quad j \geq 2, \quad \mathcal{P}_1 = \frac{1}{|\Lambda_1 - \Lambda_2|} \quad (3.11)$$

Now, we claim that there exist  $C > 0$  and  $r > 2$ , depending only on  $d$ , satisfying

$$\mathcal{P}_j \leq C k^{2(r-2)} \quad (3.12)$$

for each  $j \geq 1$  such that  $\Lambda_j = k^2$  or  $\Lambda_j = dk^2$ .

In order to prove (3.12), we begin by noting from Definition 1.2 that for an irrational number  $x$  with Liouville-Roth constant  $\mu$ , there exists a constant  $C > 0$  such that

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^r} \quad (3.13)$$

for each fixed  $r > \mu$  and any integers  $p, q$  with  $q > 0$ . On the other hand, any number has the same Liouville-Roth constant as their reciprocal. This is proven in Appendix C. Hence we can take

$$r > \mu(\sqrt{d}) = \mu(1/\sqrt{d}). \quad (3.14)$$

From (3.11), note that for estimating each  $\mathcal{P}_j$  we only have to bound the difference between two consecutive elements of the sequence  $\{\Lambda_j\}$ . Assuming that  $j \geq 2$ , we have four different cases:

- If  $\Lambda_j = dk^2$  and  $\Lambda_{j \pm 1} = d(k \pm 1)^2$ , then

$$|\Lambda_j - \Lambda_{j \pm 1}| = d|k^2 - (k \pm 1)^2| \geq dk \quad (3.15)$$

- If  $\Lambda_j = dk^2$  and  $\Lambda_{j \pm 1} = m^2$ , we have from (3.13) with  $x = \sqrt{d}$  that

$$\begin{aligned} |\Lambda_j - \Lambda_{j \pm 1}| &= |dk^2 - m^2| = \left| \sqrt{d}k - m \right| \left| \sqrt{d}k + m \right| = \\ &\geq \sqrt{d}k^2 \left| \sqrt{d} - \frac{m}{k} \right| \geq c_1 k^{2-r} \end{aligned} \quad (3.16)$$

where  $c_1 > 0$  depends only on  $\mu$ ,  $r$  and  $d$ .

- If  $\Lambda_j = k^2$  and  $\Lambda_{j \pm 1} = (k \pm 1)^2$ , we have

$$|\Lambda_j - \Lambda_{j \pm 1}| = |k^2 - (k \pm 1)^2| \geq k \quad (3.17)$$



- $\Lambda_j = k^2$  and  $\Lambda_{j\pm 1} = dm^2$ , we have from (3.13) for  $x = 1/\sqrt{d}$  that

$$\begin{aligned} |\Lambda_j - \Lambda_{j\pm 1}| &= |k + \sqrt{d}m| |k - \sqrt{d}m| \\ &= |k + \sqrt{d}m| k \sqrt{d} \left| \frac{1}{\sqrt{d}} - \frac{m}{k} \right| \end{aligned} \quad (3.18)$$

$$\geq \sqrt{d} k^2 \left| \frac{1}{\sqrt{d}} - \frac{m}{k} \right| \quad (3.19)$$

$$\geq c_2 k^{2-r} \quad (3.20)$$

for some  $c_2 > 0$  only depending on  $\mu$ ,  $r$  and  $d$ .

Finally, we have the corresponding estimates concerning  $|\Lambda_1 - \Lambda_2|$ , and then we have proved (3.12).

Then, for each  $k$ , we set  $q_{1,k} = q_j$  where  $j$  is the only index satisfying  $\Lambda_j = k^2$ , and  $q_{2,k} = q_l$  where  $\Lambda_l = dk^2$ . Then (3.10) follows with  $\zeta = 2(r-2)$ ,  $\delta = \delta_0 \max\{1, \sqrt{d}\}$  and  $\epsilon = \epsilon_0 \max\{1, \sqrt{d}\}$ .  $\square$

According to Proposition 2.1, for each  $y_0 \in L^2(0, \pi) \times H^{-1}(0, \pi)$  and  $h \in L^2(0, T)$ , equation (3.4) admits a unique solution  $y \in C([0, T], L^2(0, \pi) \times H^{-1}(0, \pi)) \cap L^2(0, T; (H^2(0, \pi) \cap H_0^1(0, \pi)) \times L^2(0, \pi))$  satisfying  $y(0) = y_0$ . Furthermore

$$\|y\|_{L^2(0, T; (H^2 \cap H_0^1) \times L^2)}^2 + \|y\|_{C([0, T], L^2 \times H^{-1})}^2 \leq C \left( \|h\|_{L^2(0, T)}^2 + \|y_0\|_{L^2 \times H^{-1}}^2 \right). \quad (3.21)$$

Now, we are in position to prove the main result of this section.

*Proof of Theorem 3.4.* We know that  $A$  is diagonalizable and it is the generator of a strongly continuous semigroup  $\{\exp(tA)\}_{t \geq 0}$  (see [19, Section 2.6]). Write  $X = L^2(0, \pi) \times H^{-1}(0, \pi)$ ,  $X' = L^2(0, \pi) \times H_0^1(0, \pi)$  and  $Y = ((H^2(0, \pi) \cap H_0^1(0, \pi))^2)'$ ,  $Y' = (H^2(0, \pi) \cap H_0^1(0, \pi))^2$ , with this  $\langle \cdot, \cdot \rangle_{X, X'} = \langle \cdot, \cdot \rangle_{Y, Y'}$  on  $X \times Y'$ , since  $X, X'$  are both primal-dual pairs with respect to the same pivot space (namely,  $(L^2(0, \pi))^2$ ).

By standard interpolation arguments, the semigroup  $\{\exp(tA)\}_{t \geq 0}$  can be extended to the space  $Y$  (see [15, Theorem 5.1, pp. 27] and [19, Proposition 10.2.4]). Hence, since  $Bh \in L^2(0, T; Y)$ , the mild solution formula

$$y(T) = \exp(TA)y_0 + \int_0^T \exp((T-t)A)Bh dt. \quad (3.22)$$

holds in  $Y$ , and because of [13, Prop. 0.1, pp. 4], also in  $X$ .

Similarly,  $A^*$  generates  $\{\exp(tA^*)\}_{t \geq 0} = \{\exp(tA)^*\}_{t \geq 0}$  that shares the eigenfunctions of  $A^*$  and its eigenvalues are exponentials of those of  $A^*$  (which, in this case match those of  $A$ , see [19, Sections 2.6 & 2.8]).

