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Boundary controllability of a simplified stabilized Kuramoto-Sivashinsky system**

Víctor Hernández-Santamaría* Alberto Mercado† Piero Visconti‡

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Abstract

In this paper, we study the controllability of a nonlinear system of coupled second- and fourth-order parabolic equations. This system can be regarded as a simplification of the well-known stabilized Kuramoto-Sivashinsky system. Using only one control applied on the boundary of the second-order equation, we prove that the local-null controllability of the system holds if the diffusion coefficient of the second-order equation is a quadratic irrational number.

Keywords: parabolic system, boundary controllability, source term method.

AMS subject classifications: 93B05, 93C20, 35K41.

1 Introduction

The study of controllability of systems of coupled partial differential equations has deserved a lot of attention in the literature of the past recent years. For systems of second order equations we refer the reader to the classical work [1, 2, 3], and the references within. For systems involving fourth- and second-order differential equations, we refer to [8], [9], [6], and [11]. Particularly, in [7], the authors study controllability properties of a linear cascade system coupling a bilaplacian operator to a heat equation with a single boundary control, obtaining positive and negative controllability results depending of the diffusion coefficient.

The interest in the study of systems involving fourth order parabolic equations comes from models of front propagation in reaction-diffusion phenomena. Indeed, in [14] a system was proposed as a model for such phenomena with both dissipative and dispersive features and allowing a stable solitary-pulse. The system consists in a one-dimensional Kuramoto-Sivashinsky-KdV (KS-KdV) equation, linearly coupled to an extra dissipative equation, under the name of the stabilized Kuramoto-Sivashinsky system, and given by

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + au_{xx} + uu_x = v_x, \\ v_t - dv_{xx} + cv_x = u_x, \end{cases} \quad (1.1)$$

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where $\gamma > 0$ accounts for the long-wave instability, a is the short-wave dissipation, $d > 0$ is the dissipative parameter and $c \in \mathbb{R}$ is the group-velocity mismatch between wave modes. Notice that the coupling is through first order terms, which is harder to deal with than zero order couplings.

The objective of this work is to perform one step in the direction of studying the boundary controllability of system (1.1), by considering the controllability of the non-linear version of the system studied in [7]. That is, we consider the problem given by

$$\begin{cases} u_t + u_{xxxx} + uu_x = v, & t \in (0, T), x \in (0, \pi), \\ v_t - dv_{xx} = 0, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T), \end{cases} \quad (1.2)$$

where $u = u(t, x)$ and $v = v(t, x)$ are the state variables and $h = h(t)$ is the control. Observe that the control acts only on the heat equation and influences indirectly the first equation by means of the coupling. The parameter $d > 0$ is the diffusion of the heat equation and will play a crucial role in the analysis of controllability properties for the system.

The controllability problem we are interested can be formulated as follows.

Definition 1.1. *We say system (1.2) is locally null-controllable at time T if there exists $R > 0$ such that for any state $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ satisfying*

$$\|(u_0, v_0)\|_{L^2(0, \pi) \times H^{-1}(0, \pi)} \leq R,$$

there exists a control $h \in L^2(0, T)$ such that the solution of (1.2) with initial condition

$$u(0, \cdot) = u_0 \quad \text{and} \quad v(0, \cdot) = v_0$$

satisfies

$$u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0.$$

In [7], the global null-controllability of the linear version of system (1.2) (see (3.1) below) is proved, depending on the diffusion coefficient $d > 0$. More precisely, if \sqrt{d} is not a Liouville number (see [5]), then the system (3.1) is null-controllable at time T for any $T > 0$ and any initial datum $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$.

In this spirit, the main result of this work is the following.

Theorem 1.2. *If \sqrt{d} is a quadratic irrational number, then system (1.2) is locally null controllable.*

Remark 1.3. *We recall that $x \in \mathbb{R}$ is a quadratic irrational number if it is irrational and there exists a quadratic polynomial with rational coefficients q such that $q(x) = 0$. In particular, these numbers can be written as $q_1 + q_2\sqrt{c}$ with q_1, q_2 rationals and $c > 0$ and they are a proper subset of the algebraic numbers.*

The proof of Theorem 1.2 follows from a precise estimate of the cost of the control of the linear system (3.1) and an application of the so called source term method, introduced by Y. Liu, T. Takahashi and M. Tucsnak [13]. It is important to mention that we are able to prove such precise estimate only in the case where \sqrt{d} is a quadratic irrational number. Even though a more general result is known in the linear case, where the global controllability of the linear system (3.1) holds for any irrational number with finite Liouville-Roth constant (see [7, Theorem 1.5] and Section 3 below), here we use strongly the fact that \sqrt{d} is a quadratic irrational number in order to deduce the required cost of the control.

Remark 1.4. *As in other works relying on the source term method for deducing controllability properties of nonlinear systems (see e.g., [12, 10] for recent works), the a priori well-posedness of the nonlinear system (1.2) is not a prerequisite to study the problem. In fact, while proving Theorem 1.2, the existence and uniqueness of the controlled solution will be a byproduct of the methodology.*

The rest of the paper is organized as follows. In Section 2 we establish some basic results of well-posedness and we apply the source term method to our system. In Section 3 we obtain adequate bounds for the cost of the control of the linear system under correspondent hypothesis of the diffusion coefficient. Finally, in Section 4 we prove Theorem 1.2.

2 The source term method

In [13], the authors introduced the so-called source term method, a quite general methodology to deduce local controllability results for semilinear problems based on the controllability properties of the underlying linear system perturbed by an external force with some decay properties.

By adapting this methodology to our case, we are reduced to study the controllability of the linear system

$$\begin{cases} u_t + u_{xxxx} = v + f_1, & t \in (0, T), x \in (0, \pi), \\ v_t - dv_{xx} = f_2, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T), \end{cases} \quad (2.1)$$

where $f = (f_1, f_2)$ is a suitable source term.

We begin by stating a general well-posedness result for system (2.1).

Proposition 2.1. *Let $t_1 < t_2$ be any arbitrary positive times. If $f \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')^2$, $h \in L^2(t_1, t_2)$ and $(u_1, v_1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$, then (2.1) admits a unique solution $(u, v) \in C([t_1, t_2], L^2(0, \pi) \times H^{-1}(0, \pi)) \cap L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi)) \times L^2(0, \pi))$ satisfying $(u, v)(t_1, \cdot) = (u_1, v_1)$. Moreover, we have the following bound*

$$\begin{aligned} \|(u, v)\|_{L^2(t_1, t_2; (H^2 \cap H_0^1) \times L^2)}^2 + \|(u, v)\|_{C([t_1, t_2], L^2 \times H^{-1})}^2 \\ \leq C \left(\|h\|_{L^2(t_1, t_2)}^2 + \|(u_1, v_1)\|_{L^2 \times H^{-1}}^2 + \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')^2}^2 \right), \end{aligned}$$

where C does not depend on t_1, t_2 .

