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# Convergence to a self similar solution of a one-dimensional one-phase Stefan Problem

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## Abstract

We revisit the one-dimensional one-phase Stefan problem with a Dirichlet boundary condition at  $x = 0$  as stated in the book of Avner Friedman about parabolic equations [F3]. We prove that under rather general hypotheses on the initial data, the solution converges to a self-similar profile as  $t \rightarrow +\infty$ .

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*Keywords:* Stefan problem; free-boundary problem; self-similar solution; large-time behavior.

## 1 Introduction

In this article, we revisit a standard one-dimensional one-phase Stefan problem. This free boundary problem arises in very simple physical situations and has been studied by numerous authors; in particular we should mention a chapter of the book of Avner Friedman on parabolic equations

(Chapter 8 of [F3, p.215]). This problem is given by

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ u(0, t) = h, & t > 0, \\ u(s(t), t) = 0, & t > 0, \\ \frac{ds(t)}{dt} = -u_x(s(t), t), & t > 0, \\ s(0) = b_0, \\ u(x, 0) = u_0(x), & 0 < x < b_0 \end{cases} \quad (1.1)$$

where  $x = s(t)$  is the unknown free boundary which is to be found together with  $u(x, t)$ .

Friedman [F3] proves that this problem has a unique smooth classical solution  $(u(x, t), s(t))$  in  $Q := \{(x, t), t > 0, 0 < x < s(t)\}$ . Moreover it follows from Schaeffer [S] and Friedman [F1] that  $s \in C^\infty(0, \infty)$  and that  $u$  is infinitely differentiable up to the free boundary  $s$ .

The purpose of this paper is to study the large time behavior of the solution pair  $(u, s)$ . Also let us mention some previous results from literature. Meirmanov [M] has proved that  $\frac{s(t)}{\sqrt{t}} \rightarrow a$ , where  $a$  is the unique solution of the nonlinear equation (1.3) below. Also, Ricci and Xie [R] have performed a stability analysis of some special solutions of a related one-phase Stefan problem on the semi-infinite interval  $(s(t), \infty)$ . In particular, they mention that the interface  $s(t)$  behaves as  $\beta\sqrt{t}$  for some positive constant  $\beta$  which they characterize. Moreover, Aiki and Muntean [AM1, AM2], as mentioned in [Z], have proved the existence of two positive constants  $c$  and  $C$  independent of  $t$  such that

$$c\sqrt{t} \leq s(t) \leq C\sqrt{t+1} \text{ for all } t \geq 0,$$

in the case of a more complicated system.

In this article, we will prove that the solution pair  $(u, s)$  converges to a self-similar solution as  $t \rightarrow \infty$ .

First, let us define the self-similar solution. To do so, we introduce the self-similar variable  $\eta = \frac{x}{\sqrt{t+1}}$ . Then, the self-similar solution is given by

$$u(x, t) = U\left(\frac{x}{\sqrt{t+1}}\right) = U(\eta) = h \left[ 1 - \frac{\int_0^\eta e^{-\frac{s^2}{4}} ds}{\int_0^a e^{-\frac{s^2}{4}} ds} \right] \text{ for all } \eta \in (0, a), \quad (1.2)$$

where  $a$  is characterized as the unique solution of the nonlinear equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}} ds. \quad (1.3)$$

In the first step, we will write the problem (1.1) in terms of  $\eta$  and  $t$ . To do so, we set

$$\begin{cases} V(\eta, t) = u(x, t), \\ a(t) = \frac{s(t)}{\sqrt{t+1}}. \end{cases} \quad (1.4)$$

However, the partial differential equation for  $V$  which we obtain explicitly involves the time variable  $t$ . It is given by

$$(t+1)V_t = V_{\eta\eta} + \frac{\eta}{2}V_\eta, \quad t > 0, \quad 0 < \eta < a(t). \quad (1.5)$$

This leads us to perform the change of time variable  $\tau = \ln(t+1)$ . A similar change of variables was performed by [HH]. The full time evolution problem corresponding to the system (1.1) in coordinates  $\eta$  and  $\tau$  is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \frac{\eta}{2}W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0, \tau) = h, & \tau > 0, \\ W(b(\tau), \tau) = 0, & \tau > 0, \\ \frac{db(\tau)}{d\tau} + \frac{b(\tau)}{2} = -W_\eta(b(\tau), \tau), & \tau > 0, \\ b(0) = b_0, \\ W(\eta, 0) = u_0(\eta), & 0 < \eta < b_0 \end{cases} \quad (1.6)$$

where  $b(\tau) = a(t)$ . We shall denote by  $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$  the solution pair of (1.6) with the initial conditions  $(u_0, b_0)$ .

It is in the coordinates  $\eta$  and  $\tau$  that we will rigorously characterize the large time behavior of the solution pair  $(W, b)$ . However, for technical reasons, we sometimes have to use different variables, namely  $(y, \tau)$  with  $y = \frac{\eta}{b(\tau)}$  for all  $0 < \eta < b(\tau)$ . The problem is then transformed into a problem on a fixed domain.

**Organization of the paper.** In Section 2, we introduce the Stefan problem [F2] and recall known well-posedness and regularity results [F1, S]. Using a maximum principle [F3], we show that if  $u_0$  is nonnegative and bounded then the solution  $u$  is also nonnegative and bounded.

In Section 3, we start by defining a notion of upper and lower solutions for Problem (1.1). Then, we prove a comparison principle in the  $(x, t)$  coordinates for a pair of upper and lower solutions of Problem (1.1).

In Section 4, we construct the self-similar solution  $(U, a)$ . We will show that  $U$  is as given by (1.2) and  $a$  is characterized as the unique solution of the nonlinear equation (1.3).

In Section 5, we transform Problem (1.1) in coordinates  $(x, t)$  to obtain an equivalent problem, Problem (1.6), in coordinates  $(\eta, \tau)$  where the solution pair becomes  $(W, b)$ . We present an equivalent comparison principle in these coordinates and a class of functions which include both the lower and upper-solutions. We use the notation  $(\bar{W}, \bar{b})$  for the upper-solution, respectively  $(\underline{W}_\lambda, \underline{b}_\lambda)$  for the lower-solution depending on a parameter  $\lambda \geq 0$ , and we construct a function  $(W_\lambda, b_\lambda)$  such that

$$(W_\lambda, b_\lambda) \text{ is } \begin{cases} \text{an upper solution} & \text{if } 0 \leq \lambda \leq 1, \\ \text{a lower solution} & \text{if } \lambda \geq 1. \end{cases} \quad (1.7)$$

Then, we prove the monotonicity in time of the solution pair  $(W, b)$  of the time evolution Problem (1.6) with the two initial conditions  $(\bar{W}, \bar{b})$  and  $(\underline{W}_\lambda, \underline{b}_\lambda)$ . In other words, we show that starting from a lower solution, the solution  $\underline{W}(\eta, \tau) := W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda))$  increases in time as  $\tau \rightarrow \infty$  to a limit function  $\psi$  and the corresponding moving boundary  $\underline{b}(\tau) := b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda))$  increases to a limit  $\underline{b}_\infty$ . Similarly, one can show that starting from an upper solution, the solution decreases to another limit function  $\phi$  as  $\tau \rightarrow \infty$  and the moving boundary  $\bar{b}$  converges to a limit  $\bar{b}_\infty$ .

At the end of this section, we discuss some properties of upper and lower solutions to conclude that they are ordered functions.

However, we do not know yet whether  $\psi$  and  $\phi$  coincide with the self-similar profile  $U$  and whether  $\underline{b}_\infty$  and  $\bar{b}_\infty$  coincide with the point  $a$ . In order to prove these results we first have to show extra a priori estimates which we do in the following section.

In Section 6, we prove a number of a priori estimates some in the moving domain and some in the fixed domain. Indeed, we temporarily pass to fixed domain  $(y, \tau) \in (0, 1) \times \mathbb{R}^+$  to avoid technical

problems related to the characterization of the limits  $\underline{b}_\infty$  and  $\bar{b}_\infty$ . In other words, we need to show that  $\underline{W}_\eta(\underline{b}(\tau), \tau)$  converges to  $\psi_\eta(\underline{b}_\infty)$  as  $\tau \rightarrow \infty$ . This requests to prove the uniform convergence of  $\underline{W}_\eta(\eta, \tau)$  to its limit as  $\tau \rightarrow \infty$  which we can more easily do in the fix domain coordinates. Section 7 is devoted to the study of the limits as  $\tau \rightarrow \infty$ . More precisely, we prove that  $(\psi, \underline{b}_\infty)$  verifies the following conditions

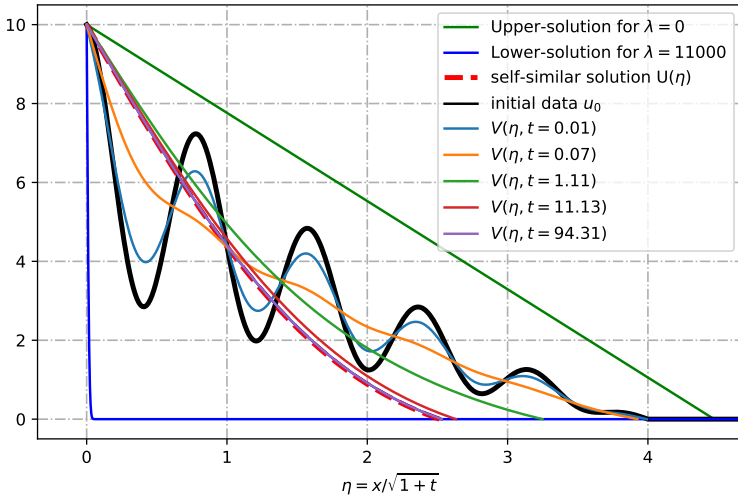
$$\psi(0) = h, \quad \psi(\underline{b}_\infty) = 0, \quad \frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty).$$

and  $\psi$  satisfies the ordinary differential equation

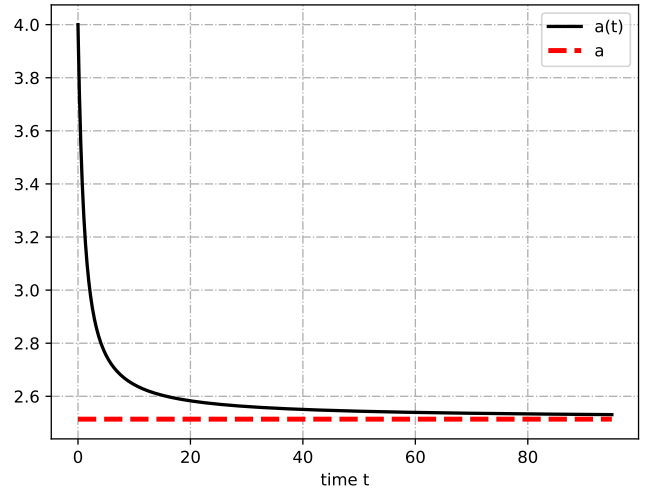
$$\psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0.$$

Similarly, it turns out that  $\left( W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})), b(\tau, (\bar{\mathcal{W}}, \bar{b})) \right)$  converges as  $\tau \rightarrow \infty$  towards the unique solution  $(\phi, \bar{b}_\infty)$  of the stationary problem corresponding to Problem (1.6). At the end of Section 7, we show that the solution pair  $(\psi, \underline{b}_\infty)$  coincides with the unique solution  $(U, a)$  of Problem (4.4) which coincides also with the solution pair  $(\phi, \bar{b}_\infty)$ .

Next, we present the results of some numerical simulations. We choose the initial data  $(u_0, b_0)$  such that  $\underline{b}_\lambda \leq b_0 \leq \bar{b}$  and  $\underline{\mathcal{W}}_\lambda \leq u_0 \leq \bar{\mathcal{W}}$ . Figure 1 shows the large behavior of the solution pair  $(V, a)$  defined in (1.4).



(a) Time evolution of the unknown function  $V(\eta, t)$ .



(b) Time evolution of the moving boundary  $a(t)$ .

Figure 1: Large time behavior of the solution pair  $(V, a)$ .

To state an exact formulation of the results of this article, it is most convenient to use the variable

$y$  lying in  $[0, 1]$ . In the variables  $(y, \eta)$ , the problem for  $(\hat{W}(y, \tau), b(\tau)) = (W(\eta, \tau), b(\tau))$  is given by

$$\begin{cases} \hat{W}_\tau(y, \tau) = \frac{1}{b^2(\tau)} \hat{W}_{yy}(y, \tau) + y \left( \frac{d \ln(b(\tau))}{d\tau} + \frac{1}{2} \right) \hat{W}_y(y, \tau), & \tau > 0, \quad 0 < y < 1, \\ \hat{W}(0, \tau) = h, & \tau > 0, \\ \hat{W}(1, \tau) = 0, & \tau > 0, \\ \frac{1}{2} \frac{db^2(\tau)}{d\tau} + \frac{b^2(\tau)}{2} = -\hat{W}_y(1, \tau), & \tau > 0, \\ b(0) = b_0, \\ \hat{W}(y, 0) = u_0(b_0 y), & 0 \leq y \leq 1. \end{cases} \quad (1.8)$$

The main result of this article is the following. We suppose that the initial data  $(u_0, b_0)$  satisfies the hypothesis:

**H<sub>0</sub>** :  $b_0 \leq \bar{b}$  and  $u_0 \in \mathbb{W}^{1,\infty}(0, b_0)$  with  $u_0(0) = h$  and

$$\begin{aligned} u_0(x) &= 0 \quad \text{for all } x \geq b_0, \\ 0 \leq u_0(x) &\leq h \left( 1 - \frac{x}{\sqrt{2h}} \right) \quad \text{for all } 0 \leq x \leq b_0. \end{aligned}$$

**Main Theorem 1.1.** Suppose that  $(u_0, b_0)$  satisfies the hypothesis **H<sub>0</sub>**. The unique solution  $(\hat{W}, b)$  of Problem (1.8) is such that

$$\lim_{\tau \rightarrow +\infty} \|\hat{W}(\cdot, \tau) - \hat{U}\|_{C([0,1])} = 0, \quad (1.9)$$

$$\lim_{\tau \rightarrow +\infty} b(\tau) = a, \quad (1.10)$$

where  $(\hat{U}, a)$  is the unique solution of the stationary problem

$$\begin{cases} \frac{1}{a^2} \hat{U}_{yy} + \frac{y}{2} \hat{U}_y = 0, & 0 < y < 1, \\ \hat{U}(0) = h, \quad \hat{U}(1) = 0, \\ \frac{a^2}{2} = -\hat{U}_y(1) \end{cases} \quad (1.11)$$

which is equivalent to the stationary problem corresponding to Problem (1.6)

$$\begin{cases} U_{\eta\eta} + \frac{\eta}{2} U_\eta = 0, & 0 < \eta < a, \\ U(0) = h, \quad U(a) = 0, \\ \frac{a}{2} = -U_\eta(a), \end{cases} \quad (1.12)$$

for the self-similar solution of Problem (1.1).

**Remark 1.2.** (1.10) is equivalent to the convergence result

$$\frac{s(t)}{\sqrt{t+1}} \rightarrow a \quad \text{as } t \rightarrow +\infty, \quad (1.13)$$

which was already proved by Meirmanov [M].

## 2 Friedman's formulation

Let  $h > 0$ ,  $b > 0$ . We define the function space

$$X^h(b) := \{u_0(x) \in C[0, \infty), \quad u_0(0) = h, \quad u_0(x) \geq 0 \text{ for } 0 \leq x < b, \quad u_0(x) = 0 \text{ for } x \geq b\}.$$

and we consider the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ u(0, t) = h, & t > 0, \\ u(s(t), t) = 0, & t > 0, \\ \frac{ds(t)}{dt} = -u_x(s(t), t), & t > 0, \\ s(0) = b_0, \\ u(x, 0) = u_0(x) \in X^h(b_0). \end{cases} \quad (2.1)$$

Problem (2.1) is a free boundary problem where  $x = s(t)$  is the free boundary to be found together with the unknown function  $u(x, t)$ .

**Definition 2.1.** Let  $T > 0$ . We say that the pair  $(u, s)$  is a classical solution of Problem (2.1) if

- (i)  $s(t)$  is continuously differentiable for  $0 \leq t \leq T$ ;
- (ii)  $u \in C(\overline{Q_T})$ , where  $Q_T := \{(x, t), t \in (0, T], 0 < x < s(t)\}$ ;
- (iii)  $u \in C^{2,1}(Q_T)$ ;
- (iv)  $u_x \in C(\overline{Q_T^\delta})$  for all  $\delta > 0$  where  $Q_T^\delta = \{(x, t), t \in (\delta, T], 0 < x < s(t)\}$ ;
- (v) the equations of Problem (2.1) are satisfied.

Let  $(u(x, t), s(t))$  be a solution of (2.1) for all  $0 \leq t \leq T$ . We extend  $u$  by:

$$u(x, t) = 0 \text{ for } x \geq s(t), \quad (2.2)$$

so that  $u(\cdot, t)$  is defined for all  $x \geq 0$ .