Now, let us write the moment problem associated with the null controllability of system (3.4). It is clear that, if  $\sqrt{d}$  is irrational, the positive real numbers  $\{k^4, dk^2 : k \in \mathbb{N}\}$  are all distinct. We denote them by  $\{\lambda_j\}_{j \in \mathbb{N}}$ , indexed in increasing order. For each  $j$ , we denote by  $\phi_j$  and  $\psi_j$  the eigenvector and the biorthogonal eigenvector, respectively, associated with the eigenvalue  $-\lambda_j$ . Since  $\{\psi_j\}_{j \in \mathbb{N}}$  is a Riesz basis of  $X'$ ,  $y(T) = 0$  if and only if for each  $j \in \mathbb{N}$  we have

$$\langle y(T), \psi_j \rangle_{X, X'} = 0,$$

i.e.

$$\left\langle \int_0^T \exp((T-t)A)Bh dt, \psi_j \right\rangle_{X, X'} = - \langle \exp(TA)y_0, \psi_j \rangle_{X, X'}. \quad (3.23)$$

In that case, since  $\{\psi_j\} \subset Y' \subset X'$  and  $\exp((T - \cdot)A)Bh \in L^2(0, T; Y)$  we have

$$\begin{aligned} \left\langle \int_0^T \exp((T-t)A)Bh dt, \psi_j \right\rangle_{X, X'} &= \left\langle \int_0^T \exp((T-t)A)Bh dt, \psi_j \right\rangle_{Y, Y'} \\ &= \int_0^T \langle \exp((T-t)A)Bh, \psi_j \rangle_{Y, Y'} dt, \end{aligned} \quad (3.24)$$

so that

$$\int_0^T \langle \exp((T-t)A)Bh, \psi_j \rangle_{Y, Y'} dt = - \langle \exp(TA)y_0, \psi_j \rangle_{X, X'}, \quad (3.25)$$

i.e.

$$\int_0^T (h, B^* \exp((T-t)A^*)\psi_j)_{\mathbb{C}} dt = - \langle y_0, \exp(TA^*)\psi_j \rangle_{X, X'}, \quad (3.26)$$

applying the spectral decomposition of  $\exp(tA^*)$

$$\overline{B^*\psi_j} \int_0^T h(T-s) \exp(-\lambda_j s) ds = - \exp(-\lambda_j T) \langle y_0, \psi_j \rangle_{X, X'}. \quad (3.27)$$

This motivates writing  $h$  as the formal sum

$$h(t) = \sum_{j \in \mathbb{N}} \frac{-\exp(-\lambda_j T)}{\overline{B^*\psi_j}} \langle y_0, \psi_j \rangle_{X, X'} q_j(T-t), \quad (3.28)$$

where  $\{q_j\}_{j \in \mathbb{N}}$  is a set of functions biorthogonal to  $\{\exp(-\lambda_j t)\}_{j \in \mathbb{N}}$  in  $L^2(0, T)$ .

By Lemma 3.5 such a family  $\{q_j\}_{j \in \mathbb{N}}$  exists. It suffices to prove that (3.28) is absolutely convergent.

Notice that

$$\begin{aligned} \|h\|_{L^2(0, T)} &\leq \sum_{j \in \mathbb{N}} \frac{\exp(-\lambda_j T)}{|\overline{B^*\psi_j}|} |\langle y_0, \psi_j \rangle_{X, X'}| \|q_j\|_{L^2(0, T)} \\ &\leq \left( \sum_{j \in \mathbb{N}} \frac{\exp(-2\lambda_j T)}{|\overline{B^*\psi_j}|^2} \|q_j\|_{L^2(0, T)}^2 \right)^{1/2} \left( \sum_{j \in \mathbb{N}} |\langle y_0, \psi_j \rangle_{X, X'}|^2 \right)^{1/2}, \end{aligned} \quad (3.29)$$

since  $\{\psi_j\}_{j \in \mathbb{N}}$  is a Riesz basis of  $L^2(0, \pi) \times H^{-1}(0, \pi)$ , there exists a constant  $Q$  such that

$$\|\langle y_0, \psi_j \rangle_{X, X'}\|_{\ell^2} \leq Q \|y_0\|_X, \quad (3.30)$$

so that

$$\|h\|_{L^2(0, T)} \leq Q \left( \sum_{j \in \mathbb{N}} \frac{\exp(-2\lambda_j T)}{|\overline{B^*\psi_j}|^2} \|q_j\|_{L^2(0, T)}^2 \right)^{1/2} \|y_0\|_X. \quad (3.31)$$

We seek to bound the series

$$\sum_{j \in \mathbb{N}} \frac{\exp(-2\lambda_j T)}{|\overline{B^*\psi_j}|^2} \|q_j\|_{L^2(0, T)}^2. \quad (3.32)$$

We explicitly compute

$$B^* \Psi_{1,k} = \sqrt{\frac{2}{\pi}} \frac{dk}{dk^2 - k^4}, \quad B^* \Psi_{2,k} = \sqrt{\frac{2}{\pi}} d. \quad (3.33)$$

Now, using the bound found in Lemma 3.5 we find that, letting  $\omega = \delta + \varepsilon + (6 + 2\zeta)$  and recalling that  $0 < T \leq 1$ , we have

