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Refined asymptotic for the blow-up solution of the Complex Ginzburg-Landau equation in the subcritical case

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Abstract

In this paper, we aim at giving a refined behavior to blow-up solutions for the Complex Ginzburg-Landau (CGL) equation in the subcritical case. More precisely, we construct blowup solutions and refine their blowup profile by a more approached accurate description.

Mathematical subject classification: 35K57, 35K40, 35B44.

Keywords: Blow-up profile, Complex Ginzburg-Landau equation.

1 Introduction

In this paper, we aim at considering the Complex Ginzburg-Landau (CGL) equation

$$\begin{aligned} u_t &= (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u + \alpha u, & (\text{CGL}) \\ u(., 0) &= u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C}) \end{aligned} \quad (1)$$

where $\delta, \beta, \alpha \in \mathbb{R}$.

The equation is better known when $p = 3$ which has a long history in physics (see Aranson and Kramer [AK02]). The CGL equation describes a lot of phenomena including nonlinear waves, second-order phase transitions, and superconductivity. We note that the CGL equation can be used to describe the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation (see for example Section VI-C, page 37 and Section VII, page 40 from [AK02] and the references cited therein). The equation can be considered as a general normal form for a large class of bifurcations and nonlinear wave phenomena in continuous media systems. More generally, the CGL equation is used to describe synchronization and collective oscillation in complex media.

The study of collapse, chaotic or blow-up solutions of equation (1) appears in many works; in the description of an unstable plane Poiseuille flow, see Stewartson and Stuart [SS71], Hocking, Stewartson, Stuart and Brown [HSSB72] or in the context of binary mixtures in Kolodner and *al.* [KBS88], [KSAL95], where the authors describe an extensive series of experiments on traveling-wave convection in an ethanol/water mixture, and they observe collapse solutions that appear experimentally.

For our purpose, we consider CGL independently from any particular physical context and investigate it as a mathematical model in partial differential equations with $p > 1$.

The Cauchy problem for equation (1) can be solved in a variety of spaces using the semigroup theory as in the case of the heat equation (see [Caz03, GV96, GV97]). The space $L^\infty(\mathbb{R}^N)$ is a convenient choice for us.

We say that $u(t)$ blows up or collapses in finite time $T < \infty$, if $u(t)$ exists for all $t \in [0, T)$ and $\lim_{t \rightarrow T} \|u(t)\|_{L^\infty} = +\infty$. In that case, T is called the blow-up time of the solution. A point $x_0 \in \mathbb{R}^N$ is said to be a blow-up point if there is a sequence $\{(x_j, t_j)\}$, such that $x_j \rightarrow x_0$, $t_j \rightarrow T$ and $|u(x_j, t_j)| \rightarrow \infty$ as $j \rightarrow \infty$. The set of all blow-up points is called the blow-up set.

Let us now introduce the following definition:

Definition 1.1 *The parameters (β, δ) are called sub-critical point if $p - \delta^2 - \beta\delta(p + 1) > 0$*

Some results are available in the *subcritical* case from Zaag [Zaa98] ($\beta = 0$) and also Masmoudi and Zaag [MZ08] ($\beta \neq 0$). More precisely, if

$$p - \delta^2 - \beta\delta(p + 1) > 0,$$

then, the authors construct a solution of equation (1), which blows up in finite time $T > 0$ only at the origin such that for all $t \in [0, T)$,

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} u(x,t) - \left(p-1 + \frac{b_{sub}|x|^2}{(T-t)|\log(T-t)|} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty} \leq \frac{C_0}{1 + \sqrt{|\log(T-t)|}}, \quad (2)$$

where

$$b_{sub} = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(1+p))} > 0 \text{ and } \mu = -\frac{2b_{sub}}{(p-1)^2}\beta(1+\delta^2). \quad (3)$$

Note that this result was previously obtained formally by Hocking and Stewartson [HS72] ($p = 3$) and mentioned later in Popp et al [PSKK98] (see those references for more blow-up results often approved numerically, in various regimes of the parameters).

In the *critical* case, there are some results concerned by the construction of a blow up solution by Nouaili and Zaag [NZ18] and Duong, Nouaili and Zaag [DNZ20]. More precisely, if

$$p - \delta^2 - \beta\delta(p+1) = 0,$$

then, the authors construct a solution of equation (1), which blows up in finite time $T > 0$ only at the origin such that for all $t \in [0, T)$, (see Theorem 2 in [DNZ20]),

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} e^{-i\nu\sqrt{|\log(T-t)|}} u(x,t) - \left(p-1 + \frac{b_{cri}|x|^2}{(T-t)|\log(T-t)|^{\frac{1}{2}}} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty} \leq \frac{C_0}{1 + |\log(T-t)|^{\frac{1}{4}}}, \quad (4)$$

where

$$b_{cri}^2 = \frac{(p-1)^2 4(p+1)^2 \delta^2}{16(1+\delta^2)(p(2p-1) - (p-2)\delta^2)((p+3)\delta^2 + p(3p+1))} > 0$$

and $\nu = \nu(\beta, p)$, $\mu = \mu(\beta, p)$ are given in [DNZ20].