**Theorem 2.2** ([F2, Theorem 1]). Let  $h > 0$ ,  $b > 0$  and  $u_0 \in X^h(b)$ . Then, there exists a unique solution  $(u(x, t), s(t))$  of (2.1) for all  $t > 0$  in the classical sense. Moreover, the solution  $(u, s)$  is such that  $s$  is infinitely differentiable on  $(0, \infty)$  and  $u$  is infinitely differentiable up to the free boundary for all  $t > 0$  [F1],[S]. Furthermore, the function  $s(t)$  is strictly increasing in  $t$ .

**Proposition 2.3.** Let  $h > 0$ ,  $b > 0$  and  $u_0 \in X^h(b)$  such that  $0 \leq u_0 \leq h$ . Then, the solution  $(u(x, t), s(t))$  of (2.1) is such that  $0 \leq u(x, t) \leq h$  for all  $(x, t) \in Q_T$ .

*Proof.* We apply the strong maximum principle (Theorem 1 of [F3, p.34]) which states that if  $u$  attains its minimum or its maximum in an interior point  $(x^0, t^0) \in Q_T$ , then  $u$  is constant in  $Q_{t^0}$ . However, since  $u(0, t) = h > 0$  for  $t \in (0, T]$  and  $u(s(t), t) = 0$ ,  $u(\cdot, t)$  cannot be constant in space on  $(0, s(t))$ , so that  $u$  attains its minimum and its maximum on the boundary  $\Gamma := \{(0, t), 0 \leq t \leq T\} \cup \{(x, 0), 0 < x < b\} \cup \{(s(t), t), 0 \leq t \leq T\}$ . As  $0 \leq u_0 \leq h$ , we conclude that  $0 \leq u(x, t) \leq h$  for all  $(x, t) \in Q_T$ .  $\square$

### 3 Comparison principle

To begin with, we define a notion of lower and upper solutions.

**Definition 3.1.** For  $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ , we define  $\mathcal{L}(u) := u_t - u_{xx}$ .  $(\underline{u}, \underline{s})$  is a lower solution of the Problem (2.1) if it satisfies

$$\begin{cases} \mathcal{L}(u) = \underline{u}_t - \underline{u}_{xx} \leq 0 & \text{in } Q_T, \\ \underline{u}(0, t) \leq h, \quad \underline{u}(\underline{s}(t), t) = 0, & t > 0, \\ \frac{d\underline{s}(t)}{dt} = -\underline{u}_x(\underline{s}(t), t), & t > 0, \\ \underline{s}(0) \leq b_0, \\ \underline{u}(x, 0) \leq u_0(x), & x \in (0, b_0). \end{cases} \quad (3.1)$$

$(\bar{u}, \bar{s})$  is an upper solution of the Problem (2.1) if it satisfies (3.1) with all  $\leq$  replaced by  $\geq$ .

**Theorem 3.2** (Comparison principle). Let  $(u_1(x, t), s_1(t))$  and  $(u_2(x, t), s_2(t))$  be respectively lower and upper solutions of (2.1) corresponding respectively to the data  $(h_1, u_{01}, b_1)$  and  $(h_2, u_{02}, b_2)$ .

If  $b_1 < b_2$ ,  $h_1 \leq h_2$  and  $u_{01} \leq u_{02}$ , then  $s_1(t) < s_2(t)$  for  $t \geq 0$  and  $u_1(x, t) \leq u_2(x, t)$  for  $x \geq 0$  and  $t \geq 0$ .

In particular,  $u_1(x, t) < u_2(x, t)$  for  $0 < x \leq s_1(t)$  and  $t > 0$ .

Before proving Theorem 3.2, we first show the following result.

**Lemma 3.3.** Any upper solution  $(\bar{u}, \bar{s})$  of Problem (2.1) is such that  $\bar{u} > 0$  in  $Q_T$ .

*Proof.* We first perform the change of function  $\bar{u}(x, t) = \bar{v}(x, t)e^{\lambda t}$  where  $\lambda$  is a strictly positive constant. The function  $\bar{v}$ , as is easily seen, satisfies the inequality

$$(\bar{v}_t - \bar{v}_{xx} + \lambda \bar{v})e^{\lambda t} \geq 0 \text{ in } Q_T, \text{ for all } \lambda > 0,$$

so that

$$\bar{v}_t - (\bar{v}_{xx} - \lambda \bar{v}) \geq 0 \text{ in } Q_T, \text{ for all } \lambda > 0.$$

Now, we prove that  $\bar{v} \geq 0$  in  $\overline{Q_T}$ . Indeed, it follows from the weak maximum principle (Lemma 1 of [F3, p.34]) that  $\bar{v}$  cannot have a negative minimum in  $Q_T$ . Then,  $\bar{v}$  attains its minimum on the boundary  $\Gamma := \{(0, t), 0 \leq t \leq T\} \cup \{(x, 0), 0 < x < b_0\} \cup \{(s(t), t), 0 \leq t \leq T\}$ ; since  $\bar{v} \geq 0$  on  $\Gamma$ , it follows that  $\bar{v} \geq 0$  in  $Q_T$  which implies that  $\bar{u} \geq 0$  in  $\overline{Q_T}$ .

Next, we apply the strong maximum principle (Theorem 1 of [F3, p.34]) which states that if  $\bar{v}$  attains its negative minimum at an interior point  $(\bar{x}, \bar{t}) \in Q_T$ , then  $\bar{v}$  is constant in  $Q_{\bar{t}}$ . However, since  $\bar{v}(0, t) \geq he^{-\lambda t} > 0$  for  $t \in (0, T]$ , we have reached a contradiction, so that we conclude that  $\bar{v} > 0$  in  $Q_T$ . Then, we conclude that  $\bar{u} > 0$  in  $Q_T$ .  $\square$

*Proof of Theorem 3.2.* Suppose that

$$\text{there exists } t_0 > 0 \text{ such that } s_1(t) < s_2(t) \text{ for } 0 \leq t < t_0 \text{ and } s_1(t_0) = s_2(t_0). \quad (3.2)$$

Let  $x_0 := s_1(t_0)$ . Since  $s_1(t) < s_2(t)$  for  $0 \leq t < t_0$ , we see that

$$s'_1(t_0) \geq s'_2(t_0). \quad (3.3)$$

Let  $D := \{(x, t) \mid 0 < t \leq t_0, 0 < x < s_1(t)\}$  and  $\Gamma := \{(0, t) \mid 0 \leq t \leq t_0\} \cup \{(x, 0) \mid 0 < x < b_1\} \cup \{(s_1(t), t) \mid 0 \leq t \leq t_0\}$ . We introduce  $w(x, t) := u_2(x, t) - u_1(x, t)$ . We shall prove that  $w > 0$  in



$D$ . Indeed,  $w_t - w_{xx} \geq 0$  in  $D$ , it follows from the weak maximum principle that  $w \geq 0$  in  $\bar{D}$ . Then, we remark that  $w(s_1(t), t) = u_2(s_1(t), t)$  and according to Lemma 3.3 we have  $u_2(s_1(t), t) > 0$ , so that we deduce from the strong maximum principle that  $w > 0$  in  $D$ .

Let  $\xi > 0$ ,  $a := \xi^{-2}$ ,

$$\varphi(x, t) := e^{-a(x-x_0+\xi)^2+a(t-t_0)} - e^{-a\xi^2} \quad (3.4)$$

and

$$\rho(x) := (x - x_0 + \xi)^2 - \xi^2 + t_0. \quad (3.5)$$

Let  $\delta > 0$  be small (to be chosen later). We define

$$D(\delta) := \{(x, t) \mid x_0 - \delta < x < x_0, \rho(x) < t < t_0\}.$$

Next we show that there exist a small  $\xi > 0$  and a small  $\delta_1 > 0$  such that  $D(\delta_1) \subset D$ , indeed since  $0 < s'_1(t_0) < \infty$  and there exists a small  $\xi > 0$  such that

$$s'_1(t_0) < \left. \frac{d\rho^{-1}(t)}{dt} \right|_{t=t_0} = \frac{1}{\rho'(x_0)} = \frac{1}{2\xi}.$$

It follows that if  $\xi < \frac{1}{2s'_1(t_0)}$  then  $D(\delta_1) \subset D$ . Indeed, suppose that

$$\rho(x) := (x - x_0 + \xi)^2 - \xi^2 + t_0 = t.$$

Then

$$\frac{d\rho^{-1}(t)}{dt} = \frac{1}{\rho'(\rho^{-1}(t))} = \frac{1}{\rho'(x)},$$

where  $\rho^{-1}(t)$  is the inverse function of  $\rho(x)$  near  $x = x_0$  and  $\rho'(x) = 2(x - x_0 + \xi)$  which implies that  $\rho'(x_0) = 2\xi$ . By direct calculation, we shall prove that

$$\varphi_t(x_0, t_0) - \varphi_{xx}(x_0, t_0) = -e^{-1}\xi^{-2} < 0. \quad (3.6)$$

Indeed, from (3.4) we deduce that

$$\varphi_t(x, t) = ae^{-a(x-x_0+\xi)^2+a(t-t_0)} = a\varphi(x, t) + ae^{-a\xi^2}.$$

We remark that since  $\varphi(x_0, t_0) = 0$ , it follows that  $\varphi_t(x_0, t_0) = ae^{-a\xi^2} = \xi^{-2}e^{-1}$ . Next, we compute the space derivatives of  $\varphi$ :

$$\varphi_x(x, t) = -2a(x - x_0 + \xi)e^{-a(x-x_0+\xi)^2+a(t-t_0)},$$

$$\varphi_{xx}(x, t) = -2ae^{-a(x-x_0+\xi)^2+a(t-t_0)} + 4a^2(x - x_0 + \xi)^2e^{-a(x-x_0+\xi)^2+a(t-t_0)}.$$

Thus,  $\varphi_{xx}(x_0, t_0) = -2ae^{-a\xi^2} + 4a^2\xi^2e^{-a\xi^2}$  and since  $a = \xi^{-2}$ , we have,

$$\varphi_{xx}(x_0, t_0) = -2\xi^{-2}e^{-1} + 4\xi^{-2}e^{-1} = 2\xi^{-2}e^{-1},$$

which implies (3.6).

Since  $\varphi$  is smooth, and since  $\varphi$  satisfies (3.6), there exists a neighborhood  $U$  of  $(x_0, t_0)$  such that  $\varphi_t - \varphi_{xx} < 0$  in  $U$ . We choose  $\delta_2 \in (0, \delta_1)$  such that  $D(\delta_2) \subset U$ .

We define  $z(x, t) := w(x, t) - \varepsilon\varphi(x, t)$ , where  $\varepsilon > 0$  will be chosen later. Then,

$$z_t - z_{xx} \geq 0 \text{ on } D(\delta_2). \quad (3.7)$$

Indeed, since  $w_t - w_{xx} \geq 0$  in  $D$ ,  $\varphi_t - \varphi_{xx} < 0$  in  $U$  and  $D(\delta_2) \subset U$ , we have

$$z_t - z_{xx} = w_t - \varepsilon\varphi_t - w_{xx} + \varepsilon\varphi_{xx} = w_t - w_{xx} + \varepsilon(\varphi_{xx} - \varphi_t) \geq 0 + \varepsilon(\varphi_{xx} - \varphi_t) > 0 \text{ in } D(\delta_2).$$

Let

$$\gamma_0 := \{(x, t) \mid x_0 - \delta_2 \leq x \leq x_0, t = \rho(x)\}$$

and

$$\gamma_1 := \{(x, t) \mid x = x_0 - \delta_2, \rho(x_0 - \delta_2) \leq t < t_0\}.$$

In what follows, we use the notation  $\partial(D(\delta_2)) := \gamma_0 \cup \gamma_1$  to denote the parabolic boundary of  $D(\delta_2)$ . Next, we show that  $\varphi = 0$  on  $\gamma_0$ . Indeed  $t = \rho(x)$  on  $\gamma_0$ , we have that

$$\varphi(x, \rho(x)) = e^{-a(x-x_0+\xi)^2+a(x-x_0+\xi)^2} e^{-a\xi^2} - e^{-a\xi^2} = 0$$

and thus,  $\varphi = 0$  on  $\gamma_0$ . Since  $w \geq 0$  in  $\bar{D}$  and  $\gamma_0 \subset \bar{D}$ , we deduce from the definition of  $z$  that  $z = w \geq 0$  on  $\gamma_0$ .

Since  $w > 0$  on  $\gamma_1$ , there exists a small  $\varepsilon > 0$  such that  $z \geq 0$  on  $\gamma_1$ . Indeed,  $w > 0$  on  $D$ , so, there exists  $\mu > 0$  such that  $w \geq \mu$  in  $\gamma_1$ . Moreover, from (3.4) we deduce that

$$\varphi(x, t) \leq e^{-a(x-x_0+\xi)^2+a(t-t_0)}$$

so that  $\varphi(x, t) \leq 1$  and  $\varepsilon\varphi(x, t) \leq \varepsilon$ . Therefore, if

$$\varepsilon \leq \frac{\mu}{2},$$

we have

$$w \geq \mu \geq \varepsilon\varphi(x, t)$$

which implies that  $z \geq 0$  on  $\gamma_1$ . Using the fact that  $\partial(D(\delta_2)) = \gamma_0 \cup \gamma_1$  and  $z \geq 0$  on  $\partial(D(\delta_2))$ , we deduce from the weak maximum principle together with (3.7) that  $z \geq 0$  in  $\bar{D}(\delta_2)$  and hence  $w(x, t_0) \geq \varepsilon\varphi(x, t_0)$  for  $x_0 - \delta_2 \leq x \leq x_0$ . Thus,

$$z(x, t_0) \geq 0 \text{ for all } x \in [x_0 - \delta_2, x_0]. \quad (3.8)$$

Moreover, since  $(x_0, t_0)$  both belongs to  $s_1$  and  $s_2$ , it follows that

$$z(x_0, t_0) = w(x_0, t_0) = u_2(x_0, t_0) - u_1(x_0, t_0) = 0. \quad (3.9)$$

We deduce from (3.8) and (3.9) that  $z_x(x_0, t_0) \leq 0$ , or else,

$$w_x(x_0, t_0) \leq \varepsilon\varphi_x(x_0, t_0) = -2\varepsilon e^{-1}\xi^{-1} < 0,$$

and hence  $u_{1x}(x_0, t_0) > u_{2x}(x_0, t_0)$ . Because of (3.1), we see that  $s'_1(t_0) < s'_2(t_0)$ . This contradicts (3.3). Since we have obtained a contradiction, (3.2) cannot occur. We see that  $s_1(t) < s_2(t)$  for all  $t \geq 0$ . By the weak maximum principle we see that  $u_1(x, t) \leq u_2(x, t)$  for  $x \geq 0$  and  $t \geq 0$ . It follows from the strong maximum principle that  $u_1(x, t) < u_2(x, t)$  for  $0 < x < s_1(t)$  and  $t > 0$ .  $\square$

Next we present an extension of Theorem 3.2 for the case that  $b_1 \leq b_2$ .

**Corollary 3.4** (Extension of the comparison principle). Let  $(u_1(x, t), s_1(t))$  and  $(u_2(x, t), s_2(t))$  be respectively lower and upper solutions of (2.1) corresponding respectively to the data  $(h_1, u_{01}, b_1)$  and  $(h_2, u_{02}, b_2)$  such that  $u_{01}$  or  $u_{02}$  is a nonincreasing function.

If  $b_1 \leq b_2$ ,  $h_1 \leq h_2$  and  $u_{01} \leq u_{02}$ , then  $s_1(t) \leq s_2(t)$  for  $t \geq 0$  and  $u_1(x, t) \leq u_2(x, t)$  for  $x \geq 0$  and  $t \geq 0$ .

*Proof.* The case where  $b_1 < b_2$  has already been studied. It only remains to study the case “ $b_1 = b_2$ ”. We start to suppose that  $u_{01}$  is nonincreasing. The case where  $u_{02}$  is nonincreasing will be considered after.

We will construct a lower solution  $(u_\varepsilon, s_\varepsilon)$ ,  $0 < \varepsilon < 1$ , of Problem (2.1) corresponding to the data  $(h_1, b_{0\varepsilon}, u_{0\varepsilon})$  such that  $(b_{0\varepsilon}, u_{0\varepsilon})$  satisfies

$$\begin{cases} b_{0\varepsilon} < b_2 & \text{and } b_{0\varepsilon} \rightarrow b_1 = b_2 \text{ as } \varepsilon \rightarrow 1, \\ u_{0\varepsilon} \leq u_{02}, \end{cases} \quad (3.10)$$

and

$$\begin{cases} s_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 1} s_1(t), & t \geq 0, \\ u_\varepsilon(x, t) \xrightarrow{\varepsilon \rightarrow 1} u_1(x, t), & x \geq 0, t \geq 0. \end{cases} \quad (3.11)$$

Then, it follows from (3.10) and Theorem 3.2 that

$$\begin{cases} s_\varepsilon(t) < s_2(t), & t \geq 0, \\ u_\varepsilon(x, t) \leq u_2(x, t), & x \geq 0, t \geq 0. \end{cases} \quad (3.12)$$

Letting  $\varepsilon \rightarrow 1$ , we obtain

$$\begin{cases} s_1(t) \leq s_2(t), & t \geq 0, \\ u_1(x, t) \leq u_2(x, t), & x \geq 0, t \geq 0. \end{cases} \quad (3.13)$$

Next, we complete the proof by the construction of a lower solution  $(u_\varepsilon, s_\varepsilon)$  which satisfies (3.11) with data  $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$  such that  $(u_{0\varepsilon}, b_{0\varepsilon})$  satisfies (3.10).

