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Almost blenders and parablenders

Sébastien Biebler

December 29, 2020

Abstract

A blender for a surface endomorphism is a hyperbolic basic set for which the union of the local unstable manifolds contains robustly an open set. Introduced by Bonatti and Díaz in the 90s, blenders turned out to have many powerful applications to differentiable dynamics. In particular, a generalization in terms of jets, called parablenders, allowed Berger to prove the existence of generic families displaying robustly infinitely many sinks. In this paper, we introduce analogous notions in a measurable point of view. We define an almost blender as a hyperbolic basic set for which a prevalent perturbation has a local unstable set having positive Lebesgue measure. Almost parablenders are defined similarly in terms of jets. We study families of endomorphisms of $\mathbb{R}^2$ leaving invariant the continuation of a hyperbolic basic set. When some inequality involving the entropy and the maximal contraction along stable manifolds is satisfied, we obtain an almost blender or parablender. This answers partially a conjecture of Berger. The proof is based on thermodynamic formalism: following works of Mihailescu, Simon, Solomyak and Urbański, we study families of fiberwise unipotent skew-products and we give conditions under which these maps have limit sets of positive measure inside their fibers.

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1 Introduction

1.1 Blenders and almost blenders

Fractal sets have played a central role in the development of differentiable dynamics. Among several examples, a central notion is that of blender, cast by Bonatti and Díaz in the 90s. It was first introduced in the invertible setting in [BD1] to construct robustly transitive nonhyperbolic diffeomorphisms. A blender is a hyperbolic basic set on which the dynamics has a special behavior: its unstable set forms an ≪impenetrable wall≫ in the sense that it intersects any perturbation of a submanifold of dimension lower than the stable dimension. In the case of surface endomorphisms, this notion takes the following simpler form:

Definition. *A C^r-**blender** for a C^r-endomorphism $\mathcal{F}$ of a surface $\mathcal{S}$ is a hyperbolic basic set $\mathcal{K}$ s.t. an union of its local unstable manifolds has C^r-robustly a non-empty interior: there exists a non-empty open set $\mathcal{U} \subset \mathcal{S}$ included in an union of local unstable manifolds of the continuation $\tilde{\mathcal{K}}$ of $\mathcal{K}$ for any map $\tilde{\mathcal{F}}$ C^r-close to $\mathcal{F}$.*

This property turned out to have many other powerful applications: for example C^1-density of stable ergodicity [ACW], robust homoclinic tangencies [BD2, Bie1] and thus Newhouse phenomenon, the existence of generic families displaying robustly infinitely many sinks [Be1], robust bifurcations in complex dynamics [Du, Taf, Bie2], fast growth of the number of periodic points [Be2, AST] ... Thus the following question is of fundamental interest: *when do blenders appear ?*

In this direction, Berger proposed the following conjecture:

Conjecture A (Berger [Be3]). *Let $\mathcal{F}$ be a C^r-local diffeomorphism of a manifold $\mathcal{M}$, for $r \geq 2$. Let $\mathcal{K}$ be a hyperbolic basic set for $\mathcal{F}$. Suppose that the topological entropy $h_{\mathcal{F}}$ of $\mathcal{F}|\mathcal{K}$ satisfies:*

$$h_{\mathcal{F}} > \dim \mathcal{E}^s \cdot |\log m(D\mathcal{F})| \qquad \textit{with} \qquad m(D\mathcal{F}) := \min_{z\in\mathcal{K}, u\in\mathcal{E}^s_z, ||u||=1} ||D_z\mathcal{F}(u)||\,,$$

and $\mathcal{E}^s$ the stable bundle of $\mathcal{K}$. Then there exists a C^r-neighborhood $\mathcal{U}$ of $\mathcal{F}$ and an infinite codimensional subset $\mathcal{N} \subset \mathcal{U}$ such that for every $\tilde{\mathcal{F}} \in \mathcal{U}\backslash\mathcal{N}$, the continuation $\tilde{\mathcal{K}}$ of $\mathcal{K}$ is a C^r-blender.

Note that we cannot hope that $\mathcal{K}$ is itself systematically a C^r-blender under the assumptions of Conjecture A. Here is an easy counterexample.

Counterexample: Let us consider the doubling map $f : x \mapsto 2x \bmod 1$ on the circle $\mathbb{S} := \mathbb{R}/\mathbb{Z}$. The whole circle is a hyperbolic basic set of repulsive type and f is a C^∞-local diffeomorphism. The topological entropy h_f is equal to $\log 2 > 0$. Let us pick $\lambda < 1$ close to 1 s.t. $h_f > |\log\lambda|$. The map $\mathcal{F} : (x, y) \in \mathbb{S}\times\mathbb{R} \mapsto (f(x), \lambda y) \in \mathbb{S}\times\mathbb{R}$ leaves invariant the hyperbolic basic set of saddle type $\mathcal{K} := \mathbb{S}\times\{0\}$. Moreover $\mathcal{F}$ is a C^∞-local diffeomorphism and $h_{\mathcal{F}} = h_f > |\log\lambda|$. However the unstable set of $\mathcal{K}$ is included in $\mathbb{S}\times\{0\}$ and thus has empty interior.

Conjecture A is also linked to a program proposed by Díaz [Di] on the thermodynamical study of blenders.

In this article, we give an answer to both questions, in a measurable point of view. We are going to define a measurable variant of the notion of blender, called ≪almost blender≫. This will be a hyperbolic basic set having its unstable set of positive Lebesgue measure instead of being with non-empty interior. Also, this property will be asked to be ≪ robust ≫ in a measurable way instead of topological. Let us precise this ≪ measurable robustness ≫. We recall that this notion is tricky since there is

no canonical measure on the space $C^r(\mathcal{M},\mathcal{M})$ of C^r-endomorphisms of a manifold $\mathcal{M}$. But we need an analogous of the finite-dimensional notion of $\ll$ Lebesgue almost every $\gg$ in an infinite-dimensional setting. Nevertheless, there are several notions of prevalence or typicity which generalize this concept. A panorama has been drawn by Hunt and Kaloshin in [HK], by Ott and Yorke in [OY] or by Ilyashenko and Li in [IL]. Here is one of these notions of prevalence, particularly adapted to our case:

Definition ([HK] P.53). *We say that a set E in a Banach space B is* finite-dimensionally prevalent *if there exists a continuous family $(v_q)_{q\in\mathcal{Q}}$ of vectors $v_q \in B$, for q varying in a neighborhood $\mathcal{Q}$ of 0 in $\mathbb{R}^m$ with $m>0$ and $v_0=0$, s.t. for every fixed $v\in B$, we have that $v+v_q\in E$ for Leb_m a.e. $q\in\mathcal{Q}$.*

In other terms, we require that for some finite-dimensional family of perturbations, if we start at any point in B, then by adding a perturbation randomly chosen with respect to the Lebesgue measure, we are in E with probability 1. A similar notion, simply called $\ll$ prevalence $\gg$, has been designed by Sauer, Ott and Casdagli (see Definition 3.5 in [OY], or also [SYC] and [HSY]), for completely metrizable topological vector spaces and with the additive condition that v_q is a linear function of q. In our results, we will have this additional linearity but since we do not need it, we will take inspiration from the above definition. See Remark 1 P.53 in [HK] for details.

We restrict ourselves in this article to the case where the manifold $\mathcal{M}$ is equal to $\mathbb{R}^2$ and endowed with its usual Euclidean metric. The vector space $C^r(\mathbb{R}^2,\mathbb{R}^2)$ of C^r-endomorphisms of $\mathbb{R}^2$ is endowed with the topology given by the uniform C^r-norm:

$$||\mathcal{F}||_{C^r} := \sup_{0\le i\le r, z\in\mathbb{R}^2} ||D^i_z\mathcal{F}||$$

when $0\le r<\infty$. The space of C^r-bounded C^r-endomorphisms endowed with this norm is a Banach space. Since we are interested in properties depending only on perturbations of a map on a compact set, we can restrict ourselves to C^r-bounded C^r-endomorphisms if necessary and so the above definition of prevalence from [HK] could fit to our setting. Similarly, for $r=\infty$, we endow $C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ with the union of uniform C^s-topologies on K_j among integers s and j, for an exhausting sequence of compact sets K_j of $\mathbb{R}^2$, which gives to $C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ a complete metrizable topology.

However we cannot hope that a blender-like property, even in a weak sense, holds true densely in $C^r(\mathbb{R}^2,\mathbb{R}^2)$, even less in a prevalent way. This is why we introduce the following immediate adaptation for perturbations of a map:

Definition. *A property (P) holds true for a* prevalent C^r-perturbation *of a C^r-endomorphism $\mathcal{F}$ of $\mathbb{R}^2$ if there exists a C^r-neighborhood $\mathcal{U}$ of $\mathcal{F}$ and a continuous family $(\Sigma_q)_{q\in\mathcal{Q}}$ of C^r-endomorphisms Σ_q of $\mathbb{R}^2$, for q varying in a neighborhood $\mathcal{Q}$ of 0 in $\mathbb{R}^m$ with $m>0$ and $\Sigma_0=0$, s.t. for every fixed $\mathcal{G}\in\mathcal{U}$, the map $\mathcal{G}+\Sigma_q$ satisfies (P) for Leb_m a.e. $q\in\mathcal{Q}$.*

In particular, property (P) holds true for an arbitrary small perturbation of $\mathcal{F}$.

Here is now a new concept, which formalizes a measurable variant of blenders.

Definition. *A hyperbolic basic set $\mathcal{K}$ for a C^r-endomorphism $\mathcal{F}$ of $\mathbb{R}^2$, with $r\ge 1$, is an* **almost C^r-blender** *if an union of local unstable manifolds of the continuation $\tilde{\mathcal{K}}$ of $\mathcal{K}$ has positive measure for a prevalent C^r-perturbation $\tilde{\mathcal{F}}$ of $\mathcal{F}$:*

$$\mathrm{Leb}_2(W^u_{\mathrm{loc}}(\tilde{\mathcal{K}}))>0\,.$$

A *hyperbolic basic set* for $\mathcal{F}$ is a compact, $\mathcal{F}$-invariant, hyperbolic, transitive set $\mathcal{K}$ s.t. periodic points of $\mathcal{F}|\mathcal{K}$ are dense in $\mathcal{K}$ (basic notions about hyperbolic sets for

endomorphisms are in the Appendix).

Our first result gives an answer to Conjecture A, in a measurable point of view:

Theorem A. *Let $\mathcal{F}$ be a C^r-local diffeomorphism of $\mathbb{R}^2$, with $2 \leq r \leq \infty$. Let $\mathcal{K}$ be a hyperbolic basic set for $\mathcal{F}$. Suppose that the topological entropy $h_{\mathcal{F}}$ of $\mathcal{F}|\mathcal{K}$ satisfies $h_{\mathcal{F}} > |\log m(D\mathcal{F})|$. Then $\mathcal{K}$ is an almost C^r-blender.*

Conjecture A seems a very difficult problem in its full generality. A related question is the following long-standing open problem:

Conjecture B ([Ho] Conj. 3.1). *Let μ be the self similar measure associated to some IFS $\Psi = (\psi^a)_{a\in\mathcal{A}}$ formed by a finite number of contracting similarities ψ^a on $\mathbb{R}$. Suppose that there are no exact overlaps and that the similarity dimension of the IFS is strictly larger than 1. Then μ is absolutely continuous with respect to* Leb_1.

One can refer to the survey of Hochman [Ho] for more details. Let us also point out that the creation of blenders had also been investigated by Moreira and Silva [MS].

1.2 Parablenders and almost parablenders

Berger introduced in [Be1] a variant of blenders, defined for families of maps this time, where not only the unstable set of a hyperbolic set, but also the set of jets of points inside unstable manifolds contains an open set. Such sets were named parablenders ($\ll$ para $\gg$ standing for $\ll$ parameter $\gg$). Parablenders were introduced to prove the existence of generic families displaying robustly infinitely many sinks, which gave a counter-example to a conjecture of Pugh and Shub from the 90s [PS].

Definition ([Be1, BCP]). *A C^r-**parablender** at $p_0 \in \mathcal{P}$ for a C^r-family $(\mathcal{F}_p)_{p\in\mathcal{P}}$ of endomorphisms of a surface $\mathcal{S}$, $r \geq 1$, parametrized by a parameter p in an open subset $\mathcal{P} \subset \mathbb{R}^d$, is a continuation $(\mathcal{K}_p)_{p\in\mathcal{P}}$ of hyperbolic basic sets $\mathcal{K}_p$ for $\mathcal{F}_p$ s.t.:*

- *for every $(\gamma_p)_{p\in\mathcal{P}}$ in a non-empty open set of C^r-families of points $\gamma_p \in \mathcal{S}$,*
- *for every C^r-family $(\tilde{\mathcal{F}}_p)_{p\in\mathcal{P}}$ of endomorphisms C^r-close to $(\mathcal{F}_p)_{p\in\mathcal{P}}$,*

there exists a C^r-family $(\zeta_p)_{p\in\mathcal{P}}$ of points $\zeta_p \in \mathcal{S}$ s.t.:

- *there is a local unstable manifold of $\mathcal{K}_{p_0}$ whose continuation for $\tilde{\mathcal{F}}_p$ contains ζ_p, for any $p \in \mathcal{P}$,*
- *the r-jets of ζ_p and γ_p at p_0 are equal:*

$$(\zeta_p, \partial_p\zeta_p, \ldots, \partial_p^r\zeta_p)_{|p=p_0} = (\gamma_p, \partial_p\gamma_p, \ldots, \partial_p^r\gamma_p)_{|p=p_0}\,.$$

In particular, $\mathcal{K}_{p_0}$ is a C^r-blender for $\mathcal{F}_{p_0}$ if $(\mathcal{K}_p)_{p\in\mathcal{P}}$ is a C^r-parablender for $(\mathcal{F}_p)_{p\in\mathcal{P}}$ at p_0. In a subsequent work [Be2], Berger used parablenders to prove the existence of generic families of maps displaying robustly a fast growth of the number of periodic points, solving a problem of Arnold [Ar] in the finitely differentiable case.

From now on, we work with C^r-families $(\mathcal{F}_p)_p$ of endomorphisms $\mathcal{F}_p$ of $\mathbb{R}^2$, with $2 \leq r \leq \infty$, parametrized by a parameter p varying in $\mathcal{P} := (-1,1)^d$ with $1 \leq d < \infty$. In fact, we will need to work with families which admit some extension on a larger parameter space. We therefore fix an open set $\mathcal{P}' \subset \mathbb{R}^d$ s.t. $\mathcal{P} \Subset \mathcal{P}'$. We then define a C^r-family $(\mathcal{F}_p)_p$ of endomorphisms $\mathcal{F}_p$ of $\mathbb{R}^2$ as an element of $C^r(\mathcal{P}' \times \mathbb{R}^2, \mathbb{R}^2)$. We endow this space with the uniform C^r-topology when $0 \leq r < \infty$, and with the union of uniform C^s-topologies on K_j among integers s and j, for an exhausting sequence of compact sets K_j of $\mathcal{P}' \times \mathbb{R}^2$ when $r = \infty$. Note that for simplicity we denote often in the following $(\mathcal{F}_p)_{p\in\mathcal{P}}$ this family since we are interested mainly in

the dynamics when $p \in \mathcal{P}$ but keep in mind that it admits such an extension. Let $(\mathcal{K}_p)_{p\in\mathcal{P}}$ be the (hyperbolic) continuation of a hyperbolic basic set (extending to $\mathcal{P}'$). Let $\mathcal{E}^s_p$ and $\mathcal{E}^u_p$ be the one-dimensional stable and unstable bundles of $\mathcal{K}_p$.

Our main result deals with jets of points inside local unstable manifolds of $\mathcal{K}_p$. Let $(M_p)_p$ be a C^r-curve of points M_p in the continuation of one local unstable manifold of $\mathcal{K}_p$. For any integer $s \leq r$, one can consider the s-jet of M_p at any $p_0 \in \mathcal{P}$:

$$\mathrm{J}^s_{p_0} M_p := (M_p, \partial_p M_p, \dots, \partial^s_p M_p)_{|p=p_0} .$$

An interesting set is then the set $\mathrm{J}^s_{p_0} W^u_{\mathrm{loc}}(\mathcal{K}_p)$ of all the s-jets among such curves $(M_p)_p$. When this set has robustly a non-empty interior, $(\mathcal{K}_p)_{p\in\mathcal{P}}$ is a C^s-parablender at p_0. Let $\delta_{d,s}$ be the dimension of the set of jets in d variables of order s in one dimension, which is that of the space $\mathbb{R}_s[X_1, \cdots, X_d]$ of polynomials in d variables of degree at most s. In particular, notice that the space of jets of order s of maps from $\mathcal{P}$ to $\mathbb{R}^2$ is of dimension $2\delta_{d,s}$.

Here is the counterpart for families of the definition of a prevalent C^r-perturbation:

Definition. *A property (P) holds true for a* prevalent C^r-perturbation *of a C^r-family $(\mathcal{F}_p)_{p\in\mathcal{P}}$ of endomorphisms of $\mathbb{R}^2$ if there exists a C^r-neighborhood $\mathcal{U}$ of $(\mathcal{F}_p)_{p\in\mathcal{P}}$ and a continuous family $(\Sigma_q)_{q\in\mathcal{Q}}$ of C^r-families Σ_q of endomorphisms of $\mathbb{R}^2$, for q in a neighborhood $\mathcal{Q}$ of 0 in $\mathbb{R}^m$ with $m > 0$ and $\Sigma_0 = (0)_{p\in\mathcal{P}}$, s.t. for every fixed family $(\mathcal{G}_p)_p \in \mathcal{U}$, the family $(\mathcal{G}_p)_p + \Sigma_q$ satisfies (P) for Leb_m a.e. $q \in \mathcal{Q}$.*

In particular, property (P) holds true for an arbitrary small perturbation of $(\mathcal{F}_p)_{p\in\mathcal{P}}$.

The following is an analogous of C^s-parablenders, in a measurable point of view:

Definition. *The continuation $(\mathcal{K}_p)_{p\in\mathcal{P}}$ of a hyperbolic basic set for a C^r-family $(\mathcal{F}_p)_{p\in\mathcal{P}}$ of endomorphisms of $\mathbb{R}^2$, with $r \geq 1$, is an* **almost $C^{r,s}$-parablender**, *with s an integer s.t. $s \leq r$, if for a prevalent C^r-perturbation $(\tilde{\mathcal{F}}_p)_{p\in\mathcal{P}}$ of $(\mathcal{F}_p)_{p\in\mathcal{P}}$, the continuation $(\tilde{\mathcal{K}}_p)_{p\in\mathcal{P}}$ of $(\mathcal{K}_p)_{p\in\mathcal{P}}$ satisfies:*

$$\mathrm{Leb}_{2\delta_{d,s}}(\mathrm{J}^s_{p_0} W^u_{\mathrm{loc}}(\tilde{\mathcal{K}}_p)) > 0 \text{ for } \mathrm{Leb}_d \text{ a.e. } p_0 \in \mathcal{P} .$$

Note that if $(\mathcal{K}_p)_{p\in\mathcal{P}}$ is an almost $C^{r,s}$-parablender and p is a parameter in $\mathcal{P}$, the set $\mathcal{K}_p$ is an almost C^r-blender.

Here is our main result, which generalizes Theorem A in terms of jets:

Theorem B. *Let $(\mathcal{F}_p)_{p\in\mathcal{P}}$ be a C^r-family of local diffeomorphisms of $\mathbb{R}^2$, with $2 \leq r \leq \infty$. Let $(\mathcal{K}_p)_{p\in\mathcal{P}}$ be the continuation of a hyperbolic basic set for $(\mathcal{F}_p)_{p\in\mathcal{P}}$. Take an integer $s \leq r-2$ and suppose that the topological entropy $h_{\mathcal{F}_p}$ of $\mathcal{F}_p|\mathcal{K}_p$ satisfies:*

$$(\star) \qquad h_{\mathcal{F}_p} > \delta_{d,s} \cdot |\log m(D\mathcal{F}_p)| \qquad \forall p \in \mathcal{P}' .$$

Then $(\mathcal{K}_p)_{p\in\mathcal{P}}$ is an almost $C^{r,s}$-parablender.

This second result also goes in the direction of Conjecture A, both in a measurable point of view and in terms of jets this time. Let us mention that both Theorems A and B still hold true if we assume that the maps involved are local diffeomorphisms only in a neighborhood of the basic sets. Last but not least, we hope to use Theorem B to solve the conjecture of Pugh and Shub [PS] in the smooth C^∞ case which is not handled by [Be1]. Finally, let us mention the following immediate question:

Question. *Is it possible to generalize Theorems A and B to the case where $\mathcal{M}$ is any surface (not necessarily equal to $\mathbb{R}^2$) and for the alternative notion of prevalence defined by Kaloshin in this context ? We recall that this latter one is defined as*

follows: a subset $E \subset C^r(\mathcal{M},\mathcal{M})$ is **strictly n-prevalent** *if there exists an open dense set of n-parameter families $(\mathcal{F}_p)_p$ s.t. $\mathcal{F}_p \in E$ for a.e. p and if for every $\mathcal{F} \in C^r(\mathcal{M},\mathcal{M})$, there exists such a family with $\mathcal{F}_0 = \mathcal{F}$. A* **$n$-prevalent** *set is a countable intersection of strictly n-prevalent sets.*

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Combinatorics and notations

Let $\mathcal{A}$ be a finite alphabet of cardinality at least 2. Let

$$\overrightarrow{\mathcal{A}} = \mathcal{A}^{\mathbb{N}}\ ,\ \overleftarrow{\mathcal{A}} = \mathcal{A}^{\mathbb{Z}^*_-} \text{ and } \overleftrightarrow{\mathcal{A}} = \mathcal{A}^{\mathbb{Z}}$$

be the sets of infinite forward, backward and bilateral words with letters in $\mathcal{A}$. We consider the left full shift on $\overrightarrow{\mathcal{A}}$ or $\overleftrightarrow{\mathcal{A}}$:

$$\sigma : \alpha = (\alpha_i)_i \in \overrightarrow{\mathcal{A}} \sqcup \overleftrightarrow{\mathcal{A}} \mapsto \sigma(\alpha) = (\alpha_{i+1})_i \in \overrightarrow{\mathcal{A}} \sqcup \overleftrightarrow{\mathcal{A}}$$

and the right full shift on $\overleftarrow{\mathcal{A}}$:

$$\sigma : \alpha = (\alpha_i)_i \in \overleftarrow{\mathcal{A}} \mapsto \sigma(\alpha) = (\alpha_{i-1})_i \in \overleftarrow{\mathcal{A}}\,.$$

In particular these full shifts are of positive entropy and topologically mixing. We also define $\mathcal{A}^*$ as the set of finite words with letters in $\mathcal{A}$ and denote by e the empty word. We endow $\overleftrightarrow{\mathcal{A}}$ with the distance given by $d_\infty(\alpha,\beta) = D^q$ for every $\alpha = (\alpha_i)_i \in \overleftrightarrow{\mathcal{A}}$ and $\beta = (\beta_i)_i \in \overleftrightarrow{\mathcal{A}}$, where $D \in (0,1)$ is a fixed number and q is the largest integer such that $\alpha_i = \beta_i$ for every $|i| < q$ if $\alpha \neq \beta$. We endow $\overrightarrow{\mathcal{A}}$ with a metric defined similarly.