The proof of this result is classical and relies on well-known arguments. For the sake of completeness, we present it in Appendix A.

Now, we recall the method introduced in [13] that will allow us to deal with the controllability of the nonlinear system (1.2). We consider a function ρ verifying the following properties

$$\rho : [0, T] \rightarrow [0, +\infty) \text{ continuous, non-increasing, positive on } [0, T), \text{ with } \rho(T) = 0.$$

We denote by $L_\rho^2(0, T; X)$ the weighted L^2 space with values on X and with measure m_T/ρ^2 , where m_T denotes the Lebesgue measure of the interval $(0, T)$. In the spirit of [13], we have the following controllability result for (2.1) in weighted spaces.

Proposition 2.2. *Assume that system (2.1) with $f \equiv 0$ is null controllable in any time τ , with control cost $K(\tau)$, which is a continuous and non increasing function of τ . Suppose that $q > 1$ and $\rho_1 : [0, T] \rightarrow \mathbb{R}$ is a continuous, non-increasing function with $\rho_1(T) = 0$ and such that*

$$\rho_0(t) := \rho_1(q^2t + (1 - q^2)T)K((q - 1)(T - t)), \quad \left(t \in \left[\left(1 - \frac{1}{q^2}\right)T, T \right] \right),$$

is non-increasing and verifies $\rho_0(T) = 0$.

Then, for each $f \in L^2_{\rho_1}(0, T; (H^2 \cap H^1_0)')^2$ and any initial condition $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$, there exists a control $h \in L^2_{\rho_0}(0, T)$ such that the solution (u, v) to (2.1) satisfies

$$\begin{aligned} \|(u, v)\|_{L^2_{\rho_0}(0, T; (H^2 \cap H^1_0) \times L^2)}^2 + \left\| \frac{(u, v)}{\rho_0} \right\|_{C([0, T], L^2 \times H^{-1})}^2 + \|h\|_{L^2_{\rho_0}(0, T)}^2 \\ \leq C_2 \left(\|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|f\|_{L^2_{\rho_1}(0, T; (H^2 \cap H^1_0)')}^2 \right), \end{aligned} \quad (2.2)$$

where $C_2 > 0$ is uniform with respect to f , y_0 and h .

The proof of this result is very close to [13, Proposition 2.3], therefore we only give a brief sketch of the proof in Appendix B.

Remark 2.3. • *Although the original weight $\rho_0(t)$ is defined only for $t \in \left[\left(1 - \frac{1}{q^2}\right)T, T \right]$, it is possible to define an extension (still denoted by ρ_0) which is continuous and non-increasing on the whole interval $(0, T)$. Thus estimate (2.2) is meaningful.*

• *Since ρ_0 verifies in particular $\rho_0(T) = 0$, it is clear that $(u, v)(T, \cdot) = 0$.*

3 Bounds on control cost for the linear system

As we anticipated in Section 1, one of the key ingredients of the proof of Theorem 1.2 is a precise bound of the cost of the control of a linear version of (1.2). Actually, the importance of this bound can be already seen in the statement of Proposition 2.2, where the cost of control is assumed to behave as a continuous and non-increasing function $K(\cdot)$. In this section, we give sufficient conditions on d such that the control cost has bounds that allow us to deal with the controllability of the non-linear problem.

We begin by recalling the known results on the linear control problem, which is given by the following system.

$$\begin{cases} u_t(t, x) + u_{xxxx}(t, x) = v(t, x), & t \in (0, T), x \in (0, \pi), \\ v_t(t, x) - dv_{xx}(t, x) = 0, & t \in (0, T), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (0, T), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (0, T), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (0, T). \end{cases} \quad (3.1)$$

Definition 3.1. *Let $T > 0$. System (3.1) is said to be null-controllable in time T if for any state $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$, there exists a control $h \in L^2(0, T)$ such that the solution of (3.1) with initial condition*

$$u(0, \cdot) = u_0 \quad \text{and} \quad v(0, \cdot) = v_0$$

satisfies

$$u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0.$$

We have the following.

Theorem 3.2 (Theorem 1.5 of [7]). *If \sqrt{d} is an irrational number with finite Liouville-Roth constant, then system (3.1) is null-controllable in time T for any $T > 0$.*

Theorem 3.2 indeed tell us that system (3.1) is null-controllable at time T for a wide class of diffusion coefficients d . In particular, this means that for every $(u_0, v_0) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ the set of admissible controls

$$\mathcal{C}(T, u_0, v_0) := \{h \in L^2(0, T) : (u, v)(T, \cdot) = 0\} \quad (3.2)$$

is non-empty. Thus the control cost in time T is given by

$$K(T) := \sup \left(\left\{ \inf_{h \in \mathcal{C}(T, u_0, v_0)} \|h\|_{L^2(0, T)} : \|(u_0, v_0)\|_{L^2 \times H^{-1}} \leq 1 \right\} \right). \quad (3.3)$$

The techniques used in [7] to prove Theorem 3.2 do not allow to directly obtain an estimation of (3.3) in all the cases where the system (3.1) is null-controllable. We will see that, under an extra assumption on the coefficient d , we can obtain a suitable estimation of the control cost for system (3.1). The result is the following.

Theorem 3.3. *If \sqrt{d} is an irrational algebraic number of degree 2, then there exist positive constants C and M such that the control cost for system (3.1) satisfies*

$$K(T) \leq C \exp(M/T).$$

In order to prove Theorem 3.3, we begin by rewriting system (3.1) in a more compact way, that is,

$$y_t = Ay + Bh, \quad (3.4)$$

where $y = (u, v)$, $A : D(A) := \{u \in H^2(0, \pi) \cap H_0^1(0, \pi) : \partial_{xx}u \in H^2(0, \pi) \cap H_0^1(0, \pi)\} \times H_0^1(0, \pi) \subset L^2(0, \pi) \times H^{-1}(0, \pi) \rightarrow L^2(0, \pi) \times H^{-1}(0, \pi)$ is given by

$$A = - \begin{pmatrix} (-\partial_{xx})^2 & -1 \\ 0 & -d\partial_{xx} \end{pmatrix}, \quad (3.5)$$

and $B \in \mathcal{L}(\mathbb{C}, ((H^2(0, \pi) \cap H_0^1(0, \pi))^2)')$ is given by

$$(Bh)(\phi_1, \phi_2) = h\phi_{2x}(0). \quad (3.6)$$

According to Proposition 2.1, for each $y_0 \in L^2(0, \pi) \times H^{-1}(0, \pi)$, equation (3.4) admits a unique solution $y \in C([0, T], L^2(0, \pi) \times H^{-1}(0, \pi)) \cap L^2(0, T; (H^2(0, \pi) \cap H_0^1(0, \pi)) \times L^2(0, \pi))$ satisfying $y(0) = y_0$. Furthermore

$$\|y\|_{L^2(0, T; (H^2 \cap H_0^1) \times L^2)}^2 + \|y\|_{C([0, T], L^2 \times H^{-1})}^2 \leq C \left(\|h\|_{L^2(0, T)}^2 + \|y_0\|_{L^2 \times H^{-1}}^2 \right). \quad (3.7)$$