Construction of the lower solution  $(u_\varepsilon, s_\varepsilon)$ . We choose

$$\begin{cases} s_\varepsilon(t) = \varepsilon \cdot s_1\left(\frac{t}{\varepsilon^2}\right), & t \geq 0, \\ u_\varepsilon(x, t) = u_1\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), & x \geq 0, t \geq 0. \end{cases} \quad (3.14)$$

We first check that  $(u_\varepsilon, s_\varepsilon)$  corresponding to the data  $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$  is a lower solution of (2.1). Indeed, since  $u_1$  is a lower solution of (2.1), it follows that

$$\mathcal{L}(u_\varepsilon) = \frac{1}{\varepsilon^2} \left( u_{1,t} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) - u_{1,xx} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \leq 0, \quad (3.15)$$

$$u_\varepsilon(0, t) = u_1\left(0, \frac{t}{\varepsilon^2}\right) = h_1 \leq h, \quad (3.16)$$

$$u_\varepsilon(s_\varepsilon(t), t) = u_1\left(s_1\left(\frac{t}{\varepsilon^2}\right), \frac{t}{\varepsilon^2}\right) = 0, \quad (3.17)$$

$$\frac{ds_\varepsilon(t)}{dt} = \frac{1}{\varepsilon} \frac{d}{dt} s_1\left(\frac{t}{\varepsilon^2}\right) = \frac{-1}{\varepsilon} u_{1,x} \left( s_1\left(\frac{t}{\varepsilon^2}\right), \frac{t}{\varepsilon^2} \right) = -u_{\varepsilon,x}(s_\varepsilon(t), t). \quad (3.18)$$

Next, we choose data  $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$  such that  $(u_{0\varepsilon}, b_{0\varepsilon})$  satisfies (3.10). We set

$$b_{0\varepsilon} := \varepsilon b_1. \quad (3.19)$$

Then, it follows from (3.19) and  $0 < \varepsilon < 1$  that

$$s_\varepsilon(0) = \varepsilon s_1(0) = \varepsilon b_1 =: b_{0\varepsilon} < b_1. \quad (3.20)$$

Finally, we should check that  $u_\varepsilon(x, 0) := u_{0\varepsilon}$  satisfies the second condition of (3.10). Indeed, we have

$$u_\varepsilon(x, 0) = u_1\left(\frac{x}{\varepsilon}, 0\right) = u_{01}\left(\frac{x}{\varepsilon}\right). \quad (3.21)$$

Since  $u_{01}$  is a nonincreasing function and  $0 < \varepsilon < 1$ , it follows that

$$u_{01}\left(\frac{x}{\varepsilon}\right) \leq u_{01}(x) \text{ for } x \geq 0. \quad (3.22)$$

We deduce from (3.22) that

$$u_\varepsilon(x, 0) := u_{0\varepsilon}(x) = u_{01}\left(\frac{x}{\varepsilon}\right) \leq u_{01}(x) \leq u_{02}(x) \text{ for } x \geq 0. \quad (3.23)$$

Therefore,  $(u_\varepsilon, s_\varepsilon)$  satisfies (3.15)-(3.18) and corresponds to data  $(h_1, u_{0\varepsilon}, b_{0\varepsilon})$  such that  $(u_{0\varepsilon}, b_{0\varepsilon})$  satisfies (3.10). Thus, it is a lower solution of (2.1).

Now, we consider the case where the function  $u_{02}$  is nonincreasing. We can proceed exactly as before by considering the upper solution  $(u_\varepsilon, s_\varepsilon)$  of Problem (2.1) with  $\varepsilon > 1$ , given by

$$\begin{cases} s_\varepsilon(t) = \varepsilon \cdot s_2\left(\frac{t}{\varepsilon^2}\right), & t \geq 0, \\ u_\varepsilon(x, t) = u_2\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), & x \geq 0, t \geq 0. \end{cases} \quad (3.24)$$

The corresponding initial datum  $b_{0\varepsilon} = s_\varepsilon(0)$  and  $u_{0\varepsilon} = u_\varepsilon(x, 0)$  verify

$$\begin{cases} b_1 < b_{0\varepsilon} \text{ and } b_{0\varepsilon} \rightarrow b_2 = b_1 \text{ as } \varepsilon \rightarrow 1, \\ u_{01} \leq u_{0\varepsilon}, \end{cases} \quad (3.25)$$

and

$$\begin{cases} s_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 1} s_2(t), & t \geq 0, \\ u_\varepsilon(x, t) \xrightarrow{\varepsilon \rightarrow 1} u_2(x, t), & x \geq 0, t \geq 0. \end{cases} \quad (3.26)$$

Then, the result follows from the use of Theorem 3.2 with (3.25) and letting  $\varepsilon \rightarrow 1$ .  $\square$

## 4 Self-similar solution

We now look for a self-similar solution of the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, \quad 0 < x < s(t), \\ u(0, t) = h, & t > 0, \\ u(s(t), t) = 0, & t > 0, \\ \frac{ds(t)}{dt} = -u_x(s(t), t), & t > 0, \end{cases} \quad (4.1)$$

in the form

$$\begin{cases} u(x, t) = U\left(\frac{x}{\sqrt{t+1}}\right), \\ s(t) = a\sqrt{t+1}, \end{cases} \quad (4.2)$$

for some positive constant  $a$  still to be determined. We set

$$\eta := \frac{x}{\sqrt{t+1}}. \quad (4.3)$$

and deduce that

$$\begin{cases} U_{\eta\eta} + \frac{\eta}{2}U_{\eta} = 0, & 0 < \eta < a, \\ U(0) = h, & U(a) = 0. \end{cases} \quad (4.4)$$

The unique solution of (4.4) is given by

$$U(\eta) = h \left[ 1 - \frac{\int_0^{\eta} e^{-\frac{s^2}{4}} ds}{\int_0^a e^{-\frac{s^2}{4}} ds} \right] \quad \text{for all } \eta \in (0, a). \quad (4.5)$$

It remains to determine the constant  $a$ . We write that

$$s'(t) = \frac{a}{2\sqrt{t+1}} = -u_x(s(t), t) = -\frac{U_{\eta}\left(\frac{s(t)}{\sqrt{t+1}}\right)}{\sqrt{t+1}}, \quad (4.6)$$

which implies that

$$\frac{a}{2} = -U_{\eta}(a), \quad (4.7)$$

so that  $a$  is characterized as the unique solution of the equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}} ds. \quad (4.8)$$

We remark that the function  $a = a(h)$  is strictly increasing, which in turn implies that the functional  $h \rightarrow U$  is strictly increasing.

We conclude that the self-similar solution of Problem (4.1) coincides with the unique solution  $(U, a)$  of Problem (1.12).

## 5 New coordinates, upper and lower solutions

We set

$$\begin{cases} V(\eta, t) = u(x, t), \\ a(t) = \frac{s(t)}{\sqrt{t+1}}, \end{cases} \quad (5.1)$$

and obtain the problem

$$\begin{cases} (t+1)V_t = V_{\eta\eta} + \frac{\eta}{2}V_{\eta}, & t > 0, \quad 0 < \eta < a(t), \\ V(0, t) = h, \quad V(a(t), t) = 0, & t > 0, \\ (t+1)\frac{da(t)}{dt} + \frac{a(t)}{2} = -V_{\eta}(a(t), t), & t > 0. \end{cases} \quad (5.2)$$

Finally we set

$$\tau = \ln(t + 1).$$

The equations in the system (5.2) read as

$$\begin{cases} W_\tau = W_{\eta\eta} + \frac{\eta}{2}W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0, \tau) = h, \quad W(b(\tau), \tau) = 0, & \tau > 0, \\ \frac{db(\tau)}{d\tau} + \frac{b(\tau)}{2} = -W_\eta(b(\tau), \tau), & \tau > 0, \end{cases} \quad (5.3)$$

where we have set

$$V(\eta, t) = W(\eta, \tau), \quad a(t) = b(\tau).$$

Next, we write the full time evolution problem corresponding to the system (5.3). It is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \frac{\eta}{2}W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0, \tau) = h, & \tau > 0, \\ W(b(\tau), \tau) = 0, & \tau > 0, \\ \frac{db(\tau)}{d\tau} + \frac{b(\tau)}{2} = -W_\eta(b(\tau), \tau), & \tau > 0, \\ b(0) = b_0, \\ W(\eta, 0) = u_0(\eta), & 0 \leq \eta \leq b_0. \end{cases} \quad (5.4)$$

Finally, we note that the stationary solution of Problem (5.4) coincides with the unique solution of Problem (1.12), or in other words, the self-similar solution of Problem (1.1).

**Definition 5.1.** We define the linear operator  $\mathcal{L}(W) := W_\tau - W_{\eta\eta} - \frac{\eta}{2}W_\eta$ .

$(\underline{W}, \underline{b})$  is a lower solution of Problem (5.4) if it satisfies:

$$\begin{cases} \mathcal{L}(W) = W_\tau - W_{\eta\eta} - \frac{\eta}{2}W_\eta \leq 0, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ \underline{W}(0, \tau) \leq h, \quad \underline{W}(\underline{b}(\tau), \tau) = 0, & \tau > 0, \\ \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau), \tau), & \tau > 0, \\ \underline{b}(0) \leq b_0, \\ \underline{W}(\eta, 0) \leq u_0(\eta), & 0 \leq \eta \leq \underline{b}(0). \end{cases} \quad (5.5)$$

$(\bar{W}, \bar{b})$  is an upper solution of the Problem (5.4) if it satisfies Problem (5.5) with all  $\leq$  replaced with  $\geq$ .

Finally, we deduce from Corollary 3.4 that the following comparison principle holds.

**Theorem 5.2.** Let  $(W_1(\eta, \tau), b_1(\tau))$  and  $(W_2(\eta, \tau), b_2(\tau))$  be respectively lower and upper solutions of (5.4) corresponding respectively to the data  $(h_1, u_{01}, b_{01})$  and  $(h_2, u_{02}, b_{02})$  such that  $u_{01}$  or  $u_{02}$  is a nonincreasing function.

If  $b_{01} \leq b_{02}$ ,  $h_1 \leq h_2$  and  $u_{01} \leq u_{02}$ , then  $b_1(\tau) \leq b_2(\tau)$  for  $\tau \geq 0$  and  $W_1(\eta, \tau) \leq W_2(\eta, \tau)$  for  $\eta \geq 0$  and  $\tau \geq 0$ .

Throughout this paper, we will also make use of the explicit notation  $W(\eta, \tau, (u_0, b_0))$  and  $b(\tau, (u_0, b_0))$  for the solution pair associated with the initial data  $(u_0, b_0)$ .

### 5.1 Construction of upper and lower solutions

In this section, we construct ordered upper and lower solutions for Problem (5.4). For  $\lambda \geq 0$ , we consider  $(W_\lambda, b_\lambda)$  the unique solution of the problem

$$\begin{cases} W_{\eta\eta} + \frac{\lambda\eta}{2}W_\eta = 0, & 0 < \eta < b, \\ W(0) = h, \quad W(b) = 0, \\ \frac{b}{2} = -W_\eta(b), \end{cases} \quad (5.6)$$

which is given by

$$W_\lambda(\eta) = h \left[ 1 - \frac{\int_0^\eta e^{-\frac{\lambda s^2}{4}} ds}{\int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds} \right] \quad \text{for all } \eta \in (0, b_\lambda) \quad (5.7)$$

and  $b_\lambda$  is the unique solution of the equation

$$h = \frac{b_\lambda}{2} e^{\frac{\lambda b_\lambda^2}{4}} \int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds. \quad (5.8)$$

We can easily show the following properties for  $(W_\lambda, b_\lambda)$ .

**Lemma 5.3.** We have that

$$0 \leq W_\lambda(\eta) \leq h \quad \text{for all } \lambda \geq 0 \text{ and } 0 \leq \eta \leq b_\lambda, \quad (5.9)$$

$$W_{\lambda,\eta}(\eta) \leq 0 \quad \text{for all } \lambda \geq 0 \text{ and } 0 \leq \eta \leq b_\lambda \quad (5.10)$$

and

$$W_{\lambda,\eta\eta}(\eta) \geq 0 \quad \text{for all } \lambda \geq 0 \text{ and } 0 \leq \eta \leq b_\lambda. \quad (5.11)$$

In particular,

$$W_\lambda \text{ is } \begin{cases} \text{a linear function} & \text{if } \lambda = 0, \\ \text{a convex function} & \text{if } \lambda > 0, \end{cases} \quad (5.12)$$

and

$$b_\lambda = \begin{cases} \sqrt{2h} & \text{if } \lambda = 0, \\ \text{satisfies the equation (5.8)} & \text{if } \lambda > 0. \end{cases} \quad (5.13)$$

**Lower solution.** We suppose that

$$\lambda \geq 1, \quad (5.14)$$

then  $(W_\lambda, b_\lambda)$  is a lower solution for Problem (5.4). Indeed, we easily check that  $W_\lambda$  satisfies the following property

$$-W_{\lambda,\eta\eta} - \frac{\eta}{2}W_{\lambda,\eta} \leq 0 \quad \text{if and only if } \lambda \geq 1. \quad (5.15)$$

We define  $(\underline{W}_\lambda, \underline{b}_\lambda)$  by

$$\underline{b}_\lambda = b_\lambda \quad \text{and} \quad \underline{W}_\lambda(\eta) := \begin{cases} W_\lambda(\eta) & \text{if } 0 \leq \eta \leq \underline{b}_\lambda, \\ 0 & \text{if } \eta > \underline{b}_\lambda. \end{cases} \quad (5.16)$$

The pair  $(\underline{W}_\lambda, \underline{b}_\lambda)$  is a lower solution for Problem (5.4).

We assume the following condition on the initial data  $(u_0, b_0)$ :

$$\mathcal{W}_\lambda(\eta) \leq u_0(\eta) \quad \text{for all } 0 \leq \eta \leq b_0, \quad \underline{b}_\lambda \leq b_0. \quad (5.17)$$

According to (5.10),  $\mathcal{W}_\lambda$  is a nonincreasing function and then, in view of the comparison principle Theorem 5.2, it follows that

$$\underline{b}_\lambda \leq b(\tau, (u_0, b_0)) \quad \text{and} \quad \mathcal{W}_\lambda(\eta) \leq W(\eta, \tau, (u_0, b_0)) \quad \text{for all } \tau \geq 0, \eta \geq 0. \quad (5.18)$$

**Upper solution.** Now, we suppose that

$$0 \leq \lambda \leq 1. \quad (5.19)$$

We define  $(\bar{\mathcal{W}}_\lambda, \bar{b}_\lambda)$  by

$$\bar{b}_\lambda = b_\lambda \quad \text{and} \quad \bar{\mathcal{W}}_\lambda(\eta) := \begin{cases} W_\lambda(\eta) & \text{if } 0 \leq \eta \leq \bar{b}_\lambda, \\ 0 & \text{if } \eta > \bar{b}_\lambda. \end{cases} \quad (5.20)$$

In view of (5.15), the pair  $(\bar{\mathcal{W}}_\lambda, \bar{b}_\lambda)$  is an upper solution for Problem (5.4). We now suppose that  $\lambda = 0$  and define the corresponding upper solution by  $(\bar{\mathcal{W}}, \bar{b})$

$$\bar{b} = \sqrt{2h} \quad \text{and} \quad \bar{\mathcal{W}}(\eta) := \begin{cases} W_0(\eta) & \text{if } 0 \leq \eta \leq \bar{b}, \\ 0 & \text{if } \eta > \bar{b}, \end{cases} \quad (5.21)$$

where  $W_0(\eta) = h(1 - \frac{\eta}{\sqrt{2h}})$  for all  $0 < \eta < \bar{b}$ .

We assume the following condition on the initial data  $(u_0, b_0)$ :

$$u_0(\eta) \leq \bar{\mathcal{W}}(\eta) \quad \text{for all } 0 \leq \eta \leq \bar{b}, \quad b_0 \leq \bar{b}. \quad (5.22)$$

According to (5.10),  $\bar{\mathcal{W}}$  is a nonincreasing function and then, in view of the comparison principle Theorem 5.2, it follows that

$$b(\tau, (u_0, b_0)) \leq \bar{b} \quad \text{and} \quad W(\eta, \tau, (u_0, b_0)) \leq \bar{\mathcal{W}}(\eta) \quad \text{for all } \tau \geq 0, \eta \geq 0. \quad (5.23)$$

Next, we prove the monotonicity in time of the solution pair  $(W, b)$  of the time evolution Problem (5.4) with the two initial conditions  $(\bar{\mathcal{W}}, \bar{b})$  and  $(\mathcal{W}_\lambda, \underline{b}_\lambda)$ .