For $\alpha \in \mathcal{A}^* \cup \overrightarrow{\mathcal{A}} \cup \overleftarrow{\mathcal{A}} \cup \overleftrightarrow{\mathcal{A}}$, let $|\alpha| \in \mathbb{N} \cup \{+\infty\}$ be the number of letters in α. When $|\alpha| > n$ for some integer $n > 0$, we call α_i the ith letter of α and denote $\alpha_{|n} := (\alpha_0, \cdots, \alpha_{n-1})$ when $\alpha \in \mathcal{A}^* \cup \overrightarrow{\mathcal{A}}$ and $\alpha_{|n} := (\alpha_{-n}, \cdots, \alpha_{-1})$ when $\alpha \in \overleftarrow{\mathcal{A}} \cup \overleftrightarrow{\mathcal{A}}$. Finally, for $\alpha = (\alpha_{-n}, \cdots, \alpha_{-1}) \in \mathcal{A}^*$, let $[\alpha]$ be the corresponding cylinder in $\overleftrightarrow{\mathcal{A}}$:

$$[\alpha] := \{\beta \in \overleftrightarrow{\mathcal{A}} : \beta_i = \alpha_i\ \forall -n \leq i \leq -1\}\,.$$

We define similarly cylinders in $\overleftarrow{\mathcal{A}}$ and $\overrightarrow{\mathcal{A}}$ and use the same notation. **Greek (resp. gothic) letters will be used for finite or backward infinite (resp. forward infinite) words.** For $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha \in \overleftarrow{\mathcal{A}}$, $\beta \in \mathcal{A}^*$, we denote by $\alpha\mathfrak{a}, \beta\mathfrak{a}, \alpha\beta$ their concatenations. The topological closure of a set relatively to the Euclidean distance is denoted with an overline. The notation $\preceq$ stands for the usual domination relation and $f \asymp g$ means that $f \preceq g$ and $g \preceq f$.

2 Example

We give here an application of our results. More precisely, we provide simple examples of almost blenders and parablenders.

Let us consider the segment $X = [-1,1]$. We pick three integers $n \geq 2$, $d \geq 1$ and $s \geq 0$. We choose $n' := (n+1)^{\lceil \delta_{d,s} \rceil}$ disjoint subsegments $X_j \Subset X$ and n' numbers $0 < r_j < 1 - 1/n$, for $1 \leq j \leq n'$. Let g_j be the affine preserving order map sending X_j onto X. We pick a C^∞-map $g : \mathbb{R} \to \mathbb{R}$ which is equal to g_j on a small

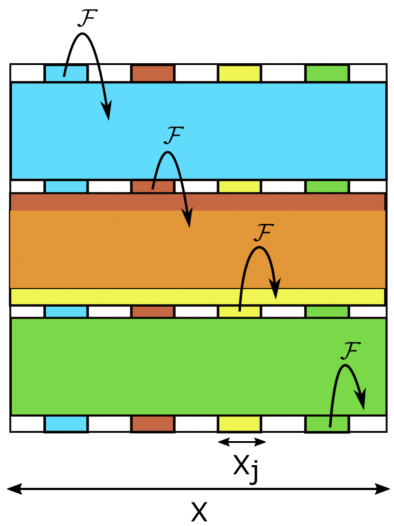
neighborhood of each interval X_j and also a C^∞-map $h : \mathbb{R} \to \mathbb{R}$ which is equal to r_j on a small neighborhood of each interval X_j. The following C^∞-endomorphism

$$\mathcal{F} : (x, y) \in \mathbb{R}^2 \mapsto (g(x), \frac{y}{n} + h(x)) \in \mathbb{R}^2$$

is a local diffeomorphism on a small neighborhood of $U := \bigsqcup_{1 \le j \le n'} X_j \times X$. It is easy to verify that the set

$$\mathcal{K} := \bigcap_{n \in \mathbb{Z}} \mathcal{F}^n(U) \tag{1}$$

is a compact, hyperbolic, invariant, locally maximal set, with stable and unstable dimensions equal to 1. This remains true for any C^∞-endomorphism $\tilde{\mathcal{F}}$ which is C^∞-close to $\mathcal{F}$, with the same formula.



We set $\mathcal{A} := \{1, \cdots, n'\}$ and call $\mathcal{F}_j$ the restriction of $\mathcal{F}$ on $X_j \times X$. For $\mathfrak{a} = (\mathfrak{a}_i)_{i \ge 0} \in \overrightarrow{\mathcal{A}}$ and $\alpha = (\alpha_i)_{i<0} \in \overleftarrow{\mathcal{A}}$, the following are local stable and unstable manifolds of $\mathcal{K}$:

$$W^{\mathfrak{a}} = \bigcap_{j \ge 0} \mathrm{Dom}(\mathcal{F}_{\mathfrak{a}_j} \circ \cdots \circ \mathcal{F}_{\mathfrak{a}_0}) \text{ and } W^{\alpha} = \bigcap_{j<0} \mathrm{Im}(\mathcal{F}_{\alpha_j} \circ \cdots \circ \mathcal{F}_{\alpha_{-1}}), \tag{2}$$

where the domains $\mathrm{Dom}(\mathcal{F}_{\mathfrak{a}_j} \circ \cdots \circ \mathcal{F}_{\mathfrak{a}_0})$ and $\mathrm{Dom}(\mathcal{F}_{\alpha_j} \circ \cdots \circ \mathcal{F}_{\alpha_{-1}})$ of $\mathcal{F}_{\mathfrak{a}_j} \circ \cdots \circ \mathcal{F}_{\mathfrak{a}_0}$ and $\mathcal{F}_{\alpha_j} \circ \cdots \circ \mathcal{F}_{\alpha_{-1}}$ are $(g_{\mathfrak{a}_j} \circ \cdots \circ g_{\mathfrak{a}_0})^{-1}(X) \times X$ and $(g_{\alpha_j} \circ \cdots \circ g_{\alpha_{-1}})^{-1}(X) \times X$.

It is immediate that $W^{\mathfrak{a}}$ is a vertical segment of second coordinate projection X. By hyperbolic continuation, for every C^∞-endomorphism $\tilde{\mathcal{F}}$ which is C^∞-close to $\mathcal{F}$ and every $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, we can define a local stable manifold $\tilde{W}^{\mathfrak{a}}$ which is a vertical graph of class C^∞ over $y \in X$ with small slope. We notice that these local stable manifolds are pairwise disjoint. We have analogous properties for local unstable manifolds (except their disjointness) and $\tilde{W}^{\mathfrak{a}}$ and $\tilde{W}^{\alpha}$ intersect in exactly one point.

Let us now consider any d-unfolding $(\mathcal{F}_p)_{p \in \mathcal{P}}$ of $\mathcal{F}$, i.e. a C^∞-family of endomorphisms $\mathcal{F}_p$ s.t. $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{P} = (-1, 1)^d$. Up to restricting and then rescaling the parameter space, this family leaves invariant the continuation $(\mathcal{K}_p)_{p \in \mathcal{P}}$ of the hyperbolic set $\mathcal{K}$ and $\mathcal{F}_p$ is a local diffeomorphism on a neighborhood of $\mathcal{K}_p$. We can define families of local stable and unstable manifolds $W_p^{\mathfrak{a}}$ and W_p^{α} as above, and we denote by ϕ_p the map sending $\beta = \alpha\mathfrak{a} \in \overleftrightarrow{\mathcal{A}}$ (with $\alpha \in \overleftarrow{\mathcal{A}}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$) to the unique intersection point $\phi_p(\alpha\mathfrak{a}) \in \mathcal{K}_p$ between $W_p^{\mathfrak{a}}$ and W_p^{α}. We denote:

$$\Phi_p : \beta \in \overleftrightarrow{\mathcal{A}} \mapsto \left(\phi_p(\sigma^i(\beta))\right)_i \in \overleftrightarrow{\mathcal{K}_p}.$$

We notice that Φ_p conjugates the full shift $(\overleftrightarrow{\mathcal{A}}, \sigma)$ to the dynamics $(\overleftrightarrow{\mathcal{K}_p}, \overleftrightarrow{\mathcal{F}_p})$ on the inverse limit and so periodic points are dense in $\mathcal{K}_p$ and $\mathcal{K}_p$ is transitive, and thus a hyperbolic basic set. The entropy $h_{\mathcal{F}_p}$ of $\mathcal{F}_p|\mathcal{K}_p$ is equal to $\log(n')$ and its stable contraction is close to $1/n$. We recall that:

$$\frac{\log(n')}{|\log(1/n)|} = \lceil \delta_{d,s} \rceil \cdot \frac{\log(n+1)}{\log(n)} > \delta_{d,s} \geq 1 \,.$$

Thus $h_{\mathcal{F}} > |\log(1/n)|$ and assumption $(\star)$ holds true for the family $(\mathcal{F}_p)_{p\in\mathcal{P}}$. By Theorems A and B, we conclude the following:

Proposition. *The set $\mathcal{K}$ is an almost C^∞-blender and for any d-unfolding $(\mathcal{F}_p)_{p\in\mathcal{P}}$ of $\mathcal{F}$, $(\mathcal{K}_p)_{p\in\mathcal{P}}$ is an almost $C^{\infty,s}$-parablender, up to restricting and rescaling $\mathcal{P}$.*

3 Skew-product Formalism and Strategy

Our method is based on a method introduced by Mihailescu, Simon, Solomyak and Urbański. Let us give some details. For IFS without overlaps, the Hausdorff dimension of the limit set is given by Bowen's formula [Bo2]. In [SSU], Simon, Solomyak and Urbański introduced a method to compute it even in the presence of overlaps. The key ingredient in their proofs (see Section 4) is a transversality property (see also [So] and [PeSo] for more on transversality). This also allows to get parameters for which the limit set has positive measure, which is our interest. Later these results were extended by Mihailescu and Urbański to the case of hyperbolic and fiberwise conformal skew-products in [MU]. Here we extend these to the setting of families of skew-products fiberwise unipotent.

3.1 Skew-products

We work with (N-dimensional) C^r-skew-products acting on $\overrightarrow{\mathcal{A}} \times [-1,1]^N$, where $N > 0$. Here $\mathcal{A}$ is a fixed finite alphabet of cardinality at least 2. For simplicity, we denote $X := [-1,1]^N$. The regularity r of the maps is given by either an integer at least 2 or $+\infty$.

Definition 3.1.1. *A* **pre-C^r-skew-product** *is a map of the form:*

$$F : (\mathfrak{a}, x) \in \overrightarrow{\mathcal{A}} \times X \mapsto (\sigma(\mathfrak{a}), f_{\mathfrak{a}}(x)) \in \overrightarrow{\mathcal{A}} \times X$$

satisfying that there exists an open set $X' \subset \mathbb{R}^N$ independent of $\mathfrak{a}$ s.t. $X \Subset X'$ and s.t. $f_{\mathfrak{a}} : X \to X$ extends to a C^r-diffeomorphism from X' to $f_{\mathfrak{a}}(X') \Subset X$ for every $\mathfrak{a}$.

The map F is a **C^r-skew-product** *if moreover the two following maps*

$$\mathfrak{a} \in \overrightarrow{\mathcal{A}} \mapsto f_{\mathfrak{a}} \in C^0(X', \mathbb{R}^N) \text{ and } \mathfrak{a} \in \overrightarrow{\mathcal{A}} \mapsto Df_{\mathfrak{a}} \in C^0(X', \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N))$$

are Hölder with positive exponent, and the following third map is continuous:

$$\mathfrak{a} \in \overrightarrow{\mathcal{A}} \mapsto D^2 f_{\mathfrak{a}} \in C^0(X', \mathcal{L}^2(\mathbb{R}^N, \mathbb{R}^N)) \,.$$

In the latter definition, $\overrightarrow{\mathcal{A}}$ is endowed with its distance and the spaces of C^0-maps from X' to $\mathbb{R}^N$, to the space $\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ of linear maps from $\mathbb{R}^N$ to $\mathbb{R}^N$ and to the space $\mathcal{L}^2(\mathbb{R}^N, \mathbb{R}^N)$ of bilinear maps from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$ endowed with the uniform C^0-metric. In the following, we suppose that the extensions of the maps $f_{\mathfrak{a}}$ are fixed.

We are even more interested in families $(F_p)_p$ of (pre-)C^r-skew-products, indexed by p varying in $[-1,1]^d$, for $1 \leq d < \infty$. We define such a family as a family of maps:

$$F_p : (\mathfrak{a}, x) \in \overrightarrow{\mathcal{A}} \times X \mapsto (\sigma(\mathfrak{a}), f_{p,\mathfrak{a}}(x)) \in \overrightarrow{\mathcal{A}} \times X$$

s.t. $\hat{F} : (\mathfrak{a}, (p, x)) \mapsto (\sigma(\mathfrak{a}), (p, f_{p,\mathfrak{a}}(x)))$ is a ($N + d$-dimensional) (pre-)C^r-skew-product. In particular, there exist open neighborhoods X' and $\mathcal{P}'$ of X and $\overline{\mathcal{P}}$ in $\mathbb{R}^N$ and $\mathbb{R}^d$ s.t. $(p, x) \mapsto (p, f_{p,\mathfrak{a}}(x))$ extends to a diffeomorphism from $\mathcal{P}' \times X'$ into $\mathcal{P} \times X$ for each $\mathfrak{a}$ and the map F_p is a (N-dimensional) (pre-)C^r-skew-product for every $p \in \mathcal{P}'$. Again, we denote $(F_p)_{p\in\mathcal{P}}$ this family since we are interested mainly on the dynamics when $p \in \mathcal{P}$ but still keep in mind that it admits such an extension.

We will say that such a family $(F_p)_{p\in\mathcal{P}}$ of (pre-)C^r-skew-products satisfies the **Unipotent** assumption **(U)** when the following is satisfied:

(U): For any $p \in \mathcal{P}'$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $x \in X'$, the differential $Df_{p,\mathfrak{a}}(x)$ is inferior unipotent, that is an inferior triangular matrix with all its diagonal coefficients equal the one other, and its unique eigenvalue is strictly bounded between 0 and 1 in modulus.

We will adopt the following formalism. For every $p \in \mathcal{P}'$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n > 0$ and $\alpha = (\alpha_{-n}, \dots, \alpha_{-1}) \in \mathcal{A}^*$, we set:

$$\forall x \in X',\ \psi^{\alpha}_{p,\mathfrak{a}}(x) := f_{p,\alpha_{-1}\mathfrak{a}} \circ \dots \circ f_{p,\alpha_{-n}\cdots\alpha_{-1}\mathfrak{a}}(x)\,.$$

We show below that a consequence of **(U)** is that $\psi^{\alpha}_{p,\mathfrak{a}}$ is a C^r-contraction from X' to $\psi^{\alpha}_{p,\mathfrak{a}}(X') \Subset X$ when $|\alpha|$ is large enough. If we now take an infinite backward sequence $\alpha = (\dots, \alpha_{-n}, \dots, \alpha_{-1}) \in \overleftarrow{\mathcal{A}}$, we see that the points $\psi^{\alpha_{|n}}_{p,\mathfrak{a}}(0)$ converge to a point $\pi_{p,\mathfrak{a}}(\alpha) \in X$. This defines a C^0-map $\pi_{p,\mathfrak{a}} : \overleftarrow{\mathcal{A}} \to X$.

Definition 3.1.2. *The* **limit set** $K_{p,\mathfrak{a}}$ *of the skew-product* F_p *inside the* $\mathfrak{a}$*-fiber is:*

$$K_{p,\mathfrak{a}} := \pi_{p,\mathfrak{a}}(\overleftarrow{\mathcal{A}})\,.$$

We will give conditions under which this set has positive measure.

For a C^1-map $f : X \to X$, let $m(Df)$ and $M(Df)$ be the respective minimum and maximum of $||Df(x) \cdot u||$ among $x \in X$ and $u \in \mathbb{R}^N$ s.t. $||u|| = 1$. We need the following thermodynamical formalism:

Definition 3.1.3. *The pressure at the parameter* p *in the* $\mathfrak{a}$*-fiber is the map:*

$$\Pi_{p,\mathfrak{a}} : s \in \mathbb{R}_+ \mapsto \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\alpha\in\mathcal{A}^n} M(D\psi^{\alpha}_{p,\mathfrak{a}})^s\,.$$

When $\Pi_{p,\mathfrak{a}}$ *has a unique zero, we call it the* **similarity dimension** *inside the* $\mathfrak{a}$*-fiber.*

In Proposition 5.1.4, we show that both the pressure and the similarity dimension are well-defined and independent of $\mathfrak{a}$. In particular, we denote them by Π_p and $\Delta(p)$.

We adopt the following terminology to denote perturbations with special properties:

Definition 3.1.4. *Let* $(F_p)_p$ *be a family of (pre-)*C^r*-skew-products and let us fix neighborhoods* X' *and* $\mathcal{P}'$ *of* X *and* $\overline{\mathcal{P}}$ *s.t.* $(p, x) \mapsto (p, f_{p,\mathfrak{a}}(x))$ *extends to a diffeomorphism from* $\mathcal{P}' \times X'$ *into* $\mathcal{P} \times X$ *for every* $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$*. For any* $\vartheta > 0$*, a* ϑ**-perturbation** *of* $(F_p)_p$ *is a family of pre-*C^r*-skew-products* $(\tilde{F}_p)_p$ *s.t. the map* $(p, x) \mapsto (p, \tilde{f}_{p,\mathfrak{a}}(x))$ *extends to a diffeomorphism from* $\mathcal{P}' \times X'$ *into* $\mathcal{P} \times X$ *for every* $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ *and:*

$$\sup_{\mathfrak{a}\in\overrightarrow{\mathcal{A}}} ||(p, x) \in X' \times \mathcal{P}' \mapsto (f_{p,\mathfrak{a}} - \tilde{f}_{p,\mathfrak{a}})(x)||_{C^r} < \vartheta\,.$$

The family $(\tilde{F}_p)_p$ *is a* ϑ**-**U**-perturbation** *when it satisfies assumption* (U)*.*

For ϑ-U-perturbations with small ϑ, we will show that for any $\alpha \in \overleftarrow{\mathcal{A}}$, the points

$$\tilde{f}_{p,\alpha_{-1}\mathfrak{a}} \circ \ldots \circ \tilde{f}_{p,\alpha_{-n}\cdots\alpha_{-1}\mathfrak{a}}(0)$$

converge to $\tilde{\pi}_{p,\mathfrak{a}}(\alpha) \in X$ s.t. $p \mapsto \tilde{\pi}_{p,\mathfrak{a}}(\alpha)$ is a C^r-map C^r-close to $p \mapsto \pi_{p,\mathfrak{a}}(\alpha)$.

We will consider also **parameterized families of ϑ-perturbations** $(\tilde{F}_{t,p})_p$:

$$\mathbb{F} := ((\tilde{F}_{t,p})_p)_{t\in\mathcal{T}} .$$

Here t varies in $\mathcal{T} := (-1,1)^\tau$ with $\tau > 0$ and $(t,p,x) \mapsto \tilde{f}_{t,p,\mathfrak{a}}(x)$ is C^r for every $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$. When each $(\tilde{F}_{t,p})_p$ is a ϑ-U-perturbation, we say that $\mathbb{F}$ is a parameterized families of ϑ-U-perturbations. When ϑ is small enough, we will denote by $\tilde{\pi}_{t,p,\mathfrak{a}}(\alpha)$ the limit point corresponding to any $\alpha \in \overleftarrow{\mathcal{A}}$ and $\tilde{K}_{t,p,\mathfrak{a}} := \tilde{\pi}_{t,p,\mathfrak{a}}(\overleftarrow{\mathcal{A}})$, and then the C^r-maps $p \mapsto \tilde{\pi}_{t,p,\mathfrak{a}}(\alpha)$ will be C^r-close to $p \mapsto \pi_{p,\mathfrak{a}}(\alpha)$, uniformly in (t,α). We will set conditions under which $\tilde{K}_{t,p,\mathfrak{a}}$ has positive Lebesgue measure for a.e. $t \in \mathcal{T}$.

3.2 Strategy and Organization of the paper

The strategy will be to focus on the dynamics restricted to the local stable manifolds, which allows us to reduce the dynamics to that one of a C^r-skew-product.

Hence we forget for some time families of endomorphisms and we work with families of (pre-)C^r-skew-products. We say that such a family of (pre-)C^r-skew-products $(F_p)_{p\in\mathcal{P}}$ satisfying assumption **(U)** satisfies also the **Transversality** assumption **(T)** when the following is satisfied:

(T): There exists $C > 0$ such that for every sequences $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $\alpha, \beta \in \overleftarrow{\mathcal{A}}$ satisfying $\alpha_{-1} \neq \beta_{-1}$, and for every $r > 0$, we have:

$$\mathrm{Leb}_d\{p \in \mathcal{P} : ||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| < r\} \leq Cr^N ,$$

and moreover for every small $\vartheta > 0$ and every family $\mathbb{F}$ of ϑ-U-perturbations, for any $t \in \mathcal{T}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha, \beta \in \overleftarrow{\mathcal{A}}$ s.t. $\alpha_{-1} \neq \beta_{-1}$ and $r > 0$ we have:

$$\mathrm{Leb}_d\{p \in \mathcal{P} : ||\tilde{\pi}_{t,p,\mathfrak{a}}(\alpha) - \tilde{\pi}_{t,p,\mathfrak{a}}(\beta)|| < r\} \leq Cr^N .$$

The main technical result to prove Theorem B is the following. It sets conditions under which a given family of skew-products intersects its fibers into a set of positive measure, up to perturbations.