The authors obtain in fact a more refined description showing some first order terms of the blow-up solution. On contrary, in the subcritical case,

few results are available showing first order terms in the blow-up solution (2).

Up to our knowledge, there are only formal results given by Berger and Kohn [BK88] and Velázquez, Galaktionov, and Herrero [VGH91] when $\beta = \delta = 0$, which corresponds to the nonlinear heat equation (NLH) .

1.1 Statement of our result

Our main concern is to give a refined asymptotic description of the blow-up solution given by Masmoudi and Zaag. Rather than considering that solution and refining its description, we will instead start over from the beginning, and we construct a solution $u(x, t)$ of (1) in the subcritical case ($p - \beta\delta(p + 1) - \delta^2 > 0$) that blows up in some finite time T , in the sense that

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_{L^\infty} = +\infty,$$

and which has the same zero order description as the solution of Masmoudi and Zaag [MZ08], with a more accurate description showing next order terms in the expansion. Let us first recall the Theorem 1 from Masmoudi and Zaag [MZ08]:

Blow-up profiles for equation (1). Let us consider the subcritical case where $p - \delta^2 - \beta\delta(p + 1) > 0$. Then, there exists a unique constant η depending on p , δ and β such that equation (1) has a solution $u(x, t)$, which blows up in finite time T , only at the origin. Moreover:

(i) For all $t \in [0, T)$,

$$\left\| (T - t)^{\frac{1+i\delta}{p-1}} |\log(T - t)|^{-i\mu} u(\cdot, t) - \varphi_0 \left(\frac{\cdot}{\sqrt{(T - t)|\log(T - t)|^{1/2}}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C_0}{1 + |\log(T - t)|^{\frac{1}{2}}}, \quad (5)$$

where

$$\varphi_0(z) = (p - 1 + bz^2)^{-\frac{1+i\delta}{p-1}}, \quad (6)$$

with $b(= b_{sub})$ is defined as in (5),

$$\mu = -\frac{4b\beta(1 + \delta^2)}{(p - 1)^2} \quad (7)$$

(ii) For all $x \neq 0$, $u(x, t) \rightarrow u^*(x) \in C^2(\mathbb{R}^N \setminus \{0\})$ and

$$u^*(x) \sim |2 \log |x||^{i\mu} \left[\frac{b_{sub}|x|^2}{\sqrt{2}|\log |x||} \right]^{-\frac{1+i\delta}{p-1}} \text{ as } x \rightarrow 0. \quad (8)$$

Let us introduce the main result of this work.

Theorem 1 (First order terms) *Following Theorem 1.1, we claim that the solution decomposes in self similar variables*

$$W(y, t) = (T - t)^{\frac{1+i\delta}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T - t}},$$

as follows: For $M > 0$

$$\begin{aligned} & \sup_{|y| < M |\log(T-t)|^{\frac{1}{2}}} \left| W(y, t) e^{-i\eta \frac{\log(|\log(T-t)|)}{|\log(T-t)|}} |\log(T-t)|^{-i\mu} e^{i\theta(t)} \right. \\ & - \left. \left\{ \varphi_0 \left(\frac{y}{|\log(T-t)|^{1/2}} \right) + \frac{a(1+i\delta)}{|\log(T-t)|} + \frac{\log |\log(T-t)|}{|\log(T-t)|^2} \mathcal{E}(y) + \frac{1}{|\log(T-t)|^2} \mathcal{F}(y) \right\} \right| \\ & \leq \frac{C}{|\log(T-t)|^3} (1 + |y|^5), \end{aligned} \quad (9)$$

and $\theta(t) \rightarrow \theta_0$ as $t \rightarrow T$, such that

$$|\theta(t) - \theta_0| \leq C \frac{\log(|\log(T-t)|)^2}{|\log(T-t)|^2}$$

with

$$\varphi_0(z) = (p - 1 + bz^2)^{-\frac{1+i\delta}{p-1}}, \quad (10)$$

where b is defined as in (5), ν and a are given by (7) and \mathcal{E} and $\mathcal{F}(y)$ is a function defined as follows

$$\mathcal{E}(y) = \tilde{\mathcal{A}}_2 \tilde{h}_2(y), \quad (11)$$

$$\mathcal{F}(y) = \tilde{\mathcal{B}}_0 \tilde{h}_0(y) + \mathcal{B}_2 h_2(y) + \tilde{\mathcal{B}}_2 \tilde{h}_2(y), \quad (12)$$

where $\tilde{\mathcal{B}}_0$, \mathcal{B}_2 and $\tilde{\mathcal{A}}_2$ depend only on β and δ and are given by (47) in Definition 3.1 and $h_0(y)$, $h_2(y)$ and $\tilde{h}_2(y)$ will be given in Lemma 2.2.