**Lemma 5.4.** Suppose that the initial data  $(u_0, b_0)$  satisfies (5.17) and (5.22). Let  $(\mathcal{W}_\lambda, \underline{b}_\lambda)$  and  $(\bar{\mathcal{W}}, \bar{b})$  be defined by (5.16) and (5.21).

- (i) The functions  $W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b}))$  and  $b(\tau, (\bar{\mathcal{W}}, \bar{b}))$  are nonincreasing in time. Furthermore, there exist a positive constant  $\bar{b}_\infty$  and a function  $\phi \in L^\infty(0, \bar{b}_\infty)$  such that

$$\lim_{\tau \rightarrow +\infty} W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})) = \phi(\eta) \quad \text{for all } \eta \in (0, \bar{b}_\infty), \quad (5.24)$$

$$\lim_{\tau \rightarrow +\infty} b(\tau, (\bar{\mathcal{W}}, \bar{b})) = \bar{b}_\infty. \quad (5.25)$$

- (ii) The function  $W(\eta, \tau, (\mathcal{W}_\lambda, \underline{b}_\lambda))$  and  $b(\tau, (\mathcal{W}_\lambda, \underline{b}_\lambda))$  are nondecreasing in time. Furthermore, there exist a positive constant  $\underline{b}_\infty$  and a function  $\psi \in L^\infty(0, \underline{b}_\infty)$  such that

$$\lim_{\tau \rightarrow +\infty} W(\eta, \tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) = \psi(\eta) \quad \text{for all } \eta \in (0, \underline{b}_\infty), \quad (5.26)$$

$$\lim_{\tau \rightarrow +\infty} b(\tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) = \underline{b}_\infty. \quad (5.27)$$



*Proof.* Applying repeatedly Theorem 5.2, one can show that  $W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b}))$  and  $b(\tau, (\bar{\mathcal{W}}, \bar{b}))$  are nonincreasing in time and that  $W(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  and  $b(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  are nondecreasing in time. Indeed, from (5.23) we have that

$$b(\tau, (u_0, b_0)) \leq \bar{b} \text{ and } W(\eta, \tau, (u_0, b_0)) \leq \bar{\mathcal{W}}(\eta) \text{ for all } \tau \geq 0 \text{ and } \eta \geq 0.$$

In particular, with  $u_0 = \bar{\mathcal{W}}$  and  $b_0 = \bar{b}$ , we get

$$b(\tau, (\bar{\mathcal{W}}, \bar{b})) \leq \bar{b} \text{ and } W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})) \leq \bar{\mathcal{W}}(\eta) \text{ for all } \tau \geq 0 \text{ and } \eta \geq 0. \quad (5.28)$$

From (5.9), we have that

$$0 \leq \bar{\mathcal{W}}(\eta) \leq h.$$

Then, it follows from Proposition 2.3 that

$$0 \leq W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})) \leq h \text{ for all } \tau \geq 0 \text{ and } \eta \geq 0. \quad (5.29)$$

Let  $\sigma > 0$  be fixed. Due to (5.10),  $\bar{\mathcal{W}}$  is a nonincreasing function, then we apply Theorem 5.2 for (5.28) to obtain

$$b(\tau + \sigma, (\bar{\mathcal{W}}, \bar{b})) \leq b(\sigma, (\bar{\mathcal{W}}, \bar{b})) \text{ and } W(\eta, \tau + \sigma, (\bar{\mathcal{W}}, \bar{b})) \leq W(\eta, \sigma, (\bar{\mathcal{W}}, \bar{b})) \text{ for all } \tau \geq 0 \text{ and } \eta \geq 0.$$

Thus for each  $\eta$ ,  $W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b}))$  is nonincreasing in  $\tau$  and from (5.29), it is bounded from below by zero. Therefore it has a limit  $\phi$  as  $\tau \rightarrow \infty$ .

Also  $b(\tau, (\bar{\mathcal{W}}, \bar{b}))$  is nonincreasing in  $\tau$  and from (5.18) we deduce that it is bounded from below by  $\underline{b}_\lambda$ . Therefore it has a limit  $\bar{b}_\infty$  as  $\tau \rightarrow \infty$ .

The same reasoning can be applied to prove that  $W(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  and  $b(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  are nondecreasing in time. Thus for each  $\eta$ ,  $W(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  is nondecreasing in  $\tau$  and it is bounded from above by the constant function  $h$  as follows from Proposition 2.3. Therefore it has a limit  $\psi$  as  $\tau \rightarrow \infty$ . Also,  $b(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda))$  is nondecreasing in  $\tau$  and bounded from above by  $\bar{b}$  thanks to (5.23). Therefore it has a limit  $\underline{b}_\infty$  as  $\tau \rightarrow \infty$ .  $\square$

Later we will show that  $\phi$  and  $\psi$  coincide with the unique solution of Problem (1.12). To that purpose, we will derive in the Section 6 estimates for the free boundary Problem (5.4) in both moving and fixed domains.

## 5.2 Properties of a family of upper and lower solutions

In this subsection, we establish some further properties of upper and lower solutions through successive lemmas.

**Lemma 5.5.** The following properties hold for  $b_\lambda$  satisfying (5.8).

- (i)  $b_\lambda$  is a decreasing function of  $\lambda$ .
- (ii)  $b_\lambda \rightarrow 0$  as  $\lambda \rightarrow +\infty$ .

*Proof.* We start to prove (i). We define  $\mathcal{F}$  as the function given by

$$\mathcal{F}(\lambda, b_\lambda) = \frac{b_\lambda}{2} \int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds - h \quad (5.30)$$

and consider the equation  $\mathcal{F}(\lambda, b_\lambda) = 0$ . We compute the differential of  $\mathcal{F}$  through partial derivatives given by

$$d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}}{\partial b_\lambda} db_\lambda. \quad (5.31)$$

From (5.30), it follows that

$$\frac{\partial \mathcal{F}}{\partial \lambda} = \frac{b_\lambda}{2} \int_0^{b_\lambda} \frac{(b_\lambda^2 - s^2)}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds > 0 \text{ for all } b_\lambda > 0, \quad (5.32)$$

and

$$\frac{\partial \mathcal{F}}{\partial b_\lambda} = \frac{1}{2} \int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds + \frac{b_\lambda}{2} \left( 1 + \int_0^{b_\lambda} \frac{2b_\lambda \lambda}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds \right) > 0 \text{ for all } b_\lambda > 0. \quad (5.33)$$

Since  $\mathcal{F}(\lambda, b_\lambda) = 0$ , it follows from (5.31) that

$$\frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda} db_\lambda = 0. \quad (5.34)$$

Thus, since  $\frac{\partial \mathcal{F}}{\partial b_\lambda} \neq 0$ , it follows from (5.32), (5.33) and (5.34) that

$$\frac{db_\lambda}{d\lambda} = - \frac{\frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda}}{\frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda}} < 0, \quad (5.35)$$

which completes the proof of (i).

Now, we turn to the proof of (ii). For  $\lambda \geq 0$ , we have  $b_\lambda > 0$  and  $b_\lambda$  is a decreasing function of  $\lambda$ . Hence, there exists  $\alpha \geq 0$  such that  $b_\lambda \rightarrow \alpha$  as  $\lambda \rightarrow +\infty$  and  $b_\lambda \geq \alpha$  for all  $\lambda \geq 0$ . We shall prove that  $\alpha = 0$ . This fact mainly relies on the following inequality which will be proved later on. Let  $a \geq 0$ . For  $\lambda \geq 0$  large enough, the following inequality holds:

$$\int_0^a e^{-\frac{\lambda s^2}{4}} ds \geq a \left( 1 + \frac{\lambda}{4} a^2 \right) e^{-\frac{\lambda a^2}{4}}. \quad (5.36)$$

Since  $b_\lambda \geq \alpha$  for all  $\lambda \geq 0$ , we deduce from (5.8) that

$$h \geq \frac{\alpha}{2} e^{\frac{\lambda \alpha^2}{4}} \int_0^\alpha e^{-\frac{\lambda s^2}{4}} ds. \quad (5.37)$$

For  $\lambda$  large enough we infer from the estimate (5.36) that

$$h \geq \frac{\alpha^2}{2} \left( 1 + \frac{\lambda}{4} \alpha^2 \right). \quad (5.38)$$

Letting  $\lambda \rightarrow +\infty$  in (5.38), we see that we necessarily have  $\alpha = 0$ . It remains to prove that the inequality (5.36) holds for  $\lambda$  large enough. We only have to consider the case where  $a > 0$  since (5.36) is trivially true for  $a = 0$ . Let us introduce  $f(x) = e^{-\frac{\lambda x^2}{4}}$ . We have  $f''(x) = \frac{\lambda}{2} (\frac{\lambda}{2} x^2 - 1) e^{-\frac{\lambda x^2}{4}}$ . We choose  $\lambda > 0$  large enough to have  $0 < \sqrt{\frac{2}{\lambda}} < a$  and then  $f$  is convex in  $[\sqrt{\frac{2}{\lambda}}, a]$ . Therefore, for all  $x \in [\sqrt{\frac{2}{\lambda}}, a]$  we have

$$f(x) \geq g(x) := f(a) + (x - a) f'(a) \quad (5.39)$$

that is

$$e^{-\frac{\lambda x^2}{4}} \geq \left(1 + \frac{\lambda}{2}a(a-x)\right) e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in \left[\sqrt{\frac{2}{\lambda}}, a\right]. \quad (5.40)$$

Next we prove that (5.39) also holds for  $x \in [0, \sqrt{\frac{2}{\lambda}}]$ . Indeed, we have

$$\max_{x \in [0, \sqrt{\frac{2}{\lambda}}]} g(x) = g(0) = \left(1 + \frac{\lambda}{2}a^2\right) e^{-\frac{\lambda a^2}{4}}$$

and

$$\min_{x \in [0, \sqrt{\frac{2}{\lambda}}]} f(x) = f\left(\sqrt{\frac{2}{\lambda}}\right) = e^{-\frac{1}{2}}.$$

Since  $g(0) \rightarrow 0$  as  $\lambda \rightarrow +\infty$ , we get, for  $\lambda$  large enough

$$\max_{[0, \sqrt{\frac{2}{\lambda}}]} g = g(0) \leq \min_{[0, \sqrt{\frac{2}{\lambda}}]} f = e^{-\frac{1}{2}} \quad (5.41)$$

and then

$$g(x) \leq f(x), \quad \text{for all } x \in [0, \sqrt{\frac{2}{\lambda}}] \quad (5.42)$$

Combining (5.39) with (5.42) leads to  $f(x) \geq g(x)$  for all  $x \in [0, a]$ , that is

$$e^{-\frac{\lambda x^2}{4}} \geq \left(1 + \frac{\lambda}{2}a(a-x)\right) e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in [0, a]. \quad (5.43)$$

Integrating (5.43) over  $[0, a]$  leads to the desired inequality (5.36).  $\square$

Next, we prove the following result.

**Lemma 5.6.** Let  $\lambda_1$  and  $\lambda_2$  be such that  $\lambda_1 < \lambda_2$ , then it follows that

$$b_{\lambda_1} > b_{\lambda_2}, \quad (5.44)$$

and

$$W_{\lambda_1}(\eta) \geq W_{\lambda_2}(\eta) \quad \text{for all } 0 \leq \eta \leq b_{\lambda_2}. \quad (5.45)$$

*Proof.* From Lemma 5.5, since  $\lambda_1 < \lambda_2$ , it follows that  $b_{\lambda_1} > b_{\lambda_2}$ . Then, (5.44) holds. Next, we show (5.45). To do so, let the pair  $(W_{\lambda_i}, b_{\lambda_i})_{i \in \{1,2\}}$  be the unique solution of the problem

$$\begin{cases} W_{\lambda_i, \eta \eta} + \frac{\lambda_i \eta}{2} W_{\lambda_i, \eta} = 0, & 0 < \eta < b_{\lambda_i} \\ W_{\lambda_i}(0) = h, \quad W_{\lambda_i}(b_{\lambda_i}) = 0. \end{cases} \quad (5.46)$$

Then, we recall that for  $i \in \{1, 2\}$  the solution pair  $(W_{\lambda_i}, b_{\lambda_i})$  is given by

$$W_{\lambda_i}(\eta) = h \left[ 1 - \frac{\int_0^\eta e^{-\frac{\lambda_i s^2}{4}} ds}{\int_0^{b_{\lambda_i}} e^{-\frac{\lambda_i s^2}{4}} ds} \right] \quad \text{for all } 0 \leq \eta \leq b_{\lambda_i}, \quad (5.47)$$

with also

$$W_{\lambda_i, \eta}(\eta) = \frac{-h e^{-\frac{\lambda_i \eta^2}{4}}}{\int_0^{b_{\lambda_i}} e^{-\frac{\lambda_i s^2}{4}} ds} \quad \text{for all } 0 \leq \eta \leq b_{\lambda_i}. \quad (5.48)$$

Next, we define the linear operator  $\mathcal{L}(W) := W_{\eta\eta} + \frac{\lambda_1\eta}{2}W_\eta$  for all  $0 \leq \eta \leq b_{\lambda_2}$ . We compute  $\mathcal{L}(W_{\lambda_2} - W_{\lambda_1})$  to obtain

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) = W_{\lambda_2,\eta\eta} + \frac{\lambda_1\eta}{2}W_{\lambda_2,\eta} - W_{\lambda_1,\eta\eta} - \frac{\lambda_1\eta}{2}W_{\lambda_1,\eta} \text{ for all } 0 \leq \eta \leq b_{\lambda_2}. \quad (5.49)$$

Then, from (5.46), we have that

$$W_{\lambda_2,\eta\eta}(\eta) = -\frac{\lambda_2\eta}{2}W_{\lambda_2,\eta} \text{ for all } 0 \leq \eta \leq b_{\lambda_2}. \quad (5.50)$$

We substitute (5.50) in (5.49). Then, since  $(W_{\lambda_1}, b_{\lambda_1})$  is a solution of problem (5.46), (5.49) becomes

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) = \frac{(\lambda_1 - \lambda_2)\eta}{2}W_{\lambda_2,\eta}. \quad (5.51)$$

Since  $\lambda_1 < \lambda_2$ , by (5.48) and (5.51), we deduce that

$$\mathcal{L}(W_{\lambda_2} - W_{\lambda_1}) \geq 0 \text{ for all } 0 \leq \eta \leq b_{\lambda_2}. \quad (5.52)$$

Then, from (5.46), since

$$(W_{\lambda_2} - W_{\lambda_1})(0) = h - h = 0$$

and

$$(W_{\lambda_2} - W_{\lambda_1})(b_{\lambda_2}) = 0 - W_{\lambda_1}(b_{\lambda_2}) < 0,$$

we deduce from the one-dimensional maximum principle (Theorem 1 of [PW, p.2]) that the function  $W_{\lambda_2} - W_{\lambda_1}$  attains its maximum on the boundary. This implies that

$$W_{\lambda_2}(\eta) - W_{\lambda_1}(\eta) \leq 0 \text{ for all } 0 \leq \eta \leq b_{\lambda_2},$$

which completes the proof of Lemma 5.6.  $\square$

The next result ensures that the assumption made in (5.17) on the initial datum is fulfilled for  $\lambda$  large enough.

**Lemma 5.7.** Let  $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}(0, b_0)$  and  $(\mathcal{W}_\lambda, \underline{b}_\lambda)$  defined by (5.16). There exists  $\lambda \geq 1$  large enough such that  $\mathcal{W}_\lambda \leq u_0$  in  $[0, b_0]$  and  $\underline{b}_\lambda \leq b_0$ .

*Proof.* According to (5.12),  $W_\lambda$  is a convex function. Thus, we have

$$W_\lambda(\eta) \leq \frac{h}{b_\lambda}(b_\lambda - \eta) \text{ for all } 0 \leq \eta \leq b_\lambda. \quad (5.53)$$

From the identity  $u_0(\eta) = h + \int_0^\eta \frac{du_0}{d\eta}(s)ds$  for  $0 \leq \eta \leq b_0$ , we deduce that

$$u_0(\eta) \geq h - M\eta \text{ for all } 0 \leq \eta \leq b_0 \quad (5.54)$$

where  $M = \left\| \frac{du_0}{d\eta} \right\|_{L^\infty(0, b_0)}$ . From Lemma 5.5 (ii),  $b_\lambda \rightarrow 0$  as  $\lambda \rightarrow +\infty$ . Then we can choose  $\lambda \geq 1$  large enough so that

$$b_\lambda \leq \min\left(\frac{h}{M}, b_0\right). \quad (5.55)$$

Estimate (5.54) then becomes

$$u_0(\eta) \geq h - \frac{h}{b_\lambda} \eta \quad \text{for all } 0 \leq \eta \leq b_0$$

and we deduce from (5.53) that

$$u_0(\eta) \geq W_\lambda(\eta) \quad \text{for all } 0 \leq \eta \leq b_\lambda. \quad (5.56)$$

Defining  $\underline{W}_\lambda = W_\lambda$  and  $\underline{b}_\lambda = b_\lambda$  as in (5.16), we deduce that the pair  $(\underline{W}_\lambda, \underline{b}_\lambda)$  is a lower solution for Problem (5.4).  $\square$

## 6 A priori estimates for the solution of Problem (5.4)

### 6.1 A priori estimates for the solution of Problem (5.4) on the moving domain

**Definition 6.1.** We define

$$\underline{b}(\tau) := b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \text{ and } \underline{W}(\eta, \tau) := W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \text{ for all } \tau > 0, 0 \leq \eta \leq \underline{b}(\tau).$$

We start by showing successive lemmas for the function pair  $(\underline{W}, \underline{b})$ .