Theorem C. *Let $(F_p)_{p\in\mathcal{P}}$ be a family of C^r-skew-products satisfying* (**U**), (**T**) *and $\Delta(p) > N$ for any $p \in \overline{\mathcal{P}}$. Then for every $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, we have:*

$$\mathrm{Leb}_N(K_{p,\mathfrak{a}}) > 0 \textit{ for } \mathrm{Leb}_d \textit{ a.e. } p \in \mathcal{P} ,$$

and for every family $\mathbb{F}$ of ϑ-U-perturbations of $(F_p)_{p\in\mathcal{P}}$ with small ϑ, it holds:

$$\mathrm{Leb}_N(\tilde{K}_{t,p,\mathfrak{a}}) > 0 \textit{ for } \mathrm{Leb}_d \textit{ a.e. } p \in \mathcal{P} \textit{ and } \mathrm{Leb}_\tau \textit{ a.e. } t \in \mathcal{T} .$$

Here is the strategy to prove Theorem C. For every parameter p and $\mathfrak{a}$-fiber, we define a probability measure $\nu_{p,\mathfrak{a}}$ supported on the limit set $K_{p,\mathfrak{a}}$. To show that $K_{p,\mathfrak{a}}$ has positive measure, it is enough to show that $\nu_{p,\mathfrak{a}}$ is absolutely continuous relatively to the N-dimensional Lebesgue measure, and so to prove that its density is finite almost everywhere. We compute the integral of the density relatively to the parameter and the phase space. The trick is to use the Fubini Theorem to integer first relatively to p. The finiteness of the integral is implied by the transversality assumption **(T)** and the inequality $\Delta(p) > N$. The same method will give the same

results for families of ϑ-U-perturbations, with additional integration relatively to t.

To prove Theorem B, we go back to C^r-families $(\mathcal{F}_p)_p$ of endomorphisms and we restrict the dynamics to the local stable manifolds $W_p^{\mathfrak{a}}$, which are tagged in exponent by infinite forward sequences $\mathfrak{a}$ in letters in an alphabet $\mathcal{A}$. Since $\dim \mathcal{E}_p^s = 1$, we are led to study families $(F_p)_p$ of C^r-skew-products acting on fibers which are segments.

We then look at the action induced by $(F_p)_p$ on s-jets (with $s \leq r-2$) and this gives a new family of C^2-skew-products $(G_{p_0})_{p_0}$ acting on fibers of dimension $\delta_{d,s}$. This new family satisfies assumption $(\mathbf{U})$ (the assumption $\dim \mathcal{E}_p^s = 1$ is used here), its similarity dimension is larger than $\delta_{d,s}$ by $(\star)$. We extend $(\mathcal{F}_p)_p$ into a larger C^r-family of endomorphisms so that the associated extended family of C^2-skew-products acting on s-jets satisfies moreover the transversality assumption $\mathbf{(T)}$.

To conclude, we pick a family $(\Gamma_{t,p})_{t,p}$ of parallel segments $\Gamma_{t,p}$ close to a local stable manifold $W_p^{\mathfrak{a}}$ and s.t. the set of s-jets at any $p_0 \in \mathcal{P}$ of the projection of $\Gamma_{t,p}$ on a fixed direction transversal to $\Gamma_{t,p}$ when varying t has positive measure. The local unstable set intersects each segment $\Gamma_{t,p}$ in a set which is the limit set of a ϑ-U-perturbation with small ϑ, at every parameter p. We then apply the second part of Theorem C to get positive sets of s-jets for the intersection points between the local unstable set and $\Gamma_{t,p}$ for a.e. t at a.e. p_0: in other terms we get positive sets of s-jets in the direction of $\Gamma_{t,p}$ for these values of t and p_0. To conclude, we apply the Fubini Theorem to find positive sets of bidimensional s-jets for points inside local unstable manifolds at a.e. p_0. The same extension scheme works for $(\mathcal{G}_p)_p$ close to $(\mathcal{F}_p)_p$, which proves that $(\mathcal{K}_p)_p$ is an almost $C^{r,s}$-parablender.

Finally, Theorem A is an immediate consequence of Theorem B together with Remark 6.0.4, by taking the constant family $(\mathcal{F})_{p\in(-1,1)}$ and the order s of the jets equal to 0 (remark in particular that $\delta_{1,0} = 1$).

In Section 4, we study a model given by families of IFS of affine maps on an interval. We simplify the proof of Simon, Solomyak and Urbański [SSU] in this context and introduce the strategy of the proof of Theorem C here. In Section 5, we prove Theorem C. In each fiber, the behavior of the dynamics looks like the model. Finally, we prove Theorem B in Section 6.

4 Model: IFS of affine maps on the interval

4.1 Setting and results

In this Section, we simplify the proof of a result of Simon, Solomyak and Urbański about IFS on an interval. This can be seen as a model for the behavior of the dynamics inside the fibers of a skew-product, as we will see in Section 5.

Let us fix $X := [-1,1]$ and $\mathcal{P} := (-1,1)$. We consider families $(\Psi_p)_{p\in\mathcal{P}}$, where, for every $p \in \mathcal{P}$, the IFS Ψ_p is a finite family $\Psi_p = (\psi_p^a)_{a\in\mathcal{A}}$ of affine contractions $\psi_p^a : X \to X$ s.t. $\psi_p^a(X) \Subset X$. The absolute value of the linear coefficient of ψ_p^a is denoted by $\Lambda_{p,a}$. We suppose that for every $a \in \mathcal{A}$, the map ψ_p^a depends continuously on p. In fact, we even suppose that the affine contraction ψ_p^a is still defined for p in some open neighborhood of $[-1,1]$ and still depends continuously on p.

For every $p \in \overline{\mathcal{P}}$ and $\alpha = (\alpha_{-n}, \cdots, \alpha_{-1}) \in \mathcal{A}^*$, we denote by $\psi_p^\alpha = \psi_p^{\alpha_{-1}} \circ \cdots \circ \psi_p^{\alpha_{-n}}$ the corresponding composition, which is an affine contraction of the segment X into its own interior. When $\alpha = e$, the map ψ_p^e is simply the identity. The absolute value of the linear coefficient of ψ_p^α is denoted by $\Lambda_{p,\alpha}$.

By continuity of the derivative of ψ_p^a relatively to $p \in \overline{\mathcal{P}}$, there exist $0 < \gamma' < \gamma < 1$ s.t. $\gamma' < \Lambda_{p,a} < \gamma$ for every $p \in \overline{\mathcal{P}}$ and $a \in \mathcal{A}$. Then $\Lambda_{p,\alpha} \leq \gamma^{|\alpha|}$ for every $\alpha \in \mathcal{A}^*$. If we now take $\alpha \in \overleftarrow{\mathcal{A}}$, the sequence of points $\psi_p^{\alpha_{|n}}(0)$ tends to a point $\pi_p(\alpha)$. This defines for every $p \in \overline{\mathcal{P}}$ a C^0-map $\pi_p : \overleftarrow{\mathcal{A}} \to X$. Since this convergence is uniform in p, the map $p \mapsto \pi_p(\alpha)$ is C^2 for any α. Moreover the map $p \mapsto \pi_p$ is continuous, the set of C^0-maps from $\overleftarrow{\mathcal{A}}$ to $\mathbb{R}$ being endowed with the uniform C^0-norm. We set:

$$K_p := \pi_p(\overleftarrow{\mathcal{A}})\,.$$

We also suppose that the following assumption $(\mathbf{T_{aff}})$ is satisfied by $(\Psi_p)_{p\in\mathcal{P}}$.

$(\mathbf{T_{aff}})$: There exists $C > 0$ s.t. for every $\alpha, \beta \in \overleftarrow{\mathcal{A}}$ satisfying $\alpha_{-1} \neq \beta_{-1}$, we have:

$$\mathrm{Leb}_1\{p \in \mathcal{P} : |\pi_p(\alpha) - \pi_p(\beta)| < r\} \leq Cr \text{ for any } r > 0\,.$$

It is immediate that for any $p \in \overline{\mathcal{P}}$, there exists exactly one number $\Delta(p) \geq 0$ s.t.:

$$\sum_{a\in\mathcal{A}} \Lambda_{p,a}^{\Delta(p)} = 1\,.$$

This is the similarity dimension of the IFS Ψ_p. We are now in position to state the following result, which is a direct consequence of Theorem 3.1 of [SSU]:

Theorem D. *(Simon, Solomyak, Urbański) Let $(\Psi_p)_{p\in\mathcal{P}}$ be a family of IFS of affine contractions satisfying $(\mathbf{T_{aff}})$ and $\Delta(p) > 1$ for any $p \in \overline{\mathcal{P}}$. Then it holds:*

$$\mathrm{Leb}_1(K_p) > 0 \textit{ for } \mathrm{Leb}_1 \textit{ a.e. } p \in \mathcal{P}\,.$$

4.2 Proof of Theorem D

Proof of Theorem D. Here is the strategy. Let $p_0 \in \mathcal{P}$ such that $\Delta(p_0) > 1 + \epsilon$ for a small $\epsilon > 0$. To prove the result, it is enough to show that there exists $\delta > 0$ s.t. $\mathcal{B} := (p_0 - \delta, p_0 + \delta)$ is included in $\mathcal{P}$ and $\mathrm{Leb}_1(K_p) > 0$ for Leb_1 a.e. $p \in \mathcal{B}$.

We define a probability measure μ on $\overleftarrow{\mathcal{A}}$ by setting $\mu[\alpha] = \Lambda_{p_0,\alpha}^{\Delta(p_0)}$ for every cylinder defined by $\alpha \in \mathcal{A}^*$. For any $p \in \mathcal{P}$, let ν_p be the pushforward of μ by π_p, which is supported on K_p. To conclude, it is enough to show that there exists $\delta > 0$ s.t. for Leb_1 a.e. $p \in \mathcal{B}$, the measure ν_p is absolutely continuous relatively to Leb_1. We set

$$D(\nu_p, x) := \liminf_{r\to 0} \frac{\nu_p(x - r, x + r)}{2r}$$

for every $p \in \mathcal{P}$ and $x \in \mathbb{R}$, which is the lower density of the measure ν_p at x.

Lemma 4.2.1. *The map $(p, x) \in \mathcal{P} \times \mathbb{R} \mapsto D(\nu_p, x)$ is Borel measurable.*

Proof. Since $p \in \mathcal{P} \mapsto \pi_p$ is continuous, it is also the case for $p \in \mathcal{P} \mapsto \nu_p$ (the set of probability measures being endowed with the weak-$*$ topology). Using this and since $x \in \mathbb{R} \mapsto \nu_p(x - r, x + r)$ is Borel measurable for every $p \in \mathcal{P}$ and $r > 0$, the map $(p, x) \in \mathcal{P} \times \mathbb{R} \mapsto \nu_p(x - r, x + r)$ is Borel measurable for every $r > 0$. Since $r \mapsto \nu_p(x - r, x + r)$ is monotonic and $r \mapsto 2r$ continuous, the lower limit $D(\nu_p, x)$ does not change if r is restricted to positive rationals. Thus the measurability of $(p, x) \in \mathcal{P} \times \mathbb{R} \mapsto D(\nu_p, x)$ reduces to that of the lower limit of countably many measurable maps. □

We prove below:

Proposition 4.2.2. *There exists $\delta > 0$ s.t. $\mathcal{B} \subset \mathcal{P}$ and the following is finite:*

$$\mathcal{I} := \int_{p\in\mathcal{B}} \int_{x\in\mathbb{R}} D(\nu_p, x) d\nu_p d\mathrm{Leb}_1 < +\infty\,.$$

This is enough to show that for Leb_1 a.e. $p \in \mathcal{B}$, ν_p is absolutely continuous relatively to Leb_1. Indeed, then, for Leb_1 a.e. $p \in \mathcal{B}$, we will have $D(\nu_p, x) < +\infty$ for ν_p a.e. $x \in \mathbb{R}$ and we apply the following result from the third item of Lemma 2.12 in [Ma].

Proposition 4.2.3. *Let ν be a Radon measure on $\mathbb{R}^n$, where $n > 0$, s.t. the density $D(\nu, x)$ of ν relatively to Leb_n is finite for ν a.e. $x \in \mathbb{R}^n$. Then ν is absolutely continuous relatively to Leb_n.*

This concludes the proof of Theorem D. □

Proof of Proposition 4.2.2. For δ small enough, the interval $\mathcal{B} := (p_0 - \delta, p_0 + \delta)$ is included in $\mathcal{P}$. If necessary, we reduce δ so that $\Lambda_{p_1,a}^{1+\epsilon/2} \leq \Lambda_{p_2,a}$ for every $p_1, p_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. In particular, this implies the following:

$$\forall p_1, p_2 \in \mathcal{B},\ \forall \alpha \in \mathcal{A}^*,\ \Lambda_{p_1,\alpha}^{1+\frac{\epsilon}{2}} \leq \Lambda_{p_2,\alpha}\,. \tag{3}$$

The strategy is to bound $\mathcal{I}$ by a new integral which will be easily shown to be finite using $(\mathbf{T_{aff}})$, for this specific choice of δ. First, by Fatou's lemma, it holds:

$$\mathcal{I} \leq \liminf_{r\to 0} \frac{1}{2r} \int_{p\in\mathcal{B}} \int_{x\in\mathbb{R}} \nu_p(x-r, x+r) d\nu_p d\mathrm{Leb}_1\,. \tag{4}$$

We can write $\nu_p(x-r, x+r) = \int_{y\in\mathbb{R}} 1_{\{|x-y|<r\}} d\nu_p$ as the integral of the indicator function $1_{\{|x-y|<r\}}$, equal to 1 if $|x-y| < r$, and 0 if not. Using this and then the definition of ν_p as the pushforward of μ by π_p, we have:

$$\int_{x\in\mathbb{R}} \nu_p(x-r, x+r) d\nu_p = \int_{(\alpha,\beta)\in\overleftarrow{\mathcal{A}}\times\overleftarrow{\mathcal{A}}} 1_{\{|\pi_p(\alpha)-\pi_p(\beta)|<r\}} d\mu\times\mu\,, \tag{5}$$

where $1_{\{|\pi_p(\alpha)-\pi_p(\beta)|<r\}}$ is equal to 1 if $|\pi_p(\alpha) - \pi_p(\beta)| < r$ and 0 if not. Then, we inject Eq. (5) into Eq. (4) and use Fubini's Theorem to reverse the order of integration:

$$\mathcal{I} \leq \liminf_{r\to 0} \frac{1}{2r} \int_{(\alpha,\beta)\in\overleftarrow{\mathcal{A}}\times\overleftarrow{\mathcal{A}}} \mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\alpha)-\pi_p(\beta)| < r\} d\mu\times\mu\,. \tag{6}$$

We are going to write the latter integral as a sum whose terms are all easier to bound. For every finite word $\rho \in \mathcal{A}^*$, we denote by $\mathcal{C}_\rho$ the set of pairs $(\alpha, \beta) \in \overleftarrow{\mathcal{A}} \times \overleftarrow{\mathcal{A}}$ such that $\alpha_{-|\rho|} \cdots \alpha_{-1} = \beta_{-|\rho|} \cdots \beta_{-1} = \rho$ but $\alpha_{-|\rho|-1} \neq \beta_{-|\rho|-1}$. We notice that $\overleftarrow{\mathcal{A}} \times \overleftarrow{\mathcal{A}} = \bigsqcup_{n\geq 0} \bigsqcup_{\rho\in\mathcal{A}^n} \mathcal{C}_\rho$ and so by Eq. (6) we have:

$$\mathcal{I} \leq \liminf_{r\to 0} \frac{1}{2r} \sum_{n\geq 0} \sum_{\rho\in\mathcal{A}^n} \int_{(\alpha,\beta)\in\mathcal{C}_\rho} \mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\alpha)-\pi_p(\beta)| < r\} d\mu\times\mu\,. \tag{7}$$

We show below that a consequence of the transversality assumption $(\mathbf{T_{aff}})$ is:

Lemma 4.2.4. *For every $n \geq 0$, $\rho \in \mathcal{A}^n$ and $(\alpha, \beta) \in \mathcal{C}_\rho$, we have:*

$$\mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\alpha)-\pi_p(\beta)| < r\} \preceq r \cdot \Lambda_{p_0,\rho}^{-1-\epsilon/2}\,.$$

We can inject the bound of Lemma 4.2.4 into Eq. (7):

$$\mathcal{I} \preceq \sum_{n\geq 0} \sum_{\rho\in\mathcal{A}^n} \int_{(\alpha,\beta)\in\mathcal{C}_\rho} \Lambda_{p_0,\rho}^{-1-\epsilon/2} d\mu\times\mu\,. \tag{8}$$

We use the equality $\mu[\rho] = \Lambda_{p_0,\rho}^{\Delta(p_0)}$, the inequality $(1+\frac{\epsilon}{2})/\Delta(p_0) < (1+\frac{\epsilon}{2})/(1+\epsilon) < 1 - \frac{\epsilon}{3}$ and finally the inequality $\gamma' < \Lambda_{p,a} < \gamma$ to get:

$$\Lambda_{p_0,\rho}^{-1-\epsilon/2} \asymp \mu[\rho]^{\frac{-1-\epsilon/2}{\Delta(p_0)}} \leq \mu[\rho]^{-(1-\frac{\epsilon}{3})} \preceq \gamma^{\frac{n\epsilon}{3}} \cdot \mu[\rho]^{-1}\,. \tag{9}$$

We now inject this bound into Eq. (8) to find:

$$\mathcal{I} \preceq \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{3}} \sum_{\rho\in\mathcal{A}^n} \frac{(\mu\times\mu)(\mathcal{C}_\rho)}{\mu[\rho]} \leq \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{3}} \sum_{\rho\in\mathcal{A}^n} \mu[\rho] = \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{3}} < +\infty\,,$$

where we used the inequality $(\mu\times\mu)(\mathcal{C}_\rho) \leq \mu[\rho]^2$ (coming from $\mathcal{C}_\rho \subset [\rho]^2$) to prove the second inequality. This concludes the proof of Proposition 4.2.2. □

Proof of Lemma 4.2.4. For every $p \in \mathcal{P}$, $n \geq 0$, $\rho \in \mathcal{A}^n$ and $(\alpha,\beta) \in \mathcal{C}_\rho$, it holds:

$$|\pi_p(\alpha) - \pi_p(\beta)| = \Lambda_{p,\rho} \cdot |\pi_p(\sigma^n(\alpha)) - \pi_p(\sigma^n(\beta))|\,. \tag{10}$$

Indeed, the points $\pi_p(\alpha)$ and $\pi_p(\beta)$ are the respective images of $\pi_p(\sigma^n(\alpha))$ and $\pi_p(\sigma^n(\beta))$ by the map ψ_p^ρ which is an affine contraction, and the absolute value of the linear coefficient of ψ_p^ρ is $\Lambda_{p,\rho}$. Thus, using Eq. (3), it holds:

$$\mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\alpha)-\pi_p(\beta)| < r\} = \mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\sigma^n(\alpha))-\pi_p(\sigma^n(\beta))| < \frac{r}{\Lambda_{p,\rho}}\}$$

$$\mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\alpha)-\pi_p(\beta)| < r\} \leq \mathrm{Leb}_1\{p\in\mathcal{B} : |\pi_p(\sigma^n(\alpha))-\pi_p(\sigma^n(\beta))| < \frac{r}{\Lambda_{p_0,\rho}^{1+\frac{\epsilon}{2}}}\}$$

To conclude, by $(\mathbf{T_{aff}})$ and since $\mathcal{B}\subset\mathcal{P}$ the right-hand term of the latter is smaller than $C\cdot r\cdot \Lambda_{p_0,\rho}^{-1-\epsilon/2}$. □

Example. Let us give a simple example of application of Theorem D. Let $n \geq 2$ be an integer. We set $\mathcal{A} := \{0,1,2,\dots,n\}$, $X := [0,1]$ and $\mathcal{P} := (1/n, 1-1/n)$. Let $c < 1/n$ be a real number close to $1/n$. For $a \in \mathcal{A}$, we put $\psi_p^a(x) := cx + \frac{1}{2}(1/n - c) + a/n$ if $0 \leq a < n$ and $\psi_p^n(x) := cx + p$ if $a = n$. Condition $(\mathbf{T_{aff}})$ is clearly satisfied when n is large. Moreover trivial computations show that the similarity dimension is $\Delta(p) = -\log(n+1)/\log(c) > 1$ for any $p \in \overline{\mathcal{P}}$. By Theorem D, K_p has positive one-dimensional Lebesgue measure for a.e. $p \in \mathcal{P}$.

5 The unipotent case: Proof of Theorem C

We now extend Theorem D to the case of families of fiberwise unipotent skew-products. The fibers are indexed by $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and the dynamics on each fiber will look like the one of the model previously introduced. Here are some differences:

- We will not restrain ourselves to fibers of dimension 1 and we will not suppose that the dynamics on each fiber is conformal but we will suppose that its differentials are unipotent with contracting eigenvalue (assumption $(\mathbf{U})$).
- We will need distortion results (Lemmas 5.2.1, 5.2.2, 5.2.3 and 5.2.4) since the dynamics will not supposed to be affine this time.