The constant $\tilde{\mathcal{B}}_2$ depend only on β and δ when $\beta \neq 0$. When $\beta = 0$, we can choose arbitrary $\tilde{\mathcal{B}}_2$.

Remark 1.2 *For technical reasons, the proof of Theorem 1 must be done separately for $\beta \neq 0$ and $\beta = 0$.*

Remark 1.3 *We will consider CGL, given by (1), only when $\alpha = 0$. The case $\alpha \neq 0$ can be done as in [EZ11]. In fact, when $\alpha \neq 0$, exponentially small terms will be added to our estimates in self-similar variable (see (13) in the below), and that will be absorbed in our error terms, since our trap $\mathcal{V}_A(s)$ defined in Definition 3.1 is given in polynomial scales.*

Let us give an idea of the method used to prove the results. We construct the blow-up solution with the profile in Theorem 1, by following the method of [MZ97], [BK94]. This kind of methods has been applied for various nonlinear evolution equations. For hyperbolic equations, it has been successfully used for the construction of multi-solitons for semilinear wave equation in one space dimension (see [CZ13]). For parabolic equations, it has been used in [MZ08] and [Zaa01] for the Complex Ginzburg Landau (CGL) equation with no gradient structure, the critical harmonic heat flow in [RS13], the two dimensional Keller-Segel equation in [RS14] and the nonlinear heat equation involving nonlinear gradient term in [EZ11], [TZ19]. Recently, this method has been applied for various non variational parabolic system in [NZ15] and [GNZ17, GNZ18b, GNZ18a, GNZ19], for a logarithmically perturbed nonlinear equation in [NZ16, Duo19b, Duo19a, DNZ19]. We also mention a result for a higher order parabolic equation [GNZ20], two more results for equation involving non local terms in [DZ19, AZ19].

Following [MZ97], [NZ18] and [DNZ20], the proof is divided in two steps. First, we reduce the problem to a finite dimensional case. Second, we solve the finite time dimensional problem and conclude by contradiction using index theory. More precisely, the proof is performed in the framework of the similarity variables defined below in (13). We linearize the self-similar solution around the profile φ_0 and we obtain q (see (15) below). Our goal is to guarantee that $q(s)$ belongs to some set $\mathcal{V}_A(s)$ (introduced in Definition 3.1), which shrinks to 0 as $s \rightarrow +\infty$. The proof relies on two arguments:

- The linearized equation gives two positives mode; \tilde{Q}_0 and \tilde{q}_1 , one zero modes (\tilde{q}_2) and an infinite dimensional negative part. The negative part is easily controlled by the effect of the heat kernel. The control of the zero mode is quite delicate (see Part 2: Proof of Proposition 3.9, page 34). Consequently, the control of q is reduced to the control of its positive modes.

- The control of the positive modes \tilde{Q}_0 and \tilde{q}_1 is handled thanks to a topological argument based on index theory (see the argument at page 19).

The organization of the rest of this paper is as follows. Section 3 is divided in three subsections. In Subsection 3 we give the proof of the existence of the profile assuming technical details when $\beta \neq 0$. In particular, we construct a shrinking set and give an example of initial data giving rise to the blow-up profile. Subsection 3.1 is devoted to the proof of technical results which are needed in the proof of existence. In Subsection 3.3, we explain the case $\beta = 0$.

2 Formulation of the problem

We here consider the CGL equation, introduced in (1), with $\alpha = 0$. Since, as we mentioned before in Remark 1.3 that the perturbation of αu is quite small. Now, let us introduce the similarity variable

$$u(x, t) = (T - t)^{-\frac{1}{p-1}} w(y, s), \quad y = \frac{x}{\sqrt{T-t}} \text{ and } s = -\ln(T-t). \quad (13)$$

Hence, w reads

$$\partial_s w = (1 + i\beta)\Delta w - \frac{1}{2}y \cdot \nabla w - \frac{w}{p-1} + (1 + i\beta)|w|^{p-1}w. \quad (14)$$

Using the idea from [MZ08], we will introduce $q(y, s)$ and $\theta(s)$ satisfying

$$w(y, s) = e^{i(\mu \log s + \eta \frac{\ln s}{s} + \theta(s))} (\varphi(y, s) + q(y, s)), \quad (15)$$

where

$$\varphi(y, s) = \varphi_0 \left(\frac{y}{s^{1/2}} \right) + (1 + i\delta) \frac{a}{s} \equiv \kappa^{-i\delta} \left(p - 1 + b \frac{|y|^2}{s} \right)^{-\frac{1+i\delta}{p-1}} + (1 + i\delta) \frac{a}{s}, \quad (16)$$

where μ and a and b are well known in [MZ08]

$$\mu = -\frac{2b\beta(1 + \delta^2)}{(p-1)^2}, \quad \text{and } a = 2\kappa(1 - \beta\delta) \frac{b}{(p-1)^2},$$

and

$$b = \frac{(p-1)^2}{4(p - \delta^2 - (p+1)\delta\beta)}.$$

We will explain how we choose these constant the proof. In particular, η is new constant add for the refinement the behavior of w . Note that $\varphi_0(z)$ satisfies the following equation

$$-\frac{1}{2}z \cdot \nabla \varphi_0 - \frac{1+i\delta}{p-1}\varphi_0 + (1+i\delta)|\varphi_0|^{p-1}\varphi_0 = 0. \quad (17)$$