**Lemma 6.2.** We have the following uniform bounds in time

$$\underline{b}_\lambda \leq \underline{b}(\tau) := b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \leq \underline{b}_\infty \leq \bar{b} \quad \text{for all } \tau \geq 0. \quad (6.1)$$

and

$$0 \leq \underline{W}(\eta, \tau) := W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \leq h \text{ for all } \tau \geq 0, 0 \leq \eta \leq \bar{b}. \quad (6.2)$$

*Proof.* It follows from (5.18) and (5.23) that

$$\underline{b}_\lambda \leq b(\tau, (u_0, b_0)) \leq \bar{b} \text{ for all } \tau \geq 0.$$

In particular, for  $(u_0, b_0) = (\underline{W}_\lambda, \underline{b}_\lambda)$ , we obtain

$$\underline{b}_\lambda \leq \underline{b}(\tau) := b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \leq \bar{b} \text{ for all } \tau \geq 0.$$

We conclude from (5.27) that

$$\underline{b}_\lambda \leq \underline{b}(\tau) \leq \underline{b}_\infty \leq \bar{b},$$

then (6.1) holds.

Now we prove (6.2). Indeed, we know from (5.9) and (5.16) that  $0 \leq \underline{W}_\lambda(\eta) \leq h$  for all  $\eta \in (0, \bar{b})$ , which by Proposition 2.3 implies that

$$0 \leq \underline{W}(\eta, \tau) := W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \leq h \quad \text{for all } \tau \geq 0, 0 \leq \eta \leq \bar{b},$$

so that (6.2) holds.  $\square$

Next we prove the following result.

**Lemma 6.3.** Let  $\sigma > 0$ . For all  $\tau > 0$ , we have that

$$\|\underline{W}_\eta(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C(\sigma), \quad (6.3)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$  and where

$$\Omega_{\sigma, \tau} := \{(\eta, S); 0 < \eta < \underline{b}(S + \tau), S \in (0, \sigma)\}. \quad (6.4)$$

*Proof.* We have

$$W_\tau(\eta, \tau) = W_{\eta\eta}(\eta, \tau) + \frac{\eta}{2} W_\eta(\eta, \tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau),$$

Then,

$$(W - h)_\tau(\eta, \tau)(W - h)(\eta, \tau) = W_{\eta\eta}(\eta, \tau)(W - h)(\eta, \tau) + \frac{\eta}{2} W_\eta(\eta, \tau)(W - h)(\eta, \tau). \quad (6.5)$$

A direct computations yields

$$\frac{d}{d\tau} \int_0^{\underline{b}(\tau)} (W(\eta, \tau) - h)^2 d\eta = \frac{d\underline{b}(\tau)}{d\tau} \left( W(\underline{b}(\tau), \tau) - h \right)^2 + 2 \int_0^{\underline{b}(\tau)} (W - h)_\tau(\eta, \tau)(W - h)(\eta, \tau) d\eta.$$

Since  $W(\underline{b}(\tau), \tau) = 0$ , we obtain

$$\int_0^{\underline{b}(\tau)} (W - h)_\tau(\eta, \tau)(W - h)(\eta, \tau) d\eta = \frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} (W(\eta, \tau) - h)^2 d\eta - \frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} h^2. \quad (6.6)$$

Then, we deduce from (6.5) and (6.6) that

$$\begin{aligned} \int_0^{\underline{b}(\tau)} W_{\eta\eta}(\eta, \tau)(W - h)(\eta, \tau) d\eta + \int_0^{\underline{b}(\tau)} \frac{\eta}{2} W_\eta(\eta, \tau)(W - h)(\eta, \tau) d\eta = \\ \frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} (W(\eta, \tau) - h)^2 d\eta - \frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} h^2. \end{aligned} \quad (6.7)$$

Next, we integrate by parts the first term on the left-hand-side of (6.7) and we use  $W(\underline{b}(\tau), \tau) = 0$  and  $W(0, \tau) = h$  to obtain

$$\int_0^{\underline{b}(\tau)} W_{\eta\eta}(\eta, \tau)(W - h)(\eta, \tau) d\eta = -W_\eta(\underline{b}(\tau), \tau)h - \int_0^{\underline{b}(\tau)} |W_\eta|^2 d\eta. \quad (6.8)$$

Due to Lemma 6.2, we have  $|W(\eta, \tau) - h| \leq h$  and  $0 \leq \eta \leq \bar{b}$ . It follows that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2} W_\eta(\eta, \tau)(W - h)(\eta, \tau) d\eta \leq \frac{\bar{b}}{2} h \int_0^{\underline{b}(\tau)} |W_\eta| d\eta. \quad (6.9)$$

Then, we deduce from (6.7), (6.8) and (6.9) that

$$\frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} (W(\eta, \tau) - h)^2 d\eta - \frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} h^2 \leq -W_\eta(\underline{b}(\tau), \tau)h - \int_0^{\underline{b}(\tau)} |W_\eta|^2 d\eta + \frac{\bar{b}}{2} h \int_0^{\underline{b}(\tau)} |W_\eta| d\eta. \quad (6.10)$$

Moreover, it follows from Cauchy-Schwarz's and Young's inequalities that

$$\int_0^{\underline{b}(\tau)} |W_\eta| |1| d\eta \leq \frac{1}{2\varepsilon} \int_0^{\underline{b}(\tau)} |W_\eta|^2 d\eta + \frac{\varepsilon}{2} \int_0^{\underline{b}(\tau)} |1|^2 d\eta \quad (6.11)$$

for all  $\varepsilon > 0$ . Since  $-W_\eta(\underline{b}(\tau), \tau) = \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2}$  and in view of (6.11), (6.10) becomes

$$\frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} (W - h)^2 d\eta - \frac{h^2}{2} \frac{d\underline{b}(\tau)}{d\tau} + \int_0^{\underline{b}(\tau)} |W_\eta|^2 d\eta \leq \left( \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} \right) h + \frac{\bar{b}}{4} h \left( \frac{1}{\varepsilon} \int_0^{\underline{b}(\tau)} |W_\eta|^2 d\eta + \varepsilon \underline{b}(\tau) \right). \quad (6.12)$$

Let  $\sigma > 0$ ; we integrate both sides of the inequality (6.12) on  $(\tau, \tau + \sigma)$  to obtain

$$\begin{aligned} & \frac{1}{2} \int_{\tau}^{\tau+\sigma} \frac{d}{ds} \int_0^{\underline{b}(s)} (W - h)^2 d\eta ds - \frac{h^2}{2} \int_{\tau}^{\tau+\sigma} \frac{d\underline{b}(s)}{ds} ds + \int_{\tau}^{\tau+\sigma} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds \leq \\ & \int_{\tau}^{\tau+\sigma} \left( \frac{d\underline{b}(s)}{ds} + \frac{\underline{b}(s)}{2} \right) h ds + \frac{\bar{b} h}{4} \int_{\tau}^{\tau+\sigma} \frac{1}{\varepsilon} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds + \frac{\bar{b} h \varepsilon}{4} \int_{\tau}^{\tau+\sigma} \underline{b}(s) ds. \end{aligned}$$

Then, it follows that

$$\begin{aligned} & \frac{1}{2} \int_0^{\underline{b}(\tau+\sigma)} \left( W(\eta, \tau + \sigma) - h \right)^2 d\eta - \frac{1}{2} \int_0^{\underline{b}(\tau)} \left( W(\eta, \tau) - h \right)^2 d\eta - \frac{h^2}{2} \left( \underline{b}(\tau + \sigma) - \underline{b}(\tau) \right) + \\ & \left( 1 - \frac{\bar{b} h}{4 \varepsilon} \right) \int_{\tau}^{\tau+\sigma} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds \leq \left( \underline{b}(\tau + \sigma) - \underline{b}(\tau) \right) h + \frac{h(2 + \varepsilon \bar{b})}{4} \int_{\tau}^{\tau+\sigma} \underline{b}(s) ds. \end{aligned}$$

For  $\varepsilon = \frac{\bar{b}h}{2}$ , we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\tau}^{\tau+\sigma} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds \leq \\ & \frac{1}{2} \int_0^{\underline{b}(\tau)} \left( W(\eta, \tau) - h \right)^2 d\eta + \left( h + \frac{h^2}{2} \right) \left( \underline{b}(\tau + \sigma) - \underline{b}(\tau) \right) + \frac{h(4 + \bar{b}^2 h)}{8} \int_{\tau}^{\tau+\sigma} \underline{b}(s) ds. \end{aligned}$$

Since  $\underline{b}(\tau) \leq \bar{b}$  for all  $\tau > 0$  and  $|W(\eta, \tau) - h| \leq h$ , it follows that

$$\int_{\tau}^{\tau+\sigma} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds \leq h^2 \bar{b} + \left( 2h + h^2 \right) \bar{b} + \frac{h(4 + \bar{b}^2 h)}{4} \sigma \bar{b}.$$

We conclude that

$$\int_{\tau}^{\tau+\sigma} \int_0^{\underline{b}(s)} |W_{\eta}|^2 d\eta ds \leq C(\sigma) \text{ for some positive constant } C(\sigma). \quad (6.13)$$

Next we perform the change of variable  $S = s - \tau$ ; then

$$\int_0^{\sigma} \int_0^{\underline{b}(S+\tau)} |W_{\eta}(\eta, S + \tau)|^2 d\eta dS \leq C(\sigma) \text{ for all } \tau \geq 0,$$

which implies that

$$\|W_{\eta}(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C(\sigma), \quad (6.14)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$ . This completes the proof of Lemma 6.3.  $\square$

Next we show the following result.

**Lemma 6.4.** Let  $\sigma > 0$ . For all  $\tau > 0$ , we have that

$$\|W_{\eta}(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C(\sigma), \quad (6.15)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$ .

Before proving Lemma 6.4, we will recall the following result.

**Lemma 6.5** (The Uniform Gronwall Lemma (Lemma 1.1 of [T, p.89])). Let  $g$  and  $y$  be two positive locally integrable functions on  $(0, +\infty)$  such that  $\frac{dy}{dt}$  is locally integrable on  $(0, \infty)$ , which satisfy the inequalities

$$\frac{dy}{dt} \leq g y \quad \text{for all } t \geq 0, \quad (6.16)$$

$$\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} y(s)ds \leq a_2 \quad \text{for all } t \geq 0, \quad (6.17)$$

where  $r, a_1, a_2$ , are positive constants which do not depend in  $t$ . Then

$$y(t+r) \leq \frac{a_2}{r} \exp(a_1), \quad \text{for all } t \geq 0. \quad (6.18)$$

*Proof of Lemma 6.4.* We define

$$Z(\eta, \tau) := \underline{W}_\eta(\eta, \tau) \quad \text{for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau), \quad (6.19)$$

where  $\underline{W}(\eta, \tau)$  is defined in Definition 6.1. From Problem (5.4), we have

$$(\underline{W}_\eta)_\tau = (\underline{W}_\eta)_{\eta\eta} + \frac{\eta}{2}(\underline{W}_\eta)_\eta + \frac{\underline{W}_\eta}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau),$$

so that

$$Z_\tau = Z_{\eta\eta} + \frac{\eta}{2}Z_\eta + \frac{Z}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau). \quad (6.20)$$

Next we show that  $Z_\eta(0, \tau) = 0$  and that  $Z_\eta(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau)$ . Indeed, we have that  $(\underline{W}(0, \tau))_\tau = (h)_\tau = 0$  and  $(\underline{W}(0, \tau))_\tau = \underline{W}_\tau(0, \tau) = (\underline{W}_\eta)_\eta(0, \tau) + \frac{0}{2}\underline{W}_\eta(0, \tau)$ . Then

$$Z_\eta(0, \tau) = 0. \quad (6.21)$$

Moreover, we have

$$\left( \underline{W}(\underline{b}(\tau), \tau) \right)_\tau = \frac{d\underline{b}(\tau)}{d\tau} \underline{W}_\eta(\underline{b}(\tau), \tau) + \underline{W}_\tau(\underline{b}(\tau), \tau) = 0, \quad (6.22)$$

and

$$\underline{W}_\tau(\underline{b}(\tau), \tau) = (\underline{W}_\eta)_\eta(\underline{b}(\tau), \tau) + \frac{\underline{b}(\tau)}{2} \underline{W}_\eta(\underline{b}(\tau), \tau). \quad (6.23)$$

We substitute (6.23) in (6.22) to obtain

$$(\underline{W}_\eta)_\eta(\underline{b}(\tau), \tau) + \frac{\underline{b}(\tau)}{2} \underline{W}_\eta(\underline{b}(\tau), \tau) + \frac{d\underline{b}(\tau)}{d\tau} \underline{W}_\eta(\underline{b}(\tau), \tau) = 0,$$

so that

$$Z_\eta(\underline{b}(\tau), \tau) + \frac{\underline{b}(\tau)}{2} Z(\underline{b}(\tau), \tau) + \frac{d\underline{b}(\tau)}{d\tau} Z(\underline{b}(\tau), \tau) = 0. \quad (6.24)$$

Since  $\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau), \tau) = -Z(\underline{b}(\tau), \tau)$  then (6.24) becomes

$$Z_\eta(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau). \quad (6.25)$$



Therefore, from (6.20), (6.21) and (6.25), the time evolution Problem (5.4) leads to

$$\begin{cases} Z_\tau = Z_{\eta\eta} + \frac{\eta}{2}Z_\eta + \frac{Z}{2}, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ Z_\eta(0, \tau) = 0, & \tau > 0, \\ Z_\eta(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau), & \tau > 0, \\ \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -Z(\underline{b}(\tau), \tau), & \tau > 0, \\ \underline{b}(0) = \underline{b}, \\ Z(\eta, 0) = W_{\lambda, \eta}(\eta), & 0 \leq \eta \leq \underline{b}_\lambda, \end{cases} \quad (6.26)$$

where  $W_{\lambda, \eta}(\eta) = \frac{-h e^{-\frac{\lambda\eta^2}{4}}}{\int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}} ds}$  with  $\lambda \geq 1$ . We consider the function  $F$  defined by

$$F(\tau) = \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta. \quad (6.27)$$

Then, we compute  $\frac{dF(\tau)}{d\tau} = \frac{d\underline{b}(\tau)}{d\tau} Z^2(\underline{b}(\tau), \tau) + 2 \int_0^{\underline{b}(\tau)} Z_\tau(\eta, \tau) Z(\eta, \tau) d\eta$ , so that

$$\int_0^{\underline{b}(\tau)} Z_\tau(\eta, \tau) Z(\eta, \tau) d\eta = \frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta - \frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} Z^2(\underline{b}(\tau), \tau). \quad (6.28)$$

We multiply (6.20) by  $Z$  and integrate in space between 0 en  $\underline{b}(\tau)$  to obtain

$$\int_0^{\underline{b}(\tau)} Z_\tau(\eta, \tau) Z(\eta, \tau) d\eta = \int_0^{\underline{b}(\tau)} Z_{\eta\eta}(\eta, \tau) Z(\eta, \tau) d\eta + \int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta, \tau) Z(\eta, \tau) d\eta + \int_0^{\underline{b}(\tau)} \frac{Z^2(\eta, \tau)}{2} d\eta. \quad (6.29)$$

We integrate by parts the first term on the right-hand-side of (6.29) and using  $Z_\eta(0, \tau) = 0$  and  $Z_\eta(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau)$ , we deduce that

$$\int_0^{\underline{b}(\tau)} Z_{\eta\eta}(\eta, \tau) Z(\eta, \tau) d\eta = Z^3(\underline{b}(\tau), \tau) - \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta. \quad (6.30)$$

Next, since  $0 \leq \eta \leq \bar{b}$ , it follows that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta, \tau) Z(\eta, \tau) d\eta \leq \frac{\bar{b}}{2} \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)| |Z(\eta, \tau)| d\eta. \quad (6.31)$$