5.1 Notations and immediate facts

We adopt from now the following formalism in order to prove Theorem C. Let $(F_p)_{p\in\mathcal{P}}$ be a family of C^r-skew-products satisfying $(\mathbf{U})$ and $(\mathbf{T})$. We recall that there exist open neighborhoods X' and $\mathcal{P}'$ of X and $\overline{\mathcal{P}}$ in $\mathbb{R}^N$ and $\mathbb{R}^d$ s.t. each map $(p,x)\mapsto(p, f_{p,\mathfrak{a}}(x))$ extends to a diffeomorphism from $X'\times\mathcal{P}'$ into $X\times\mathcal{P}$ and the map F_p is a (N-dimensional) C^r-skew-product for every $p\in\mathcal{P}'$. We set:

$$\forall p\in\mathcal{P}', \mathfrak{a}\in\overrightarrow{\mathcal{A}}, a\in\mathcal{A}, x\in X',\ \psi_{p,\mathfrak{a}}^a(x) := f_{p,a\mathfrak{a}}(x) \tag{11}$$

and notice that $\psi_{p,\mathfrak{a}}^a : X' \to X$ is a C^2-map depending continuously on $(p,\mathfrak{a})$. The C^2-norm of $\psi_{p,\mathfrak{a}}^a$ on X is then bounded independently of $p\in\overline{\mathcal{P}}$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$ and $a\in\mathcal{A}$. We now define for any $p\in\mathcal{P}'$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$, $n>0$ and $\alpha = (\alpha_{-n},\cdots,\alpha_{-1}) \in \mathcal{A}^*$:

$$\forall x\in X',\ \psi_{p,\mathfrak{a}}^\alpha(x) := \psi_{p,\mathfrak{a}}^{\alpha_{-1}} \circ\cdots\circ \psi_{p,\alpha_{-n+1}\cdots\alpha_{-1}\mathfrak{a}}^{\alpha_{-n}}(x) = f_{p,\alpha_{-1}\mathfrak{a}}\circ\cdots\circ f_{p,\alpha_{-n}\cdots\alpha_{-1}\mathfrak{a}}(x)\,.$$

In particular, assumption **(U)** implies that for every $p \in \mathcal{P}'$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha \in \mathcal{A}^*$ and $x \in X'$, the differential $D\psi^{\alpha}_{p,\mathfrak{a}}(x)$ is unipotent inferior and thus has a unique eigenvalue. This motivates the definition of the following contraction rate:

Definition 5.1.1. *For any $p \in \mathcal{P}'$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha \in \mathcal{A}^*$ and $x \in X'$, let $\lambda_{p,\mathfrak{a},\alpha}(x)$ be the absolute value of the unique eigenvalue of the differential $D\psi^{\alpha}_{p,\mathfrak{a}}(x)$ and:*

$$\Lambda_{p,\mathfrak{a},\alpha} := \max_{x\in X} \lambda_{p,\mathfrak{a},\alpha}(x)\,. \tag{12}$$

For any $p \in \mathcal{P}'$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha = (\alpha_{-n}, \dots, \alpha_{-1}) \in \mathcal{A}^*$ and $x \in X'$, we then have:

$$\lambda_{p,\mathfrak{a},\alpha}(x) = \prod_{k=1}^{n} \lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\big(\psi^{\alpha_{-k-1}}_{p,\mathfrak{a}_{k+1}} \circ \cdots \circ \psi^{\alpha_{-n}}_{p,\mathfrak{a}_n}(x)\big) \text{ with } \mathfrak{a}_k := \alpha_{|k-1}\mathfrak{a}\,. \tag{13}$$

By continuity of $D\psi^{a}_{p,\mathfrak{a}}(x)$ relatively to $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $x \in X$ and by compactness of $\overline{\mathcal{P}}$, $\overrightarrow{\mathcal{A}}$ and X, there exist $0 < \gamma' < \gamma < 1$ s.t. for any p, $\mathfrak{a}$, a, it holds:

$$\forall x \in X,\ \gamma' < \lambda_{p,\mathfrak{a},a}(x) < \gamma\,. \tag{14}$$

In particular for every $\alpha \in \mathcal{A}^*$ it holds:

$$\gamma'^{|\alpha|} < \Lambda_{p,\mathfrak{a},\alpha} < \gamma^{|\alpha|}\,. \tag{15}$$

We will need later the following, whose proof is in the Appendix:

Lemma 5.1.2. *There exists a real polynomial P positive on $\mathbb{R}_+$ s.t. for any $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha \in \mathcal{A}^*$, $x \in X$ and $(i,j) \in \{1, \cdots, N\}^2$ with $i > j$, the modulus of the $(i,j)^{th}$ coefficient of the differential $D\psi^{\alpha}_{p,\mathfrak{a}}(x)$ is smaller than $P(|\alpha|) \cdot \lambda_{p,\mathfrak{a},\alpha}(x)$.*

In particular, for any $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $\alpha \in \mathcal{A}^*$ of length sufficiently large, the map $\psi^{\alpha}_{p,\mathfrak{a}}$ is a contraction. For $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $\alpha \in \overleftarrow{\mathcal{A}}$, the diameter of $\psi^{\alpha_{|n}}_{p,\mathfrak{a}}(X)$ is then small when n is large. Thus the sequence of points $\psi^{\alpha_{|n}}_{p,\mathfrak{a}}(0)$ converges to $\pi_{p,\mathfrak{a}}(\alpha) \in X$. This defines for every $p \in \overline{\mathcal{P}}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ a C^0-map $\pi_{p,\mathfrak{a}} : \overleftarrow{\mathcal{A}} \to X$. The map $(p,\mathfrak{a}) \in \overline{\mathcal{P}} \times \overrightarrow{\mathcal{A}} \mapsto \pi_{p,\mathfrak{a}}$ is then continuous, the set of C^0-maps from $\overleftarrow{\mathcal{A}}$ to $\mathbb{R}^N$ being endowed with the uniform C^0-norm. We set:

$$K_{p,\mathfrak{a}} := \pi_{p,\mathfrak{a}}(\overleftarrow{\mathcal{A}})\,.$$

For any family $\mathbb{F} := ((\tilde{F}_{t,p})_p)_{t\in\mathcal{T}}$ of ϑ-U-perturbations with small $\vartheta > 0$, the map $\psi^{\alpha}_{t,p,\mathfrak{a}} := \tilde{f}_{t,p,\alpha_{-1}\mathfrak{a}} \circ \cdots \circ \tilde{f}_{t,p,\alpha_{-n}\cdots\alpha_{-1}\mathfrak{a}}$ is also a contraction when $|\alpha|$ is large and so the sequence of points $\psi^{\alpha_{|n}}_{t,p,\mathfrak{a}}(0)$ still converges to $\tilde{\pi}_{t,p,\mathfrak{a}}(\alpha) \in X$. This allows us to define a C^0-map $\tilde{\pi}_{t,p,\mathfrak{a}} : \overleftarrow{\mathcal{A}} \to X$ for any t and p, and its limit set $\tilde{K}_{t,p,\mathfrak{a}} := \tilde{\pi}_{t,p,\mathfrak{a}}(\overleftarrow{\mathcal{A}})$.

Lemma 5.1.3. *The map $p \mapsto \pi_{p,\mathfrak{a}}(\alpha)$ is C^r for every $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $\alpha \in \overleftarrow{\mathcal{A}}$. Moreover, for every family $\mathbb{F}$ of ϑ-U-perturbations with small $\vartheta > 0$, the map $(t,p) \mapsto \tilde{\pi}_{t,p,\mathfrak{a}}(\alpha)$ is C^r and $p \mapsto \tilde{\pi}_{t,p,\mathfrak{a}}(\alpha)$ is C^r-close to $p \mapsto \tilde{\pi}_{p,\mathfrak{a}}(\alpha)$ uniformly in $t \in \mathcal{T}$.*

Proof. These maps are the respective uniform limits of the following C^r-maps

$$p \mapsto \psi^{\alpha_{|n}}_{p,\mathfrak{a}}(0) \text{ and } (t,p) \mapsto \psi^{\alpha_{|n}}_{t,p,\mathfrak{a}}(0)\,.$$

To conclude, we remark that these convergences are both exponential and that the maps $f_{p,\mathfrak{a}}$ and $\tilde{f}_{t,p,\mathfrak{a}}$ are (uniformly in t and $\mathfrak{a}$) C^r-close when ϑ is small. □

As an immediate consequence of Lemma 5.1.2, the pressure function $\Pi_{p,\mathfrak{a}}$ defined in Definition 3.1.3 is equal to:

$$\Pi_{p,\mathfrak{a}}(s) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\alpha\in\mathcal{A}^n} \Lambda^s_{p,\mathfrak{a},\alpha} \text{ for any } s \geq 0\,. \tag{16}$$

Moreover this map satisfies the following nice properties:

Proposition 5.1.4. *The map $s \in \mathbb{R}_+ \mapsto \Pi_{p,\mathfrak{a}}(s) \in \mathbb{R}$ is well-defined, strictly decreasing, continuous, independent of $\mathfrak{a}$, $\Pi_{p,\mathfrak{a}}(0) > 0$ and $\lim_{s\to+\infty}\Pi_{p,\mathfrak{a}}(s) = -\infty$. In particular, it has exactly one zero denoted by $\Delta(p)$, depending continuously on p.*

From now on we suppose that $\Delta(p) > N$ for any $p \in \overline{\mathcal{P}}$.

5.2 Distortion lemmas

We now state distortion results, whose proofs are given in the Appendix:

Lemma 5.2.1. *(Bounded distortion w.r.t. x) There exists $D_1 > 1$ s.t. for every $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $\alpha \in \mathcal{A}^*$ and $x, y \in X$, it holds:*

$$1/D_1 < \frac{\lambda_{p,\mathfrak{a},\alpha}(x)}{\lambda_{p,\mathfrak{a},\alpha}(y)} < D_1 \,.$$

Lemma 5.2.2. *(Distortion w.r.t. p) For every $\eta > 0$, there exists $\delta(\eta) > 0$ and $D_2 = D_2(\eta) > 1$ s.t. for every $p_1, p_2 \in \overline{\mathcal{P}}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, it holds the following:*

$$||p_1 - p_2|| \leq \delta(\eta) \implies \forall \alpha \in \mathcal{A}^*,\ D_2^{-1} e^{-\eta|\alpha|} < \frac{\Lambda_{p_1,\mathfrak{a},\alpha}}{\Lambda_{p_2,\mathfrak{a},\alpha}} < D_2 e^{\eta|\alpha|} \,.$$

Lemma 5.2.3. *(Bounded distortion w.r.t. $\mathfrak{a}$) There exists $D_3 > 1$ s.t. for any $p \in \overline{\mathcal{P}}$, $\mathfrak{a}, \mathfrak{a}' \in \overrightarrow{\mathcal{A}}$ and $\alpha \in \mathcal{A}^*$, it holds:*

$$1/D_3 < \frac{\Lambda_{p,\mathfrak{a},\alpha}}{\Lambda_{p,\mathfrak{a}',\alpha}} < D_3 \,.$$

Lemma 5.2.4. *(Distortion w.r.t. ϑ-perturbations) For every $\epsilon' > 1$, there exists $D_4 > 1$ s.t. for every family $\mathbb{F}$ of ϑ-U-perturbations with ϑ small enough, we have for every $t \in \mathcal{T}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $p \in \overline{\mathcal{P}}$ and $\alpha \in \mathcal{A}^*$:*

$$\Lambda_{p,\mathfrak{a},\alpha}^{\epsilon'}/D_4 < \tilde{\Lambda}_{t,p,\mathfrak{a},\alpha} < D_4 \Lambda_{p,\mathfrak{a},\alpha}^{1/\epsilon'}$$

where $\tilde{\Lambda}_{t,p,\mathfrak{a},\alpha}$ is the maximum among $x \in X$ of the absolute value $\tilde{\lambda}_{t,p,\mathfrak{a},\alpha}(x)$ of the unique eigenvalue of the differential $D\psi^{\alpha}_{t,p,\mathfrak{a}}(x)$.

5.3 Choice of a probability measure μ

We will first need the following result of Bowen ([Bo1] thm 1.4 P7 and its proof P19) about the existence of a Gibbs measure. We recall that σ is a full shift. We state the result in this case but it remains true for subshifts of finite type topologically mixing. We suppose that a parameter p_0 has been fixed (the precise choice will be made in the next subsection). We fix an arbitrary $\mathfrak{a}_0 \in \overrightarrow{\mathcal{A}}$.

Theorem E (Bowen). *Let $\phi : \overleftrightarrow{\mathcal{A}} \to \mathbb{R}$ be a Hölder map with positive exponent (the set $\overleftrightarrow{\mathcal{A}}$ being endowed with the distance d_∞). Then there exists a unique σ-invariant measure μ on $\overleftrightarrow{\mathcal{A}}$ s.t. for every $A \in \overleftrightarrow{\mathcal{A}}$, it holds:*

$$\mu[A_{|n}] \asymp \exp\Big(-\Pi n + \sum_{k=0}^{n-1} \phi(\sigma^k(A)) \Big)$$

where $\Pi = \Pi(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi)$, with

$$Z_n(\phi) := \sum_{x \in \mathcal{A}^n} \exp(S_x) \text{ and } S_x = \sup\{\sum_{k=0}^{n-1} \phi(\sigma^k(y)) : y \in [x]\} \,.$$

Writing $A = \alpha\mathfrak{a} \in \overleftrightarrow{\mathcal{A}}$ as the concatenation of $\alpha \in \overleftarrow{\mathcal{A}}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, we can apply the previous result with the map

$$\phi : A \in \overleftrightarrow{\mathcal{A}} \mapsto \Delta(p_0) \cdot \log \lambda_{p_0,\mathfrak{a},\alpha_{-1}} \left(\pi_{p_0,\alpha_{-1}\mathfrak{a}}(\sigma(\alpha)) \right) .$$

We show in the Appendix:

Lemma 5.3.1. *The map ϕ is Hölder with positive exponent.*

Using Lemma 5.2.1 and 5.2.3, we note that $\Pi = \Pi(\phi)$ coincides with $\Pi_{p_0}(\Delta(p_0))$ and thus vanishes by definition of $\Delta(p_0)$ (see Proposition 5.1.4). Moreover, by Eq. (13), for any $A \in \overleftrightarrow{\mathcal{A}}$ the sum $\sum_{k=0}^{n-1} \phi(\sigma^k(A))$ is equal to:

$$\Delta(p_0) \cdot \log \prod_{k=1}^{n} \lambda_{p_0,\mathfrak{a}_k,\alpha_{-k}} \left(\pi_{p_0,\mathfrak{a}_{k+1}}(\sigma^k(\alpha))\right) = \Delta(p_0) \cdot \log \lambda_{p_0,\mathfrak{a},\alpha_{|n}} \left(\pi_{p_0,\alpha_{|n}\mathfrak{a}}(\sigma^n(\alpha))\right)$$

with $\mathfrak{a}_k := \alpha_{|k-1}\mathfrak{a}$. By Theorem E, this gives us a σ-invariant measure μ on $\overleftrightarrow{\mathcal{A}}$ such that for every $A = \alpha\mathfrak{a} \in \overleftrightarrow{\mathcal{A}}$, we have:

$$\mu[A_{|n}] \asymp \lambda^{\Delta(p_0)}_{p_0,\mathfrak{a},\alpha_{|n}} \left(\pi_{p_0,\alpha_{|n}\mathfrak{a}}(\sigma^n(\alpha))\right) \text{ when } n \to +\infty\,.$$

Using successively Lemmas 5.2.1 and 5.2.3, for any $A = \alpha\mathfrak{a} \in \overleftrightarrow{\mathcal{A}}$, it holds:

$$\mu[A_{|n}] \asymp \Lambda^{\Delta(p_0)}_{p_0,\mathfrak{a},\alpha_{|n}} \asymp \Lambda^{\Delta(p_0)}_{p_0,\mathfrak{a}_0,\alpha_{|n}} \text{ when } n \to +\infty\,. \tag{17}$$

We then define a σ-invariant probability measure on $\overleftarrow{\mathcal{A}}$, still denoted μ, by giving to each cylinder in $\overleftarrow{\mathcal{A}}$ the same measure than the corresponding one in $\overleftrightarrow{\mathcal{A}}$. Then:

$$\mu[\rho] \asymp \Lambda^{\Delta(p_0)}_{p_0,\mathfrak{a}_0,\rho} \text{ when } \rho \in \mathcal{A}^n \text{ and } n \to +\infty\,. \tag{18}$$

Remark 5.3.2. *We will not need the σ-invariance property of μ in the following but only the estimation from Eq.* (18).

5.4 Proof of Theorem C

The strategy is the same as for the proof of Theorem D. Let us consider $p_0 \in \mathcal{P}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$. We have $\Delta(p_0) > N + \epsilon$, where $\epsilon := \frac{1}{2}(\min_{p\in\overline{\mathcal{P}}} \Delta(p) - N) > 0$. To prove the result, we show that there exists $\delta > 0$ s.t. the d-dimensional ball $\mathcal{B}$ of center p_0 of radius δ is included in $\mathcal{P}$ with $\mathrm{Leb}_N(K_{p,\mathfrak{a}}) > 0$ and $\mathrm{Leb}_N(\tilde{K}_{t,p,\mathfrak{a}}) > 0$ for Leb_d a.e. $p \in \mathcal{B}$ and Leb_τ a.e. $t \in \mathcal{T}$, for every family $\mathbb{F}$ of ϑ-U-perturbations with small ϑ.

We endow $\overleftarrow{\mathcal{A}}$ with the probability measure μ defined in Subsection 5.3. For any $p \in \mathcal{P}$ and $t \in \mathcal{T}$, let $\nu_{p,\mathfrak{a}}$ and $\nu_{t,p,\mathfrak{a}}$ be the images of μ by the maps $\pi_{p,\mathfrak{a}}$ and $\tilde{\pi}_{t,p,\mathfrak{a}}$.

As Proposition 4.2.2 implies Theorem D, Theorem C is a consequence of:

Proposition 5.4.1. *There exists $\delta > 0$ s.t. the ball $\mathcal{B}$ of center p_0 and radius δ is included in $\mathcal{P}$ and the two following integrals are finite:*

$$\mathcal{I} := \int_{p\in\mathcal{B}} \int_{x\in\mathbb{R}^N} \liminf_{r\to 0} \frac{\nu_{p,\mathfrak{a}}(x + B(r))}{c_N r^N} d\nu_{p,\mathfrak{a}} d\mathrm{Leb}_d < +\infty\,,$$

$$\mathcal{I}' := \int_{p\in\mathcal{B}} \int_{t\in\mathcal{T}} \int_{x\in\mathbb{R}^N} \liminf_{r\to 0} \frac{\nu_{t,p,\mathfrak{a}}(x + B(r))}{c_N r^N} d\nu_{t,p,\mathfrak{a}} d\mathrm{Leb}_\tau d\mathrm{Leb}_d < +\infty\,,$$

for any family $\mathbb{F}$ of ϑ-U-perturbations with small ϑ, where $B(r) \subset \mathbb{R}^N$ is the ball of center 0 and radius r and the constant c_N is defined by $c_N r^N := \mathrm{Leb}_N(B(r))$.

Proof. Let us take a small δ s.t. $\mathcal{B} \subset \mathcal{P}$. The radius δ will be reduced one time so that $\mathcal{I}$ and $\mathcal{I}'$ are finite. We first begin by bounding $\mathcal{I}$. The proof is similar to the one of Proposition 4.2.2: we begin by using Fatou's Lemma, the definition of $\nu_{p,\mathfrak{a}}$ and the Fubini-Tonelli Theorem to find the following bound:

$$\mathcal{I} \leq \liminf_{r\to 0} \frac{1}{c_N r^N} \int_{(\alpha,\beta)\in\overleftarrow{\mathcal{A}}\times\overleftarrow{\mathcal{A}}} \mathrm{Leb}_d\{p \in \mathcal{B} : ||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| < r\} d\mu \times \mu\,.$$

We write $\overleftarrow{\mathcal{A}} \times \overleftarrow{\mathcal{A}} = \bigsqcup_{n\geq 0} \bigsqcup_{\rho\in\mathcal{A}^n} \mathcal{C}_\rho$ where $\mathcal{C}_\rho$ is the set of pairs $(\alpha,\beta) \in \overleftarrow{\mathcal{A}} \times \overleftarrow{\mathcal{A}}$ s.t. $\alpha_{-|\rho|}\cdots\alpha_{-1} = \beta_{-|\rho|}\cdots\beta_{-1} = \rho$ but $\alpha_{-|\rho|-1} \neq \beta_{-|\rho|-1}$. Thus $\mathcal{I}$ is smaller than:

$$\liminf_{r\to 0} \frac{1}{c_N r^N} \sum_{n\geq 0} \sum_{\rho\in\mathcal{A}^n} \int_{(\alpha,\beta)\in\mathcal{C}_\rho} \mathrm{Leb}_d\{p\in\mathcal{B} : ||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| < r\} d\mu\times\mu\,. \quad (19)$$

We show below that a consequence of the transversality assumption (**T**) is:

Lemma 5.4.2. *We fix $\eta := \frac{-\epsilon\log\gamma}{2N+\epsilon}$ and reduce δ if necessary s.t. $\delta < \delta(\eta)$ (where $\delta(\eta)$ is defined in Lemma 5.2.2). Then for any $n \geq 0$, $\rho\in\mathcal{A}^n$, $(\alpha,\beta)\in\mathcal{C}_\rho$, we have:*

$$\mathrm{Leb}_d\{p\in\mathcal{B} : ||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| < r\} \preceq r^N \cdot \Lambda_{p_0,\mathfrak{a}_0,\rho}^{-N-2\epsilon/3}\,.$$

For any family $\mathbb{F}$ of ϑ-U-perturbations of $(F_p)_p$ with small ϑ, for any $t\in\mathcal{T}$, we have:

$$\mathrm{Leb}_d\{p\in\mathcal{B} : ||\tilde{\pi}_{t,p,\mathfrak{a}}(\alpha) - \tilde{\pi}_{t,p,\mathfrak{a}}(\beta)|| < r\} \preceq r^N \cdot \Lambda_{p_0,\mathfrak{a}_0,\rho}^{-N-2\epsilon/3}\,.$$

Notice that when the similarity dimension is close to the dimension N of the fibers (and so ϵ small), we need to work with a ball of small radius η. We can inject the first bound of Lemma 5.4.2 into Eq. (19):

$$\mathcal{I} \preceq \sum_{n\geq 0} \sum_{\rho\in\mathcal{A}^n} \int_{(\alpha,\beta)\in\mathcal{C}_\rho} \Lambda_{p_0,\mathfrak{a}_0,\rho}^{-N-2\epsilon/3} d\mu\times\mu\,. \quad (20)$$

We use successively Eq. (18), the inequality $(N+\frac{2\epsilon}{3})/\Delta(p_0) < (N+\frac{2\epsilon}{3})/(N+\epsilon) < 1-\frac{\epsilon}{4N}$ and finally Eq. (15) (which gives $\mu[\rho]\preceq\gamma^n$) to get:

$$\Lambda_{p_0,\mathfrak{a}_0,\rho}^{-N-2\epsilon/3} \asymp \mu[\rho]^{\frac{-N-2\epsilon/3}{\Delta(p_0)}} \leq \mu[\rho]^{-(1-\frac{\epsilon}{4N})} \preceq \gamma^{\frac{n\epsilon}{4N}} \cdot \mu[\rho]^{-1}\,. \quad (21)$$

We now inject this bound into Eq. (20) to find:

$$\mathcal{I} \preceq \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{4N}} \sum_{\rho\in\mathcal{A}^n} \frac{(\mu\times\mu)(\mathcal{C}_\rho)}{\mu[\rho]} \leq \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{4N}} \sum_{\rho\in\mathcal{A}^n} \mu[\rho] = \sum_{n\geq 0} \gamma^{\frac{n\epsilon}{4N}} < +\infty\,,$$

where we used the inequality $(\mu\times\mu)(\mathcal{C}_\rho) \leq \mu[\rho]^2$ (coming from $\mathcal{C}_\rho \subset [\rho]^2$) to prove the second inequality. To bound $\mathcal{I}'$ for every family $\mathbb{F}$ of ϑ-U-perturbations of $(F_p)_p$ with small ϑ, we just remark that the same proof works when ϑ is small enough, with an additional integration relatively to $t\in\mathcal{T}$. □

Proof of Lemma 5.4.2. Let us begin with the following distortion lemma:

Lemma 5.4.3. *There exists $D_5 > 0$ s.t. for every $p_1, p_2 \in \overline{\mathcal{P}}$, $\mathfrak{a}_1, \mathfrak{a}_2 \in \overrightarrow{\mathcal{A}}$ and $\rho\in\mathcal{A}^*$, the following holds true:*

$$||p_1 - p_2|| < \delta \implies \Lambda_{p_1,\mathfrak{a}_1,\rho}^{1+\frac{\epsilon}{2N}} \leq D_5 \cdot \Lambda_{p_2,\mathfrak{a}_2,\rho}\,.$$