$$\begin{aligned} & \sum_{k \in \mathbb{N}} \frac{\exp(-2k^4 T)}{|B^* \Psi_{1,k}|^2} \|q_{1,k^2}\|_{L^2(0,T)}^2 \\ & \leq C \sum_{k \in \mathbb{N}} \exp(-2k^4 T) \exp(2\gamma/T) \exp(2(\delta\sqrt{T} + \varepsilon)k^2) k^{6+2\zeta} \\ & \leq \hat{C} \exp(2\gamma/T) \sum_{k \in \mathbb{N}} \exp(2(\omega k^2 - k^4 T)) \\ & \leq \hat{C} \exp(2\gamma/T) \left[ \sum_{k^2 < \lceil \frac{\omega+1}{T} \rceil} \exp(2(\omega k^2 - k^4 T)) + \sum_{k^2 \geq \lceil \frac{\omega+1}{T} \rceil} (\exp(2(\omega k - k^3 T)))^k \right] \\ & \leq \hat{C} \exp(2\gamma/T) \left[ \left\lceil \sqrt{\frac{\omega+1}{T}} \right\rceil \exp(\omega^2/2T) + \frac{e}{1-e} \right] \\ & \leq \tilde{C} \exp(2(\gamma + \omega^2/4 + 1)/T) \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} & \sum_{k \in \mathbb{N}} \frac{\exp(-2dk^2 T)}{|B^* \Psi_{2,k}|^2} \|q_{2,k}\|_{L^2(0,T)}^2 \\ & \leq C \sum_{k \in \mathbb{N}} \exp(-2dk^2 T) \exp(2\gamma/T) \exp(2(\delta\sqrt{T} + \varepsilon)k) k^{2\zeta} \\ & \leq \hat{C} \exp(2\gamma/T) \sum_{k \in \mathbb{N}} \exp(2(\omega k - dk^2 T)) \\ & \leq \hat{C} \exp(2\gamma/T) \left[ \sum_{k < \lceil \frac{\omega+1}{dT} \rceil} \exp(2(\omega k - dk^2 T)) + \sum_{k \geq \lceil \frac{\omega+1}{dT} \rceil} (\exp(2(\omega - dkT)))^k \right] \\ & \leq \hat{C} \exp(2\gamma/T) \left[ \left\lceil \frac{\omega+1}{dT} \right\rceil \exp(\omega^2/2dT) + \frac{e}{1-e} \right] \\ & \leq \tilde{C} \exp(2(\gamma + \omega^2/4d + 1)/T) \end{aligned} \quad (3.35)$$

plugging (3.34) and (3.35) into (3.31) we get Theorem 3.4 with  $M = \gamma + 1 + \frac{\omega^2}{4} \max\{1, \frac{1}{d}\}$ .  $\square$

## 4 Controllability of the nonlinear system

We will obtain a weak formulation for system (1.2), which, using the notation introduced in Section 3, has to be something like

$$y_t = Ay + Bh - (uu_x, 0).$$

Taking into account that  $uu_x = \left(\frac{u^2}{2}\right)_x$ , formally multiplying the previous equation by a test function  $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  (which is  $\mathbb{R}^2$  valued) and integrating by parts we get

$$\int_0^\pi y_t \cdot \Phi \, dx = \int_0^\pi (Ay + Bh) \cdot \Phi \, dx + \int_0^\pi \left(\frac{u^2}{2}\right) \phi_{1x} \, dx.$$

This motivates us to define  $F : L^2(0, \pi) \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$  by

$$\langle F(u), \phi \rangle_{(H^2(0,\pi) \cap H_0^1(0,\pi))', H^2(0,\pi) \cap H_0^1(0,\pi)} = \int_0^\pi \left(\frac{u^2}{2}\right) \phi_x \, dx,$$

and we would like to define that  $y = (u, v)$  is a solution of (1.2) if it satisfies the system (2.1) with  $f = F(u)$ . In order to do this, we show that  $F(u)$  has enough regularity such that the corresponding mild solution (3.22) is well defined.

Using Hölder inequality and since  $H^1(0, \pi) \hookrightarrow L^\infty(0, \pi)$ , we deduce that, for all  $\phi \in H^2(0, \pi) \cap H_0^1(0, \pi)$ , we have

$$\begin{aligned} \langle F(u), \phi \rangle_{(H^2(0,\pi) \cap H_0^1(0,\pi))', H^2(0,\pi) \cap H_0^1(0,\pi)} &\leq \left\| \frac{u^2}{2} \right\|_{L^1(0,\pi)} \|\phi_x\|_{L^\infty(0,\pi)} \\ &\leq \frac{1}{2} \|u^2\|_{L^1(0,\pi)} C_3 \|\phi\|_{H^2(0,\pi) \cap H_0^1(0,\pi)} \\ &\leq \frac{C_3}{2} \|u\|_{L^2(0,\pi)}^2 \|\phi\|_{H^2(0,\pi) \cap H_0^1(0,\pi)}, \end{aligned}$$

meaning  $\|F(u)\|_{(H^2(0,\pi) \cap H_0^1(0,\pi))'} \leq \frac{C_3}{2} \|u^2\|_{L^1(0,\pi)} = \frac{C_3}{2} \|u\|_{L^2(0,\pi)}^2$  for some  $C_3 > 0$ , by definition of the dual norm.

Moreover, note that

$$\begin{aligned} \|(F(u), 0)\|_{L^2(0,T; (H^2 \cap H_0^1)')}^2 &= \|F(u)\|_{L^2(0,T; (H^2 \cap H_0^1)')}^2 = \int_0^T \|F(u)\|_{(H^2 \cap H_0^1)'}^2 \\ &\leq \frac{C_3^2}{4} \int_0^T \|u\|_{L^2(0,\pi)}^4 \leq \frac{TC_3^2}{4} \|u\|_{C([0,T], L^2(0,\pi))}^4 \\ &\leq \frac{TC_3^2}{4} \|y\|_{C([0,T], L^2(0,\pi) \times H^{-1}(0,\pi))}^4 < \infty. \end{aligned}$$

Therefore  $F(u) \in L^2(0, T; (H^2(0, \pi) \cap H_0^1(0, \pi))')$ , and it makes sense to put  $f = F(u)$  in (2.1). Accordingly, we say that  $y \in C([0, T], L^2(0, \pi) \times H^{-1}(0, \pi))$  is a solution of (1.2) if it satisfies

$$y_t = Ay + Bh + (F(u), 0).$$

In the following result, we show that the local existence of solutions of the nonlinear problem (1.2): for data with small enough norms, there exists a solution.

**Proposition 4.1.** *Let  $T > 0$ . There exists  $R_1 > 0$  such that for all  $(u_0, v_0) \in L^2 \times H^{-1}$  and  $h \in L^2(0, T)$  satisfying*

$$\|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|h\|_{L^2(0,T)}^2 \leq R_1^2, \quad (4.1)$$

*system (1.2) has a solution  $(u, v) \in C([0, T], L^2 \times H^{-1}) \cap L^2(0, T; H^2(0, \pi) \cap H_0^1(0, \pi) \times L^2(0, \pi))$  such that  $u(0, \cdot) = u_0$  and  $v(0, \cdot) = v_0$ .*

*Proof.* Fix  $T > 0$  and let  $(u_0, v_0) \in L^2 \times H^{-1}$  and  $h \in L^2(0, T)$  verifying (4.1) for some  $R_1 > 0$  to be determined. Denoting  $\mathcal{Z} = L^2(0, T; (H^2 \cap H_0^1)')$ , we define

$$\begin{aligned} \mathcal{F} : \mathcal{Z} &\longrightarrow \mathcal{Z}, \\ f &\longmapsto F(u) \end{aligned}$$

where  $(u, v)$  solves (2.1). By definition, the fixed points of  $\mathcal{F}$  correspond exactly to the solutions of (1.2).