In [7, Section 3.2] it was found that there is exists a Riesz basis made up of eigenfunctions of A , these are

$$\Phi_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k, \quad \Phi_{2,k} = k \begin{pmatrix} (k^4 - dk^2)^{-1} \\ 1 \end{pmatrix} \varphi_k, \quad (3.8)$$

where $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$. The eigenfunction $\Phi_{1,k}$ is associated to the eigenvalue $-k^4$ while $\Phi_{2,k}$ is associated to the eigenvalue $-dk^2$.

The biorthogonal basis corresponding to this Riesz basis is given by

$$\Psi_{1,k} = \begin{pmatrix} 1 \\ (dk^2 - k^4)^{-1} \end{pmatrix} \varphi_k, \quad \Psi_{2,k} = k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k. \quad (3.9)$$

It is clear that, if \sqrt{d} is irrational, the positive real numbers $\{k^4, dk^2 : k \in \mathbb{N}\}$ are all distinct. We denote them by $\{\lambda_j\}_{j \in \mathbb{N}}$, indexed in increasing order. For each j , we denote by ϕ_j and ψ_j the eigenvector and the biorthogonal eigenvector, respectively, associated with the eigenvalue $-\lambda_j$.

We have the following auxiliary result.

Lemma 3.4. *If \sqrt{d} is an irrational algebraic number of degree 2, there exist positive constants C, M (independent of T) such that*

$$\sum |a_j|^2 \exp(-\lambda_j T) \leq C \exp(M/T) \int_0^T \left| \sum a_j \exp(-\lambda_j t) \right|^2 dt \quad (3.10)$$

for any $\{a_j\}_{j \in \mathbb{N}} \in \ell^2(\mathbb{C})$.

Proof. Notice that if the inequality is true for a set of exponents larger than $\{k^4, dk^2 : k \in \mathbb{N}\}$, it is in particular true for $\{k^4, dk^2 : k \in \mathbb{N}\}$ (just by making the remaining coefficients null). We will show that we have (3.10) for $\{k^2, dk^2 : k \in \mathbb{N}\}$.

Hence, we denote $\{\mu_j\}_{j \in \mathbb{N}} = \{k^2, dk^2 : k \in \mathbb{N}\}$ indexed in increasing order. By [16, Corollary 3.6], in order to prove Lemma 3.4 it is enough to show

- the gap property $|\mu_{j+1} - \mu_j| \geq \gamma > 0$ for each $j \in \mathbb{N}$, and that
- there exist constants r, R such that

$$|\mu_j - rj^2| \leq Rj. \quad (3.11)$$

Having shown this, [16, Corollary 3.6] yields (3.10) with any $M > \frac{3\pi^2}{2r}$ and some K depending on R and γ , however not explicitly.

We first show that $\{k^2, dk^2 : k \in \mathbb{N}\}$ admits a gap. Since \sqrt{d} is an irrational algebraic number of degree 2, Liouville's Theorem implies that there exists a positive constant δ such that

$$\left| \sqrt{d} - \frac{p}{q} \right| \geq \frac{\delta}{q^2} \quad (3.12)$$

for all integers p, q with $q > 0$. Hence

$$|dk^2 - m^2| = \left| \sqrt{d} - \frac{m}{k} \right| k \left| \sqrt{d}k + m \right| \geq \left| \sqrt{d} - \frac{m}{k} \right| \sqrt{d}k^2 \geq \sqrt{d}\delta. \quad (3.13)$$

Since $|dk^2 - dm^2| \geq d$ and $|k^2 - m^2| \geq 1$ for $k \neq m$, a gap condition is satisfied, with gap $\gamma = \min\{1, d, \sqrt{d}\delta\}$.

In order to prove (3.11), we write $\lfloor x \rfloor$ for the floor function of x , given by $\max\{n : n \in \mathbb{Z}, n \leq x\}$. Suppose $\mu_j = k^2$, then $dm^2 \leq \mu_j \iff m \leq \frac{k}{\sqrt{d}}$, which is true for $m = 1, \dots, \left\lfloor \frac{k}{\sqrt{d}} \right\rfloor$. With this

$j = k + \left\lfloor \frac{k}{\sqrt{d}} \right\rfloor = \left\lfloor k \frac{\sqrt{d} + 1}{\sqrt{d}} \right\rfloor$. Notice that

$$\begin{aligned} j = \left\lfloor k \frac{\sqrt{d} + 1}{\sqrt{d}} \right\rfloor &\iff j \leq \frac{\sqrt{d} + 1}{\sqrt{d}} k < j + 1 \\ &\iff k < (j + 1) \frac{\sqrt{d}}{\sqrt{d} + 1} < j \frac{\sqrt{d}}{\sqrt{d} + 1} + 1 \leq k + 1, \end{aligned} \quad (3.14)$$

with this $k = \left\lfloor (j + 1) \frac{\sqrt{d}}{\sqrt{d} + 1} \right\rfloor$, take $r = \frac{d}{(\sqrt{d} + 1)^2}$, we have

$$\begin{aligned} |\mu_j - rj^2| &= \left| k^2 - \frac{dj^2}{(\sqrt{d} + 1)^2} \right| = \left| k - j \frac{\sqrt{d}}{\sqrt{d} + 1} \right| \cdot \left| k + j \frac{\sqrt{d}}{\sqrt{d} + 1} \right| \\ &\leq \left(1 + \frac{\sqrt{d}}{\sqrt{d} + 1} \right) \left(\frac{2\sqrt{d}}{\sqrt{d} + 1} + 1 \right) j. \end{aligned} \quad (3.15)$$

On the other hand, if $\mu_j = dk^2$, $m^2 \leq \mu_j \iff m \leq \sqrt{dk}$, which is true for $m = 1, \dots, \lfloor \sqrt{dk} \rfloor$. With this $j = k + \lfloor \sqrt{dk} \rfloor = \lfloor (\sqrt{d} + 1)k \rfloor$.