Proof. If $||p_1-p_2|| < \delta$, then $||p_1-p_2|| < \delta(\eta)$. By Lemma 5.2.2, it holds:

$$\Lambda_{p_1,\mathfrak{a}_1,\rho}^{1+\epsilon_0} \leq D_2^{1+\epsilon_0} e^{|\rho|\eta(1+\epsilon_0)} \Lambda_{p_2,\mathfrak{a}_1,\rho}^{1+\epsilon_0} \leq D_2^{1+\epsilon_0} e^{|\rho|\eta(1+\epsilon_0)} \gamma^{|\rho|\epsilon_0} \Lambda_{p_2,\mathfrak{a}_1,\rho}\,,$$

with $\epsilon_0 := \epsilon/(2N)$. By Lemma 5.2.3, $\Lambda_{p_2,\mathfrak{a}_1,\rho} \asymp \Lambda_{p_2,\mathfrak{a}_2,\rho}$ when $|\rho|\to+\infty$, with bounds independent of p_2. The result follows since, by definition of η, it holds:

$$e^{|\rho|\eta(1+\epsilon_0)}\gamma^{|\rho|\epsilon_0} = e^{|\rho|(\eta(1+\epsilon_0)+\epsilon_0\log\gamma)} \text{ with } \eta(1+\epsilon_0)+\epsilon_0\log\gamma = 0\,.$$

□

Lemma 5.4.4. *There exists a real polynomial R positive on $\mathbb{R}_+$ s.t. for every $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n \geq 0$, $\rho \in \mathcal{A}^n$ and $(\alpha, \beta) \in \mathcal{C}_\rho$, it holds:*

$$||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| \geq \frac{\Lambda_{p,\mathfrak{a},\rho}}{R(n)} \cdot ||\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha)) - \pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))|| \,. \tag{22}$$

Moreover for any $\epsilon' > 1$, for every family $\mathbb{F}$ of ϑ-U-perturbations of $(F_p)_p$ with $\vartheta > 0$ small enough we have for every $t \in \mathcal{T}$, $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n \geq 0$, $\rho \in \mathcal{A}^n$ and $(\alpha, \beta) \in \mathcal{C}_\rho$:

$$||\tilde{\pi}_{t,p,\mathfrak{a}}(\alpha) - \tilde{\pi}_{t,p,\mathfrak{a}}(\beta)|| \geq \frac{\Lambda^{\epsilon'}_{p,\mathfrak{a},\rho}}{R(n)} \cdot ||\tilde{\pi}_{t,p,\rho\mathfrak{a}}(\sigma^n(\alpha)) - \tilde{\pi}_{t,p,\rho\mathfrak{a}}(\sigma^n(\beta))|| \,. \tag{23}$$

The proof is in the Appendix. Given $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n \geq 0$, $\rho \in \mathcal{A}^n$, $(\alpha, \beta) \in \mathcal{C}_\rho$, using Eq. (22) and then Lemma 5.4.3 together with the fact that $\mathcal{B}$ is the ball of center p_0 and radius δ, it holds:

$$\mathrm{Leb}_d\{p \in \mathcal{B} : ||\pi_{p,\mathfrak{a}}(\alpha) - \pi_{p,\mathfrak{a}}(\beta)|| < r\} \leq \mathrm{Leb}_d\{p \in \mathcal{B} : ||\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha)) - \pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))|| < \frac{R(n)r}{\Lambda_{p,\mathfrak{a},\rho}}\} \leq$$

To conclude, by **(T)** and since $\mathcal{B} \subset \mathcal{P}$ the latter is smaller than:

$$r^N \cdot Q(n) \cdot \Lambda_{p_0,\mathfrak{a}_0,\rho}^{-N-\epsilon/2} \text{ with } Q(n) := C \cdot D_5^N \cdot (R(n))^N \,.$$

The result follows since $\Lambda_{p_0,\mathfrak{a}_0,\rho}$ decreases exponentially with n, and $\epsilon/2 < 2\epsilon/3$. The proof of the second item is similar, by taking ϵ' close to 1 in Eq. (23). □

6 Jets: Proof of Theorem B

We now prove Theorem B. The strategy is to study the dynamics of the family $(\mathcal{F}_p)_p$ inside the local stable manifolds to reduce the problem to the dynamics of a family $(F_p)_p$ of C^r-skew-products (Step 1) with one-dimensional fibers, from which we construct a family of C^2-skew-products $(G_{p_0})_{p_0}$ acting on s-jets (Step 2). Then we extend the latter one into a larger family $(G_{q_0})_{q_0}$ to satisfy the transversality assumption **(T)** (Step 3). Finally, we look at the intersection between the unstable set and a family of curves all close to a stable manifold. In each curve, this intersection is equal to the limit set of a perturbation of the skew-product. We then apply successively Theorem C and the Fubini Theorem to conclude to a set of jets of positive measure at a.e. parameter, which gives the parablender property (Step 4).

Step 1: Dynamically defined family of skew-products. We first need to define local stable and unstable manifolds. Let us fix a small $\varepsilon > 0$ and an arbitrary parameter in $\overline{\mathcal{P}}$, taken arbitrarily equal to 0 for simplicity. We recall that $\mathcal{K}_0$ is a hyperbolic basic set for $\mathcal{F}_0$. Up to a change of metric on the stable (resp. unstable) bundles of $\mathcal{K}_0$, we suppose that $D\mathcal{F}_0$ strictly contracts (resp. expands) the stable (resp. unstable) bundle by a factor $\lambda < 1$ (resp. $1/\lambda$) uniform over $z \in \mathcal{K}_0$.

It has been shown by Qian and Zhang (see section 4 of [QZ]) that the limit inverse $\overleftrightarrow{\mathcal{K}_0}$ can be endowed with a map $[\cdot]$ defined on a subset of $\overleftrightarrow{\mathcal{K}_0} \times \overleftrightarrow{\mathcal{K}_0}$ with values in $\overleftrightarrow{\mathcal{K}_0}$ so that for every sufficiently closed orbits $x, y \in \overleftrightarrow{\mathcal{K}_0}$, the orbit $z = [x, y]$ is well-defined and the 0-coordinate projection $\pi(z) \in \mathcal{K}_0$ of z is the intersection of the local unstable manifold of x and the local stable manifold of y_0. This map endows $\overleftrightarrow{\mathcal{K}_0}$ with a structure of Smale space (see Ruelle [Ru], Chapter 7, for the definition of a Smale space). This implies that $\overleftrightarrow{\mathcal{K}_0}$ admits Markov partitions of arbitrarily small diameter (see again Ruelle [Ru], Chapter 7, for this result and the definition and properties of a Markov partition for a Smale space). We pick such a partition of $\overleftrightarrow{\mathcal{K}_0}$ by a finite number of compact rectangles $\overline{\mathcal{R}}_1, \ldots, \overline{\mathcal{R}}_M$ of diameter small compared to ε. Up to

reducing ε, we can suppose that $\mathcal{F}_0$ is a local diffeomorphism when restricted to the 0-coordinate projection $\mathcal{R}_i := \pi(\overline{\mathcal{R}}_i) \subset \mathcal{K}_0$ of any rectangle $\overline{\mathcal{R}}_i$.

The topological entropy of $\mathcal{F}_0|\mathcal{K}_0$ is larger than $\delta_{d,s} \cdot |\log m(D\mathcal{F}_0)|$ by $(\star)$. Then for large ℓ we can pick $N_{\varepsilon,\ell}$ orbits $x_a = (x_a^i)_{0\leq i\leq \ell-1}$ of length ℓ (with $a \in \mathcal{A} := \{1, 2, \dots, N_{\varepsilon,\ell}\}$) under $\mathcal{F}_0$ which are (ℓ, ε)-separated (in the sense of Bowen) with:

$$(\diamond) \qquad \frac{\log N_{\varepsilon,\ell}}{\ell} > \delta_{d,s} \cdot |\log m(D\mathcal{F}_0)| \,.$$

In particular the cardinality $N_{\varepsilon,\ell}$ of $\mathcal{A}$ is at least 2. We can extend each of these orbits x_a of length ℓ into an infinite orbit in $\overleftrightarrow{\mathcal{K}_0}$, still denoted x_a. Up to slightly perturbating x_a, we can suppose that for every $0 \leq k \leq \ell - 1$, the orbit $\overleftrightarrow{\mathcal{F}_0}^k(x_a)$ is in the interior of some rectangle $\overline{\mathcal{R}}_i$ that we denote $\overline{\mathcal{R}}_{a,k}$. In particular $x_a^k = \mathcal{F}_0^k(x_a^0)$ is in $\mathcal{R}_{a,k} = \pi(\overline{\mathcal{R}}_{a,k})$. Since the rectangles $\overline{\mathcal{R}}_i$ are in finite number M independent of ℓ, up to modifying $N_{\varepsilon,\ell}$ by a multiplicative constant (independent of ℓ large), we can suppose that all orbits x_a $(a \in \mathcal{A})$ begin in the same $\mathcal{R}_i$ and end in the same $\mathcal{R}_j$, with $(\diamond)$ still true. For any $a \in \mathcal{A}$, let us define:

$$R_a := \mathcal{R}_{a,0} \cap \mathcal{F}_0^{-1}(\mathcal{R}_{a,1}) \cap \cdots \cap \mathcal{F}_0^{-(\ell-1)}(\mathcal{R}_{a,\ell-1}) \,.$$

which is not empty by assumption. Since the ℓ-orbits $(x_a^i)_{0\leq i\leq \ell-1}$ (with $a \in \mathcal{A}$) under $\mathcal{F}_0$ are (ℓ, ε)-separated and since the diameter of each rectangle of the partition is small compared to ε, the sets R_a $(a \in \mathcal{A})$ are pairwise disjoint compact subsets of $\mathcal{R}_i$ having their images by $\mathcal{F}_0^\ell$ included in $\mathcal{R}_j$. Up to adding a constant (independent of ℓ large) to ℓ, using the properties of Markov partitions, we can suppose moreover that $\mathcal{R}_i$ and $\mathcal{R}_j$ are equal. We denote this set by R in the following and thus the sets R_a are non empty pairwise disjoint compact subsets of R with images under $\mathcal{F}_0^\ell$ included in R. This does not change the number $N_{\varepsilon,\ell}$ of such sets R_a and so inequality $(\diamond)$ is still true. For every non empty finite word $\beta = \beta_0 \cdots \beta_p \in \mathcal{A}^{p+1}$, we now set

$$R_\beta := R_{\beta_0} \cap \mathcal{F}_0^{-\ell}(R_{\beta_1}) \cap \cdots \cap \mathcal{F}_0^{-\ell p}(R_{\beta_p}) \,.$$

which is a non empty compact set by the properties of Markov partitions, and the sets R_β among $\beta \in \mathcal{A}^{p+1}$ are pairwise disjoint subsets of R for a fixed value of p. Finally, for $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, we define:

$$R_{\mathfrak{a}} := \bigcap_{n>0} R_{\mathfrak{a}_{|n}}$$

which is also a non empty compact subset of R, included in $\mathcal{K}_0$, and the sets $R_{\mathfrak{a}}$ $(\mathfrak{a} \in \overrightarrow{\mathcal{A}})$ are pairwise disjoint. Finally, we notice that since the partition admits a continuation in a neighborhood of 0 in $\mathcal{P}$, the sets R_a, R_β and $R_{\mathfrak{a}}$ admit continuations $R_{p,a}$, $R_{p,\beta}$ and $R_{p,\mathfrak{a}}$ with the same dynamical properties when p varies in a neighborhood of 0 in the parameter space. Up to taking a finite covering of $\overline{\mathcal{P}}$ by such neighborhoods and extending a finite number of times $(\mathcal{F}_p)_p$ in Steps 2-3-4, we can suppose that $R_{p,a}$, $R_{p,\beta}$ and $R_{p,\mathfrak{a}}$ vary continuously in a neighborhood of $\overline{\mathcal{P}}$, that we can suppose equal to $\mathcal{P}'$ up to reducing it, and inequality $(\diamond)$ is satisfied in $\overline{\mathcal{P}}$.

Let us take a small $\varepsilon' > 0$. For any infinite forward sequence $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and any $p \in \mathcal{P}'$, we notice that all the points in $R_{p,\mathfrak{a}}$ are asymptotic (using the hyperbolicity) and thus belong to a same stable manifold W of $\mathcal{K}_p$ (which is one-dimensional). We define $W_p^{\mathfrak{a}}$ as the ε'-neighborhood in W of the maximal arc of W bounded by points of $R_{p,\mathfrak{a}}$. We parameterize $W_p^{\mathfrak{a}}$ with $X := [-1, 1]$ via a C^r-map $s_p^{\mathfrak{a}}$ s.t. $(s_p^{\mathfrak{a}})_p$ depends Hölder in the C^{r-1}-topology on $\mathfrak{a}$ by Remark 7.0.3.

Since $\mathcal{F}_p$ is a local diffeomorphism and since $W_p^{\mathfrak{a}}$ is injectively immersed, up to decreasing ε', the restriction of $\mathcal{F}_p^\ell$ to each $W_p^{\mathfrak{a}}$ is a diffeomorphism satisfying:

$$\forall p \in \mathcal{P}', \forall \mathfrak{a} \in \overrightarrow{\mathcal{A}},\ \mathcal{F}_p^\ell(W_p^{\mathfrak{a}}) \subset \mathring{W}_p^{\sigma(\mathfrak{a})} := s_p^{\sigma(\mathfrak{a})}((-1, 1))$$

by hyperbolicity. We define the C^r-diffeomorphism $f_{p,\mathfrak{a}} := (s_p^{\sigma(\mathfrak{a})})^{-1} \circ \mathcal{F}_p^\ell \circ s_p^{\mathfrak{a}}$ on a small neighborhood X' of X independent of $(p, \mathfrak{a})$. Its image is s.t. $f_{p,\mathfrak{a}}(X') \Subset X$. The differential $D\mathcal{F}_p^\ell$ strictly contracts (resp. expands) $\mathcal{E}_{p,z}^s$ (resp. $\mathcal{E}_{p,z}^u$) by a factor $\lambda^\ell < \lambda < 1$ (resp. $1/\lambda$) uniform over $p \in \overline{\mathcal{P}}$ and $z \in \mathcal{K}_p$. Up to modifying $\mathcal{F}_p^\ell$ outside a neighborhood of $\mathcal{K}_p$ (this operation does not modify the local stable/unstable sets of $\mathcal{K}_p$ in a neighborhood of $\mathcal{K}_p$ nor jets of points inside it), we can suppose that $\mathcal{F}_p^\ell$ contracts strictly each $W_p^{\mathfrak{a}}$ by λ and so each map $f_{p,\mathfrak{a}}$ contracts by $< \lambda$. Moreover the map $(p, x) \mapsto f_{p,\mathfrak{a}}(x)$ is C^r. It extends on $\mathcal{P}' \times X'$, up to reducing $\mathcal{P}'$ and X'. Easy computations show that the map $\mathfrak{a} \mapsto ((p, x) \mapsto f_{p,\mathfrak{a}}(x))$ is Hölder for the C^{r-1}-topology and continuous for the C^r-topology. We set

$$F_p : (\mathfrak{a}, x) \in \overrightarrow{\mathcal{A}} \times X \mapsto (\sigma(\mathfrak{a}), f_{p,\mathfrak{a}}(x)) \in \overrightarrow{\mathcal{A}} \times X \,,$$

and $(F_p)_p$ is a family of fiberwise λ-contracting C^r-skew-products satisfying the preliminary assumptions of Theorem C. We now show:

Lemma 6.0.1. *The similarity dimension $\Delta(p)$ is larger than $\delta_{d,s}$ for any $p \in \overline{\mathcal{P}}$.*

Proof. The fact that the similarity dimension does not depend on the fiber is a consequence of Proposition 5.1.4. Moreover for a fixed fiber $\mathfrak{a}$, there are $N_{\varepsilon,\ell}$ contractions $f_{p,\mathfrak{a}}$ which contract by at most $m(D\mathcal{F}_p)^\ell$. Thus the result follows by the definition of the similarity dimension and inequality $(\diamond)$. □

Using backward sequences instead of forward sequences, we can define similarly families $(W_p^\alpha)_p$ of local unstable manifolds of $\mathcal{K}_p$ parameterized by maps s_p^α s.t. the map $\mathcal{F}_p^\ell$ restricted to some subset of each $\mathring{W}_p^{\sigma(\alpha)} := s_p^{\sigma(\alpha)}((-1, 1))$ is a diffeomorphism onto W_p^α which expands strictly by $1/\lambda$. Moreover the local unstable manifold W_p^α intersects the local stable manifold $W_p^{\mathfrak{a}}$ at a unique point.

Remark 6.0.2. *For any C^r-family $(\mathcal{F}_q)_{q\in\mathcal{Q}}$ of endomorphisms which extends $(\mathcal{F}_p)_{p\in\mathcal{P}}$ (by this, we mean that $\mathcal{Q}$ is a neighborhood of $\mathcal{P}$), by hyperbolic continuation, we can extend $(\mathcal{K}_p)_{p\in\mathcal{P}}$ into the continuation $(\mathcal{K}_q)_{q\in\mathcal{Q}}$ of a hyperbolic basic set of stable dimension 1, up to reducing $\mathcal{Q}$. Thus we can also extend the local stable and unstable manifolds (using Theorem F in the Appendix) and so the family $(F_p)_p$ into a family $(F_q)_q$ of fiberwise λ-contracting C^r-skew-products.*

In the following, we prove Theorem B in the case $\ell = 1$. The proof is the same when $\ell > 1$, with a variant in the proof of Proposition 6.0.3 described in Remark 6.0.6.

Step 2: Maps acting on jets. From $(F_p)_p$, we define a family of C^2-skew-products $(G_{p_0})_{p_0}$ acting on s-jets at any p_0. We begin by performing this Step for $d = 1$ for the sake of simplicity concerning the notations, and then treat the general case $d \geq 2$.

For any $p_0 \in \mathcal{P}'$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, we define from $(f_{p,\mathfrak{a}})_p$ the following map $g_{p_0,\mathfrak{a}}$ acting on s-jets:

$$g_{p_0,\mathfrak{a}} : (x_p, \partial_p x_p, \ldots, \partial_p^s x_p)_{|p=p_0} \mapsto (f_{p,\mathfrak{a}}(x_p), \partial_p(f_{p,\mathfrak{a}}(x_p)), \ldots, \partial_p^s(f_{p,\mathfrak{a}}(x_p)))_{|p=p_0} \,,$$

when defined. Since $(f_{p,\mathfrak{a}})_p$ is a C^r-family of maps with $s \leq r-2$, the family $(g_{p_0,\mathfrak{a}})_{p_0}$ is itself a C^2-family of maps. It depends continuously on $\mathfrak{a}$. Notice that

$$\partial_p(f_{p,\mathfrak{a}}(x_p))_{|p=p_0} = Df_{p_0}(x_{p_0}) \cdot \partial_p(x_p)_{|p=p_0} + \partial_p(f_p(x_{p_0}))_{|p=p_0} \,.$$

We notice that $\partial_p(f_{p,\mathfrak{a}}(x_p))_{|p=p_0}$ has possibly a non zero derivative relatively to x_{p_0} (independent of $\partial_p(x_p)_{|p=p_0}$), a derivative $Df_{p_0}(x_{p_0})$ relatively to $\partial_p(x_p)_{|p=p_0}$ and derivatives equal to zero relatively to each $\partial_p^j(x_p)_{|p=p_0}$ for $2 \leq j \leq s$.

When $1 < k \leq s$, the term $\partial_p^k(f_{p,\mathfrak{a}}(x_p))_{|p=p_0}$ has possibly a non zero derivative relatively to $\partial_p^j(x_p)_{|p=p_0}$ for $0 \leq j \leq k-1$ (independent of $\partial_p^k(x_p)_{|p=p_0}$), a derivative

$Df_{p_0}(x_{p_0})$ relatively to $\partial_p^k(x_p)_{|p=p_0}$ and derivatives equal to zero relatively to each $\partial_p^j(x_p)_{|p=p_0}$ for $k < j \leq s$. To see this, it is enough to write the map $p \mapsto f_{p,\mathfrak{a}}(x_p)$ as the composition of $p \mapsto (p, x_p)$ and $(p, x) \mapsto f_{p,\mathfrak{a}}(x)$ and to apply the multidimensional Faà di Bruno formula (see for example Theorem 2.1 in [CS]).

We recall that by assumption we have $0 < |Df_{p_0}(x_{p_0})| < \lambda < 1$. Then for any $p_0 \in \mathcal{P}'$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, the map $g_{p_0,\mathfrak{a}}$ has inferior unipotent differentials with eigenvalues uniformly bounded between 0 and 1 in modulus.

We recall that $\delta_{1,s} = s + 1$. We define a set of s-jets Y, identified with a subset of $\mathbb{R}^{s+1}$ as follows (the term of degree i corresponding to the ith-coordinate). We set:

$$Y := X \times [-R_1, R_1] \times \cdots \times [-R_s, R_s] \,.$$

We can choose R_1 large compared to the diameter of X and R_{i+1} large compared to R_i for $1 \leq i \leq s-1$. Since its differentials are unipotent with non zero contracting diagonal coefficients, an immediate induction shows that $g_{p_0,\mathfrak{a}}$ is a C^2-diffeomorphism from a small neighborhood Y' of Y onto $g_{p_0,\mathfrak{a}}(Y') \Subset Y$. Up to rescaling, we can suppose that $Y := [-1, 1]^{s+1}$. The map $\mathfrak{a} \mapsto ((p, x) \mapsto f_{p,\mathfrak{a}}(x))$ is Hölder for the C^{r-1}-topology and continuous for the C^r-topology and we have $s \leq r - 2$. This implies that the map $\mathfrak{a} \mapsto ((p_0, y) \mapsto g_{p_0,\mathfrak{a}}(y))$ is Hölder for the C^1-topology and continuous for the C^2-topology. To summarize, we just prove that $(G_{p_0})_{p_0}$, with

$$G_{p_0} : (\mathfrak{a}, y) \in \overrightarrow{\mathcal{A}} \times Y \mapsto (\sigma(\mathfrak{a}), g_{p_0,\mathfrak{a}}(y)) \in \overrightarrow{\mathcal{A}} \times Y \,,$$

is a family of C^2-skew-products satisfying the preliminary assumptions of Theorem C, **(U)** and $\Delta(p_0) > s + 1 = \delta_{1,s}$ for any $p_0 \in \overline{\mathcal{P}}$.

As already said, we get the same result with $d \geq 2$ but with painful notations. Indeed, it is enough to remark that the action of $(f_{p,\mathfrak{a}})_p$ on jets of multiorder $(k_1, \ldots, k_d)$ with $\sum_i k_i \leq s$ only depends on jets of the same multiorder, with a linear coefficient $Df_{p_0}(x_{p_0})$, and on jets of multiorder $(k'_1, \ldots, k'_d)$ with $\sum_i k'_i < \sum_i k_i$. In particular, its differentials still satisfy assumption **(U)**.