From the previous estimations of  $F$ , we obtain

$$\begin{aligned} \|\mathcal{F}(f)\|_{\mathcal{Z}}^2 &= \|F(u)\|_{L^2(0, T; (H^2 \cap H_0^1)')}^2 = \int_0^T \|F(u)\|_{(H^2 \cap H_0^1)'}^2 dt \\ &\leq \frac{C_3^2}{4} \int_0^T \|u(t)\|_{L^2(0, \pi)}^4 dt \leq \frac{TC_3^2}{4} \|u\|_{C([0, T], L^2)}^4 \leq \frac{TC_3^2}{4} \|(u, v)\|_{C([0, T], L^2 \times H^{-1})}^4 \\ &\leq \frac{TC_3^2 C_1^2}{4} \left( \|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|h\|_{L^2(0, T)}^2 + \|f\|_{\mathcal{Z}}^2 \right)^2 < +\infty, \end{aligned} \quad (4.2)$$

and taking  $C_4 = \frac{\sqrt{T}C_1C_3}{2}$ , we get that

$$\|\mathcal{F}(f)\|_{\mathcal{Z}} \leq C_4 \left( \|y_0\|_{L^2 \times H^{-1}}^2 + \|h\|_{L^2(0, T)}^2 + \|f\|_{\mathcal{Z}}^2 \right),$$

and then  $\mathcal{F}$  is well defined.

Moreover, if we define  $R_1 = \frac{1}{4C_4}$ , we have

$$\|\mathcal{F}(f)\|_{\mathcal{Z}} \leq 2C_4 R_1^2 \leq \frac{R_1}{2} \leq R_1 \quad \forall f \in B(0, R_1) \subset \mathcal{Z}.$$

Therefore,  $\mathcal{F}$  maps  $B(0, R_1) \subset \mathcal{Z}$  into itself.

To conclude, we will show that  $\mathcal{F}$  is a contraction on  $B(0, R_1)$  and use the Banach fixed point theorem. To this end, take  $f^{(1)}, f^{(2)} \in B(0, R_1)$  and consider  $y^{(1)} = (u^{(1)}, v^{(1)})$ ,  $y^{(2)} = (u^{(2)}, v^{(2)})$  the respective solutions.

A straightforward computation yields

$$\begin{aligned} \|\mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)})\|_{\mathcal{Z}}^2 &= \|F(u^{(1)}) - F(u^{(2)})\|_{L^2(0, T; (H^2 \cap H_0^1)')}^2 \\ &= \int_0^T \|F(u^{(1)}) - F(u^{(2)})\|_{(H^2 \cap H_0^1)'}^2 dt \leq \frac{C_3^2}{4} \int_0^T \|(u^{(1)})^2 - (u^{(2)})^2\|_{L^1(0, \pi)}^2 dt \\ &\leq \frac{C_3^2}{4} \int_0^T \|u^{(1)} + u^{(2)}\|_{L^2(0, \pi)}^2 \|u^{(1)} - u^{(2)}\|_{L^2(0, \pi)}^2 dt. \end{aligned}$$

Whence, for  $f^{(1)}, f^{(2)}$  such that  $\|f^{(1)}\|_{\mathcal{Z}}, \|f^{(2)}\|_{\mathcal{Z}} \leq R_1$ , we get

$$\begin{aligned} &\|\mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)})\|_{\mathcal{Z}}^2 \\ &\leq \frac{TC_3^2}{4} \|u^{(1)} + u^{(2)}\|_{C([0, T], L^2)}^2 \|u^{(1)} - u^{(2)}\|_{C([0, T], L^2)}^2 \\ &\leq \frac{TC_3^2 C_2^2}{4} \left( \|2y_0\|_{L^2 \times H^{-1}}^2 + \|2h\|_{L^2(0, T)}^2 + \|f^{(1)} + f^{(2)}\|_{\mathcal{Z}}^2 \right) \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}}^2 \\ &\leq 8C_4^2 R_1^2 \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}}^2, \end{aligned}$$

where we have taken into account the smallness of the initial condition, the control and the source terms.

From the above expression and using  $R_1 = \frac{1}{4C_4}$ , we get

$$\left\| \mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)}) \right\|_{\mathcal{Z}} \leq 2\sqrt{2}C_4R_1 \left\| f^{(1)} - f^{(2)} \right\|_{\mathcal{Z}} = \frac{\sqrt{2}}{2} \left\| f^{(1)} - f^{(2)} \right\|_{\mathcal{Z}},$$

and the existence of a unique fixed point of  $\mathcal{F}$  in  $B(0, R_1)$  follows from the Banach fixed point theorem.  $\square$

The following result gives us the global uniqueness of the nonlinear problem.

**Proposition 4.2.** *Let  $y^{(1)} \in C([0, T^{(1)}], L^2 \times H^{-1})$  and  $y^{(2)} \in C([0, T^{(2)}], L^2 \times H^{-1})$  be solutions of (1.2) with the same data  $(u_0, v_0)$  and  $h$ . Let  $\tilde{T} = \min\{T^{(1)}, T^{(2)}\}$ . Then  $y^{(1)}|_{[0, \tilde{T}]} = y^{(2)}|_{[0, \tilde{T}]}$ .*

*Proof.* Assume that  $y^{(1)}|_{[0, \tilde{T}]} \neq y^{(2)}|_{[0, \tilde{T}]}$ , we may define  $T^{(0)} = \inf\{t \in [0, \tilde{T}] : y^{(1)}(t) \neq y^{(2)}(t)\}$ , we then have  $y^{(1)}(T^{(0)}) = y^{(2)}(T^{(0)})$  and  $T^{(0)} < \tilde{T}$ , let  $0 < \epsilon < \tilde{T} - T^{(0)}$ .