$$\begin{aligned} j = \lfloor (\sqrt{d} + 1)k \rfloor &\iff j \leq (\sqrt{d} + 1)k < j + 1 \\ &\iff k < \frac{j + 1}{\sqrt{d} + 1} < \frac{j}{\sqrt{d} + 1} + 1 \leq k + 1 \end{aligned} \quad (3.16)$$

meaning $k = \left\lfloor \frac{j + 1}{\sqrt{d} + 1} \right\rfloor$, with this

$$\begin{aligned} |\mu_j - rj^2| &= \left| dk^2 - \frac{dj^2}{(\sqrt{d} + 1)^2} \right| = \left| \sqrt{dk} - j \frac{\sqrt{d}}{\sqrt{d} + 1} \right| \cdot \left| \sqrt{dk} + j \frac{\sqrt{d}}{\sqrt{d} + 1} \right| \\ &\leq \left(\sqrt{d} + \frac{\sqrt{d}}{\sqrt{d} + 1} \right) \left(\frac{2\sqrt{d}}{\sqrt{d} + 1} + \sqrt{d} \right) j. \end{aligned} \quad (3.17)$$

To conclude, taking

$$R = \max \left\{ \left(1 + \frac{\sqrt{d}}{\sqrt{d} + 1} \right) \left(\frac{2\sqrt{d}}{\sqrt{d} + 1} + 1 \right), \left(\sqrt{d} + \frac{\sqrt{d}}{\sqrt{d} + 1} \right) \left(\frac{2\sqrt{d}}{\sqrt{d} + 1} + \sqrt{d} \right) \right\}, \quad (3.18)$$

we deduce (3.11). \square

Proof of theorem 3.3. We begin by writing the moment problem associated to the null controllability of system (3.4). We know that A is diagonalizable and it is the generator of a strongly continuous semigroup $\{\exp(tA)\}_{t \geq 0}$. Then

$$y(T) = \exp(TA)y_0 + \int_0^T \exp((T-t)A)Bh dt. \quad (3.19)$$

Since $\{\psi_j\}_{j \in \mathbb{N}}$ is a Riesz basis, $y(T) = 0$ if and only if, for each $j \in \mathbb{N}$ we have

$$\int_0^T (\exp((T-t)A)Bh, \psi_j) dt = -(\exp(TA)y_0, \psi_j), \quad (3.20)$$

i.e.

$$\int_0^T (h, B^* \exp((T-t)A^*)\psi_j) dt = -(y_0, \exp(TA^*)\psi_j), \quad (3.21)$$

applying the spectral decomposition

$$\overline{B^*\psi_j} \int_0^T h(T-s) \exp(-\lambda_j s) dt = -\exp(-\lambda_j T)(y_0, \psi_j). \quad (3.22)$$

This motivates writing h as the formal sum

$$h(t) = \sum_{j \in \mathbb{N}} \frac{-\exp(-\lambda_j T)}{\overline{B^*\psi_j}} (y_0, \psi_j) q_j(T-t), \quad (3.23)$$

where $\{q_j\}_{j \in \mathbb{N}}$ is a set of functions biorthogonal to $\{\exp(-\lambda_j t)\}_{j \in \mathbb{N}}$ in $L^2(0, T)$.

It suffices to prove that such family $\{q_j\}_{j \in \mathbb{N}}$ exists and (3.23) is absolutely convergent.

We denote $p_j = \exp(-\lambda_j t)$ and $\Pi_j = \overline{\text{span}\{p_i : i \neq j\}}$ for each $j \in \mathbb{N}$, and by P_j the orthogonal projection on Π_j .

For each $f \in \text{span}\{p_i : i \neq j\}$ we can write $p_j - f$ as a finite linear combination $\sum_{i \in I} a_i p_i$ of exponentials $\{p_i\}_{i \in \mathbb{N}^*}$ with I finite and $j \in I$, such that $a_j = 1$, therefore, due to lemma 3.4, we have

$$\|p_j - f\|_{L^2(0, T)} \geq C^{-1/2} \exp(-\lambda_j T/2 - M/2T). \quad (3.24)$$

By the continuity of the norm, we have (3.24) for $f \in \Pi_j$, in particular, this shows us that $p_j \notin \Pi_j$.

We define

$$q_j := \frac{p_j - P_j p_j}{\|p_j - P_j p_j\|_{L^2(0, T)}^2}. \quad (3.25)$$

With this $\{q_j\}_{j \in \mathbb{N}}$ is biorthogonal to $\{p_j\}_{j \in \mathbb{N}}$, and

$$\|q_j\|_{L^2(0, T)} = \frac{1}{\|p_j - P_j p_j\|_{L^2(0, T)}} \leq C^{1/2} \exp(\lambda_j T/2 + M/2T). \quad (3.26)$$

We seek to bound the series

$$\sum_{j \in \mathbb{N}} \frac{\exp(-\lambda_j T)}{|B^*\psi_j|^2}. \quad (3.27)$$

We explicitly compute

$$B^*\Psi_{1,k} = \sqrt{\frac{2}{\pi}} \frac{dk}{dk^2 - k^4}, \quad B^*\Psi_{2,k} = \sqrt{\frac{2}{\pi}} dk^2. \quad (3.28)$$

On one side, since $k \leq k^2$ and $\exp(-dT x)$ is decreasing

$$\sum_{k \in \mathbb{N}} \frac{\exp(-dk^2 T)}{|B^* \Psi_{2,k}|^2} \leq \frac{\pi}{2d^2} \sum_{k \in \mathbb{N}} \exp(-dkT) \leq \frac{\pi}{2d^2} \int_0^\infty \exp(-dT x) dx = \frac{\pi}{2d^3 T} =: \frac{R_1}{T}, \quad (3.29)$$

for the other branch of eigenvalues we need to be more careful, since $|B^* \Psi_{1,k}|$ is not bounded by below.