Using equation (14), we derive that q solves the following equation

$$\frac{\partial q}{\partial s} = \mathcal{L}_\beta q - \frac{(1+i\delta)}{p-1}q + L(q, \theta', y, s) + R^*(\theta', y, s). \quad (18)$$

where

$$\begin{aligned} \mathcal{L}_\beta q &= (1+i\beta)\Delta q - \frac{1}{2}y \cdot \nabla q, \\ L(q, \theta', y, s) &= (1+i\delta) \left\{ |\varphi + q|^{p-1}(\varphi + q) - |\varphi|^{p-1}\varphi - i \left(\eta \left(\frac{1}{s^2} - \frac{\ln s}{s^2} \right) + \frac{\mu}{s} + \theta'(s) \right) q \right\}, \\ R^*(\theta', y, s) &= R(y, s) - i \left(\eta \left(\frac{1}{s^2} - \frac{\ln s}{s^2} \right) + \frac{\mu}{s} + \theta'(s) \right) \varphi, \\ R(y, s) &= -\frac{\partial \varphi}{\partial s} + (1+i\beta)\Delta \varphi - \frac{1}{2}y \cdot \nabla \varphi - \frac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi. \end{aligned} \quad (19)$$

Our aim is to find a $\theta \in C^1([-\log T, \infty), \mathbb{R})$ such that equation (22) has a solution $q(y, s)$ defined for all $(y, s) \in \mathbb{R}^N \times [-\log T, \infty)$ such that

$$q(y, s) = \frac{\mathcal{F}(y) \log s}{s^2} + v(y, s),$$

where \mathcal{F} is defined by (12) in Theorem 1 and

$$\|v(s)\|_{L^\infty} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

From (17), one sees that the variable $z = \frac{y}{s^{1/2}}$ plays a fundamental role. Thus, we will consider the dynamics for $|z| > K$, and $|z| < 2K$ separately for some $K > 0$ to be fixed large.

2.1 The outer region where $|y| > Ks^{1/2}$

Let us consider a non-increasing cut-off function $\chi_0 \in C^\infty(\mathbb{R}^+, [0, 1])$ such that $\chi_0(\xi) = 1$ for $\xi < 1$ and $\chi_0(\xi) = 0$ for $\xi > 2$ and introduce

$$\chi(y, s) = \chi_0 \left(\frac{|y|}{Ks^{1/2}} \right), \quad (20)$$

where K will be fixed large. Let us define

$$q_e(y, s) = e^{\frac{i\delta}{p-1}s} q(y, s) (1 - \chi(y, s)), \quad (21)$$

and note that q_e is the part of $q(y, s)$, corresponding to the non-blowup region $|y| > Ks^{1/2}$. As we will explain in subsection (3.2.3), the linear operator of the equation satisfied by q_e is negative, which makes it easy to control $\|q_e(s)\|_{L^\infty}$. This is not the case for the part of $q(y, s)$ for $|y| < 2Ks^{1/2}$, where the linear operator has two positive eigenvalues, a zero eigenvalue in addition to infinitely many negative ones. Therefore, we have to expand q with respect to these eigenvalues in order to control $\|q(s)\|_{L^\infty(|y| < 2Ks^{1/2})}$. This requires more work than for q_e . The following subsection is dedicated to that purpose. From now on, K will be fixed constant which is chosen such that $\|\varphi(s')\|_{L^\infty(|y| > Ks^{1/2})}$ is small enough, namely $\|\varphi_0(z)\|_{L^\infty(|z| > K)}^{p-1} \leq \frac{1}{C(p-1)}$ (see subsection (3.2.3) below, for more details).

2.2 The inner region where $|y| < 2Ks^{1/2}$

If we linearize the term $L(q, \theta', y, s)$ in equation (18), then we can write (18) as

$$\frac{\partial q}{\partial s} = \mathcal{L}_{\beta, \delta} q - i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q + V_1 q + V_2 \bar{q} + B(q, y, s) + R^*(\theta', y, s), \quad (22)$$

where

$$\begin{aligned} \mathcal{L}_{\delta, \beta} q &= (1 + i\beta) \Delta q - \frac{1}{2} y \cdot \nabla q + (1 + i\delta) \Re q, \\ V_1(y, s) &= (1 + i\delta) \frac{p+1}{2} \left(|\varphi|^{p-1} - \frac{1}{p-1} \right), \quad V_2(y, s) = (1 + i\delta) \frac{p-1}{2} \left(|\varphi|^{p-3} \varphi^2 - \frac{1}{p-1} \right), \\ B(q, y, s) &= (1 + i\delta) \left(|\varphi + q|^{p-1} (\varphi + q) - |\varphi|^{p-1} \varphi - |\varphi|^{p-1} q - \frac{p-1}{2} |\varphi|^{p-3} \varphi (\varphi \bar{q} + \bar{\varphi} q) \right), \\ R^*(\theta', y, s) &= R(y, s) - i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \varphi, \\ R(y, s) &= -\frac{\partial \varphi}{\partial s} + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{(1+i\delta)}{p-1} \varphi + (1 + i\delta) |\varphi|^{p-1} \varphi. \end{aligned} \quad (23)$$