Due to Proposition 2.1, for  $t \in [T^{(0)}, T^{(0)} + \epsilon]$  we have

$$\begin{aligned} \left\| y^{(1)}(t) - y^{(2)}(t) \right\|_{L^2 \times H^{-1}}^2 &\leq C_1 \left\| F(u^{(1)}) - F(u^{(2)}) \right\|_{L^2(T^{(0)}, t; (H^2 \cap H_0^1)')}^2 \\ &\leq \frac{C_1 C_3^2}{2} \int_{T^{(0)}}^t \left\| (u^{(1)}(s))^2 - (u^{(2)}(s))^2 \right\|_{L^1(0, \pi)}^2 ds \\ &\leq \frac{C_1 C_3^2}{2} \int_{T^{(0)}}^t \left\| u^{(1)}(s) + u^{(2)}(s) \right\|_{L^2(0, \pi)}^2 \left\| u^{(1)}(s) - u^{(2)}(s) \right\|_{L^2(0, \pi)}^2 ds \\ &\leq \frac{C_1 C_3^2}{2} \left\| u^{(1)} + u^{(2)} \right\|_{C([T^{(0)}, T^{(0)} + \epsilon], L^2)}^2 \int_{T^{(0)}}^t \left\| u^{(1)}(s) - u^{(2)}(s) \right\|_{L^2(0, \pi)}^2 ds, \\ &\leq \frac{C_1 C_3^2}{2} \left\| u^{(1)} + u^{(2)} \right\|_{C([T^{(0)}, T^{(0)} + \epsilon], L^2)}^2 \int_{T^{(0)}}^t \left\| y^{(1)}(s) - y^{(2)}(s) \right\|_{L^2 \times H^{-1}}^2 ds, \end{aligned}$$

so that the Grönwall Lemma implies  $y^{(1)}(t) = y^{(2)}(t)$  on  $[T^{(0)}, T^{(0)} + \epsilon]$ , which contradicts the definition of  $T^{(0)}$ .  $\square$

**Remark 4.3.** *As a consequence of Propositions 4.1 and 4.2, we deduce the local well-posedness of the nonlinear system (1.2): for data with small enough norm, there exists a unique solution. Hence, the property of local null-controllability given in Definition 1.1 is a-priori well defined, in the sense that, given an initial condition and a control (with small enough norms), there is only one solution that needs to vanish at  $t = T$ . Although the technique of the proof of Proposition 4.1 is quite similar to that of the proof of Theorem 4.4, where we will prove the existence of both a null-control  $h$  and a controlled solution, we include both results and their proofs for the sake of completeness.*

**Theorem 4.4.** *Let  $0 < T \leq 1$ . Suppose (3.1) is null-controllable in time  $\tau$  for all  $\tau > 0$ , and also suppose that there are uniform constants (w.r.t.  $\tau$ )  $C_5, M > 0$  such that the control cost is bounded by  $K(\tau) = C_5 \exp(M/\tau)$  for all  $\tau \in (0, 1]$ . Then (1.2) is locally null-controllable at time  $T$ .*

*Proof.* We consider  $\rho_0$  and  $\rho_1$ , as in [18], given by

$$\begin{aligned} \rho_1(t) &= \exp\left(-\frac{\alpha}{(T-t)^2}\right), \\ \rho_0(t) &= C_5 \exp\left(-\frac{\alpha}{q^4(T-t)^2} + \frac{M}{(q-1)(T-t)}\right), \end{aligned} \tag{4.3}$$

where  $K(T) = C_5 \exp(M/T)$  is an upper bound for the control cost,

$$1 < q < 2^{1/4}, \quad \text{and} \quad \alpha > \frac{2MTq^4}{(q-1)(2-q^4)}.$$

Then we may directly check that (4.3) define continuous and nonincreasing functions, which are positive in  $[0, T)$ , vanish at  $t = T$ , and verify

$$\frac{1}{\rho_1} \leq \frac{C_5^2}{\rho_0^2}, \quad (4.4)$$

and for  $t \in \left[ \left(1 - \frac{1}{q^2}\right)T, T \right]$ , we have

$$\rho_0(t) = \rho_1(q^2t + (1 - q^2)T)K((q-1)(T-t)).$$

In this way, the weights (4.3) verify the hypotheses in Proposition 2.2.

We now prove that if the initial data  $(u_0, v_0)$  is sufficiently small, there exists a control  $h$  and a solution  $(u, v)$  of (1.2) such that  $(u, v)(T, \cdot) = 0$ . Denoting  $\mathcal{Z}_{\rho_1} = L_{\rho_1}^2(0, T; (H^2 \cap H_0^1)')$ , and given  $y_0 \in L^2 \times H^{-1}$ , we define

$$\begin{aligned} \mathcal{F}_{\rho_1} : \mathcal{Z}_{\rho_1} &\longrightarrow \mathcal{Z}_{\rho_1}, \\ f &\mapsto F(u) \end{aligned}$$

where  $(u, v)$  solves (2.1) with  $h$  given by Proposition 2.2. Clearly, in view of Proposition 2.2, finding a fixed point of the map  $\mathcal{F}_{\rho_1}$  will gives us a controlled solution to (1.2) which in addition satisfies  $(u, v)(T, \cdot) = 0$ .

We begin by showing that  $\mathcal{F}_{\rho_1}$  is well defined.

$$\begin{aligned} \|\mathcal{F}_{\rho_1}(f)\|_{\mathcal{Z}_{\rho_1}}^2 &= \|F(u)\|_{L_{\rho_1}^2(0, T; (H^2 \cap H_0^1)')}^2 = \int_0^T \frac{\|F(u)\|_{(H^2 \cap H_0^1)'}^2}{\rho_1(t)^2} dt \\ &\leq \frac{C_5^4 C_3^2}{4} \int_0^T \frac{\|u(t)\|_{L^2(0, \pi)}^4}{\rho_0(t)^4} dt \leq \frac{TC_5^4 C_3^2}{4} \left\| \frac{u}{\rho_0} \right\|_{C([0, T], L^2)}^4 \leq \frac{TC_5^4 C_3^2}{4} \left\| \frac{(u, v)}{\rho_0} \right\|_{C([0, T], L^2 \times H^{-1})}^4 \\ &\leq \frac{TC_5^4 C_3^2 C_2^2}{4} \left( \|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|f\|_{\mathcal{Z}_{\rho_1}}^2 \right)^2 < +\infty, \end{aligned} \quad (4.5)$$

where  $C_2 > 0$  is the constant coming from Proposition 2.2.