We define $L = \frac{\pi}{2d^2} \max_{k < \sqrt{d}} \frac{(d - k^2)^2}{k^4}$, then $|B^* \Psi_{1,k}|^2 \geq \frac{1}{Lk^6}$. Denoting $m = \left\lfloor \left(\frac{12}{7T} \right)^{2/7} \right\rfloor$, we get

$$\begin{aligned} \sum_{k \in \mathbb{N}} \frac{\exp(-k^4 T)}{|B^* \Psi_{2,k}|^2} &\leq L \sum_{k \in \mathbb{N}} \exp(-k^{7/2} T) k^6 \\ &= L \left(\sum_{k < m} \exp(-k^{7/2} T) k^6 + \exp(-m^{7/2} T) m^6 + \sum_{k \geq m+1} \exp(-k^{7/2} T) k^6 \right), \end{aligned} \quad (3.30)$$

since the function $\exp(-Tx^{7/2})x^6$ is increasing for $x < \left(\frac{12}{7T} \right)^{2/7}$ and decreasing for $x > \left(\frac{12}{7T} \right)^{2/7}$, we have

$$\begin{aligned} &\sum_{k < m} \exp(-k^{7/2} T) k^6 + \exp(-m^{7/2} T) m^6 + \sum_{k \geq m+1} \exp(-k^{7/2} T) k^6 \\ &\leq \int_0^m \exp(-Tx^{7/2}) x^6 + \exp(-Tm^{7/2}) m^6 + \int_m^\infty \exp(-Tx^{7/2}) x^6 \\ &\leq \int_0^\infty \exp(-Tx^{7/2}) x^6 + \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7} \\ &= \frac{2}{7T} \int_0^\infty \exp(-Tx^{7/2}) 7T/2x^{5/2} x^{7/2} + \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7} \\ &= \frac{2}{7T} \int_0^\infty \exp(-Tx^{7/2}) 7/2x^{5/2} + \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7} \\ &= \frac{2}{7T^2} \int_0^\infty \exp(-Tx^{7/2}) 7T/2x^{5/2} + \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7} \\ &= \frac{2}{7T^2} + \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7}, \end{aligned} \quad (3.31)$$

which means

$$\sum_{k \in \mathbb{N}} \frac{\exp(-k^4 T)}{|B^* \Psi_{2,k}|^2} \leq \frac{L}{7/2T^2} + L \exp(-12/7) \left(\frac{12}{7T} \right)^{12/7}. \quad (3.32)$$

Taking $R_2 = \frac{2L}{7}$ and $R_3 = L \exp(-12/7)$ we have

$$\begin{aligned} \sum_{j \in \mathbb{N}} \frac{\exp(-\lambda_j T)}{|B^* \psi_j|^2} &\leq R_1 \frac{1}{T} + R_2 \left(\frac{1}{T} \right)^2 + R_3 \left(\frac{1}{T} \right)^{12/7} \\ &\leq R_1 \exp(1/T) + R_2 \exp(2/T) + R_3 \exp(12/7T) \\ &\leq (R_1 + R_2 + R_3) \exp(2/T) =: R_0^2 \exp(2/T). \end{aligned} \quad (3.33)$$

Finally, since $\{\psi_j\}_{j \in \mathbb{N}}$ is a Riesz basis of $L^2(0, \pi) \times H^{-1}(0, \pi)$, there exists a constant Q such that

$$|(y_0, \psi_j)|_{\ell^2} \leq Q |y_0|_{L^2 \times H^{-1}}. \quad (3.34)$$

Putting this all together we have

$$\begin{aligned} & \sum_{j \in \mathbb{N}} \left| \frac{-\exp(-\lambda_j T)}{B^* \psi_j} (y_0, \psi_j) q_j(T-t) \right|_{L^2(0, T)} \\ &= \sum_{j \in \mathbb{N}} \left| \frac{\exp(-\lambda_j T/2)}{B^* \psi_j} (y_0, \psi_j) \right| |\exp(-\lambda_j T/2) q_j(T-t)|_{L^2(0, T)} \\ &\leq C^{1/2} \exp M/2T \sum_{j \in \mathbb{N}} \left| \frac{\exp(\lambda_j T/2)}{B^* \psi_j} (y_0, \psi_j) \right| \leq C^{1/2} \exp M/2T \left| \frac{\exp(-\lambda_j T/2)}{B^* \psi_j} \right|_{\ell^2} \cdot |(y_0, \psi_j)|_{\ell^2} \\ &\leq C^{1/2} R_0 Q \exp((M/2 + 1)/T) |y_0|_{L^2 \times H^{-1}}. \end{aligned} \quad (3.35)$$

Taking new constants $\tilde{C} = C^{1/2} R_0 Q$ and $\tilde{M} = M/2 + 1$, this concludes the proof. \square

4 Controllability of the nonlinear system

We introduce a weak formulation for system (1.2).

Taking into account that $uu_x = \left(\frac{u^2}{2}\right)_x$, formally multiplying by a test function φ (which is \mathbb{R}^2 valued) and integrating by parts we get

$$\int_0^\pi y_t \cdot \varphi \, dx = \int_0^\pi (Ay + Bh) \cdot \varphi \, dx + \int_0^\pi \left(\frac{u^2}{2}\right) \varphi_{1x} \, dx,$$

where we have used the notation introduced in Section 3. This motivates us to define $F : L^2(0, \pi) \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$ in the following way

$$\langle F(u), \phi \rangle_{(H^2(0, \pi) \cap H_0^1(0, \pi))', H^2(0, \pi) \cap H_0^1(0, \pi)} = \int_0^\pi \left(\frac{u^2}{2}\right) \phi_x \, dx,$$

Using Hölder inequality and since $H^1(0, \pi) \hookrightarrow L^\infty(0, \pi)$, we deduce

$$\begin{aligned} \langle F(u), \phi \rangle_{(H^2(0, \pi) \cap H_0^1(0, \pi))', H^2(0, \pi) \cap H_0^1(0, \pi)} &\leq \left\| \frac{u^2}{2} \right\|_{L^1(0, \pi)} \|\phi_x\|_{L^\infty(0, \pi)} \\ &\leq \frac{C_3}{2} \|u\|_{L^2(0, \pi)}^2 \|\phi\|_{H^2(0, \pi) \cap H_0^1(0, \pi)}, \end{aligned}$$

meaning $\|F(u)\|_{(H^2(0, \pi) \cap H_0^1(0, \pi))'} \leq \frac{C_3}{2} \|u\|_{L^2(0, \pi)}^2$ for some $C_3 > 0$.

We say that $y \in C([0, T], L^2(0, \pi) \times H^{-1}(0, \pi)) \cap L^2(0, T; H^2(0, \pi) \cap H_0^1(0, \pi) \times L^2(0, \pi))$ is a solution of (1.2) if it satisfies

$$y_t = Ay + Bh + (F(u), 0).$$

Note that

$$\begin{aligned}
\|(F(u), 0)\|_{L^2(0,T;(H^2 \cap H_0^1)')}^2 &= \|F(u)\|_{L^2(0,T;(H^2 \cap H_0^1)')}^2 = \int_0^T \|F(u)\|_{(H^2 \cap H_0^1)'}^2 \\
&\leq \frac{C_3^2}{4} \int_0^T \|u\|_{L^2(0,\pi)}^4 \leq \frac{TC_3^2}{4} \|u\|_{C([0,T],L^2(0,\pi))}^4 \\
&\leq \frac{TC_3^2}{4} \|y\|_{C([0,T],L^2(0,\pi) \times H^{-1}(0,\pi))}^4 < \infty.
\end{aligned}$$

Therefore $(F(u), 0) \in L^2(0, T; (H^2(0, \pi) \cap H_0^1(0, \pi))')$, and it makes sense to put $f = (F(u), 0)$ in (2.1).