Note that the term $B(q, y, s)$ is built to be quadratic in the inner region $|y| \leq Ks^{\frac{1}{2}}$. Indeed, we have for all $K \geq 1$ and $s \geq 1$,

$$\sup_{|y| \leq 2Ks^{\frac{1}{2}}} |B(q, y, s)| \leq C(K) |q|^2. \quad (24)$$

Note also that $R(y, s)$ measures the defect of $\varphi(y, s)$ from being an exact solution of (14). However, since $\varphi(y, s)$ is an approximate solution of (14),

one easily derives the fact that

$$\|R(s)\|_{L^\infty} \leq \frac{C}{s}. \quad (25)$$

Therefore, if $\theta'(s)$ goes to zero as $s \rightarrow \infty$, we expect the term $R^*(\theta', y, s)$ to be small, since (22) and (25) yield

$$|R^*(\theta', y, s)| \leq \frac{C}{s} + |\theta'(s)|. \quad (26)$$

Therefore, since we would like to make q go to zero as $s \rightarrow \infty$, the dynamics of equation (22) are influenced by the asymptotic limit of its linear term,

$$\mathcal{L}_{\beta, \delta} q + V_1 q + V_2 \bar{q},$$

as $s \rightarrow \infty$. In the sense of distribution (see the definitions of V_1 and V_2 in (22) and φ (16)) this limit is $\mathcal{L}_{\beta, \delta} q$.

2.3 Spectral properties of \mathcal{L}_β

Here, we will restrict to $N = 1$. We consider the Hilbert space $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ which is the set of all $f \in L^2_{loc}(\mathbb{R}^N, \mathbb{C})$ such that

$$\int_{\mathbb{R}^N} |f(y)|^2 |\rho_\beta(y)| dy < +\infty,$$

where

$$\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{N/2}} \text{ and } |\rho_\beta(y)| = \frac{e^{-\frac{|y|^2}{4(1+\beta^2)}}}{(4\pi\sqrt{1+\beta^2})^{N/2}}. \quad (27)$$

We can diagonalize \mathcal{L}_β in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$. Indeed, we can write

$$\mathcal{L}_\beta q = \frac{1}{\rho_\beta} \operatorname{div}(\rho_\beta \nabla q).$$

We notice that \mathcal{L}_β is formally “self-adjoint” with respect to the weight ρ_β . Indeed, for any v and w in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ satisfying $\mathcal{L}_\beta v$ and $\mathcal{L}_\beta w$ in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$, it holds that

$$\int v \mathcal{L}_\beta w \rho_\beta dy = \int w \mathcal{L}_\beta v \rho_\beta dy. \quad (28)$$

If we introduce for each $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ the polynomial

$$f_\alpha(y) = c_\alpha \prod_{i=1}^N H_{\alpha_i} \left(\frac{y_i}{2\sqrt{1+i\beta}} \right), \quad (29)$$

where H_n is the standard one dimensional Hermite polynomial and $c_\alpha \in \mathbb{C}$ is chosen so that the term of highest degree in f_α is $\prod_{i=1}^N y_i^{\alpha_i}$, then, we get a family of eigenfunction of \mathcal{L}_β , “orthogonal“ with respect to the weight ρ_β , in the sense that for any different α and $\sigma \in \mathbb{N}^N$

$$\begin{aligned} \mathcal{L}_\beta f_\alpha &= -\frac{\alpha}{2} f_\alpha, \\ \int_{\mathbb{R}} f_\alpha(y) f_\sigma(y) \rho_\beta(y) dy &= 0. \end{aligned} \quad (30)$$

2.4 Spectral properties of $\mathcal{L}_{\beta,\delta}$

In the sequel, we will assume $N = 1$. Now, with the explicit basis diagonalizing \mathcal{L}_β , we are able to write $\mathcal{L}_{\beta,\delta}$ in a Jordan’s block’s. More precisely, we recall Lemma 3.1 from [MZ08]