Taking  $C_6 = \frac{\sqrt{TC_5^2 C_2 C_3}}{2}$ , we get that

$$\|\mathcal{F}_{\rho_1}(f)\|_{\mathcal{Z}_{\rho_1}} \leq C_6 \left( \|y_0\|_{L^2 \times H^{-1}}^2 + \|f\|_{\mathcal{Z}_{\rho_1}}^2 \right).$$

We define  $R_2 = \min\left\{ \frac{1}{4C_6}, \frac{R_1}{\sqrt{2C_2\rho_0(0)^2 + 1}} \right\}$ . For any  $y_0$  such that  $\|y_0\|_{L^2 \times H^{-1}} \leq R_2$  we have

$$\|\mathcal{F}_{\rho_1}(f)\|_{\mathcal{Z}_{\rho_1}} \leq 2C_6 R_2^2 \leq \frac{R_2}{2} \leq R_2 \quad \forall f \in B(0, R_2) \subset \mathcal{Z}_{\rho_1}.$$

Therefore,  $\mathcal{F}_{\rho_1}$  maps  $B(0, R_2) \subset \mathcal{Z}_{\rho_1}$  into itself.

To conclude, we will show that  $\mathcal{F}_{\rho_1}$  is a contraction on  $B(0, R_2)$  and use the Banach fixed point theorem. To this end, take  $f^{(1)}, f^{(2)} \in B(0, R_2)$  and consider  $y^{(1)} = (u^{(1)}, v^{(1)})$ ,  $y^{(2)} = (u^{(2)}, v^{(2)})$  and  $h^{(1)}, h^{(2)}$  the respective solutions and controls given by Proposition 2.2.

Since  $y$  and  $h$  depend linearly and continuously on the pair  $(y_0, f)$ ,  $y^{(1)} - y^{(2)}$  and  $h^{(1)} - h^{(2)}$  are the solution and control associated with  $f = f^{(1)} - f^{(2)}$  and  $y_0 = 0$ . Using property (4.4), straightforward computations yields

$$\begin{aligned} \left\| \mathcal{F}_{\rho_1}(f^{(1)}) - \mathcal{F}_{\rho_1}(f^{(2)}) \right\|_{\mathcal{Z}_{\rho_1}}^2 &= \left\| F(u^{(1)}) - F(u^{(2)}) \right\|_{L_{\rho_1}^2(0,T;(H^2 \cap H_0^1)')}^2 \\ &= \int_0^T \frac{\|F(u^{(1)}) - F(u^{(2)})\|_{(H^2 \cap H_0^1)'}^2}{\rho_1(t)^2} dt \leq \frac{C_3^2}{4} \int_0^T \frac{\|(u^{(1)})^2 - (u^{(2)})^2\|_{L^1(0,\pi)}^2}{\rho_1(t)^2} dt \\ &\leq \frac{C_5^4 C_3^2}{4} \int_0^T \frac{\|u^{(1)} + u^{(2)}\|_{L^2(0,\pi)}^2 \|u^{(1)} - u^{(2)}\|_{L^2(0,\pi)}^2}{\rho_0(t)^4} dt. \end{aligned}$$

Whence, for  $f^{(1)}, f^{(2)}$  such that  $\|f^{(1)}\|_{\mathcal{Z}_{\rho_1}}, \|f^{(2)}\|_{\mathcal{Z}_{\rho_1}} \leq R$ ,

$$\begin{aligned} &\left\| \mathcal{F}_{\rho_1}(f^{(1)}) - \mathcal{F}_{\rho_1}(f^{(2)}) \right\|_{\mathcal{Z}_{\rho_1}}^2 \\ &\leq \frac{TC_5^4 C_3^2}{4} \left\| \frac{u^{(1)} + u^{(2)}}{\rho_0} \right\|_{C([0,T],L^2)}^2 \left\| \frac{u^{(1)} - u^{(2)}}{\rho_0} \right\|_{C([0,T],L^2)}^2 \\ &\leq \frac{TC_5^4 C_3^2 C_2^2}{4} \left( \|2y_0\|_{L^2 \times H^{-1}}^2 + \|f^{(1)} + f^{(2)}\|_{\mathcal{Z}_{\rho_1}}^2 \right) \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}_{\rho_1}}^2 \\ &\leq 8C_6^2 R_2^2 \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}_{\rho_1}}^2, \end{aligned}$$

where we have taken into account the smallness of the initial datum and the source terms. From the above expression and using  $R_2 \leq \frac{1}{4C_6}$ , we get

$$\left\| \mathcal{F}_{\rho_1}(f^{(1)}) - \mathcal{F}_{\rho_1}(f^{(2)}) \right\|_{\mathcal{Z}_{\rho_1}} \leq 2\sqrt{2}C_6 R_2 \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}_{\rho_1}} = \frac{\sqrt{2}}{2} \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}_{\rho_1}},$$

and the existence of a unique fixed point  $f$  of  $\mathcal{F}_{\rho_1}$  in  $B(0, R_2)$  follows from the Banach fixed point theorem.

Notice that Proposition 2.2 implies

$$\|h\|_{L_{\rho_0}^2(0,T)}^2 \leq C_2 \left( \|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|f\|_{\mathcal{Z}_{\rho_1}}^2 \right)$$

so that  $R_2 \leq \frac{R_1}{\sqrt{1 + 2C_2\rho_0(0)^2}}$  yields

$$\|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|h\|_{L^2(0,T)}^2 \leq \|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \rho_0(0)^2 \|h\|_{L_{\rho_0}^2(0,T)}^2 \leq (1 + 2C_2\rho_0(0)^2)R_2^2 \leq R_1,$$

so that  $(u_0, v_0)$  and  $h$  satisfy the hypotheses of Proposition 4.1.  $\square$

Once we have established Theorem 4.4, the proof of Theorem 1.4 for  $T \leq 1$  is a direct application of such result and the bound for the control cost obtained in Theorem 3.4. For  $T > 1$ , we can write  $[0, T] = [0, 1] \cup [1, T]$ . For the first time interval, we control as in the previous case obtaining that  $u(\cdot, 1) = v(\cdot, 1) = 0$ . Then, setting  $h(t) \equiv 0$  for  $[1, T]$ , we extend the solution  $(u, v)$  by zero, thus yielding the desired control result.