Step 3: Extending the family. We now extend $(G_{p_0})_{p_0}$ to satisfy the transversality assumption **(T)** inside a larger family. But we have to ensure that this extension comes from an extension of $(F_p)_p$, itself coming from an extension of $(\mathcal{F}_p)_p$.

Proposition 6.0.3. *There exists a family of C^2-skew-products $(G_{q_0})_{q_0 \in \mathcal{Q}}$:*

$$G_{q_0} : (\mathfrak{a}, y) \in \overrightarrow{\mathcal{A}} \times Y \mapsto (\sigma(\mathfrak{a}), g_{q_0,\mathfrak{a}}(y)) \in \overrightarrow{\mathcal{A}} \times Y \,,$$

with $\mathcal{Q} := \mathcal{P} \times (-1, 1)^m$ for some $m > 0$, satisfying the preliminary assumptions of Theorem C, **(U)**, **(T)**, *$\Delta(q_0) > \delta_{d,s}$ for $q_0 \in \overline{\mathcal{Q}}$ and:*

$$G_{(p_0,0)} = G_{p_0} \text{ and } g_{(p_0,0),\mathfrak{a}} = g_{p_0,\mathfrak{a}} \text{ for every } p_0 \in \overline{\mathcal{P}} \text{ and } \mathfrak{a} \in \overrightarrow{\mathcal{A}} \,.$$

Moreover there exists a C^r-family $(\mathcal{F}_q)_{q \in \mathcal{Q}}$ of local diffeomorphisms extending $(\mathcal{F}_p)_{p \in \mathcal{P}}$ s.t. if $(F_q)_q = (f_{q,\mathfrak{a}})_q$ is its associated family of C^r-skew-products, we have $f_{(p_0,0),\mathfrak{a}} = f_{p_0,\mathfrak{a}}$ and $g_{q_0,\mathfrak{a}}$ is the map acting on the s-jets at p_0 derived from $(f_{q,\mathfrak{a}})_q$ (here the jets at p_0 are taken varying p for fixed q'_0 where $q_0 := (p_0, q'_0)$). The family $(\mathcal{F}_q)_{q \in \mathcal{Q}}$ is of the form $\mathcal{F}_q = \mathcal{F}_{(p,q')} = \mathcal{F}_p + \Sigma_{p,q'}$ where $(\Sigma_{p,q'})_{(p,q')}$ is a C^r-family s.t. $\Sigma_{p,0} = 0$.

We now finish the proof of Theorem B. We postpone the proof of Proposition 6.0.3 after it since it is technical.

Step 4: Conclusion. Let us pick the family $(G_{q_0})_{q_0}$ of C^2-skew-products satisfying the preliminary assumptions of Theorem C, **(U)**, **(T)** and $\Delta(q_0) > \delta_{d,s}$ for every $q_0 \in \overline{\mathcal{Q}}$ given by Proposition 6.0.3. We will pick well-chosen ϑ-U-perturbations and

apply the second part of Theorem C. We will conclude by using the Fubini Theorem.

We pick the continuation $(k_q)_q$ of a point $k_q \in \mathcal{K}_q$ at the intersection of local stable and unstable manifolds $W^{\mathfrak{a}}_q$ and W^{α}_q for arbitrary $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$ and $\alpha \in \overleftarrow{\mathcal{A}}$. Up to working locally in the parameter space $\mathcal{Q}$, we can pick a C^r-family $(\Gamma_{t,q})_{t,q}$ of segments of same direction u_Γ (these segments remain all parallel when varying t and q), which do not intersect $W^s(\mathcal{K}_q)$ and which intersect $(W^{\alpha}_q)_{t,q}$ in a curve $(z_{t,q})_{t,q}$ C^0-close to $(k_q)_{t,q}$. Moreover we choose $(\Gamma_{t,q})_{t,q}$ s.t. the v_Γ-coordinate $\mathcal{X}(q,t)$ in the basis (u_Γ, v_Γ) of $\Gamma_{t,q}$, for a fixed direction v_Γ transversal to $\Gamma_{t,q}$, is of the form

$$\mathcal{X}(q,t) := \sum_{|i|\leq s} t_i p^i \text{ for } t = (t_1, \cdots, t_{\delta_{d,s}}) \in \mathcal{T} := (-1,1)^{\delta_{d,s}}$$

which does not depend on q' but only on p for $q = (p, q')$. In particular, for any $q_0 \in \mathcal{Q}$, the map $t \in \mathcal{T} \mapsto \mathrm{J}^s_{q_0}\mathcal{X}(q,t)$ sends diffeomorphically $\mathcal{T}$ to a non empty set of s-jets in p (and thus a set of s-jets in p of positive $\delta_{d,s}$-dimensional measure).

By the parametric inclination Lemma 7.0.4 (in the Appendix), up to taking an inverse iterate, for any $t \in \mathcal{T}$, we can suppose that $(\Gamma_{t,q})_q$ is (uniformly in $t \in \mathcal{T}$) close to the continuation $(W^{\mathfrak{a}}_q)_q$ of a local stable manifold for some $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$. We can parameterize it by the segment X with a C^r-family of charts C^r-close to $(s^{\mathfrak{a}}_q)_q$. Still iterating backwards, for every $a \in \mathcal{A}$, there is a family of submanifolds $(\Gamma^a_{t,q})_q$ close to $(W^{a\mathfrak{a}}_q)_q$ (uniformly in $t \in \mathcal{T}$) s.t. $\Gamma^a_{t,q}$ is sent by $\mathcal{F}_q$ into the interior of $\Gamma_{t,q}$. We can parameterize $(\Gamma^a_{t,q})_q$ by X with a family of charts close to $(s^{a\mathfrak{a}}_q)_q$ and in these charts the restriction of $\mathcal{F}_q$ from $\Gamma^a_{t,q}$ into $\Gamma_{t,q}$ defines a family of maps $(\tilde{f}_{t,q,a\mathfrak{a}})_q$ C^r-close to $(f_{q,a\mathfrak{a}})_q$. Iterating backwards by induction, we can define for every $\alpha \in \mathcal{A}^*$ families of curves $(\Gamma^{\alpha}_{t,q})_q$ which are (uniformly in α) C^r-close to $(W^{\alpha\mathfrak{a}}_q)_q$ by Lemma 7.0.4 and s.t. each $\Gamma^{a\alpha}_{t,q}$ is sent into the interior of $\Gamma^{\alpha}_{t,q}$. This defines families of maps $(\tilde{f}_{t,q,\alpha\mathfrak{a}})_q$ C^r-close to $(f_{q,\alpha\mathfrak{a}})_q$ (uniformly in $t \in \mathcal{T}$ and $\alpha \in \mathcal{A}^*$). We set $\tilde{f}_{t,q,\mathfrak{b}} := f_{q,\mathfrak{b}}$ when $\mathfrak{b}$ is not of the form $\alpha\mathfrak{a}$ with α a non empty word. When looking at the action of the families of maps $(\tilde{f}_{t,q,\mathfrak{a}})_q$ on s-jets (w.r.t. p) for fixed values of t and $\mathfrak{a}$, one obtains a family of maps $(\tilde{g}_{t,q_0,\mathfrak{a}})_{q_0}$ from Y into itself C^2-close to $(g_{q_0,\mathfrak{a}})_{q_0}$ (uniformly in t and $\mathfrak{a}$). This defines a family (indexed by t) of ϑ-U-perturbations of the family $(G_{q_0})_{q_0}$ of C^2-skew-products, with arbitrarily small ϑ.

By the second part of Theorem C, we have

$$\mathrm{Leb}_{\delta_{d,s}}(\tilde{K}_{t,q_0,\mathfrak{a}}) > 0 \text{ for } \mathrm{Leb}_{d+m} \text{ a.e. } q_0 \in \mathcal{Q} \text{ and } \mathrm{Leb}_{\delta_{d,s}} \text{ a.e. } t \in \mathcal{T}\,,$$

where the limit set $\tilde{K}_{t,q_0,\mathfrak{a}}$ is formed by jets at p_0 taken while varying p for fixed q'_0 where $q_0 := (p_0, q'_0)$. By the Fubini Theorem, for a.e. q'_0 we have:

$$\mathrm{Leb}_{\delta_{d,s}}(\tilde{K}_{t,(p_0,q'_0),\mathfrak{a}}) > 0 \text{ for } \mathrm{Leb}_d \text{ a.e. } p_0 \in \mathcal{P} \text{ and for } \mathrm{Leb}_{\delta_{d,s}} \text{ a.e. } t \in \mathcal{T}\,.$$

We fix such a q'_0. We consider $(\tilde{\mathcal{F}}_p)_p := (\mathcal{F}_{(p,q'_0)})_p$ and $\tilde{\Gamma}_{t,p} := \Gamma_{t,(p,q'_0)}$. We notice that $\tilde{K}_{t,(p_0,q'_0),\mathfrak{a}}$ is (in the charts) the set of the s-jets at p_0 of the intersection points between the local unstable set of $\tilde{\mathcal{K}}_p$ and $\tilde{\Gamma}_{t,p}$. Moreover the set of s-jets at any p_0 of the v_Γ-coordinate of $\tilde{\Gamma}_{t,p}$ in the basis (u_Γ, v_Γ) when varying t in $\mathcal{T}$ has positive $\delta_{d,s}$-dimensional Lebesgue measure. Since we have a positive set of one-dimensional s-jets in the direction of u_Γ for a.e. $t \in \mathcal{T}$, we just have to use the Fubini Theorem to conclude to a set of bidimensional s-jets of positive measure for a.e. $p_0 \in \mathcal{P}$.

The same proof works for every family $(\mathcal{G}_p)_p$ which is C^r-close to $(\mathcal{F}_p)_p$ with the extension $(\mathcal{G}_q)_{q\in\mathcal{Q}}$ given by $\mathcal{G}_q = \mathcal{G}_p + \Sigma_{p,q'}$ where $(\Sigma_{p,q'})_{(p,q')}$ is the C^r-family given by Proposition 6.0.3. Indeed, the preliminary conditions of Theorem C, **(U)**, $\Delta(q_0) > \delta_{d,s}$ are open conditions and the extension in Proposition 6.0.3 in order to get **(T)** works for nearby families with the same additive perturbation since having

a relative positive speed is an open property. Thus we can apply Theorem C to the family of C^2-skew-products derived from $(\mathcal{G}_q)_q$. This achieves the proof that $(\mathcal{K}_p)_p$ is an almost $C^{r,s}$-parablender and thus shows Theorem B.

Remark 6.0.4. *When the family $(\mathcal{F}_p)_p$ is of the form $(\mathcal{F})_p$ with $\mathcal{F}$ independent of p and when the order s of the jets is equal to 0, the extension $(\mathcal{F}_q)_q$ satisfying the conclusions of Theorem B can be taken of the form $(\mathcal{F}+\Sigma_{q'})_{(p,q')}$ with $(\Sigma_{q'})_{q'}$ independent of p. This will help to prove Theorem A from Theorem B.*

Proof of Proposition 6.0.3. The proof is divided in two Steps. The goal and main difficulty of the extension is about satisfying **(T)**: we want to extend the family $(\mathcal{F}_p)_{p\in\mathcal{P}}$ by adding finitely many new parameters, which will give a positive relative speed to pairs of limit points with different combinatorics, inside each fiber. We set

$$\Omega := \{(\mathfrak{a},\alpha,\beta)\in \overrightarrow{\mathcal{A}}\times\overleftarrow{\mathcal{A}}\times\overleftarrow{\mathcal{A}} : \alpha_{-1}\neq\beta_{-1}\}$$

and by introducing new parameters, we will give a positive relative speed to the points coded by α and β inside the fiber coded by $\mathfrak{a}$ for any $(\mathfrak{a},\alpha,\beta)\in\Omega$, and this will give the transversality assumption **(T)** inside the extended family.

We begin by choosing a covering of Ω by small products of cylinders. The reason why we are going to work with a covering by small products of cylinders is that this will allow us to control precisely the relative movement of the two limit points. We will then extend iteratively the original family $(\mathcal{F}_p)_{p\in\mathcal{P}}$ by adding new parameters for each set of the covering.

We first pick an arbitrary number $h>0$. The extended family $(\mathcal{F}_q)_{q\in\mathcal{Q}}$ will be taken s.t. the families $(\mathcal{F}_{(p,q')})_{p\in\mathcal{P}}$ are (uniformly in q') h-C^r-close to $(\mathcal{F}_p)_{p\in\mathcal{P}}$. We pick an integer M large enough so that $\lambda^M/(1-\lambda)$ is small.

Step 1: Working locally. Let us pick any $(\mathfrak{a},\alpha,\beta)\in\Omega$. If $\mathfrak{a}$ is periodic of minimal period $\mathfrak{p}\in\mathcal{A}^*$, since $\alpha_{-1}\neq\beta_{-1}$, we can choose β s.t. β_{-1} is different from the last letter of $\mathfrak{p}$.

Fact 1. *The sequences $\sigma^k(\mathfrak{a})$ where $0\leq k\leq M$ are all distinct from $\beta_{-1}\mathfrak{a}$.*

Fact 2. *The sequences $\beta_{|k}\mathfrak{a}$ where $2\leq k\leq M$ are all distinct from $\beta_{-1}\mathfrak{a}$.*

Proof of Facts 1 and 2. If $\mathfrak{a}$ is not periodic, this is immediate. If $\mathfrak{a}$ is periodic, this is due to the fact that β_{-1} is different from the last letter of its minimal period $\mathfrak{p}$. □

From now, we distinguish two disjoint Cases which cover all possible tuples in Ω:

Case 1: the sequences $\alpha_{|k}\mathfrak{a}$ where $1\leq k\leq M$ are all distinct from $\beta_{-1}\mathfrak{a}$. In this case, we consider the two cylinders $[\rho_\alpha]$ and $[\rho_\beta]$ of $\overleftarrow{\mathcal{A}}$ of length M defined by $\rho_\alpha := \alpha_{|M}$ and $\rho_\beta := \beta_{|M}$. We pick a small cylinder $[\rho_\mathfrak{a}]$ of $\overrightarrow{\mathcal{A}}$ s.t. $\mathfrak{a}\in[\rho_\mathfrak{a}]$ and:

- The cylinders $[\alpha_{|k}\rho_\mathfrak{a}]$ where $1\leq k\leq M$ are all disjoint from $[\beta_{-1}\rho_\mathfrak{a}]$.
- The cylinders $[\beta_{|k}\rho_\mathfrak{a}]$ where $2\leq k\leq M$ are all disjoint from $[\beta_{-1}\rho_\mathfrak{a}]$.

Case 2: $\mathfrak{a}$ is periodic of minimal period $\mathfrak{p}$ and there exists $f>0$ s.t. the last letters of α are of the form $\beta_{-1}\cdot\mathfrak{p}^f$. We consider two small cylinders $[\rho_\alpha]$ and $[\rho_\beta]$ of $\overleftarrow{\mathcal{A}}$ containing α and β, and a small cylinder $[\rho_\mathfrak{a}]$ of $\overrightarrow{\mathcal{A}}$ s.t. $\mathfrak{a}\in[\rho_\mathfrak{a}]$ and:

- The cylinders $\sigma^k([\rho_\mathfrak{a}])$ where $0\leq k\leq M$ are all disjoint from $[\beta_{-1}\rho_\mathfrak{a}]$.
- The cylinders $[\beta_{|k}\rho_\mathfrak{a}]$ where $2\leq k\leq M$ are all disjoint from $[\beta_{-1}\rho_\mathfrak{a}]$.

We take these cylinders so that their lengths are large compared to M.

We notice that the compact set Ω is covered by the union of the (open) products of cylinders $[\rho_{\mathfrak{a}}] \times [\rho_\alpha] \times [\rho_\beta]$ associated to any $(\mathfrak{a}, \alpha, \beta) \in \Omega$. We can find then a finite covering of Ω by such products.

Step 2: Extension. We now construct an extension of $(\mathcal{F}_p)_p$ obtained by extending successively a finite number of times the family, adding at each step $\delta_{d,s}$ parameters corresponding to a product of cylinders $[\rho] \times [\rho'] \times [\rho''] \subset \Omega$ in the finite covering defined in Step 1. Here $[\rho]$ is a cylinder in $\overrightarrow{\mathcal{A}}$ and $[\rho']$ and $[\rho'']$ are cylinders in $\overleftarrow{\mathcal{A}}$ s.t. $\rho'_{-1} \neq \rho''_{-1}$. The $\delta_{d,s}$ parameters are intended to move the s-jet of the limit point associated to α relatively to the one corresponding to β, inside the fiber encoded by $\mathfrak{a}$, where $(\mathfrak{a}, \alpha, \beta)$ is any tuple in $[\rho] \times [\rho'] \times [\rho'']$.

The union of the points of $R_{p,\mathfrak{a}}$ on the local stable manifolds $W^{\mathfrak{a}}_p$ for $\mathfrak{a} \in [\rho''_{-1}\rho]$ is disjoint from the union of the points of $R_{p,\mathfrak{a}}$ on the local stable manifolds $W^{\mathfrak{a}}_p$ for $\mathfrak{a} \notin [\rho''_{-1}\rho]$ at every parameter $p \in \overline{\mathcal{P}}$. We then pick a C^r-family $(h_p)_p$ of bump functions h_p equal to h in a neighborhood of the first ones and equal to 0 in a neighborhood of the second ones for every $p \in \overline{\mathcal{P}}$. We also pick a C^r-family of maps v^s_p s.t. $v^s_p(z)$ is close to the stable direction of $\mathcal{K}_p$ at $\mathcal{F}_p(z)$ for any point $z \in \mathcal{K}_p$.

We now extend the family by setting for every $q := (p, q')$ with $p \in \mathcal{P}'$ and $q' = (q_i)_i \in \mathbb{R}^{\delta_{d,s}}$ small:

$$\mathcal{F}_q(z) = \mathcal{F}_p(z) + h_p(z) \cdot \Big(\sum_i q_i \cdot p^i \Big) \cdot v^s_p(z) \qquad \forall z \in \mathbb{R}^2 \,, \tag{24}$$

where we sum over $i = (i_1, \dots, i_d)$ s.t. $\sum_k i_k \leq s$ with $p^i = p_1^{i_1} \cdots p_d^{i_d}$.

This defines a C^r-family of endomorphisms $(\mathcal{F}_q)_q$. For small values of q', let us say for q in some open neighborhood of $\mathcal{P} \times \{0\}$, these are still local diffeomorphisms. Moreover, the family of hyperbolic basic sets admits a continuation as a family $(\mathcal{K}_q)_q$ (see the Appendix).

By Remark 6.0.2, we can define the associated family of C^r-skew-products $(F_q)_q$ which extends $(F_p)_p$ and then the associated map $(G_{q_0})_{q_0}$ acting on the s-jets derived from $(F_q)_q$, defined as in Step 2 of the proof of Theorem B (let us recall that the jets are taken while varying only p for fixed q' if we set $q := (p, q')$). In particular, $(G_{q_0})_{q_0}$ is a family of C^2-skew-products satisfying the preliminary assumptions of Theorem C and also (**U**). By Proposition 5.1.4, the map $q \mapsto \Delta(q)$ is continuous. Thus, up to restricting the parameter space, we have $\Delta(q_0) > \delta_{d,s}$ for any q_0.

We now prove that $(G_{q_0})_{q_0}$ satisfies the property **(T)** restricted to any $(\mathfrak{a}, \alpha, \beta) \in [\rho] \times [\rho'] \times [\rho'']$. More precisely we show below the following technical lemma:

Lemma 6.0.5. *Up to reducing $\mathcal{Q}$, for every $\mathfrak{a} \in [\rho]$, $\alpha \in [\rho']$, $\beta \in [\rho'']$, $p_0 \in \mathcal{P}$ and $r > 0$, with $q := (p, q')$, the set of q'_0 s.t. $\mathrm{J}^s_{p_0}\pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^s_{p_0}\pi_{q,\mathfrak{a}}(\beta)$ are r-close is of Lebesgue measure dominated by $r^{\delta_{d,s}}$, with an independent constant. Moreover for every family of ϑ-perturbations of $(G_{q_0})_{q_0}$ with ϑ small enough, $t \in \mathcal{T}$, $\mathfrak{a} \in [\rho]$, $\alpha \in [\rho']$, $\beta \in [\rho'']$, $p_0 \in \mathcal{P}$ and $r > 0$, the set of q'_0 s.t. $\mathrm{J}^s_{p_0}\tilde{\pi}_{t,q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^s_{p_0}\tilde{\pi}_{t,q,\mathfrak{a}}(\beta)$ are r-close is of Lebesgue measure dominated by $r^{\delta_{d,s}}$, with the same constant.*

We recall that the s-jets at p_0 are taken while varying p around p_0 for a fixed value of q' equal to q'_0. In particular, by the Fubini Theorem, the set of (p_0, q'_0) s.t. $\mathrm{J}^s_{p_0}\pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^s_{p_0}\pi_{q,\mathfrak{a}}(\beta)$ are r-close is also of measure dominated by $r^{\delta_{d,s}}$ and the same holds for ϑ-perturbations. The proof of Lemma 6.0.5 is below. We first finish the proof of Proposition 6.0.3.

We extend iteratively the initial family $(\mathcal{F}_p)_p$ a finite number of times by the same method. At each step, we extend by adding $\delta_{d,s}$ new parameters corresponding to a new product of cylinders $[\rho] \times [\rho'] \times [\rho'']$ in the finite covering of Ω defined in Step 1. The adaptation of Lemma 6.0.5 is straightforward. This proves property **(T)** and so $(G_{q_0})_{q_0 \in \mathcal{Q}}$ is a family of C^2-skew-products, with $\mathcal{Q} := \mathcal{P} \times (-1,1)^m$ for some $m > 0$ (up to rescaling). This concludes the proof. □

Proof of Lemma 6.0.5. We first prove the result for the Case 1 in Step 1 of the proof of Proposition 6.0.3. Let $\Pi_{q,\mathfrak{a}}(\alpha)$ and $\Pi_{q,\mathfrak{a}}(\beta)$ be the points of the phase space $\mathbb{R}^2$ equal to $\pi_{q,\mathfrak{a}}(\alpha)$ and $\pi_{q,\mathfrak{a}}(\beta)$ in the parametrization of $W^{\mathfrak{a}}_q$. Both belong to $W^{\mathfrak{a}}_q$.

We pick local coordinates in a neighborhood of $\Pi_{0,\mathfrak{a}}(\alpha)$ centered at $\Pi_{0,\mathfrak{a}}(\alpha)$ with a basis given by $\mathbb{R} \cdot e^s_0 + \mathbb{R} \cdot e^u_0$, where we denoted by $\mathbb{R} \cdot e^s_0$ and $\mathbb{R} \cdot e^u_0$ the stable and unstable directions of $\overleftrightarrow{\mathcal{K}_0}$ at $\Pi_{0,\mathfrak{a}}(\alpha)$. We write:

$$\mathrm{J}^s_{p_0} \Pi_{q,\mathfrak{a}}(\alpha) =: \mathcal{J}^s_{p_0} \Pi_{q,\mathfrak{a}}(\alpha) \cdot e^s_0 + \mathcal{J}^u_{p_0} \Pi_{q,\mathfrak{a}}(\alpha) \cdot e^u_0 \,.$$

Take care that the s on the left hand term is the order of the jet and the s on the right hand term just means ≪ stable ≫. We can proceed similarly for $\Pi_{0,\mathfrak{a}}(\beta)$.