Theorem 4.1. *Suppose (3.1) is null-controllable in time τ for all $\tau > 0$, and also suppose that there exist constants $C_1, M > 0$ such that the control cost is bounded by $K(T) = C_1 \exp(M/T)$. Then (1.2) is locally null-controllable at time T .*

Proof. We consider ρ_0 and ρ_1 , as in [15], given by

$$\begin{aligned}
\rho_1(t) &= \exp\left(-\frac{\alpha}{(T-t)^2}\right), \\
\rho_0(t) &= C_1 \exp\left(-\frac{\alpha}{q^4(T-t)^2} + \frac{M}{(q-1)(T-1)}\right),
\end{aligned} \tag{4.1}$$

where $K(T) = C_1 \exp(M/T)$ is a strict upper bound for the control cost, $1 < q < 2^{1/4}$ and $\alpha > \frac{MTq^4}{2(q-1)}$. Then we may directly check that (4.1) are continuous, non increasing functions, which vanish at $t = T$, and verify

$$\frac{1}{\rho_1} \leq \frac{C_1}{\rho_0}, \tag{4.2}$$

and for $t \in \left[\left(1 - \frac{1}{q^2}\right)T, T\right]$, we can write

$$\rho_0(t) = \rho_1(q^2t + (1 - q^2)T)K((q-1)(T-t)).$$

In this way, the weights (4.1) verify the hypotheses in Proposition 2.2. Also, as we have mentioned in Remark 2.3, it is possible to extend the weight ρ_0 to the whole interval $[0, T]$. We still denote it by ρ_0 .

We now prove that if the initial data (u_0, v_0) is sufficiently small, there exists a control h and a solution (u, v) of (1.2) such that $(u, v)(T, \cdot) = 0$. Denoting $\mathcal{Z} = L_{\rho_1}^2(0, T; (H^2 \cap H_0^1)')$, we define

$$\begin{aligned}
\mathcal{F} : \mathcal{Z} &\longrightarrow \mathcal{Z}, \\
f &\mapsto (F(u), 0)
\end{aligned}$$

where (u, v) solves (2.1). Clearly, in view of Proposition 2.2, finding a fixed point of the map \mathcal{F} will give us a controlled solution to (1.2) which in addition satisfies $(u, v)(T, \cdot) = 0$.

We begin by showing that \mathcal{F} is well defined.

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\mathcal{Z}}^2 &= \|F(u)\|_{L_{\rho_1}^2(0,T;(H^2 \cap H_0^1)')}^2 = \int_0^T \frac{\|F(u)\|_{(H^2 \cap H_0^1)'}^2}{\rho_1(t)^2} dt \\
&\leq \frac{C_1 C_3^2}{4} \int_0^T \frac{\|u(t)\|_{L^2(0,\pi)}^4}{\rho_0(t)^4} dt \leq \frac{TC_1 C_3^2}{4} \left\| \frac{u}{\rho_0} \right\|_{C([0,T],L^2)}^4 \leq \frac{TC_1 C_3^2}{4} \left\| \frac{(u, v)}{\rho_0} \right\|_{C([0,T],L^2 \times H^{-1})}^4 \\
&\leq \frac{TC_1 C_3^2 C_2^2}{4} \left(\|(u_0, v_0)\|_{L^2 \times H^{-1}}^2 + \|f\|_{\mathcal{Z}}^2 \right)^2 < +\infty,
\end{aligned} \tag{4.3}$$

where $C_2 > 0$ is the constant coming from Proposition 2.2.

Taking $C_4 = \frac{\sqrt{TC_1C_2C_3}}{2}$, we get that

$$\|\mathcal{F}(f)\|_{\mathcal{Z}} \leq C_4 \left(\|y_0\|_{L^2 \times H^{-1}}^2 + \|f\|_{\mathcal{Z}}^2 \right).$$

We define $R = \frac{1}{4C_4}$. For any y_0 such that $\|y_0\|_{L^2 \times H^{-1}} \leq R$ we have

$$\|\mathcal{F}(f)\|_{\mathcal{Z}} \leq 2C_4R^2 \leq \frac{R}{2} \leq R.$$

Therefore, \mathcal{F} maps $B(0, R) \subset \mathcal{Z}$ into itself.

To conclude, we need to show that \mathcal{F} is a contraction on $B(0, R)$. To this end, take $f^{(1)}, f^{(2)} \in B(0, R)$ and consider $y^{(1)} = (u^{(1)}, v^{(1)})$, $y^{(2)} = (u^{(2)}, v^{(2)})$ and $h^{(1)}, h^{(2)}$ the respective solutions and controls given by Proposition 2.2.

Since y and h depend continuously on the pair (y_0, f) , $y^{(1)} - y^{(2)}$ and $h^{(1)} - h^{(2)}$ are the solution and control associated with $f = f^{(1)} - f^{(2)}$ and $y_0 = 0$. A straightforward computation yields

$$\begin{aligned} \left\| \mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)}) \right\|_{\mathcal{Z}}^2 &= \left\| F(u^{(1)}) - F(u^{(2)}) \right\|_{L^2_{\rho_1}(0, T; (H^2 \cap H_0^1)')}^2 \\ &= \int_0^T \frac{\|F(u^{(1)}) - F(u^{(2)})\|_{(H^2 \cap H_0^1)'}}{\rho_1(t)^2} dt = \frac{C_3^2}{4} \int_0^T \frac{\|(u^{(1)})^2 - (u^{(2)})^2\|_{L^2(0, \pi)}}{\rho_1(t)^2} dt \\ &\leq \frac{C_1 C_3^2}{4} \int_0^T \frac{\|u^{(1)} + u^{(2)}\|_{L^2(0, \pi)}^2 \|u^{(1)} - u^{(2)}\|_{L^2(0, \pi)}^2}{\rho_0(t)^4} dt, \end{aligned}$$

where we have used property (4.2). Whence,

$$\begin{aligned} &\left\| \mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)}) \right\|_{\mathcal{Z}}^2 \\ &\leq \frac{TC_1C_3^2}{4} \left\| \frac{u^{(1)} + u^{(2)}}{\rho_0} \right\|_{C([0, T], L^2)}^2 \left\| \frac{u^{(1)} - u^{(2)}}{\rho_0} \right\|_{C([0, T], L^2)}^2 \\ &\leq \frac{TC_1C_3^2C_2^2}{4} \left(\|2y_0\|_{L^2 \times H^{-1}}^2 + \|f^{(1)} + f^{(2)}\|_{\mathcal{Z}}^2 \right) \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}}^2 \\ &\leq 8C_4^2R^2 \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}}^2, \end{aligned}$$

where we have taken into account the smallness of the initial datum and the source terms. From the above expression and using the definition of R , we get

$$\left\| \mathcal{F}(f^{(1)}) - \mathcal{F}(f^{(2)}) \right\|_{\mathcal{Z}} \leq 2\sqrt{2}C_4R \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}} = \frac{\sqrt{2}}{2} \|f^{(1)} - f^{(2)}\|_{\mathcal{Z}},$$

and the proof follows immediately. \square

Once we have established Theorem 4.1, the proof of Theorem 1.2 is a direct application of such result and the bound for the control cost obtained in Theorem 3.3.