## A A well-posedness result

Here, we give a proof of Proposition 2.1.

*Proof.* Since  $(-\partial_{xx})^2 : H^2(0, \pi) \cap H_0^1(0, \pi) \subset (H^2(0, \pi) \cap H_0^1(0, \pi))' \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$  and  $-d\partial_{xx} : L^2(0, \pi) \subset (H^2(0, \pi) \cap H_0^1(0, \pi))' \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$  are strictly positive self-adjoint operators, we have the following two results due to [4, Thm. 3.1, pp. 143].

If  $f \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')$  and  $u_1 \in L^2(0, \pi)$ , then

$$\begin{cases} u_t(t, x) + u_{xxxx}(t, x) = f, & t \in (t_1, t_2), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (t_1, t_2), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (t_1, t_2), \end{cases} \quad (\text{A.1})$$

admits a unique solution  $u \in C([t_1, t_2], L^2(0, \pi)) \cap L^2(t_1, t_2; H^2(0, \pi) \cap H_0^1(0, \pi))$  satisfying  $u(t_1, \cdot) = u_1$ , and we have

$$\|u\|_{L^2(t_1, t_2; H^2 \cap H_0^1)}^2 + \|u\|_{C([t_1, t_2], L^2)}^2 \leq K_1 \left( \|u_1\|_{L^2}^2 + \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right), \quad (\text{A.2})$$

for some  $K_1 > 0$  independent of  $t_1, t_2$ .

The second result tells that if  $h \in L^2(t_1, t_2)$  and  $v_1 \in H^{-1}(0, \pi)$ , then

$$\begin{cases} v_t(t, x) - dv_{xx}(t, x) = 0, & t \in (t_1, t_2), x \in (0, \pi), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (t_1, t_2), \end{cases} \quad (\text{A.3})$$

admits a unique solution  $v \in C([t_1, t_2], H^{-1}(0, \pi)) \cap L^2(t_1, t_2; L^2(0, \pi))$  satisfying  $v(t_1, \cdot) = v_1$ . Moreover, we have the following energy estimate

$$\|v\|_{L^2(t_1, t_2; L^2)}^2 + \|v\|_{C([t_1, t_2], H^{-1})}^2 \leq K_2 \left( \|v_1\|_{H^{-1}}^2 + \|h\|_{L^2(t_1, t_2)}^2 \right),$$

with  $K_2 > 0$  independent of  $t_1, t_2$ .

We will now exploit the cascade structure of the system. Notice that if  $v \in L^2(t_1, t_2; L^2(0, \pi))$ , then in particular  $v \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')$  and

$$\|v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \leq \|v\|_{L^2(t_1, t_2; L^2)}^2.$$

We take  $v$  to be the solution of (A.3), and  $u$  to be the solution of (A.1) with right hand side  $f + v$ . Since

$$\|v + f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \leq 2 \left( \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 + \|v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right),$$

we get

$$\|v\|_{L^2(t_1, t_2; L^2)}^2 + \|v\|_{C([t_1, t_2], H^{-1})}^2 \leq K_2 \left( \|h\|_{L^2(t_1, t_2)}^2 + \|v_1\|_{H^{-1}}^2 \right), \quad (\text{A.4})$$

and

$$\begin{aligned} \|u\|_{L^2(t_1, t_2; H^2 \cap H_0^1)}^2 + \|u\|_{C([t_1, t_2], L^2)}^2 &\leq K_1 \left( \|u_1\|_{L^2}^2 + \|f + v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right) \\ &\leq 2K_1 \left( \|u_1\|_{L^2}^2 + \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right) \\ &\quad + 2K_2K_1 \left( \|h\|_{L^2(t_1, t_2)}^2 + \|v_1\|_{H^{-1}}^2 \right), \end{aligned} \quad (\text{A.5})$$

where we have used (A.4) in the last line. Writing  $y = (u, v)$  and  $y_1 = (u_1, v_1)$ , we can combine (A.4) and (A.5) and take  $C_1 = \max\{K_2, (2K_1 + 1)K_2\}$  to deduce

$$\begin{aligned} & \|y\|_{L^2(t_1, t_2; (H^2 \cap H_0^1) \times L^2)}^2 + \|y\|_{C([t_1, t_2], L^2 \times H^{-1})}^2 \\ & \leq C \left( \|h\|_{L^2(t_1, t_2)}^2 + \|y_1\|_{L^2 \times H^{-1}}^2 + \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right). \end{aligned}$$

This concludes the proof.  $\square$

## B Sketch of the proof of Proposition 2.2

We define  $T_k = T \left( 1 - \frac{1}{q^k} \right)$ . Notice that with this

$$\rho_0(T_{k+2}) = \rho_1(T_k)K(T_{k+2} - T_{k+1}).$$

Consider the notation  $y = (u, v)$ . We recursively define  $y_{k+1}$  as  $z(T_{k+1})$ , where  $z = (z_1, z_2)$  solves (2.1) on  $(T_k, T_{k+1})$  with initial condition  $z(T_k) = 0$  and control  $h = 0$ . By Proposition 2.1 we have

$$\|z\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1) \times L^2)}^2 + \|z\|_{C([T_k, T_{k+1}], L^2 \times H^{-1})}^2 \leq C_1 \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2,$$

with this

$$\|y_{k+1}\|_{L^2 \times H^{-1}}^2 = \|z(T_{k+1})\|_{L^2 \times H^{-1}}^2 \leq \|z\|_{C([T_k, T_{k+1}], L^2 \times H^{-1})}^2 \leq C_1 \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2.$$

Define  $h$  on  $(T_k, T_{k+1})$  as the minimal norm control that drives (2.1) to zero at time  $T_{k+1}$  with initial condition  $y_k$  at time  $T_k$  and source term  $f \equiv 0$ . We denote the corresponding solution by  $w$ . Since we have assumed that system (2.1) with  $f \equiv 0$  is null-controllable with control cost  $K(\cdot)$ , we have, because of Remark 3.3,

$$\|h\|_{L^2(T_k, T_{k+1})}^2 \leq K^2(T_{k+1} - T_k) \|y_k\|_{L^2 \times H^{-1}}^2$$

and therefore

$$\|h\|_{L^2(T_{k+1}, T_{k+2})}^2 \leq K^2(T_{k+2} - T_{k+1}) \|y_{k+1}\|_{L^2 \times H^{-1}}^2 \leq C_1 K^2(T_{k+2} - T_{k+1}) \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2.$$

We define  $y = z + w$ . Since  $z(T_k^-) + w(T_k^-) = z(T_k^+) + w(T_k^+)$ ,  $y$  is continuous at  $T_k$  for every  $k \geq 0$ . With this definition, it is also clear that  $y$  solves (2.1) for  $t \in [0, T)$ .