Using the Cauchy-Schwarz inequality with the Young inequality, we obtain that

$$\int_0^{\underline{b}(\tau)} \frac{\eta}{2} Z_\eta(\eta, \tau) Z(\eta, \tau) d\eta \leq \frac{\bar{b}}{4\varepsilon} \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta + \frac{\bar{b}\varepsilon}{4} \int_0^{\underline{b}(\tau)} |Z(\eta, \tau)|^2 d\eta \quad (6.32)$$

for all  $\varepsilon > 0$ . Next, combining (6.28), (6.29), (6.30) and (6.32) we deduce that

$$\frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta - \frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} Z^2(\underline{b}(\tau), \tau) \leq \quad (6.33)$$

$$Z^3(\underline{b}(\tau), \tau) - \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta + \frac{\bar{b}}{4\varepsilon} \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta + \frac{\bar{b}\varepsilon}{4} \int_0^{\underline{b}(\tau)} |Z(\eta, \tau)|^2 d\eta + \int_0^{\underline{b}(\tau)} \frac{Z^2(\eta, \tau)}{2} d\eta.$$

Since  $\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -Z(\underline{b}(\tau), \tau)$ , then

$$-\frac{1}{2} \frac{d\underline{b}(\tau)}{d\tau} Z^2(\underline{b}(\tau), \tau) = \frac{1}{2} Z^3(\underline{b}(\tau), \tau) + \frac{\underline{b}(\tau)}{4} Z^2(\underline{b}(\tau), \tau).$$

So, (6.33) becomes

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta + \frac{4\varepsilon - \bar{b}}{4\varepsilon} \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta \leq \\ \frac{1}{2} Z^3(\underline{b}(\tau), \tau) - \frac{\underline{b}(\tau)}{4} Z^2(\underline{b}(\tau), \tau) + \frac{\bar{b}\varepsilon + 2}{4} \int_0^{\underline{b}(\tau)} |Z(\eta, \tau)|^2 d\eta. \end{aligned} \quad (6.34)$$

From (6.49) below, we have  $Z(\underline{b}(\tau), \tau) = W_\eta(\underline{b}(\tau), \tau) \leq 0$ ; setting  $\varepsilon = \frac{\bar{b}}{2}$  then yields

$$\frac{1}{2} \frac{d}{d\tau} \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta + \frac{1}{2} \int_0^{\underline{b}(\tau)} |Z_\eta(\eta, \tau)|^2 d\eta \leq \frac{\bar{b}^2 + 4}{8} \int_0^{\underline{b}(\tau)} |Z(\eta, \tau)|^2 d\eta. \quad (6.35)$$

It follows from (6.35), (6.13) and the uniform Gronwall Lemma 6.5 that there exists some positive constant  $C(\sigma)$  which does not depend on  $\tau$  such that

$$\int_0^{\underline{b}(\tau+\sigma)} Z^2(\eta, \tau + \sigma) d\eta \leq C(\sigma) \text{ for all } \tau > 0. \quad (6.36)$$

Next, we integrate both sides of the inequality (6.35) on  $(\tau, \tau + \sigma)$  to obtain

$$\frac{1}{2} \int_\tau^{\tau+\sigma} \frac{d}{ds} \int_0^{\underline{b}(s)} Z^2(\eta, s) d\eta ds + \frac{1}{2} \int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} |Z_\eta(\eta, s)|^2 d\eta ds \leq \frac{\bar{b}^2 + 4}{8} \int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} |Z(\eta, s)|^2 d\eta ds. \quad (6.37)$$

Then, in view of (6.36) and the fact that  $\underline{b}$  is nondecreasing, (6.37) becomes

$$\frac{1}{2} \int_0^{\underline{b}(\tau+\sigma)} Z^2(\eta, \tau + \sigma) d\eta - \frac{1}{2} \int_0^{\underline{b}(\tau)} Z^2(\eta, \tau) d\eta + \frac{1}{2} \int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} |Z_\eta(\eta, s)|^2 d\eta ds \leq \frac{(\bar{b}^2 + 4) \sigma C(\sigma)}{8}, \quad (6.38)$$

so that also

$$\int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} |Z_\eta(\eta, s)|^2 d\eta ds \leq C(\sigma) + \frac{(\bar{b}^2 + 4) \sigma C(\sigma)}{4}. \quad (6.39)$$

Next, we perform the change of variable  $S = s - \tau$ , then

$$\int_0^\sigma \int_0^{\underline{b}(S+\tau)} |Z_\eta(\eta, S + \tau)|^2 d\eta dS \leq C(\sigma) + \frac{(\bar{b}^2 + 4) \sigma C(\sigma)}{4} \text{ for all } \tau > 0$$

which implies that

$$\|Z_\eta(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C(\sigma) + \frac{(\bar{b}^2 + 4) \sigma C(\sigma)}{4}. \quad (6.40)$$

This completes the proof of Lemma 6.4.  $\square$

Next we deduce the following Corollary.

**Corollary 6.6.** Let  $\sigma > 0$ . For all  $\tau > 0$ , we have that

$$\|W_\eta(\cdot, \cdot + \tau)\|_{L^2(0, \sigma; C^{\frac{1}{2}}(\bar{\Omega}_\tau))} \leq C(\sigma), \quad (6.41)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$  and where

$$\Omega_\tau := \{\eta; 0 < \eta < \underline{b}(S + \tau), S \in (0, \sigma)\}. \quad (6.42)$$

*Proof.* From Lemmas 6.3 and 6.4, we deduce that there exists some positive constant  $\tilde{C}(\sigma)$  which does not depend on  $\tau$  such that

$$\|W_\eta(\cdot, \cdot + \tau)\|_{L^2(0, \sigma; H^1(\Omega_\tau))}^2 \leq \tilde{C}(\sigma). \quad (6.43)$$

Then, since  $H^1(\Omega_\tau) \subset C^{\frac{1}{2}}(\overline{\Omega}_\tau)$ , (6.41) follows from (6.43).  $\square$

Uniform estimate of  $W(\cdot, \cdot + \tau)$  in  $C^{\frac{1}{2}, \frac{1}{4}}(\overline{\Omega}_{\sigma, \tau})$ .

**Lemma 6.7.** There exists some positive constant  $C$  which does not depend on  $\tau$  such that

$$\|W(\cdot, \cdot + \tau)\|_{C^{\frac{1}{2}, \frac{1}{4}}(\overline{\Omega}_{\tau, \sigma})} \leq C, \quad (6.44)$$

where  $\Omega_{\sigma, \tau} := \{(\eta, S); 0 < \eta < \underline{b}(S + \tau), S \in (0, \sigma)\}$ .

*Proof.* There exists some positive constant  $C_1(\sigma)$  which does not depend on  $\tau$  such that

$$\|W_\tau(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})} \leq C_1(\sigma). \quad (6.45)$$

Indeed, we have that

$$W_\tau(\eta, \tau) = W_{\eta\eta}(\eta, \tau) + \frac{\eta}{2}W_\eta(\eta, \tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau),$$

and from Lemmas 6.3 and 6.4, we have that

$$\|W_\eta(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C_2(\sigma) \text{ for some positive constant } C_2(\sigma),$$

and

$$\|W_{\eta\eta}(\cdot, \cdot + \tau)\|_{L^2(\Omega_{\sigma, \tau})}^2 \leq C_3(\sigma) \text{ for some positive constant } C_3(\sigma).$$

Since  $\eta \leq \bar{b}$ , it follows that

$$\frac{\eta}{2}W_\eta(\cdot, \cdot + \tau) \in L^2(\Omega_{\sigma, \tau}).$$

Finally, we conclude from the partial differential equation for  $W$  that the estimate (6.45) holds, so that  $W(\cdot, \cdot + \tau) \in \mathbb{W}_2^{2,1}(\Omega_{\sigma, \tau})$ . From (Lemma 3.5 of [BHC, p.207]), we have that

$$\mathbb{W}_2^{2,1}(\Omega_{\sigma, \tau}) \subset C^{\frac{1}{2}, \frac{1}{4}}(\overline{\Omega}_{\sigma, \tau}), \quad (6.46)$$

so that (6.44) holds.  $\square$

Next we show the following result.

**Lemma 6.8.** The function  $W_\eta$  is such that  $W_\eta(\eta, \tau) \leq 0$  for all  $\tau > 0$  and  $0 < \eta < \underline{b}(\tau)$ .

*Proof.* We recall that  $Z(\eta, \tau) := W_\eta(\eta, \tau)$  for all  $\tau > 0$  and  $0 < \eta < \underline{b}(\tau)$  as defined in (6.19). From (6.26),  $Z$  satisfies the partial differential equation

$$Z_\tau = Z_{\eta\eta} + \frac{\eta}{2}Z_\eta + \frac{Z}{2}, \quad \tau > 0, \quad 0 < \eta < \underline{b}(\tau).$$

We also have that

$$Z(0, \tau) \leq 0 \text{ for all } \tau > 0. \quad (6.47)$$

Indeed, since  $0 \leq \underline{W}(\eta, \tau) \leq h$  and  $\underline{W}(0, \tau) = h$ , it follows that

$$\underline{W}_\eta(0, \tau) \leq 0 \text{ for all } \tau > 0. \quad (6.48)$$

Next, we prove that

$$Z(\underline{b}(\tau), \tau) \leq 0. \quad (6.49)$$

Indeed, from Problem (5.4) and Lemma 5.4, we deduce that  $\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau), \tau)$  and  $\frac{d\underline{b}(\tau)}{d\tau} \geq 0$  for all  $\tau > 0$ ; it follows that  $\underline{W}_\eta(\underline{b}(\tau), \tau) \leq 0$  for all  $\tau > 0$ . Next from (6.26), we have that

$$Z(\eta, 0) = \underline{W}_{\lambda, \eta}(\eta) = \frac{-h e^{-\frac{\lambda \eta^2}{4}}}{\int_0^{\underline{b}\lambda} e^{-\frac{\lambda s^2}{4}} ds} \leq 0 \text{ for } 0 \leq \eta \leq \underline{b}\lambda \text{ with } \lambda \geq 1. \quad (6.50)$$

Let  $T > 0$ , we define

$$\mathbf{Q}_T := \{(\eta, \tau), \tau \in (0, T), 0 < \eta < \underline{b}(\tau)\}. \quad (6.51)$$

Next, we perform the change of function  $Z(\eta, \tau) = \tilde{Z}(\eta, \tau)e^{\alpha\tau}$  where  $\alpha > \frac{1}{2}$ . The function  $\tilde{Z}$  satisfies the equality

$$\tilde{Z}_\tau e^{\alpha\tau} = \tilde{Z}_{\eta\eta} e^{\alpha\tau} + \frac{\eta}{2} \tilde{Z}_\eta e^{\alpha\tau} + \left(\frac{1}{2} - \alpha\right) \tilde{Z} e^{\alpha\tau} \text{ in } \mathbf{Q}_T, \text{ for all } \alpha > \frac{1}{2},$$

so that

$$\tilde{Z}_\tau - \left( \tilde{Z}_{\eta\eta} + \frac{\eta}{2} \tilde{Z}_\eta + \left(\frac{1}{2} - \alpha\right) \tilde{Z} \right) = 0 \text{ in } \mathbf{Q}_T, \text{ for all } \alpha > \frac{1}{2}.$$

Now, we prove that  $\tilde{Z} \leq 0$  in  $\overline{\mathbf{Q}_T}$ . Indeed, it follows from the weak maximum principle (Lemma 1 of [F3, p.34]) that  $\tilde{Z}$  cannot have a positive maximum in  $\mathbf{Q}_T$ . Then,  $\tilde{Z}$  attains its maximum on the boundary  $\Gamma := \{(0, \tau), 0 \leq \tau \leq T\} \cup \{(\eta, 0), 0 < \eta < \underline{b}(0)\} \cup \{(\underline{b}(\tau), \tau), 0 \leq \tau \leq T\}$ .

Then, it follows from (6.47), (6.49) and (6.50) that  $\tilde{Z} \leq 0$  on  $\Gamma$ , so that  $\tilde{Z} \leq 0$  in  $\overline{\mathbf{Q}_T}$  which implies that  $Z \leq 0$  in  $\overline{\mathbf{Q}_T}$ . Thus, we deduce that

$$\underline{W}_\eta(\eta, \tau) \leq 0, \text{ for all } \tau \geq 0, 0 \leq \eta \leq \underline{b}(\tau). \quad (6.52)$$

which completes the proof of Lemma 6.8.  $\square$

**Lemma 6.9.** Let  $\tau \geq 0$  be arbitrary. The function  $\eta \rightarrow \underline{W}_\eta(\eta, \tau)$  is nondecreasing.

*Proof.* To prove Lemma 6.9, we need to show that  $\underline{W}_{\eta\eta}(\eta, \tau) \geq 0$  for each  $\tau \geq 0$ . Indeed, we define

$$G(\eta, \tau) := Z_\eta(\eta, \tau) \text{ for all } \tau > 0 \text{ and } 0 < \eta < \underline{b}(\tau).$$

We recall that  $Z(\eta, \tau) := \underline{W}_\eta(\eta, \tau)$  for all  $\tau > 0$  and  $0 < \eta < \underline{b}(\tau)$  as defined in (6.19).

Now we derive the time evolution problem satisfied by  $G$  from the time evolution Problem (6.26) satisfied by  $Z$ . First,  $G$  satisfied the following boundary conditions

$$G(0, \tau) = 0, \quad G(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau) \text{ for all } \tau > 0. \quad (6.53)$$

From Lemma (6.8), we have that  $Z(\underline{b}(\tau), \tau) = \underline{W}_\eta(\underline{b}(\tau), \tau) \leq 0$  for all  $\tau > 0$ . It follows that

$$Z(\underline{b}(\tau), \tau) = -\sqrt{G(\underline{b}(\tau), \tau)} \text{ for all } \tau > 0. \quad (6.54)$$

Straightforward computations give

$$\begin{cases} G_\tau = G_{\eta\eta} + \frac{\eta}{2}G_\eta + G, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ G(0, \tau) = 0, & \tau > 0, \\ G(\underline{b}(\tau), \tau) = Z^2(\underline{b}(\tau), \tau), & \tau > 0, \\ \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = \sqrt{G(\underline{b}(\tau), \tau)}, & \tau > 0, \\ \underline{b}(0) = \underline{b}, \\ G(\eta, 0) = W_{\lambda, \eta\eta}(\eta), & 0 \leq \eta \leq \underline{b}_\lambda. \end{cases} \quad (6.55)$$

where  $G(\eta, 0) = W_{\lambda, \eta\eta}(\eta) = \frac{\lambda \eta h e^{-\frac{\lambda\eta^2}{4}}}{2 \int_0^{\underline{b}_\lambda} e^{-\frac{\lambda s^2}{4}} ds} \geq 0$  with  $\lambda \geq 1$ .

Finally, we use similar arguments as in the proof of Lemma 6.8 to deduce that

$$W_{\eta\eta}(\eta, \tau) \geq 0 \text{ for all } \tau \geq 0, \quad 0 \leq \eta \leq \underline{b}(\tau). \quad (6.56)$$

This completes the proof of Lemma 6.9.  $\square$

**Lemma 6.10.** Let  $\sigma > 0$ . For all  $\tau > 0$ , we have that

$$\left\| \frac{d\underline{b}(\cdot + \tau)}{d\tau} \right\|_{L^2(0, \sigma)} \leq C(\sigma), \quad (6.57)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$ , so that

$$\|\underline{b}(\cdot + \tau)\|_{C^{0, \frac{1}{2}}([0, \sigma])} \leq \hat{C}(\sigma), \quad (6.58)$$

for some positive constant  $\hat{C}(\sigma)$  which does not depend on  $\tau$ .

*Proof.* We only have to show (6.57). We recall that  $\underline{b}(\cdot + \tau)$  satisfies the ODE

$$\frac{d\underline{b}(\cdot + \tau)}{d\tau} + \frac{\underline{b}(\cdot + \tau)}{2} = -W_\eta(\underline{b}(\cdot + \tau), \cdot + \tau) \quad \text{for all } \tau > 0.$$

We have that

$$\int_0^\sigma |W_\eta(\underline{b}(s + \tau), s + \tau)|^2 ds \leq \int_0^\sigma \sup_{0 \leq \eta \leq \underline{b}(\cdot + \tau)} |W_\eta(\eta, s + \tau)|^2 ds = \|W_\eta(\cdot, \cdot + \tau)\|_{L^2(0, \sigma; C([0, \underline{b}(\cdot + \tau)]))}^2. \quad (6.59)$$

From (6.59) and Corollary 6.6, we deduce that there exists some positive constant  $C(\sigma)$  which does not depend on  $\tau$  such that

$$\int_0^\sigma |W_\eta(\underline{b}(s + \tau), s + \tau)|^2 ds \leq C(\sigma),$$

which together with Lemma 6.2 implies that (6.57) holds.  $\square$

In the following subsection, we derive estimates for the free boundary Problem (5.4) in a fixed domain.