A A general well-posedness result

Here, we give a proof of Proposition 2.1.

Proof. Since $(-\partial_{xx})^2 : H^2(0, \pi) \cap H_0^1(0, \pi) \subset (H^2(0, \pi) \cap H_0^1(0, \pi))' \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$ and $-d\partial_{xx} : L^2(0, \pi) \subset (H^2(0, \pi) \cap H_0^1(0, \pi))' \rightarrow (H^2(0, \pi) \cap H_0^1(0, \pi))'$ are strictly positive self-adjoint operators, we have the following two results due to [4, Thm. 3.1, pp. 143].

If $f_1 \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')$ and $u_1 \in L^2(0, \pi)$, then

$$\begin{cases} u_t(t, x) + u_{xxxx}(t, x) = f_1, & t \in (t_1, t_2), x \in (0, \pi), \\ u(t, 0) = u_{xx}(t, 0) = 0, & t \in (t_1, t_2), \\ u(t, \pi) = u_{xx}(t, \pi) = 0, & t \in (t_1, t_2), \end{cases} \quad (\text{A.1})$$

admits a unique solution $u \in C([t_1, t_2], L^2(0, \pi)) \cap L^2(t_1, t_2; H^2(0, \pi) \cap H_0^1(0, \pi))$ satisfying $u(t_1) = u_1$, and we have

$$\|u\|_{L^2(t_1, t_2; H^2 \cap H_0^1)}^2 + \|u\|_{C([t_1, t_2], L^2)}^2 \leq K_1 \left(\|u_1\|_{L^2}^2 + \|f_1\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right), \quad (\text{A.2})$$

for some $K_1 > 0$ independent of t_1, t_2 .

The second result tells that if $f_2 \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')$ and $v_1 \in H^{-1}(0, \pi)$, then

$$\begin{cases} v_t(t, x) - dv_{xx}(t, x) = f_2, & t \in (t_1, t_2), x \in (0, \pi), \\ v(t, 0) = h(t), v(t, \pi) = 0, & t \in (t_1, t_2), \end{cases} \quad (\text{A.3})$$

admits a unique solution $v \in C([t_1, t_2], H^{-1}(0, \pi)) \cap L^2(t_1, t_2; L^2(0, \pi))$ satisfying $v(t_1) = v_1$. Moreover, we have the following energy estimate

$$\|v\|_{L^2(t_1, t_2; L^2)}^2 + \|v\|_{C([t_1, t_2], H^{-1})}^2 \leq K_2 \left(\|h\|_{L^2(t_1, t_2)}^2 + \|v_1\|_{H^{-1}}^2 + \|f_2\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right),$$

with $K_2 > 0$ independent of t_1, t_2 .

We will now exploit the cascade structure of the system. Notice that if $v \in L^2(t_1, t_2; L^2(0, \pi))$, then in particular $v \in L^2(t_1, t_2; (H^2(0, \pi) \cap H_0^1(0, \pi))')$ and

$$\|v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \leq \|v\|_{L^2(t_1, t_2; L^2)}^2.$$

We take v to be the solution of (A.3), with right hand side f_2 , and u to be the solution of (A.1) with right hand side $f_1 + v$, since

$$\|v + f_1\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \leq 2 \left(\|f_1\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 + \|v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right),$$

We get

$$\|v\|_{L^2(t_1, t_2; L^2)}^2 + \|v\|_{C([t_1, t_2], H^{-1})}^2 \leq K_2 \left(\|h\|_{L^2(t_1, t_2)}^2 + \|v_1\|_{H^{-1}}^2 + \|f_2\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right), \quad (\text{A.4})$$

and

$$\begin{aligned} \|u\|_{L^2(t_1, t_2; H^2 \cap H_0^1)}^2 + \|u\|_{C([t_1, t_2], L^2)}^2 &\leq K_1 \left(\|u_1\|_{L^2}^2 + \|f_1 + v\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right) \\ &\leq 2K_1 \left(\|u_1\|_{L^2}^2 + \|f_1\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right) \\ &\quad + 2K_2K_1 \left(\|h\|_{L^2(t_1, t_2)}^2 + \|v_1\|_{H^{-1}}^2 + \|f_2\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right), \end{aligned} \quad (\text{A.5})$$

where we have used (A.4) in the last line. Writing $y = (u, v)$ and $y_1 = (u_1, v_1)$, we can combine (A.4) and (A.5) and take $C = \max\{K_2, (2K_1 + 1)K_2\}$ to deduce

$$\begin{aligned} & \|y\|_{L^2(t_1, t_2; (H^2 \cap H_0^1) \times L^2)}^2 + \|y\|_{C([t_1, t_2], L^2 \times H^{-1})}^2 \\ & \leq C \left(\|h\|_{L^2(t_1, t_2)}^2 + \|y_1\|_{L^2 \times H^{-1}}^2 + \|f\|_{L^2(t_1, t_2; (H^2 \cap H_0^1)')}^2 \right). \end{aligned}$$

This concludes the proof. \square

B Sketch of the proof of Proposition 2.2

We define $T_k = T \left(1 - \frac{1}{q^k}\right)$. Notice that with this

$$\rho_0(T_{k+2}) = \rho_1(T_k)K(T_{k+2} - T_{k+1}).$$

We recursively define y_{k+1} as $z(T_{k+1})$, where $z = (z_1, z_2)$ solves (2.1) on (T_k, T_{k+1}) with initial condition $z(T_k) = 0$ and control $h = 0$. By Proposition 2.1 we have

$$\|z\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1) \times L^2)}^2 + \|z\|_{C([T_k, T_{k+1}], L^2 \times H^{-1})}^2 \leq C \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2,$$

with this

$$\|y_{k+1}\|_{L^2 \times H^{-1}}^2 = \|z(T_{k+1})\|_{L^2 \times H^{-1}}^2 \leq \|z\|_{C([T_k, T_{k+1}], L^2 \times H^{-1})}^2 \leq C \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2.$$

Define h on (T_k, T_{k+1}) as the control that drives (2.1) to zero in time T_{k+1} with initial condition y_k at time T_k and source term $f \equiv 0$. We denote the corresponding solution by w . Since we have assumed that system (2.1) with $f \equiv 0$ is null-controllable with control cost $K(\cdot)$, we have

$$\|h\|_{L^2(T_k, T_{k+1})}^2 \leq K^2(T_{k+1} - T_k) \|y_k\|_{L^2 \times H^{-1}}^2$$

and therefore

$$\|h\|_{L^2(T_{k+1}, T_{k+2})}^2 \leq K^2(T_{k+2} - T_{k+1}) \|y_{k+1}\|_{L^2 \times H^{-1}}^2 \leq CK^2(T_{k+2} - T_{k+1}) \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2.$$

We define $y = z + w$. Since $z(T_k^-) + w(T_k^-) = z(T_k^+) + w(T_k^+)$, y is continuous at T_k for every $k \geq 0$. With this definition, it is also clear that y solves (2.1) for $t \in [0, T)$.