Notice that, due to the fact that  $\rho_0$  and  $\rho_1$  are nonincreasing

$$\begin{aligned} \|h\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2})}^2 & \leq \frac{1}{\rho_0^2(T_{k+2})} \|h\|_{L^2(T_{k+1}, T_{k+2})}^2 \leq C_1 \frac{K^2(T_{k+2} - T_{k+1})}{\rho_0^2(T_{k+2})} \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \\ & \leq \frac{C_1}{\rho_1^2(T_k)} \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \leq C_1 \|f\|_{L_{\rho_1}^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2, \end{aligned} \quad (\text{B.1})$$

and

$$\|h\|_{L_{\rho_0}^2(0, T_1)}^2 \leq \frac{1}{\rho_0^2(T_1)} \|h\|_{L^2(0, T_1)}^2 \leq \frac{K^2(T_1)}{\rho_0^2(T_1)} \|y_0\|_{L^2 \times H^{-1}}^2. \quad (\text{B.2})$$

Proposition 2.1 yields

$$\begin{aligned}
& \|y\|_{L^2_{\rho_0}(T_{k+1}, T_{k+2}; (H^2 \cap H_0^1) \times L^2)}^2 + \left\| \frac{y}{\rho_0} \right\|_{C([T_{k+1}, T_{k+2}], L^2 \times H^{-1})}^2 \\
& \leq C_1^2 \left( 1 + \frac{1}{K^2(T_{k+2} - T_{k+1})} \right) \|f\|_{L^2_{\rho_1}(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \\
& \quad + \frac{C_1}{K^2(T_{k+2} - T_{k+1})} \|f\|_{L^2_{\rho_1}(T_{k+1}, T_{k+2}; (H^2 \cap H_0^1)')}^2
\end{aligned} \tag{B.3}$$

and also

$$\begin{aligned}
& \|y\|_{L^2_{\rho_0}(0, T_1; (H^2 \cap H_0^1) \times L^2)}^2 + \left\| \frac{y}{\rho_0} \right\|_{C([0, T_1], L^2 \times H^{-1})}^2 \\
& \leq \frac{C_1(K^2(T_1) + 1)}{\rho_0^2(T_1)} \|y_0\|_{L^2 \times H^{-1}}^2 + \frac{C_1 \rho_1^2(0)}{\rho_0^2(T_1)} \|f\|_{L^2_{\rho_1}(0, T_1; (H^2 \cap H_0^1)')}^2.
\end{aligned} \tag{B.4}$$

It follows from Proposition 2.1 that  $y$  is continuous on  $[0, T]$ , with values in  $L^2 \times H^{-1}$ . Because  $\rho_0$  is positive on  $[0, T)$ ,  $\frac{y}{\rho_0}$  is also continuous on  $[0, T)$ . Since the right hand side of (B.3) converges to 0 as  $k \rightarrow \infty$ , it follows that  $\frac{y(t)}{\rho_0} \rightarrow 0$  in  $L^2 \times H^{-1}$  as  $t \rightarrow T^-$ . With this, we may extend  $\frac{y}{\rho_0}$  to  $[0, T]$  and it remains continuous, in particular, this implies that  $(u(T), v(T)) = 0$ . This tells us that the control  $h$  constructed drives the state to zero, and also gives a bound on how quickly the convergence takes place, namely, faster than  $\rho_0$  approaches 0.

Combining estimates (B.1)–(B.2) with (B.3)–(B.4) we can deduce the existence of a constant  $C_2 > 0$  such that (2.2) holds. This ends the proof.  $\square$

## C Liouville-Roth constant of the reciprocal of a number

**Lemma C.1.** *Let  $x > 0$ . Then  $x$  and  $1/x$  have the same Liouville-Roth constant.*

*Proof.* Assume without loss of generality that the Liouville-Roth constant of  $1/x$  is strictly greater than that of  $x$  (this implies  $x$  has finite Liouville-Roth constant). Fix  $r$  strictly in between both Liouville-Roth constants and  $\varepsilon > 0$  such that  $r + \varepsilon$  is also in this interval. By definition, there exist infinitely many pairs  $(p, q) \in \mathbb{Z}^2$  with  $q > 0$  such that

$$0 < \left| \frac{1}{x} - \frac{p}{q} \right| < \frac{1}{q^{r+\varepsilon}},$$

so we may construct a sequence  $(p_n, q_n)$  of distinct such pairs. Since for a given  $q_n$ , only finitely many values of  $p_n$  can satisfy the equation, we know by the pigeonhole principle that  $q_n \rightarrow \infty$  as  $n \rightarrow \infty$ . With this,  $\frac{p_n}{q_n} \rightarrow \frac{1}{x}$ , which implies  $p_n \rightarrow \infty$  and  $\frac{q_n}{p_n} \rightarrow x$ . Notice that

$$0 < \left| x - \frac{q_n}{p_n} \right| p_n^r = \frac{x q_n}{p_n} \left| \frac{p_n}{q_n} - \frac{1}{x} \right| q_n^{r+\varepsilon} \left( \frac{p_n}{q_n} \right)^{r+\varepsilon} \frac{1}{p_n^\varepsilon} \rightarrow 0,$$

since  $\frac{x q_n}{p_n} \left( \frac{p_n}{q_n} \right)^{r+\varepsilon} \rightarrow x^{2-r-\varepsilon}$ ,  $\left| \frac{p_n}{q_n} - \frac{1}{x} \right| q_n^{r+\varepsilon} < 1$  and  $\frac{1}{p_n^\varepsilon} \rightarrow 0$ . It follows that there are infinitely many pairs  $(p, q)$  such that

$$0 < \left| x - \frac{q}{p} \right| < \frac{1}{p^r},$$

which contradicts the fact that  $r$  is greater than the Liouville-Roth constant of  $x$ .  $\square$

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