Lemma 2.1 (Jordan block’s decomposition of $\mathcal{L}_{\beta,\delta}$) *For all $n \in \mathbb{N}$, there exists two polynomials*

$$\begin{aligned} h_n &= i f_n + \sum_{j=0}^{n-1} d_{j,n} f_j, \text{ where } d_{j,n} \in \mathbb{C} \\ \tilde{h}_n &= (1 + i\delta) f_n + \sum_{j=0}^{n-1} \tilde{d}_{j,n} f_j, \text{ where } \tilde{d}_{j,n} \in \mathbb{C}, \end{aligned} \quad (31)$$

of degree n such that

$$\begin{aligned} \mathcal{L}_{\beta,\delta} h_n &= -\frac{n}{2} h_n, \\ \mathcal{L}_{\beta,\delta} \tilde{h}_n &= \left(1 - \frac{n}{2}\right) \tilde{h}_n + c_n h_{n-2}, \end{aligned} \quad (32)$$

with $c_n = n(n-1)\beta(1+\delta)^2$ (and we take $h_k \equiv 0$ for $k < 0$). The term of highest of h_n (resp. \tilde{h}_n) is iy^n (resp. $(1+i\delta)y^n$).

Proof : See the proof of Lemma 3.1 in [MZ08]. For the explicit formulation of c_n , we look at the imaginary part of order $n-1$ in the equation $\mathcal{L}_{\beta,\delta} \tilde{h}_n = \left(1 - \frac{n}{2}\right) \tilde{h}_n + c_n h_{n-2}$.

In addition to that, we have the formulas of eigen-functions $h_j, \tilde{h}_j, j \in \{1, 2, \dots, 6\}$ in [DNZ20]:

Lemma 2.2 (The basis vectors of degree less or equal to 6) *we have*

$$\begin{aligned} h_0(y) &= i, & \tilde{h}_0 &= (1 + i\delta), \\ h_1(y) &= iy, & \tilde{h}_1 &= (1 + i\delta)y, \\ h_2(y) &= iy^2 + \beta - i(2 + \delta\beta), & \tilde{h}_2 &= (1 + i\delta)(y^2 - 2 + 2\beta\delta), \end{aligned}$$

$$\begin{aligned} h_4(y) &= iy^4 + y^2(c_{4,2} + id_{4,2}) + c_{4,0} + id_{4,0}, \\ c_{4,2} &= 6\beta, & d_{4,2} &= -6(2 + \beta\delta) = -18 - 6(\beta\delta - 1), \\ c_{4,0} &= -4\beta(3 + \beta\delta), & d_{4,0} &= 12 - 6\beta^2 + 12\beta\delta + 2\beta^2\delta^2, \end{aligned}$$

$$\begin{aligned} \tilde{h}_4(y) &= (1 + i\delta)y^4 + y^2(12(\beta\delta - 1) + i\tilde{d}_{4,2}) + \tilde{c}_{4,0} + i\tilde{d}_{4,0}. \\ \tilde{c}_{4,2} &= 12(\beta\delta - 1), & \tilde{d}_{4,2} &= 0, \\ \tilde{c}_{4,0} &= 6\beta^2(1 + \delta^2) - 12(\beta\delta - 1), & \tilde{d}_{4,0} &= -6\beta^2\delta(3\delta^2 + 7) - 12\delta(\beta\delta + 1) \end{aligned}$$

$$\begin{aligned} h_6(y) &= iy^6 + y^4(c_{6,4} + id_{6,4}) + y^2(c_{6,2} + id_{6,2}) + c_{6,0} + id_{6,0}, \\ c_{6,4} &= 15\beta, & d_{6,4} &= -15(2 + \beta\delta), \\ c_{6,2} &= -60\beta(3 + \delta\beta), & d_{6,2} &= -90\beta^2 + 180 + 180\beta\delta + 30\beta^2\delta^2, \\ c_{6,0} &= 180\beta + 120\delta\beta^2 - 45\beta^3 + 15\beta^3\delta, \\ d_{6,0} &= -180\beta\delta + 55\delta\beta^3 - 60\delta^2\beta^2 - 5\beta^3\delta^2 + 180\beta^2 - 120, \end{aligned}$$

$$\begin{aligned} \tilde{h}_6(y) &= (1 + i\delta)y^6 + y^4(\tilde{c}_{6,4} + i\tilde{d}_{6,4}) + y^2(\tilde{c}_{6,2} + i\tilde{d}_{6,2}) + \tilde{c}_{6,0} + i\tilde{d}_{6,0}, \\ \tilde{c}_{6,4} &= 30(\beta\delta - 1), & \tilde{d}_{6,4} &= 0, \\ \tilde{c}_{6,2} &= 90\beta^2(1 + \delta^2) - 180(\beta\delta - 1), \\ \tilde{d}_{6,2} &= -90\beta(1 + \delta^2)(3\beta\delta + 4) + 180(\beta\delta - 1)(\delta - 2\beta), \\ \tilde{c}_{6,0} &= -20\beta^2(1 + \delta^2)(11\beta\delta + 21) + 120(\beta\delta - 1)(-2\beta^2 + \beta\delta + 1), \\ \tilde{d}_{6,0} &= 270\beta(1 + \delta^2)(2 + \beta\delta) + \beta^2(1 + \delta^2)(140\beta\delta^2 - 180\beta\delta + 390\delta) \\ &\quad + 60(\beta\delta - 1)(2\beta^2\delta - \beta\delta^2 + 9\beta - 4\delta), \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathcal{L}_{\beta,\delta}\tilde{h}_0 &= \tilde{h}_0, \\ \mathcal{L}_{\beta,\delta}\tilde{h}_1 &= \frac{1}{2}\tilde{h}_1, \\ \mathcal{L}_{\beta,\delta}\tilde{h}_2 &= 2\beta(1 + \delta^2)h_0 = 2i\beta(1 + \delta^2), \\ \mathcal{L}_{\beta,\delta}\tilde{h}_4 &= -\tilde{h}_4 + 12\beta(1 + \delta^2)h_2, \\ \mathcal{L}_{\beta,\delta}\tilde{h}_6 &= -2\tilde{h}_6 + 30\beta(1 + \delta^2)h_4. \end{aligned}$$