## 6.2 A Priori Estimates for the solution of Problem (5.4) on the fixed domain.

It will be necessary in the sequel to reason on a fixed domain. To do so, we start by giving the transformation to the fixed domain  $\hat{\Omega} := \{(y, \tau) \in (0, 1) \times (0, \infty)\}$ . We set

$$y = \frac{\eta}{\underline{b}(\tau)}, \quad \hat{W}(y, \tau) = W(\eta, \tau) \quad \text{for all } \tau \geq 0, 0 \leq \eta \leq \underline{b}(\tau). \quad (6.60)$$

Using this change of variable in the estimates obtained in Lemmas 6.3 and 6.4, with the bounds on  $\underline{b}$  in Lemma 6.2, we readily get the following uniform estimates for the function  $\hat{W}$ .

**Lemma 6.11.** Let  $\sigma > 0$ . For all  $\tau \geq 0$ , we have that

$$\|\hat{W}_y(\cdot, \cdot + \tau)\|_{L^2((0,1) \times (0,\sigma))} \leq C(\sigma), \quad (6.61)$$

$$\|\hat{W}_{yy}(\cdot, \cdot + \tau)\|_{L^2((0,1) \times (0,\sigma))} \leq C(\sigma), \quad (6.62)$$

for some positive constant  $C(\sigma)$  which does not depend on  $\tau$ .

Next, we show the following result.

**Lemma 6.12.** We have that

$$\hat{W}_{yy}(y, \tau) \geq 0 \quad \text{for all } \tau \geq 0, 0 \leq y \leq 1, \quad (6.63)$$

and the function  $y \mapsto \hat{W}_y(y, \tau)$  is nondecreasing for all  $\tau \geq 0$ . Moreover, there exists a positive constant  $C$  which does not depend on  $\tau$  such that

$$\|\hat{W}_{yy}(\cdot, \tau)\|_{L^1(0,1)} \leq C \quad \text{for all } \tau \geq 0. \quad (6.64)$$

*Proof.* From (6.56), we have that  $W_{\eta\eta}(\eta, \tau) \geq 0$ , for all  $\tau \geq 0$ ,  $0 \leq \eta \leq \underline{b}(\tau)$ . Since  $W_{\eta\eta}(\eta, \tau) = \frac{1}{\underline{b}^2(\tau)} \hat{W}_{yy}(y, \tau)$ , we deduce that (6.63) holds. Next, we prove that

$$W_{\eta\eta}(\eta, \tau) \text{ is uniformly bounded on } L^1([0, \underline{b}(\tau)]) \text{ for all } \tau \geq 0. \quad (6.65)$$

From Lemma (6.8), we have that  $Z(\underline{b}(\tau), \tau) = W_\eta(\underline{b}(\tau), \tau) \leq 0$  for all  $\tau > 0$ . Thus, we have

$$0 \leq \int_0^{\underline{b}(\tau)} W_{\eta\eta}(\eta, s) ds = W_\eta(\underline{b}(\tau), \tau) - W_\eta(0, \tau) \leq -W_\eta(0, \tau). \quad (6.66)$$

We shall prove that  $W_\eta(0, \tau)$  is bounded below for  $\tau \geq 0$ . Indeed, from Lemma 5.4, we know that  $W$  is nondecreasing in time and since  $W(0, \tau) = h$  for all  $\tau \geq 0$ , it follows that

$$W_\eta(0, 0) \leq W_\eta(0, \tau) \quad \text{for all } \tau \geq 0. \quad (6.67)$$

We have that  $W_\eta(0, 0) = W_{\lambda, \eta}(0) = \frac{-h}{\int_0^{\underline{b}\lambda} e^{-\frac{\lambda s^2}{4}} ds}$  with  $\lambda \geq 1$ , which implies together with (6.67)

that  $-W_\eta(0, \tau) \leq \frac{h}{\int_0^{\underline{b}\lambda} e^{-\frac{\lambda s^2}{4}} ds}$ , which in turn implies that

$$\int_0^{\underline{b}(\tau)} W_{\eta\eta}(\eta, s) ds \leq \frac{h}{\int_0^{\underline{b}\lambda} e^{-\frac{\lambda s^2}{4}} ds}. \quad (6.68)$$

Since  $W_{\eta\eta}(\eta, s) = \frac{1}{\underline{b}^2(\tau)} \hat{W}_{yy}(y, s)$  and  $\underline{b}(\tau) \leq \bar{b}$ , we deduce that

$$\int_0^1 \hat{W}_{yy}(y, s) dy = \|\hat{W}_{yy}(\cdot, s)\|_{L^1(0,1)} \leq \frac{\bar{b} h}{\int_0^{\underline{b}\lambda} e^{-\frac{\lambda s^2}{4}} ds}. \quad (6.69)$$

This complete the proof of (6.64).  $\square$

Now, we prove the following result.

**Lemma 6.13.** There exists a positive constant  $C$  which does not depend on  $\tau$  such that

$$\|\hat{W}_y(\cdot, \tau)\|_{L^1(0,1)} \leq C \text{ for all } \tau \geq 0. \quad (6.70)$$

*Proof.* From Lemma 6.12, we have that the function  $y \mapsto \hat{W}_y(y, \tau)$  is nondecreasing for all  $\tau \geq 0$ . Then, it follows that

$$\|\hat{W}_y(\cdot, \tau)\|_{L^1(0,1)} = - \int_0^1 \hat{W}_y(y, \tau) dy = \hat{W}(0, \tau) - \hat{W}(1, \tau) = h \text{ for all } \tau \geq 0. \quad (6.71)$$

Indeed, from Problem (5.4), we have that  $W(0, \tau) = h$  and  $W(\underline{b}(\tau), \tau) = 0$  which implies that  $\hat{W}(0, \tau) = h$  and  $\hat{W}(1, \tau) = 0$  for all  $\tau \geq 0$ . This complete the proof of Lemma 6.13.  $\square$

## 7 Limit Problem as $\tau \rightarrow \infty$ .

**Theorem 7.1.** Let  $(\psi, \underline{b}_\infty)$  be defined in Lemma 5.4. Then  $(\psi, \underline{b}_\infty)$  is the unique stationary solution of Problem (1.12).

Before proving this theorem, we need to show some preliminary results. Let  $\hat{W}$  be defined as in (6.60). We also define

$$\hat{\psi}(y) = \psi(\eta), \quad y = \frac{\eta}{\underline{b}_\infty} \in [0, 1] \text{ for } 0 \leq \eta \leq \underline{b}_\infty. \quad (7.1)$$

We will derive estimates for  $\hat{\psi}$ . We start by showing the following result.

**Lemma 7.2.** We have  $\hat{\psi}, \hat{\psi}_y \in H^1(0, 1) \subset C^{0, \frac{1}{2}}([0, 1])$ .

*Proof.* Since  $0 \leq W(\eta, \tau) \leq h$  for all  $\tau \geq 0$  and  $\eta \in [0, \underline{b}(\tau)]$ , we have that

$$0 \leq \hat{W}(y, \tau) \leq h \text{ for all } \tau \geq 0, \quad y \in [0, 1]. \quad (7.2)$$

We deduce from (6.61) and (6.62) in Lemma 6.11 that there exists a constant  $C(\sigma) > 0$  such that

$$\|\hat{W}(\cdot, \cdot + \tau)\|_{L^2(0, \sigma; H^2(0, 1))} \leq C(\sigma) \quad (7.3)$$

for all  $\tau > 0$ . Thus, there exists  $v \in L^2(0, \sigma; H^2(0, 1))$  such that

$$\hat{W}(\cdot, \cdot + \tau) \rightharpoonup v \text{ weakly in } L^2(0, \sigma; H^2(0, 1)) \text{ as } \tau \rightarrow +\infty. \quad (7.4)$$

We shall prove that  $v = \hat{\psi}$ . First, since  $\lim_{\tau \rightarrow +\infty} W(\eta, \tau) = \psi(\eta)$  for all  $\eta \in \mathbb{R}^+$ , it follows from (6.60) and (7.1) that

$$\lim_{\tau \rightarrow +\infty} \hat{W}(y, \tau) = \hat{\psi}(y) \text{ for all } y \in [0, 1], \quad (7.5)$$

and since  $0 \leq \hat{W} \leq h$ , we deduce from Lebesgue Dominated Convergence Theorem that

$$\hat{W}(\cdot, \cdot + \tau) \rightarrow \hat{\psi} \text{ in } L^1((0, 1) \times (0, \sigma)) \text{ as } \tau \rightarrow +\infty. \quad (7.6)$$

Using again the uniform boundedness of  $\hat{W}$  and  $\hat{\psi}$ , we conclude that this convergence also holds in  $L^p((0, 1) \times (0, \sigma))$  for all  $p \in [1, \infty)$ . Hence,  $v = \hat{\psi} \in H^2(0, 1)$ . This completes the proof of Lemma 7.2.  $\square$

**Proposition 7.3.** The sequence

$$\{\hat{W}_y(\cdot, \tau)\} \text{ converges to } \hat{\psi}_y \text{ in } L^2(0, 1) \text{ as } \tau \rightarrow +\infty. \quad (7.7)$$

*Proof.* From the Lemmas 6.12 and 6.13, we deduce that there exists a positive constant  $C$  independent of  $\tau$  such that

$$\|\hat{W}_y(\cdot, \tau)\|_{\mathbb{W}^{1,1}(0,1)} \leq C \text{ for all } \tau \geq 0. \quad (7.8)$$

The space  $\mathbb{W}^{1,1}(0, 1)$  is compactly embedded in  $L^2(0, 1)$  (see for instance the proof of Lemma 2.7 in [BGH, p.86]). Thus, it follows that there exist a subsequence  $\{\hat{W}_y(\cdot, \tau_n)\}_{n=0}^{\infty}$  and a function  $\chi \in L^2(0, 1)$  such that

$$\hat{W}_y(\cdot, \tau_n) \rightarrow \chi \text{ strongly in } L^2(0, 1) \text{ as } \tau \rightarrow \infty. \quad (7.9)$$

Now, we prove that  $\chi = \hat{\psi}_y$ . From (7.9), it follows that

$$\int_0^1 \hat{W}_y(y, \tau) \varphi(y) dy \rightarrow \int_0^1 \chi(y) \varphi(y) dy \text{ as } \tau \rightarrow \infty \text{ for all } \varphi \in H_0^1(0, 1). \quad (7.10)$$

We also have that for all  $\varphi \in H_0^1(0, 1)$

$$\int_0^1 \hat{W}_y(y, \tau) \varphi(y) dy = - \int_0^1 \hat{W}(y, \tau) \varphi_y(y) dy. \quad (7.11)$$

We then deduce from (7.6) that

$$- \int_0^1 \hat{W}(y, \tau) \varphi_y(y) dy \rightarrow - \int_0^1 \hat{\psi}(y) \varphi_y(y) dy = \int_0^1 \hat{\psi}_y(y) \varphi(y) dy \text{ as } \tau \rightarrow \infty \quad (7.12)$$

for all  $\varphi \in H_0^1(0, 1)$ . We finally deduce from (7.10), (7.11) and (7.12) that  $\chi = \hat{\psi}_y$  and then (7.9) becomes

$$\hat{W}_y(\cdot, \tau_n) \rightarrow \hat{\psi}_y \text{ strongly in } L^2(0, 1) \text{ as } \tau \rightarrow \infty, \quad (7.13)$$

which completes the proof of Proposition 7.3.  $\square$

Next we show the following result.

**Proposition 7.4** (Application of Second Dini's Theorem). We have that

$$\hat{W}_y(\cdot, \tau) \text{ converges uniformly to } \hat{\psi}_y \text{ as } \tau \rightarrow \infty \text{ on } [0, 1]. \quad (7.14)$$



*Proof.* From Lemma 6.12, we have that the function  $y \mapsto \hat{W}_y(y, \tau)$  is nondecreasing for all  $\tau \geq 0$ . In view of Lemma 6.7, we recall that  $\hat{W}_y(\cdot, \tau)$  is a continuous function for all  $\tau \geq 0$ . From Proposition 7.3, we have that  $\hat{W}_y(\cdot, \tau)$  converges to  $\hat{\psi}_y$ , as  $\tau \rightarrow +\infty$ , a.e. in  $(0, 1)$  and from Lemma 7.2, we have that  $\hat{\psi}_y \in C^{0, \frac{1}{2}}([0, 1])$ . It follows from applying the second Dini's Theorem (Theorem 10.32 of [WMT, p. 454]) which states that "if a sequence of monotone continuous functions converges pointwise on  $(0, 1)$  and if the limit function is continuous in  $[0, 1]$ , then the convergence is uniform", which completes the proof of Proposition 7.4.  $\square$

**Corollary 7.5.**  $\lim_{\tau \rightarrow +\infty} \|\hat{W}(\cdot, \tau) - \hat{\psi}\|_{C^1([0,1])} = 0$ .

*Proof.* It remains to show that  $\hat{W}(\cdot, \tau)$  converges uniformly to  $\hat{\psi}$  as  $\tau \rightarrow \infty$ . We have that

$$\begin{aligned} \|\hat{W}(\cdot, \tau) - \hat{\psi}\|_{C^0([0,1])} &= \sup_{y \in [0,1]} \left| \int_0^y \hat{W}_y(s, \tau) ds + h - \int_0^y \hat{\psi}_y(s) ds - h \right| \\ &= \sup_{y \in [0,1]} \left| \int_0^y \hat{W}_y(s, \tau) ds - \int_0^y \hat{\psi}_y(s) ds \right| \leq \|\hat{W}_y(\cdot, \tau) - \hat{\psi}_y\|_{L^1(0,1)} \rightarrow 0 \text{ as } \tau \rightarrow \infty. \end{aligned} \quad (7.15)$$

$\square$

Next, we prove Theorem 7.1.

*Proof of Theorem 7.1.* The proof will be done through successive Lemmas. The first step of the proof consists in showing the following result.

**Lemma 7.6.** We have  $\psi(0) = h$  and  $\psi(\underline{b}_\infty) = 0$ .

*Proof.* We start by showing that  $\psi(0) = h$ . Indeed, we have that (recall that  $W$  is nondecreasing in time)

$$\mathcal{W}_\lambda(\eta) = W(\eta, 0) \leq W(\eta, \tau) \leq h. \quad (7.16)$$

Letting  $\tau$  tend to  $+\infty$ , we deduce that

$$\mathcal{W}_\lambda(\eta) \leq \psi(\eta) \leq h \quad \text{for all } \eta \in [0, \underline{b}_\infty].$$

Then, for  $\eta = 0$ , we obtain  $\mathcal{W}_\lambda(0) = h \leq \psi(0) \leq h$ , that is  $\psi(0) = h$ .

Next, we prove that  $\psi(\underline{b}_\infty) = 0$ . We deduce from Corollary 7.5 that

$$\hat{W}(1, \tau) \rightarrow \hat{\psi}(1) \text{ as } \tau \rightarrow \infty, \quad (7.17)$$

which is equivalent to

$$W(\underline{b}(\tau), \tau) \rightarrow \psi(\underline{b}_\infty) \text{ as } \tau \rightarrow \infty. \quad (7.18)$$

Since,

$$W(\underline{b}(\tau), \tau) = 0 \text{ for all } \tau > 0, \quad (7.19)$$

we deduce that indeed  $\psi(\underline{b}_\infty) = 0$ .  $\square$

The following result holds.

**Lemma 7.7.** We have

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty). \quad (7.20)$$

*Proof.* First, we prove the corresponding relation for  $\hat{\psi}_y(1)$  and then we will conclude the result for  $\psi_\eta$ . We recall that

$$\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -W_\eta(\underline{b}(\tau), \tau) \quad \text{for all } \tau > 0. \quad (7.21)$$

In view of the change of variables (6.60) for  $\hat{W}$ , the equation (7.21) becomes

$$\frac{1}{2} \frac{d\underline{b}^2(\tau)}{d\tau} + \frac{\underline{b}^2(\tau)}{2} = -\hat{W}_y(1, \tau), \quad \text{for all } \tau > 0. \quad (7.22)$$

Integrating (7.22) in time between  $\tau$  and  $\tau + \sigma$  and performing the change of variable  $S = s - \tau$ , we obtain

$$\frac{1}{2} (\underline{b}^2(\tau + \sigma) - \underline{b}^2(\tau)) + \frac{1}{2} \int_0^\sigma \underline{b}^2(S + \tau) dS = - \int_0^\sigma \hat{W}_y(1, S + \tau) dS. \quad (7.23)$$

Then, we deduce from Proposition 7.4 that  $\hat{W}_y(1, S + \tau)$  converges to  $\hat{\psi}_y(1)$  as  $\tau \rightarrow +\infty$  and recall that  $\underline{b}(\tau) \rightarrow \underline{b}_\infty$  as  $\tau \rightarrow +\infty$ . Passing to the limit as  $\tau \rightarrow +\infty$  in (7.23), we conclude that

$$\frac{\underline{b}_\infty^2}{2} = -\hat{\psi}_y(1). \quad (7.24)$$

Now, since  $\psi_\eta(\eta) = \frac{1}{\underline{b}_\infty} \hat{\psi}_y(y)$ ,  $y = \frac{\eta}{\underline{b}_\infty}$  for all  $0 \leq \eta \leq \underline{b}_\infty$  (see (7.1)), the relation (7.24) becomes

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty), \quad (7.25)$$

which completes the proof of Lemma 7.7.  $\square$

The last step of the proof of Theorem 7.1 consists in the following result.