1. Easy case: 0-jets. We first perform the proof for 0-jets to show the general idea. We fix the parameter $p_0 = 0$. We begin by studying $\Pi_{q,\mathfrak{a}}(\alpha)$, more precisely the variations of the 0-jet of $\Pi_{(0,q'),\mathfrak{a}}(\alpha)$, i.e. $\Pi_{(0,q'),\mathfrak{a}}(\alpha)$ itself, when moving q'. In order to do this, we set

$$\Pi_0(q') := \Pi_{(0,q'),\mathfrak{a}}(\alpha)$$

and denote by $\Pi_k(q')$ its preimage by $\mathcal{F}^k_{(0,q')}$ on $W^{\alpha_{|k}\mathfrak{a}}_{(0,q')}$ (this is the point equal to $\pi_{(0,q'),\alpha_{|k}\mathfrak{a}}(\sigma^k(\alpha))$ in the parametrization of the stable manifold). We pick local coordinates centered at each $P_k := \Pi_k(0)$ with a basis given by the corresponding preimages $\mathbb{R} \cdot e^s_k$ and $\mathbb{R} \cdot e^u_k$ of $\mathbb{R} \cdot e^s_0$ and $\mathbb{R} \cdot e^u_0$ by $\mathcal{F}^k_0$ for $k > 0$. These are the stable and unstable directions of $\overleftrightarrow{\mathcal{K}_0}$ at P_k. In the decomposition $\mathbb{R} \cdot e^s_k + \mathbb{R} \cdot e^u_k$, we write

$$\Pi_k(q') =: \Pi^s_k(q') \cdot e^s_k + \Pi^u_k(q') \cdot e^u_k \,.$$

In the coordinates given by $P_{k+1} + \mathbb{R} \cdot e^s_{k+1} + \mathbb{R} \cdot e^u_{k+1}$ and $P_k + \mathbb{R} \cdot e^s_k + \mathbb{R} \cdot e^u_k$ with $P_{k+1} = P_k = (0,0)$, the map $\mathcal{F}_0$ restricted to a neighborhood of $P_{k+1} = (0,0)$ sends $P_{k+1} = (0,0)$ to $P_k = (0,0)$ and is C^1-close to its differential which is diagonal. In particular, the $(1,1)$-coefficient is a real number λ_k s.t. $|\lambda_k| < \lambda < 1$.

By hyperbolic continuation, there exists $B > 0$ independent of k and M s.t.:

$$C_k := \left| \frac{d\Pi^s_k}{dq'}(0) \right| < B \,.$$

We recall that $\Pi_{k+1}(q')$ is sent onto $\Pi_k(q')$ by $\mathcal{F}_{(0,q')}$. Moreover $\mathcal{F}_{(0,q')}(\Pi_{k+1}(q'))$ is the sum of $\mathcal{F}_0(\Pi_{k+1}(q'))$ and a term $\Sigma_k(q') = \Sigma^s_k(q') \cdot e^s_k + \Sigma^u_k(q') \cdot e^u_k$ coming from Eq. (24). In particular, we notice that we have:

$$\left| \frac{d\Sigma^s_k}{dq'}(0) \right| < 2h \,.$$

We have:

$$\frac{d\Pi^s_k}{dq'}(0) = \lambda_k \cdot \frac{d\Pi^s_{k+1}}{dq'}(0) + \frac{d\Sigma^s_k}{dq'}(0) \,.$$

$$C_k < \lambda_k \cdot C_{k+1} + 2h \,.$$

We also notice that $\Sigma_k(q')$ is equal to zero when $0 \leq k \leq M-1$ by the first item of Case 1. This gives for every $M' > M$:

$$C_0 < B \cdot \lambda^{M'} + 2h \cdot (\lambda^{M'-1} + \cdots + \lambda^M) \,.$$

By taking M' large, we get:

$$C_0 \leq 2h \frac{\lambda^M}{1-\lambda}\,,$$

which is small compared to h by assumption. We finally get that the derivative of $\Pi_{(0,q'),\mathfrak{a}}(\alpha) = \Pi_0(q')$ at $q'=0$ in the stable direction $\mathbb{R}\cdot e_0^s$ is small compared to h:

$$\frac{d}{dq'}\Pi^s_{(0,q'),\mathfrak{a}}(\alpha) \text{ is small compared to } h\,.$$

The same holds true when replacing $\Pi_{(0,q'),\mathfrak{a}}(\alpha)$ by the preimage $\Pi_1'(q')$ of $\Pi_{(0,q'),\mathfrak{a}}(\beta)$ by $\mathcal{F}_{(0,q')}$ on $W^{\beta^{-1}\mathfrak{a}}_{(0,q')}$, using the second item of Case 1. By Eq. (24) and since e_0^s is close to $v_0^s(\Pi_1'(0))$, the derivative $\frac{d}{dq'}\Pi^s_{(0,q'),\mathfrak{a}}(\beta)$ is then close to h. Thus:

$$\frac{d}{dq'}\left(\Pi^s_{(0,q'),\mathfrak{a}}(\alpha) - \Pi^s_{(0,q'),\mathfrak{a}}(\beta)\right) \text{ is bounded away from } (0,0)\,.$$

In particular, up to reducing $\mathcal{Q}$, the set of q_0' s.t. $\mathrm{J}^0_0\Pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^0_0\Pi_{q,\mathfrak{a}}(\beta)$ are r-close is of Lebesgue measure dominated by r with an independent constant. We can proceed the same way for any p_0, and the domination constant is independent. Integrating using the Fubini Theorem, the set (p_0, q_0') s.t. $\mathrm{J}^0_{p_0}\Pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^0_{p_0}\Pi_{q,\mathfrak{a}}(\beta)$ are r-close is of Lebesgue measure dominated by r, and thus it is also the case for $\mathrm{J}^0_{p_0}\pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^0_{p_0}\pi_{q,\mathfrak{a}}(\beta)$. The statement about ϑ-perturbations follows easily with the same arguments when ϑ is small.

2. General Case: s-jets. Now we go to the general but more difficult case of s-jets. This time, we vary also the parameter p. We fix $p_0 = 0$. We investigate the differentials of $\mathrm{J}^s_0\Pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}^s_0\Pi_{q,\mathfrak{a}}(\beta)$ when derivating relatively to q'.

We keep the same coordinates for each k: we keep expressing the point $\Pi_k(p,q')$ (depending also on p this time) in the coordinates $P_k + \mathbb{R}\cdot e_k^s + \mathbb{R}\cdot e_k^u$ (independent of (p,q') small). Let $\mathcal{J}_0^s\Pi_k(q')$ and $\mathcal{J}_0^u\Pi_k(q')$ be the components in this basis of the s-jet of $\Pi_k(p,q')$ at $p_0=0$ for q' fixed. Note that by hyperbolic continuation, there exists $B'>0$ independent of k and M s.t.:

$$C_k' := \left|\frac{d\mathcal{J}_0^s\Pi_k}{dq'}(0)\right| < B'\,.$$

For q' fixed the family $(\Pi_k(p,q'))_p$ is the image of $(\Pi_{k+1}(p,q'))_p$ by the family of maps $(\mathcal{F}_{(p,q')})_p$. The family $(\mathcal{F}_{(p,q')}(\Pi_{k+1}(p,q')))_p$ is the sum of $(\mathcal{F}_p(\Pi_{k+1}(p,q')))_p$ and of the family $(\Sigma_{p,k})_p$ of perturbations coming from Eq. (24). In particular:

$$\left|\frac{d\mathcal{J}_0^s\Sigma_{k,p}}{dq'}(0)\right| \preceq 2\delta_{d,s}h\,.$$

Again $\Sigma_{k,p}(q')$ is equal to zero when $0\leq k\leq M-1$ by the second item of Case 1. However this time the action of $(\mathcal{F}_p)_p$ on s-jets is more complicated than on 0-jets. For each k, we write the Taylor expansion of $\mathcal{F}_p$ in P_{k+1} of order s. Then we replace each of its coefficients by its s-jet in p, and the variables by $\mathcal{J}_0^s\Pi_{k+1}(q')$ and $\mathcal{J}_0^u\Pi_{k+1}(q')$, and we expand this expression. This shows that

$$\mathcal{J}_0^s\Pi_k(q') = \mathcal{M}_k\cdot\mathcal{J}_0^s\Pi_{k+1}(q') + \mathcal{M}_k'\cdot\mathcal{J}_0^u\Pi_{k+1}(q') + \mathcal{J}_0^s\Sigma_{k,p}(q')\,.$$

The term $\mathcal{J}_0^s\Sigma_{k,p}(q')$ has its derivative at $q'=0$ bounded by $2\delta_{d,s}h$, and its coefficients in p^i depends only on q_i. The matrix $\mathcal{M}_k$ is inferior triangular with all its diagonal coefficients equal to λ_k with $|\lambda_k|<\lambda<1$. On the other hand, the matrix $\mathcal{M}_k'$ is inferior triangular with all its diagonal coefficients equal to 0. Iterating, using the first item of Step 1 and then taking the derivative at $q'=0$, we see that

$$\frac{d}{dq'}\mathcal{J}_0^s\Pi_{(p,q'),\mathfrak{a}}(\alpha)$$

is inferior triangular with all its diagonal coefficients bounded by

$$2\delta_{d,s} h \frac{\lambda^M}{1-\lambda}\,,$$

which is small compared to h by assumption on M. Thus for every $i = (i_1, \dots, i_d)$ s.t. $\sum_k i_k \leq s$, the coordinate in p^i of the stable component of the s-jet $\mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\alpha)$ has small derivative compared to h while moving the parameter q_i around 0.

The same holds for the preimage $\Pi_1'(p, q')$ of $\Pi_{q,\mathfrak{a}}(\beta)$ by $\mathcal{F}_q$. By Eq. (24) and since $\mathbb{R} \cdot e_0^s$ is close to $\mathbb{R} \cdot v_0^s(\Pi_1'(0,0))$, for every $i = (i_1, \dots, i_d)$ s.t. $\sum_k i_k \leq s$, the coordinate in p^i of the stable component of the s-jet $\mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\beta)$ has a derivative close to h (up to a non zero independent multiplicative constant) while moving the parameter q_i. Thus the coordinate in p^i of $\mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\alpha) - \mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\beta)$ has a non zero derivative while moving q_i around 0. The same still holds for other values of p_0. Then, up to reducing $\mathcal{Q}$, the set of (p_0, q_0') s.t. $\mathrm{J}_{p_0}^s \Pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}_{p_0}^s \Pi_{q,\mathfrak{a}}(\beta)$ are r-close is of Lebesgue measure dominated by $r^{\delta_{d,s}}$, and thus it is also the case for $\mathrm{J}_{p_0}^s \pi_{q,\mathfrak{a}}(\alpha)$ and $\mathrm{J}_{p_0}^s \pi_{q,\mathfrak{a}}(\beta)$. This gives the result, using the Fubini Theorem. The statement about ϑ-perturbations follows easily with similar arguments. This ends the proof in Case 1.

In Case 2, the proof is more simple. First take the original sequences α, β and $\mathfrak{a}$ of Case 2 around which the cylinders $[\rho_\alpha]$, $[\rho_\beta]$ and $[\rho_\mathfrak{a}]$ are centered. The coordinate in p^i of the stable component of the s-jet $\mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\beta)$ has still a non zero derivative while moving the parameter q_i (this only needs the second item of Case 2 and not the first item of Case 1 not present in Case 2). On the other hand, $\Pi_{q,\mathfrak{a}}(\alpha)$ is the image of $\Pi_{q,\mathfrak{a}}(\beta)$ by some iterate of $\mathcal{F}_q$ restricted on the M first images of $W_q^\mathfrak{a}$ since $\mathfrak{a}$ is periodic of period $\mathfrak{p}$ and the last letters of α are of the form $\beta_{-1} \cdot \mathfrak{p}^f$ for some $f > 0$. These local stable manifolds do not depend on q' by the first item of Case 2. Moreover the action of $(\mathcal{F}_q)_q$ restricted to these stable manifolds on stable components of jets is inferior triangular with diagonal coefficients between 0 and 1. Thus the coordinate in p^i of $\mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\alpha) - \mathrm{J}_0^s \Pi_{q,\mathfrak{a}}(\beta)$ has again a non zero derivative while moving q_i. Since the lengths of the cylinders are large, this remains true for any $(\alpha, \beta, \mathfrak{a})$ in $[\rho_\alpha] \times [\rho_\beta] \times [\rho_\mathfrak{a}]$. We conclude as in Case 1, which ends the proof. $\square$

Remark 6.0.6. *When $\ell = 1$, to give a relative movement inside the $\mathfrak{a}$-fiber to the points encoded by α and β, we considered the preimages of these two points by $\mathcal{F}_p$ (respectively on $W_p^{\alpha_{-1}\mathfrak{a}}$ and $W_p^{\beta_{-1}\mathfrak{a}}$). The second one was not periodic and distinct from the first one. We then perturbed $\mathcal{F}_p$ in a neighborhood of this second preimage.*

In the case when $\ell > 1$, we shall look at the ℓ respective successive preimages of these two points by $\mathcal{F}_p$ (respectively on $W_p^{\alpha_{-1}\mathfrak{a}}$ and $W_p^{\beta_{-1}\mathfrak{a}}$ and their $\ell - 1$ successive images by $\mathcal{F}_p$) and take the first pair of preimages which are distinct. One of them is not periodic, and we perform the same perturbation as before in a neighborhood of this point. Then the proof is the same with the same distinction in two cases whether the M successive preimages of the other preimage intersect this neighborhood or not.

7 Appendix

Proofs of Intermediate Lemmas 5.1.2, 5.3.1 and 5.4.4

Proof of Lemma 5.1.2. Let us fix $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $x \in X$ and $\alpha = (\alpha_{-n}, \cdots, \alpha_{-1}) \in \mathcal{A}^n$ for some $n > 0$. We notice that the differential $D\psi_{p,\mathfrak{a}}^\alpha(x)$ can be written as the product of n factors:

$$D\psi_{p,\mathfrak{a}}^\alpha(x) = \prod_{k=1}^n D\psi_{p,\mathfrak{a}_k}^{\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right) \text{ with } \mathfrak{a}_k := \alpha_{|k-1}\mathfrak{a} \qquad (25)$$

By (**U**), each of these n factors is unipotent inferior and then can be written as a sum of N terms:

$$D\psi_{p,\mathfrak{a}_k}^{\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right) = M_0 + M_1 + \cdots + M_{N-1}\,. \tag{26}$$

Here M_0 is a diagonal matrix which has all its diagonal coefficients of absolute value equal to the coefficient $\lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right)$. On the other side, for every $1 \leq k \leq N-1$, all the coefficients of M_k are equal to 0 except possibly on the kth (small) diagonal line below the (great) diagonal line.

We can write each of the n factors of Eq. (25) as in Eq. (26) and then expand $D\psi_{p,\mathfrak{a}}^{\alpha}(x)$ as a sum of N^n factors of n terms. Among them, any such product with $\geq N$ matrices having zero coefficients on and above the great diagonal line vanishes. Thus, between the N^n terms whose sum equals $D\psi_{p,\mathfrak{a}}^{\alpha}(x)$, there are at most

$$\tilde{P}(n) := \binom{n}{N-1} \cdot (N-1)^{N-1} + \binom{n}{N-2} \cdot (N-1)^{N-2} + \cdots \binom{n}{1} \cdot (N-1) + 1 \tag{27}$$

which are non zero, with $\tilde{P}$ polynomial. Each of these $\leq \tilde{P}(n)$ terms $\mathcal{M}_1 \cdots \mathcal{M}_n$ is a product of n factors $\mathcal{M}_k$.

Let us consider such a product $\mathcal{M}_1 \cdots \mathcal{M}_n$. At most $N-1$ of the $\mathcal{M}_k$ have all their coefficients equal to 0 except possibly on one of the small diagonal lines below the (great) diagonal line. The $\geq n-(N-1)$ other factors are all diagonal matrices. Each coefficient of the resulting product $\mathcal{M}_1 \cdots \mathcal{M}_n$ is then either zero or equal to the product of n non zero coefficients c_k, with c_k a non zero coefficient of $\mathcal{M}_k$. If $\mathcal{M}_k$ is diagonal, $|c_k|$ is equal to $\lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right)$. If not, $|c_k|$ is bounded by some independent constant C_1 since for every $a \in \mathcal{A}$, the C^1-bounded map $\psi_{p,\mathfrak{a}}^{a}$ depends continuously on $p \in \overline{\mathcal{P}}$ and $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$. By Eq. (13), we have:

$$\lambda_{p,\mathfrak{a},\alpha}(x) = \prod_{k=1}^{n} \lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right) \text{ with } \mathfrak{a}_k := \alpha_{|k-1}\mathfrak{a}\,. \tag{28}$$

Moreover, each of the coefficients $\lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right)$ is larger than γ'. Thus the resulting coefficient of $\mathcal{M}_1 \cdots \mathcal{M}_n$ is smaller than $\lambda_{p,\mathfrak{a},\alpha}(x) \cdot (\gamma')^{-N+1} \cdot C_1^{N-1}$ in modulus. Thus any coefficient of $D\psi_{p,\mathfrak{a}}^{\alpha}(x)$ is bounded by $P(n) \cdot \lambda_{p,\mathfrak{a},\alpha}(x)$, where $P(n) := \tilde{P}(n) \cdot (\gamma')^{-N-1} \cdot C_1^{N-1}$ is a positive polynomial on $\mathbb{R}_+$. □

Proof of Lemma 5.4.4. For every $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n \geq 0$, $\rho \in \mathcal{A}^n$ and $(\alpha, \beta) \in \mathcal{C}_\rho$, the points $\pi_{p,\mathfrak{a}}(\alpha)$ and $\pi_{p,\mathfrak{a}}(\beta)$ are the respective images of $\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha))$ and $\pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))$ by $\psi_{p,\mathfrak{a}}^{\rho}$. Let us denote by $v = (v_1, \cdots, v_N)$ the vector $v := \pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha)) - \pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))$. We denote by $j \in \{1, \cdots, N\}$ the maximal index s.t. $|v_j| > 2N \cdot P(n) \cdot |v_i|$ for every $i < j$, where the polynomial P was defined in Lemma 5.1.2. Using this, it follows:

$$|v_j| \geq F \cdot (2NP(n))^{-N} \cdot ||v|| \tag{29}$$

for some positive constant F (depending only on N). The segment between the two points $\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha))$ and $\pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))$ is fully included in X since X is convex. Let $\chi : [0,1] \to \mathbb{R}$ be the C^1-map which sends $x \in [0,1]$ to the jth coordinate of $\psi_{p,\mathfrak{a}}^{\rho}(\pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta)) + xv)$. By the mean value equality, there exists $x \in (0,1)$ s.t.:

$$\chi(1) - \chi(0) = \chi'(x) = \sum_{i=1}^{j} a_{j,i} \cdot v_i \tag{30}$$

where $a_{j,i}$ is the coefficient of index (j,i) of the differential $D\psi_{p,\mathfrak{a}}^{\rho}(y)$ with $y := \pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta)) + xv$. The right-hand equality is due to the fact that $D\psi_{p,\mathfrak{a}}^{\rho}(y)$ is

unipotent inferior by (**U**). We notice that $|a_{j,j}| = \lambda_{p,\mathfrak{a},\rho}(y)$. By Lemma 5.1.2, $|a_{j,i}|$ is smaller than $P(n) \cdot \lambda_{p,\mathfrak{a},\rho}(y)$. By Eq. (30), we then have:

$$|\chi(1)-\chi(0)| \geq \lambda_{p,\mathfrak{a},\rho}(y)\cdot|v_j|-\sum_{i<j} P(n)\cdot\lambda_{p,\mathfrak{a},\rho}(y)\cdot\frac{|v_j|}{2N\cdot P(n)} \geq \frac{\lambda_{p,\mathfrak{a},\rho}(y)\cdot|v_j|}{2}\,. \quad (31)$$

Noticing that $||\pi_{p,\mathfrak{a}}(\alpha)-\pi_{p,\mathfrak{a}}(\beta)|| \geq |\chi(1)-\chi(0)|$ and injecting Eq. (29) in Eq. (31), it then holds:

$$||\pi_{p,\mathfrak{a}}(\alpha)-\pi_{p,\mathfrak{a}}(\beta)|| \geq \frac{F}{2}\cdot(2NP(n))^{-N}\cdot\lambda_{p,\mathfrak{a},\rho}(y)\cdot||\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha))-\pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))||\,. \quad (32)$$

By Lemma 5.2.1 and noting $R(n) := \frac{2D_1}{F}\cdot(2NP(n))^N$ which is positive on $\mathbb{R}_+$, we have:

$$||\pi_{p,\mathfrak{a}}(\alpha)-\pi_{p,\mathfrak{a}}(\beta)|| \geq \frac{\Lambda_{p,\mathfrak{a},\rho}}{R(n)}\cdot||\pi_{p,\rho\mathfrak{a}}(\sigma^n(\alpha))-\pi_{p,\rho\mathfrak{a}}(\sigma^n(\beta))||\,. \quad (33)$$

The proof of the second item is similar and we apply Lemma 5.2.4 to conclude. □

Proof of Lemma 5.3.1. We notice that the coefficient $\lambda_{p,\mathfrak{a},\alpha_{-1}}\left(\pi_{p,\alpha_{-1}\mathfrak{a}}(\sigma(\alpha))\right)$ is positive and uniformly distant from 0 and $+\infty$. Since log is C^1 on $]0,+\infty[$, we just have to show that the following map is Hölder with positive exponent on its domain:

$$(\alpha,\mathfrak{a}) \mapsto \lambda_{p,\mathfrak{a},\alpha_{-1}}\left(\pi_{p,\alpha_{-1}\mathfrak{a}}(\sigma(\alpha))\right).$$