Corollary 2.1 (Basis for the set of polynomials) *The family $(h_n\tilde{h}_n)_n$ is a basis of $\mathbb{C}[X]$, the \mathbb{R} vector space of complex polynomials.*

2.5 Decomposition of q

For the sake of controlling q in the region $|y| < 2K\sqrt{s}$, we will expand the unknown function q (and not just χq where χ is defined in (20)) with respect to the family f_n and the with respect to the h_n . We start by writing

We start by writing

$$q(y, s) = \sum_{n \leq M} \mathcal{Q}_n(s) f_n(y) + q_-(y, s), \quad (33)$$

where f_n is the eigenfunction of \mathcal{L}_β defined in (29), $\mathcal{Q}_n(s) \in \mathbb{C}$, q_- satisfies

$$\int q_-(y, s) h_n(y) \rho(y) dy = 0 \text{ for all } n \leq M,$$

and M is a fixed even integer satisfying

$$M \geq 4 \left(\sqrt{1 + \delta^2} + 1 + 2 \max_{i=1,2, y \in \mathbb{R}, s \geq 1} |V_i(y, s)| \right), \quad (34)$$

with $V_{i=1,2}$ defined in (23). From (33), we have

$$\mathcal{Q}_n(s) = \frac{\int q(y, s) f_n(y) \rho_\beta(y) dy}{\int f_n(y)^2 \rho_\beta(y)} \equiv F_n(q(s)), \quad (35)$$

The function $q_-(y, s)$ can be seen as the projection of $q(y, s)$ onto the spectrum of \mathcal{L}_β , which is smaller than $(1 - M)/2$. We will call it the infinite dimensional part of q and we will denote it $q_- = P_{-,M}(q)$. We also introduce $P_{+,M} = Id - P_{-,M}$. Notice that $P_{-,M}$ and $P_{+,M}$ are projections. In the sequel, we will denote $P_- = P_{-,M}$ and $P_+ = P_{+,M}$.

The complementary part $q_+ = q - q_-$ will be called the finite dimensional part of q . We will expand it as follows

$$q_+(y, s) = \sum_{n \leq M} \mathcal{Q}_n(s) f_n(y) = \sum_{n \leq M} q_n(s) h_n(y) + \tilde{q}_n(s) \tilde{h}_n(y), \quad (36)$$

where $\tilde{q}_n, q_n \in \mathbb{R}$. Finally, we notice that for all s , we have

$$\int q_-(y, s) q_+(y, s) \rho_\beta(y) dy = 0.$$

Our purpose is to project (22) in order to write an equation for q_n and \tilde{q}_n . For that we need to write down the expression of q_n and \tilde{q}_n in terms of \mathcal{Q}_n .

Since the matrix $(h_n, \tilde{h}_n)_{n \leq M}$ in the basis of (if_n, f_n) is upper triangular (see Lemma 2.2). The same holds for its inverse. Thus, we derive from (36)

$$\begin{aligned} q_n &= \operatorname{Im} \mathcal{Q}_n(s) - \delta \operatorname{Re} \mathcal{Q}_n(s) + \sum_{j=n+1}^M A_{j,n} \operatorname{Im} \mathcal{Q}_j(s) + B_{j,n} \operatorname{Re} \mathcal{Q}_j(s) \equiv P_{n,m}(q(s)), \\ \tilde{q}_n(s) &= \operatorname{Re} \mathcal{Q}_n(s) + \sum_{j=n+1}^M \tilde{A}_{j,n} \operatorname{Im} \mathcal{Q}_j(s) + \tilde{B}_{j,n} \operatorname{Re} \mathcal{Q}_j(s) \equiv \tilde{P}_{n,M}(q(s)), \end{aligned} \quad (37)$$

where all the constants are real. Moreover, the coefficient of $\operatorname{Im} \mathcal{Q}_n$ and $\operatorname{Re} \mathcal{Q}_n$ in the expression of q_n and \tilde{q}_n are explicit. This comes from the fact that the same holds for the coefficient of if_n and f_n in the expansion of h_n and \tilde{h}_n (see Lemma 2.1).