**Proposition 7.8.** The function  $\psi \in C^\infty([0, \underline{b}_\infty])$  and satisfies the equation

$$\psi_{\eta\eta} + \frac{\eta}{2} \psi_\eta = 0 \quad \text{in } (0, \underline{b}_\infty).$$

We will prove Proposition 7.8 by means of several lemmas.

**Lemma 7.9.** Let  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$  be arbitrary. Then there exists a class of functions  $\varphi$  satisfying the following properties

- (i)  $\varphi \in C^\infty([0, \underline{b}_\infty] \times \mathbb{R}^+)$ ,
- (ii)  $\varphi(0, \tau) = 0$ ,  $\varphi(\underline{b}(\tau), \tau) = 0$  and  $\varphi_\eta(0, \tau) = 0$  for all  $\tau \geq 0$ ,
- (iii)  $\lim_{\tau \rightarrow +\infty} \varphi_\tau(\eta, \tau) = 0$  for all  $\eta \in [0, \underline{b}_\infty]$ ,
- (iv)  $\lim_{\tau \rightarrow +\infty} \varphi(\eta, \tau) = \tilde{\varphi}(\eta)$  for all  $\eta \in [0, \underline{b}_\infty]$ .

*Proof.* Let  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$  be given. We define the function  $\varphi \in C^\infty([0, \underline{b}_\infty] \times \mathbb{R}^+)$  such that

$$\varphi(\eta, \tau) = \tilde{\varphi}(\underline{b}_\infty y), \quad y = \frac{\eta}{\underline{b}(\tau)} \quad \text{for all } \tau \geq 0, 0 \leq \eta \leq \underline{b}(\tau), \quad (7.26)$$

and  $\varphi(\eta, \tau) = 0$  for all  $\tau > 0$  and  $\underline{b}(\tau) \leq \eta \leq \underline{b}_\infty$ . Next, we show that  $\varphi$  satisfies the properties (i)-(iv). The function  $\varphi$  obviously satisfies (i). Property (ii) readily holds because we have  $\varphi(0, \tau) =$

$\tilde{\varphi}(0) = 0$ ,  $\varphi(\underline{b}(\tau), \tau) = \tilde{\varphi}(\underline{b}_\infty) = 0$  and  $\varphi_\eta(0, \tau) = \frac{\underline{b}_\infty}{\underline{b}(\tau)} \tilde{\varphi}_y(0) = 0$ . Now, we turn to (iii). We have that

$$\lim_{\tau \rightarrow +\infty} \varphi_\tau(\eta, \tau) = \lim_{\tau \rightarrow +\infty} \frac{-\eta \underline{b}_\infty \frac{d\underline{b}(\tau)}{d\tau}}{\underline{b}^2(\tau)} \tilde{\varphi}_y(\underline{b}_\infty y). \quad (7.27)$$

Since  $\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\frac{1}{\underline{b}(\tau)} \hat{W}_y(1, \tau)$  for all  $\tau > 0$ , we obtain

$$\lim_{\tau \rightarrow +\infty} \frac{-\eta \underline{b}_\infty \frac{d\underline{b}(\tau)}{d\tau}}{\underline{b}^2(\tau)} \tilde{\varphi}_y(\underline{b}_\infty y) = \lim_{\tau \rightarrow +\infty} \left( \frac{\eta \underline{b}_\infty}{2\underline{b}(\tau)} + \frac{\eta \underline{b}_\infty \hat{W}_y(1, \tau)}{\underline{b}^3(\tau)} \right) \tilde{\varphi}_y(\underline{b}_\infty y) \text{ for all } 0 \leq \eta \leq \underline{b}(\tau).$$

From Proposition 7.4 and (7.24), we deduce that  $\lim_{\tau \rightarrow +\infty} \hat{W}_y(1, \tau) = \hat{\psi}_y(1) = \frac{-\underline{b}_\infty^2}{2}$ , which implies that

$$\lim_{\tau \rightarrow +\infty} \left( \frac{\eta \underline{b}_\infty}{2\underline{b}(\tau)} + \frac{\eta \underline{b}_\infty \hat{W}_y(1, \tau)}{\underline{b}^3(\tau)} \right) \tilde{\varphi}_y(\underline{b}_\infty y) = \left( \frac{\eta}{2} - \frac{\eta}{2} \right) \tilde{\varphi}_y(\underline{b}_\infty y) = 0.$$

Thus, we obtain

$$\lim_{\tau \rightarrow +\infty} \varphi_\tau(\eta, \tau) = 0.$$

Finally, we show that (iv) holds; indeed we have that

$$\lim_{\tau \rightarrow +\infty} \varphi(\eta, \tau) = \lim_{\tau \rightarrow +\infty} \tilde{\varphi} \left( \frac{\eta \underline{b}_\infty}{\underline{b}(\tau)} \right) = \tilde{\varphi}(\eta).$$

This completes the proof of Lemma 7.9.  $\square$

**Lemma 7.10.** The function  $\psi$  satisfies

$$\int_0^{\underline{b}_\infty} \psi(\eta) \left( \tilde{\varphi}_{\eta\eta} - \frac{\eta}{2} \tilde{\varphi}_\eta - \frac{\tilde{\varphi}}{2} \right) (\eta) d\eta = 0 \quad (7.28)$$

for all test functions  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ .

*Proof.* Let  $\varphi$  satisfying the properties (i) – (iv) of Lemma 7.9 and let  $\sigma > 0$  be fixed. Recall that  $(W, \underline{b})$  satisfies Problem 5.4, in particular we have

$$W_\tau(\eta, \tau) = W_{\eta\eta}(\eta, \tau) + \frac{\eta}{2} W_\eta(\eta, \tau), \quad 0 < \eta < \underline{b}(\tau), \quad \tau > 0. \quad (7.29)$$

By integrations by parts, we obtain

$$\begin{aligned} \int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} \left( W_{\eta\eta}(\eta, s) + \frac{\eta}{2} W_\eta(\eta, s) \right) \varphi(\eta, s) d\eta ds \\ = \int_\tau^{\tau+\sigma} \int_0^{\underline{b}(s)} W(\eta, s) \left( \varphi_{\eta\eta} - \frac{\eta}{2} \varphi_\eta - \frac{\varphi}{2} \right) (\eta, s) d\eta ds. \end{aligned} \quad (7.30)$$

Moreover, we have, on the one hand

$$\int_\tau^{\tau+\sigma} \frac{d}{ds} \int_0^{\underline{b}(s)} W(\eta, s) \varphi(\eta, s) d\eta ds = \int_0^{\underline{b}(\tau+\sigma)} W(\eta, \tau+\sigma) \varphi(\eta, \tau+\sigma) d\eta - \int_0^{\underline{b}(\tau)} W(\eta, \tau) \varphi(\eta, \tau) d\eta \quad (7.31)$$

and on the other hand,

$$\begin{aligned} \int_{\tau}^{\tau+\sigma} \frac{d}{ds} \int_0^{b(s)} W(\eta, s) \varphi(\eta, s) d\eta ds &= \int_{\tau}^{\tau+\sigma} \left( \int_0^{b(s)} (W_s(\eta, s) \varphi(\eta, s) + W(\eta, s) \varphi_s(\eta, s)) d\eta \right. \\ &\quad \left. + W(b(s), s) \varphi(b(s), s) \frac{db}{ds}(s) \right) ds \\ &= \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} (W_s(\eta, s) \varphi(\eta, s) + W(\eta, s) \varphi_s(\eta, s)) d\eta ds. \end{aligned} \quad (7.32)$$

Combining (7.31) with (7.32) yields

$$\begin{aligned} \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W_s(\eta, s) \varphi(\eta, s) d\eta &= - \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W(\eta, s) \varphi_s(\eta, s) d\eta \\ &\quad + \int_0^{b(\tau+\sigma)} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) d\eta - \int_0^{b(\tau)} W(\eta, \tau) \varphi(\eta, \tau) d\eta. \end{aligned} \quad (7.33)$$

We deduce from (7.29), (7.30) and (7.33) that

$$\begin{aligned} \int_0^{b(\tau+\sigma)} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) d\eta - \int_0^{b(\tau)} W(\eta, \tau) \varphi(\eta, \tau) d\eta - \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W(\eta, s) \varphi_s(\eta, s) d\eta ds \\ = \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W(\eta, s) \left( \varphi_{\eta\eta} - \frac{\eta}{2} \varphi_{\eta} - \frac{\varphi}{2} \right) (\eta, s) d\eta ds. \end{aligned} \quad (7.34)$$

Thus, since  $b(\tau) \leq b_{\infty}$  for all  $\tau \geq 0$ , we can write

$$\begin{aligned} \int_0^{b_{\infty}} \chi_{[0, b(\tau+\sigma)]} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) d\eta - \int_0^{b_{\infty}} \chi_{[0, b(\tau)]} W(\eta, \tau) \varphi(\eta, \tau) d\eta \\ - \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W(\eta, s) \varphi_s(\eta, s) d\eta ds = \int_{\tau}^{\tau+\sigma} \int_0^{b(s)} W(\eta, s) \left( \varphi_{\eta\eta} - \frac{\eta}{2} \varphi_{\eta} - \frac{\varphi}{2} \right) (\eta, s) d\eta ds. \end{aligned} \quad (7.35)$$

Furthermore, according to Lemma 5.4, we recall that

$$\lim_{\tau \rightarrow +\infty} W(\eta, \tau) = \psi(\eta) \text{ for all } 0 < \eta < b_{\infty} \text{ and } \lim_{\tau \rightarrow +\infty} b(\tau) = b_{\infty}.$$

Then, since  $\varphi$  satisfies property (iv), it follows that

$$\lim_{\tau \rightarrow +\infty} \chi_{[0, b(\tau+\sigma)]} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) = \psi(\eta) \tilde{\varphi}(\eta).$$

Moreover, we have

$$\left| \chi_{[0, b(\tau+\sigma)]} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) \right| \leq h \|\varphi\|_{L^{\infty}(\mathcal{Q})}.$$

According to Lebesgue's Dominated Convergence Theorem,

$$\int_0^{b_{\infty}} \chi_{[0, b(\tau+\sigma)]} W(\eta, \tau + \sigma) \varphi(\eta, \tau + \sigma) d\eta \rightarrow \int_0^{b_{\infty}} \psi(\eta) \tilde{\varphi}(\eta) d\eta \text{ as } \tau \rightarrow \infty. \quad (7.36)$$

Similarly, we also have that

$$\int_0^{b_{\infty}} \chi_{[0, b(\tau)]} W(\eta, \tau) \varphi(\eta, \tau) d\eta \rightarrow \int_0^{b_{\infty}} \psi(\eta) \tilde{\varphi}(\eta) d\eta \text{ as } \tau \rightarrow \infty. \quad (7.37)$$

Now, we turn to the right-hand-side of (7.35). With the change of variables  $S = s - \tau$ , we obtain

$$\begin{aligned} \int_0^\sigma \int_0^{\underline{b}_\infty} \chi_{[0, \underline{b}(S+\tau)]} W(\eta, S + \tau) \left( \varphi_{\eta\eta} - \frac{\eta}{2} \varphi_\eta - \frac{\varphi}{2} \right) (\eta, S + \tau) \, d\eta \, dS \\ \rightarrow \int_0^\sigma \int_0^{\underline{b}_\infty} \psi(\eta) \left( \tilde{\varphi}_{\eta\eta} - \frac{\eta}{2} \tilde{\varphi}_\eta - \frac{\tilde{\varphi}}{2} \right) (\eta) \, d\eta \, dS \quad \text{as } \tau \rightarrow \infty. \end{aligned} \quad (7.38)$$

Then, since  $\varphi$  satisfies the property (iii), we conclude from (7.35)–(7.38) that

$$\int_0^{\underline{b}_\infty} \psi(\eta) \left( \tilde{\varphi}_{\eta\eta} - \frac{\eta}{2} \tilde{\varphi}_\eta - \frac{\tilde{\varphi}}{2} \right) (\eta) \, d\eta = 0 \quad (7.39)$$

for all test functions  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$  which yields the result of Lemma 7.10.  $\square$

Finally, we present the proof of Lemma 7.8.

*Proof of Lemma 7.8.* From Lemma 7.2, we have that  $\psi \in H^2(0, \underline{b}_\infty)$ . Then, by means of integration by parts, we obtain

$$\int_0^{\underline{b}_\infty} \psi(\eta) \tilde{\varphi}_{\eta\eta}(\eta) \, d\eta = \int_0^{\underline{b}_\infty} \psi_{\eta\eta}(\eta) \tilde{\varphi}(\eta) \, d\eta \quad (7.40)$$

and

$$\int_0^{\underline{b}_\infty} \psi(\eta) \frac{\eta}{2} \tilde{\varphi}_\eta(\eta) \, d\eta = - \int_0^{\underline{b}_\infty} \left( \psi_\eta(\eta) \frac{\eta}{2} \tilde{\varphi}(\eta) + \frac{1}{2} \psi(\eta) \tilde{\varphi}(\eta) \right) \, d\eta \quad (7.41)$$

for all test function  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ . Hence, we deduce from (7.28) that

$$\int_0^{\underline{b}_\infty} \left( \psi_{\eta\eta}(\eta) + \frac{\eta}{2} \psi_\eta(\eta) \right) \tilde{\varphi}(\eta) \, d\eta = 0, \quad (7.42)$$

for all  $\tilde{\varphi} \in \mathcal{D}(0, \underline{b}_\infty)$ . This finally implies that

$$\psi \in C^\infty([0, \underline{b}_\infty]) \quad \text{and} \quad \psi_{\eta\eta} + \frac{\eta}{2} \psi_\eta = 0 \quad \text{for all } 0 < \eta < \underline{b}_\infty. \quad (7.43)$$

This completes the proof of Lemma 7.8.  $\square$

We conclude that the pair  $\left( W(\eta, \tau) := W(\eta, \tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)), \underline{b}(\tau) := \underline{b}(\tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) \right)$  converges to  $(\psi, \underline{b}_\infty)$  as  $\tau \rightarrow \infty$ . Thanks to Lemma 7.6, Lemma 7.7 and Lemma 7.8,  $(\psi, \underline{b}_\infty)$  satisfies Problem (1.12) and thus  $(\psi, \underline{b}_\infty)$  coincides with the unique stationary solution  $(U, a)$  of Problem (1.12). This completes the proof of Theorem 7.1.  $\square$

Similarly, one can show that  $\left( W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})), \underline{b}(\tau, (\bar{\mathcal{W}}, \bar{b})) \right)$  converges as  $\tau \rightarrow \infty$  to  $(\phi, \bar{b}_\infty)$  which also coincides with the unique stationary solution  $(U, a)$  of Problem (1.12). Recalling Lemma 5.7, we obtain the following result.

**Theorem 7.11.** Let  $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}(0, b_0)$  be such that  $0 \leq u_0 \leq \bar{\mathcal{W}}$  in  $[0, b_0]$  and  $b_0 \leq \bar{b}$  where  $(\bar{\mathcal{W}}, \bar{b})$  is defined in (5.21). Let  $(W, b) = (W(\cdot, \cdot, (u_0, b_0)), b(\cdot, (u_0, b_0)))$  be the solution of Problem (5.4) with the initial data  $(u_0, b_0)$ . Then

$$\lim_{\tau \rightarrow +\infty} W(\eta, \tau) = U(\eta) \quad \text{for all } \eta \in (0, a) \quad (7.44)$$

and

$$\lim_{\tau \rightarrow +\infty} b(\tau) = a \quad (7.45)$$

where  $(U, a)$  is the unique solution of the stationary Problem (1.12).

*Proof.* For all  $\tau > 0$  and  $\eta \geq 0$ , we have that

$$W(\eta, \tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) \leq W(\eta, \tau, (u_0, b_0)) \leq W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})) \quad (7.46)$$

and

$$b(\tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) \leq b(\tau, (u_0, b_0)) \leq b(\tau, (\bar{\mathcal{W}}, \bar{b})). \quad (7.47)$$

According to Lemma 5.4 together with the fact that  $(\psi, \underline{b}_\infty) = (\phi, \bar{b}_\infty) = (U, a)$ , we deduce that

$$\lim_{\tau \rightarrow +\infty} W(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})) = \lim_{\tau \rightarrow +\infty} W(\eta, \tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) = U(\eta), \quad (7.48)$$

$$\lim_{\tau \rightarrow +\infty} b(\tau, (\bar{\mathcal{W}}, \bar{b})) = \lim_{\tau \rightarrow +\infty} b(\tau, (\mathcal{W}_\lambda, \underline{b}_\lambda)) = a. \quad (7.49)$$

The result of Theorem 7.11 then follows from (7.46) and (7.47).  $\square$

This completes the proof of Theorem 1.1 in the introduction section.

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