Let us recall that the latter is the $(1,1)$ coefficient of $D\psi^{\alpha_{-1}}_{p,\mathfrak{a}}(\pi_{p,\alpha_{-1}\mathfrak{a}}(\sigma(\alpha)))$ (up to the sign). By assumption the map $\mathfrak{a} \in \overrightarrow{\mathcal{A}} \mapsto Df_{p,\mathfrak{a}}$ is Hölder for the C^0-topology and so it is enough to show that the map $(\alpha,\mathfrak{a}) \in \overleftarrow{\mathcal{A}} \times \overrightarrow{\mathcal{A}} \mapsto \pi_{p,\mathfrak{a}}(\alpha) \in X$ is itself Hölder. By hyperbolicity, the map $\alpha \in \overleftarrow{\mathcal{A}} \mapsto \pi_{p,\mathfrak{a}}(\alpha) \in X$ is Hölder for any $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, with exponent and constant independent of $\mathfrak{a}$. Thus it is enough to show that the map $\mathfrak{a} \in \overrightarrow{\mathcal{A}} \mapsto \pi_{p,\mathfrak{a}}(\alpha) \in X$ is Hölder for any $\alpha \in \overleftarrow{\mathcal{A}}$, with independent constants. But using again both the hyperbolicity and that $\mathfrak{a} \mapsto f_{p,\mathfrak{a}}$ and $\mathfrak{a} \mapsto Df_{p,\mathfrak{a}}$ are Hölder for the C^0-topology, we see that the maps $\mathfrak{a} \mapsto \psi^{\alpha_n}_{p,\mathfrak{a}}(0)$ are Hölder, with exponent and constant independent of p, α and n. But this sequence converges uniformly to the map $\mathfrak{a} \mapsto \pi_{p,\mathfrak{a}}(\alpha)$, which concludes the proof. □

Proof of Distortion Lemmas 5.2.1, 5.2.2, 5.2.3 and 5.2.4

Proofs of Lemma 5.2.1. The Lemma will follow easily from the two following:

Sublemma 7.0.1. *There exists $A > 0$ s.t. for any $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $a \in \mathcal{A}$, it holds:*

$$1 - A|x-y| \leq \frac{\lambda_{p,\mathfrak{a},a}(x)}{\lambda_{p,\mathfrak{a},a}(y)} \leq 1 + A|x-y| \qquad \forall\, x,y \in X\,.$$

Proof. The non zero number $\lambda_{p,\mathfrak{a},a}(x)$ is the $(1,1)$ coefficient of the differential $D\psi^a_{p,\mathfrak{a}}(x)$ (up to the sign). Since the maps $\psi^a_{p,\mathfrak{a}}$ are uniformly (in p, $\mathfrak{a}$ and a) C^2 bounded, denoting by $\tilde{A}$ a uniform bound of the second differential of $\psi^a_{p,\mathfrak{a}}$ on X among $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $a \in \mathcal{A}$, the coefficient $\lambda_{p,\mathfrak{a},a}(x)$ is between $\lambda_{p,\mathfrak{a},a}(y) - \tilde{A}|x-y|$ and $\lambda_{p,\mathfrak{a},a}(y) + \tilde{A}|x-y|$. We notice that $\gamma' < \lambda_{p,\mathfrak{a},a}(y)$. Denoting $A := \tilde{A}/\gamma'$ and taking the quotient, we get the desired inequality. □

Sublemma 7.0.2. *There exists $A' > 0$ such that for every $p \in \overline{\mathcal{P}}$, $\mathfrak{a} \in \overrightarrow{\mathcal{A}}$, $n \geq 0$, $\alpha \in \mathcal{A}^n$, $x,y \in X$, the points $\psi^\alpha_{p,\mathfrak{a}}(x)$ and $\psi^\alpha_{p,\mathfrak{a}}(y)$ are $A' \cdot \gamma^{n/2}$ distant.*

Proof. It is an immediate consequence of the inequality $\Lambda_{p,\mathfrak{a},\alpha} < \gamma^{|\alpha|}$ and of Lemma 5.1.2 that the diameter of $\psi^\alpha_{p,\mathfrak{a}}(X)$ is dominated by $\gamma^{|\alpha|/2}$. □

We can now conclude. Using Eq. (13) we write both $\lambda_{p,\mathfrak{a},\alpha}(x)$ and $\lambda_{p,\mathfrak{a},\alpha}(y)$ as products of n factors and thus their quotients as:

$$\frac{\lambda_{p,\mathfrak{a},\alpha}(x)}{\lambda_{p,\mathfrak{a},\alpha}(y)} = \prod_{k=1}^{n} \frac{\lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right)}{\lambda_{p,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(y)\right)}$$

where we set again $\mathfrak{a}_k := \alpha_{|k-1}\mathfrak{a} \in \overrightarrow{\mathcal{A}}$. Using the two Sublemmas, the previous quotient is between $\prod_{k=1}^{n}(1-AA'\cdot\gamma^{n/2})$ and $\prod_{k=1}^{n}(1+AA'\cdot\gamma^{n/2})$. Since $0<\gamma<1$, the infinite products $\prod(1\pm AA'\cdot\gamma^{n/2})$ converge and their limits are respectively in $(0,1)$ and $(1,+\infty)$, which concludes the proof. □

Proof of Lemma 5.2.2. Let us fix $\eta>0$. By Sublemmas (7.0.1) and (7.0.2), there exist $n_0\in\mathbb{N}$ s.t. for every $p\in\overline{\mathcal{P}}$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$, $n>n_0$, $\alpha\in\mathcal{A}^n$, $a\in\mathcal{A}$ and $x,y\in\psi_{p,\mathfrak{a}}^{\alpha}(X)$, we have $e^{-\eta}<\lambda_{p,\mathfrak{a},a}(x)/\lambda_{p,\mathfrak{a},a}(y)<e^{\eta}$. We recall that the map $\psi_{p,\mathfrak{a}}^{a}$ depends continuously in the C^2-norm on $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$ and p in $\overline{\mathcal{P}}$ (both compact sets). Thus there exists $\delta(\eta)>0$ such that for every $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$ and $p_1,p_2\in\mathcal{P}$ with $||p_1-p_2||<\delta(\eta)$, we have $e^{-\eta}<\lambda_{p_1,\mathfrak{a},a}(x)/\lambda_{p_2,\mathfrak{a},a}(y)<e^{\eta}$ for every $n>n_0$, $\alpha\in\mathcal{A}^n$, $a\in\mathcal{A}$, $x\in\psi_{p_1,\mathfrak{a}}^{\alpha}(X)$ and $y\in\psi_{p_2,\mathfrak{a}}^{\alpha}(X)$. We denote by $D_2>0$ the maximum of the quotients $\lambda_{p_1,\mathfrak{a},\alpha}(x)/\lambda_{p_2,\mathfrak{a},\alpha}(y)$ among $p_1,p_2\in\overline{\mathcal{P}}$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$, $n\le n_0$, $\alpha\in\mathcal{A}^n$ and $x,y\in X$. We conclude by noticing that for every $n>n_0$ and $x,y\in X$, we have:

$$\frac{\lambda_{p_1,\mathfrak{a},\alpha}(x)}{\lambda_{p_2,\mathfrak{a},\alpha}(y)} = \frac{\lambda_{p_1,\mathfrak{a},\cdots\alpha_{-n_0}\cdots\alpha_{-1}}(x)}{\lambda_{p_2,\mathfrak{a},\alpha_{-n_0}\cdots\alpha_{-1}}(y)} \cdot \prod_{k=n_0+1}^{n} \frac{\lambda_{p_1,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(x)\right)}{\lambda_{p_2,\mathfrak{a}_k,\alpha_{-k}}\left(\psi_{p,\mathfrak{a}_{k+1}}^{\alpha_{-k-1}} \circ \cdots \circ \psi_{p,\mathfrak{a}_n}^{\alpha_{-n}}(y)\right)}.$$

□

Proof of Lemma 5.2.3. Let us fix $p\in\overline{\mathcal{P}}$. We saw in the proof of Lemma 5.3.1 that

$$(\mathfrak{a},\alpha)\mapsto\lambda_{p,\mathfrak{a},\alpha_{-1}}\left(\pi_{p,\alpha_{-1}\mathfrak{a}}(\sigma(\alpha))\right)$$

is Hölder with positive exponent on its domain. We now fix an arbitrary $\beta\in\overleftarrow{\mathcal{A}}$. Let us take $\mathfrak{a},\mathfrak{a}'\in\overrightarrow{\mathcal{A}}$. By Eq. (13) and proceeding as in the proof of Lemma 5.2.1, we see that the quotient

$$\frac{\lambda_{p,\mathfrak{a},\alpha}(\pi_{p,\alpha\mathfrak{a}}(\beta))}{\lambda_{p,\mathfrak{a}',\alpha}(\pi_{p,\alpha\mathfrak{a}'}(\beta))}$$

is bounded between 0 and $+\infty$, with constants independent of p, $\mathfrak{a}$, $\mathfrak{a}'$ and α. To conclude, we just have to apply Lemma 5.2.1. □

Proof of Lemma 5.2.4. By Sublemma (7.0.2), there exists $n_0\in\mathbb{N}$ s.t. for every $p\in\overline{\mathcal{P}}$, $n>n_0$, $\rho\in\mathcal{A}^n$, $a\in\mathcal{A}$ and $x,y\in\psi_{p,\mathfrak{a}}^{\rho}(X)$, we have

$$\lambda_{p,\mathfrak{a},a}(y)^{\epsilon'}<\lambda_{p,\mathfrak{a},a}(x)<\lambda_{p,\mathfrak{a},a}(y)^{1/\epsilon'}.$$

Then up to taking ϑ-perturbations for small ϑ, for every $t\in\mathcal{T}$, $p\in\overline{\mathcal{P}}$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$, $n>n_0$, $\rho\in\mathcal{A}^n$, $a\in\mathcal{A}$, $x\in\psi_{t,p,\mathfrak{a}}^{\rho}(X)$ and $y\in\psi_{p,\mathfrak{a}}^{\rho}(X)$, we have

$$\lambda_{p,\mathfrak{a},a}(y)^{\epsilon'}<\tilde{\lambda}_{t,p,\mathfrak{a},a}(x)<\lambda_{p,\mathfrak{a},a}(y)^{1/\epsilon'}.$$

Let $D_4>0$ be the maximum of $\tilde{\lambda}_{t,p,\mathfrak{a},\rho}(x)/\lambda_{p,\mathfrak{a},\rho}^{\epsilon'}(y)$ and $\lambda_{p,\mathfrak{a},\rho}^{1/\epsilon'}(y)/\tilde{\lambda}_{t,p,\mathfrak{a},\rho}(x)$ among $t\in\mathcal{T}$, $\mathfrak{a}\in\overrightarrow{\mathcal{A}}$, $p\in\overline{\mathcal{P}}$, $n\le n_0$, $\rho\in\mathcal{A}^n$, $x,y\in X$. We conclude as for Lemma 5.2.2. □

Pressure function: Proof of Proposition 5.1.4

For simplicity, we fix $p \in \overline{\mathcal{P}}$ and denote for every $s \geq 0$:

$$Z_n(s) := \sum_{\alpha \in \mathcal{A}^n} \Lambda^s_{p,\mathfrak{a},\alpha} > 0\,.$$

For every $s \geq 0$, the sequence $n \in \mathbb{N}_+ \mapsto \log Z_n(s)$ is subadditive and so the limit $\Pi_{p,\mathfrak{a}}(s) = \lim_{n\to+\infty} \frac{1}{n}\log Z_n(s)$ exists and is finite by Fekete's Lemma. We notice that $\Pi_{p,\mathfrak{a}}(0)$ is equal to the topological entropy of the shift σ which is positive since $\mathcal{A}$ has at least two letters. An immediate consequence of Lemma 5.2.3 is that the pressure $\Pi_{p,\mathfrak{a}}$ only depends on p. We denote it by Π_p. We notice that $s \in \mathbb{R}_+ \mapsto Z_n(s)$ is log convex. Thus the map $s \in \mathbb{R}_+ \mapsto \log Z_n(s)$ is convex. The limit map $s \mapsto \Pi_p(s)$ is then also convex and thus continuous. We remark that for $s, s' \geq 0$, it holds:

$$Z_n(s+s') = \sum_{\alpha \in \mathcal{A}^n} \Lambda^{s+s'}_{p,\mathfrak{a},\alpha} \leq \sum_{\alpha \in \mathcal{A}^n} \Lambda^s_{p,\mathfrak{a},\alpha} \cdot \gamma^{ns'}$$

and then $\Pi_p(s+s') \leq \Pi_p(s) + s' \cdot \log\gamma$. Since $\log\gamma < 0$, the map $s \in \mathbb{R}_+ \mapsto \Pi_p(s)$ is strictly decreasing. Tending $s \to +\infty$, we see that $\Pi_p(s)$ tends to $-\infty$. Finally, by the intermediate value theorem, the map $s \in \mathbb{R}_+ \mapsto \Pi_p(s)$ has a unique zero $\Delta(p)$. It remains to prove the continuity of $p \mapsto \Delta(p)$. By Eq. (16), we have:

$$\Pi_p(s) = \lim_{n\to+\infty} \frac{1}{n}\log \sum_{\alpha \in \mathcal{A}^n} \Lambda^s_{p,\mathfrak{a},\alpha} \text{ for any } s \geq 0\,. \tag{34}$$

By Lemma 5.4.3 (whose proof does not need the continuity of $p \mapsto \Delta(p)$), we see that for every p and $\epsilon' > 1$, there exists a neighborhood $\mathcal{U}_{p,\epsilon'}$ of p and a constant $D_5 > 0$ s.t. for every $p' \in \mathcal{U}_{p,\epsilon'}$ and $\alpha \in \mathcal{A}^*$, the term $\Lambda_{p',\mathfrak{a},\alpha}$ is bounded between $\Lambda^{\epsilon'}_{p,\mathfrak{a},\alpha}/D_5$ and $D_5\Lambda^{1/\epsilon'}_{p,\mathfrak{a},\alpha}$. Injecting this in Eq (34), this implies that $\Pi_p(s\epsilon') \leq \Pi_{p'}(s) \leq \Pi_p(s/\epsilon')$ for every $s \geq 0$ and $p' \in \mathcal{U}_{p,\epsilon'}$ and so $\Delta(p)/\epsilon' \leq \Delta(p') \leq \Delta(p)\epsilon'$ for every $p' \in \mathcal{U}_{p,\epsilon'}$. This proves the continuity of $p \mapsto \Delta(p)$.

Hyperbolicity theory

We recall here some background on hyperbolic compact sets of C^r-endomorphisms. This subsection is mainly taken from Section 1 of the article [Be1] of Berger and from Appendix C of the article [BB] of Berger and the author.

Let $\mathcal{M}$ be a manifold. A subset $\mathcal{K} \subset \mathcal{M}$ is left invariant by a C^1-endomorphim $\mathcal{F}$ from $\mathcal{M}$ into $\mathcal{M}$ if $\mathcal{F}(\mathcal{K}) = \mathcal{K}$. When $\mathcal{F}$ is a diffeomorphism, the invariant compact set $\mathcal{K} \subset \mathcal{M}$ is hyperbolic if there exists a $D\mathcal{F}$-invariant splitting $T\mathcal{M}|\mathcal{K} = \mathcal{E}^s \bigoplus \mathcal{E}^u$ so that $\mathcal{E}^s$ is contracted by $D\mathcal{F}$ and $\mathcal{E}^u$ is expanded by $D\mathcal{F}$:

$$\exists \lambda < 1,\ C > 0,\ \forall k \in \mathcal{K},\ \forall n \geq 0,\ ||D\mathcal{F}^n|\mathcal{E}^s_k|| \leq C\lambda^n \text{ and } ||(D\mathcal{F}^n|\mathcal{E}^u_k)^{-1}|| \leq C\lambda^n\,.$$

When $\mathcal{F}$ is a local diffeomorphism, we shall study the inverse limit $\overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$ of $\mathcal{K}$:

$$\overleftrightarrow{\mathcal{K}}_{\mathcal{F}} := \{(k_i)_i \in \mathcal{K}^{\mathbb{Z}} : \mathcal{F}(k_i) = k_{i+1}, \forall i \in \mathbb{Z}\}\,.$$

It is a compact space for the topology induced by the product one of $\mathcal{K}^{\mathbb{Z}}$. The dynamics induced by $\mathcal{F}$ on $\overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$ is the shift $\overleftrightarrow{\mathcal{F}}$ and is invertible. Let $\pi : \overleftrightarrow{\mathcal{K}}_{\mathcal{F}} \to \mathcal{K}$ be the zero-coordinate projection. Let $\pi^*T\mathcal{M}$ be the bundle over $\overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$ whose fiber at $\underline{k}$ is $T_{\pi(\underline{k})}\mathcal{M}$. The map $D\mathcal{F}$ acts canonically on $\pi^*T\mathcal{M}$ as $\overleftrightarrow{\mathcal{F}}$ on the basis and as the linear map $D_{\pi(\underline{k})}\mathcal{F}$ on the fiber of $\underline{k} \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$.

The compact set $\mathcal{K}$ (or $\overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$) is hyperbolic if there exists a $D\mathcal{F}$-invariant splitting $\pi^*T\mathcal{M} = \mathcal{E}^s \bigoplus \mathcal{E}^u$ s.t. $\mathcal{E}^s_{\underline{k}}$ is contracted by $D_{\pi(\underline{k})}\mathcal{F}$ and $\mathcal{E}^u_{\underline{k}}$ is expanded by $D_{\pi(\underline{k})}\mathcal{F}$:

$$\exists \lambda < 1,\ C > 0,\ \forall \underline{k} \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}},\ \forall n \geq 0,\ ||D\mathcal{F}^n|\mathcal{E}^s_{\underline{k}}|| \leq C\lambda^n \text{ and } ||(D\mathcal{F}^n|\mathcal{E}^u_{\underline{k}})^{-1}|| \leq C\lambda^n\,.$$

Actually the definition of hyperbolicity for local diffeomorphisms is consistent with the definition of hyperbolicity for diffeomorphisms when the dynamics is invertible. Here is a useful result about structural stability:

Theorem (Przytycki). *Let $\mathcal{K}$ be a hyperbolic set for a C^1-local diffeomorphism $\mathcal{F}$ of $\mathcal{M}$. Then for every C^1-local diffeomorphism $\mathcal{F}'$ which is C^1-close to $\mathcal{F}$, there exists a continuous map $i_{\mathcal{F}'} : \overleftrightarrow{\mathcal{K}}_{\mathcal{F}} \to \mathcal{M}$ which is C^0-close to π and so that:*

1. $i_{\mathcal{F}'} \circ \overleftrightarrow{\mathcal{F}} = \mathcal{F}' \circ i_{\mathcal{F}'}$,
2. $\mathcal{K}_{\mathcal{F}'} := i_{\mathcal{F}'}(\overleftrightarrow{\mathcal{K}}_{\mathcal{F}})$ *is hyperbolic for* $\mathcal{F}'$.

Let us also recall the definition of a stable manifold in this context. For every $k \in \mathcal{K}$ and $\eta > 0$, we define the stable manifold and local stable manifold of k by:

$$W^s(k;\mathcal{F}) := \{k' \in \mathcal{M} : d(\mathcal{F}^n(k), \mathcal{F}^n(k')) \underset{n\to+\infty}{\longrightarrow} 0\}.$$

$$W^s_\eta(k;\mathcal{F}) := \{k' \in \mathcal{M} : \eta > d(\mathcal{F}^n(k), \mathcal{F}^n(k')) \underset{n\to+\infty}{\longrightarrow} 0\}.$$

For $\underline{k} \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}}$ and $\eta > 0$, the unstable manifold and local unstable manifold of $\underline{k}$ are:

$$W^u(\underline{k},\mathcal{F}) = \{k'_0 \in \mathcal{M} : \exists (k'_i)_{i<0} \text{ s.t. } \mathcal{F}(k'_{i-1}) = k'_i \text{ and } d(k_n, k'_n) \underset{n\to-\infty}{\longrightarrow} 0\}.$$

$$W^u_\eta(\underline{k},\mathcal{F}) = \{k'_0 \in \mathcal{M} : \exists (k'_i)_{i<0} \text{ s.t. } \mathcal{F}(k'_{i-1}) = k'_i \text{ and } \eta > d(k_n, k'_n) \underset{n\to-\infty}{\longrightarrow} 0\}.$$

These sets are properly embedded C^r-manifolds. For simplicity, we do not denote $\mathcal{F}$ in the article when it is obvious: $W^s(k)$ for example.

Theorem F (Berger [Be1] Prop. 1.6, Theorem C.5 [BB]). *Let $r \geq 1$ and let $\mathcal{M}$ be a manifold. Suppose that $(\mathcal{F}_p)_p$ is a C^r-family of local diffeomorphisms $\mathcal{F}_p$ of $\mathcal{M}$ leaving invariant the continuation of a compact hyperbolic set $\mathcal{K}_p$. Then there exists $\eta > 0$ s.t. the families $(W^s_\eta(k_p;\mathcal{F}_p))_{p\in\mathcal{P}}$ and $(W^u_\eta(\underline{k}_p;\mathcal{F}_p))_{p\in\mathcal{P}}$ of C^r-submanifolds are of class C^r and depend C^0 on respectively $k_0 \in \mathcal{K}_0$ and $\underline{k}_0 \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}_0}$.*

Remark 7.0.3. *An immediate adaptation of the proof of Theorem C.5 in Appendix C of [BB] actually shows that the families $(W^s_\eta(k_p;\mathcal{F}_p))_{p\in\mathcal{P}}$ and $(W^u_\eta(\underline{k}_p;\mathcal{F}_p)_{p\in\mathcal{P}}$ depend Hölder for the C^{r-1}-topology on respectively $k_0 \in \mathcal{K}_0$ and $\underline{k}_0 \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}_0}$.*

We will need the following parametric inclination lemma:

Lemma 7.0.4. *Let $r \geq 1$ and $U \Subset \mathbb{R}^m$. Suppose that $(\mathcal{F}_p)_p$ is a C^r-family of local diffeomorphisms $\mathcal{F}_p$ of U leaving invariant a compact hyperbolic set $\mathcal{K}_p$. Let $\underline{k} = (k^i_0)_i \in \overleftrightarrow{\mathcal{K}}_{\mathcal{F}_0}$ and $(\Gamma_p)_p$ be a C^r-family of manifolds of the same dimension as $W^s_\eta(\underline{k}_p;\mathcal{F}_p)$. Suppose that Γ_p does not intersect the stable set of $\mathcal{K}_p$ and $(\Gamma_p)_p$ intersects transversally $(W^u_\eta(\underline{k}_p;\mathcal{F}_p))_p$ at a C^r-family of points $(z_p)_p$. Then for any $\epsilon > 0$ and n large there is a submanifold Γ^n_p C^r-close to $W^s_\eta(k_p^{-n};\mathcal{F}_p)$, whose image by $\mathcal{F}^n_p$ is in a ϵ-neighborhood of z_p in Γ_p and s.t. $(\Gamma^n_p)_p$ is C^r-close to $(W^s_\eta(k_p^{-n};\mathcal{F}_p))_p$.*

Proof. The proof is similar to the one of the parametric inclination Lemma C.6 of [BB] but for inverse iterations this time: one extends $\mathcal{F}_p$ on a neighborhood U' of U it in such a way that Γ_p is included in the stable manifold of some saddle point. Then we apply Theorem F. □

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