Notice that

$$\begin{aligned} \|h\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2})}^2 & \leq \frac{1}{\rho_0^2(T_{k+2})} \|h\|_{L^2(T_{k+1}, T_{k+2})}^2 \leq C \frac{K^2(T_{k+2} - T_{k+1})}{\rho_0^2(T_{k+2})} \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \\ & = \frac{C}{\rho_1^2(T_k)} \|f\|_{L^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \leq C \|f\|_{L_{\rho_1}^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2, \end{aligned} \quad (\text{B.1})$$

and

$$\|h\|_{L_{\rho_0}^2(0, T_1)}^2 \leq \frac{1}{\rho_0^2(T_1)} \|h\|_{L^2(0, T_1)}^2 \leq \frac{K^2(T_1)}{\rho_0^2(T_1)} \|y_0\|_{L^2 \times H^{-1}}^2. \quad (\text{B.2})$$

Proposition 2.1 yields

$$\begin{aligned} & \|(u, v)\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2}; (H^2 \cap H_0^1) \times L^2)}^2 + \left\| \frac{(u, v)}{\rho_0} \right\|_{C([T_{k+1}, T_{k+2}], L^2 \times H^{-1})}^2 \\ & \leq C^2 \left(1 + \frac{1}{K^2(T_{k+2} - T_{k+1})} \right) \|f\|_{L_{\rho_1}^2(T_k, T_{k+1}; (H^2 \cap H_0^1)')}^2 \\ & \quad + \frac{C}{K^2(T_{k+2} - T_{k+1})} \|f\|_{L_{\rho_1}^2(T_{k+1}, T_{k+2}; (H^2 \cap H_0^1)')}^2 \end{aligned} \quad (\text{B.3})$$

and also

$$\begin{aligned} & \| (u, v) \|_{L^2_{\rho_0}(0, T_1; (H^2 \cap H_0^1) \times L^2)}^2 + \left\| \frac{(u, v)}{\rho_0} \right\|_{C([0, T_1], L^2 \times H^{-1})}^2 \\ & \leq \frac{C(K^2(T_1) + 1)}{\rho_0^2(T_1)} \|y_0\|_{L^2 \times H^{-1}}^2 + \frac{C\rho_1^2(0)}{\rho_0^2(T_1)} \|f\|_{L^2_{\rho_1}(0, T_1; (H^2 \cap H_0^1)')}^2. \end{aligned} \quad (\text{B.4})$$

Combining estimates (B.1)–(B.2) with (B.3)–(B.4) and taking into account Remark 2.3, we can do a limit process to deduce that $\frac{y}{\rho_0}$ is continuous on $[0, T]$, and equals 0 at $t = T$. Furthermore, we can deduce the existence of a constant $C_2 > 0$ such that (2.2) holds. This ends the proof.

References

- [1] F. AMMAR-KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Recent results on the controllability of linear coupled parabolic problems: a survey*, Math. Control Relat. Fields, 1 (2011), pp. 267–306.
- [2] F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÁLEZ-BURGOS, AND L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal., 267 (2014), pp. 2077–2151.
- [3] ———, *New phenomena for the null controllability of parabolic systems: minimal time and geometrical dependence*, J. Math. Anal. Appl., 444 (2016), pp. 1071–1113.
- [4] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, AND S. K. MITTER, *Representation and control of infinite dimensional systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, second ed., 2007.
- [5] Y. BUGEAUD, *Distribution modulo one and Diophantine approximation*, vol. 193 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2012.
- [6] N. CARREÑO AND E. CERPA, *Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control acting on the heat equation*, J. Math. Pures Appl. (9), 106 (2016), pp. 670–694.
- [7] N. CARREÑO, E. CERPA, AND A. MERCADO, *Boundary controllability of a cascade system coupling fourth- and second-order parabolic equations*, Systems Control Lett., 133 (2019), pp. 104542, 7.
- [8] E. CERPA, A. MERCADO, AND A. F. PAZOTO, *On the boundary control of a parabolic system coupling KS-KdV and heat equations*, Sci. Ser. A Math. Sci. (N.S.), 22 (2012), pp. 55–74.
- [9] ———, *Null controllability of the stabilized Kuramoto-Sivashinsky system with one distributed control*, SIAM J. Control Optim., 53 (2015), pp. 1543–1568.
- [10] B. GESHKOVSKI, *Null-controllability of perturbed porous medium gas flow*, ESAIM Control Optim. Calc. Var., 26 (2020), pp. Paper No. 85, 30.
- [11] V. HERNÁNDEZ-SANTAMARÍA AND L. PERALTA, *Controllability results for stochastic coupled systems of fourth-and second-order parabolic equations*, arXiv preprint arXiv:2003.01334, (2020).

- [12] K. LE BALC'H, *Local controllability of reaction-diffusion systems around nonnegative stationary states*, ESAIM Control Optim. Calc. Var., 26 (2020), pp. Paper No. 55, 32.
- [13] Y. LIU, T. TAKAHASHI, AND M. TUCSNAK, *Single input controllability of a simplified fluid-structure interaction model*, ESAIM Control Optim. Calc. Var., 19 (2013), pp. 20–42.
- [14] B. A. MALOMED, B.-F. FENG, AND T. KAWAHARA, *Stabilized Kuramoto-Sivashinsky system*, Phys. Rev. E, 64 (2001), p. 046304.
- [15] T. TAKAHASHI, *Boundary local null-controllability of the Kuramoto-Sivashinsky equation*, Math. Control Signals Systems, 29 (2017), pp. Art. 2, 21.
- [16] G. TENENBAUM AND M. TUCSNAK, *New blow-up rates for fast controls of Schrödinger and heat equations*, J. Differential Equations, 243 (2007), pp. 70–100.