Note that the projector $P_{n,m}(q)$ and $\tilde{P}_{n,m}(q)$ are well-defined thanks to (35). We will project equation (22) on the different modes h_n and \tilde{h}_n . Note that from (33) and (36), that

$$q(y, s) = \left(\sum_{n \leq M} q_n(s) h_n(y) + \tilde{q}_n(s) \tilde{h}_n(y) \right) + q_-(y, s), \quad (38)$$

we should keep in mind that the presentation in (38) is unique.

3 Existence

In this section, we prove the existence of a solution $q(s), \theta(s)$ of problem (18)-(50) and describe more the asymptotic of q

$$\begin{aligned} q(y, s) &= \tilde{h}_0(y) \left(\frac{\tilde{\mathcal{A}}_0}{s^2} \right) + h_2(y) \left(\frac{\tilde{\mathcal{A}}_2}{s^2} \right) + \tilde{h}_2(y) \left(\frac{\tilde{\mathcal{A}}_2}{s^2} + \frac{\tilde{\mathcal{B}}_2 \ln s}{s^2} \right) + v(y, s), \\ \text{with, for all } M > 0 \quad \sup_{|y| < Ms^{\frac{1}{2}}} |v(y, s)| &\leq C \frac{(1 + |y|^5) \ln^2 s}{s^3}, \\ \text{and } |\theta'(s)| &\leq \frac{CA^{10} \ln^2 s}{s^3} \text{ for all } s \in [-\log T, +\infty), \end{aligned} \quad (39)$$

where $\tilde{\mathcal{A}}_0, \mathcal{A}_2, \tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{B}}_2$ in Definition 3.1 and $h_0(y), h_2(y)$ and $\tilde{h}_2(y)$ are given in Lemma 2.2.

Hereafter, we denote by C a generic positive constant, depending only on p, δ, β and K introduced in (20), itself depending on p .

As a matter of fact, we aim at control the asymptotic (39) by a shrinking set. In fact, we are inspired from the set given in [MZ08] and [DNZ20], to introduce the new one which is more sharpened:

Definition 3.1 (A set shrinking to zero) For all $K > 1$, $A \geq 1$ and $s \geq 1$, we define $\mathcal{V}_A(s)$ as the set of all $q \in L^\infty(\mathbb{R})$ such that

$$\begin{aligned} \|q_e\|_{L^\infty(\mathbb{R})} &\leq \frac{A^{M+2}}{s^{\frac{1}{2}}}, & \left\| \frac{q-(y)}{1+|y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} &\leq \frac{A^{M+1}}{s^{\frac{M+2}{2}}}, \\ |q_j|, |\tilde{q}_j| &\leq \frac{A^j}{s^{\frac{j+1}{2}}} \text{ for all } 5 \leq j \leq M, & |q_0| &\leq \frac{1}{s^2}, |\tilde{q}_1| \leq \frac{A}{s^3}, |q_1| \leq \frac{A^4}{s^3}. \end{aligned}$$

In addition to the the other modes will satisfy the following condition:

$$\begin{aligned} |Q_4| &\leq \frac{A^7 \ln^2 s}{s^4} \text{ and } |\tilde{Q}_4| \leq \frac{A^4 \ln^2 s}{s^4}, \\ |q_3| &\leq \frac{A^3}{s^4} \text{ and } |\tilde{q}_3| \leq \frac{A^3}{s^4}, \\ |Q_2| &\leq \frac{A^8 \ln^2 s}{s^4} \text{ and } |\tilde{Q}_2| \leq \frac{A^{10} \ln^2 s}{s^3}, \end{aligned}$$

and

$$|\tilde{Q}_0| \leq \frac{A \ln^2 s}{s^4},$$

where

$$Q_4 = q_4 - \left(\frac{1}{2} D_{4,2} \frac{\tilde{q}_2}{s} + \left[\frac{C_{4,2} R_{2,1}^*}{2} + \frac{R_{4,2}^*}{2} \right] \frac{1}{s^3} \right), \quad (40)$$

$$= q_4 - \left(\mathcal{A}_4 \frac{\tilde{q}_2}{s} + \frac{\mathcal{B}_4}{s^3} \right),$$

$$\tilde{Q}_4 = \tilde{q}_4 - \left(\tilde{D}_{4,2} \frac{\tilde{q}_2}{s} + \frac{1}{s^3} \left[\tilde{C}_{4,2} R_{2,1}^* + \tilde{R}_{4,2}^* \right] \right), \quad (41)$$

$$= \tilde{q}_4 - \left(\tilde{\mathcal{A}}_4 \frac{\tilde{q}_2}{s} + \frac{\tilde{\mathcal{B}}_4}{s^3} \right),$$

$$\tilde{Q}_0 = \tilde{q}_0 - \left(\frac{\tilde{q}_2}{s} \left[\mu \tilde{L}_{0,2} - \tilde{D}_{0,2} - \frac{\tilde{\Theta}_{0,0}^* c_2}{\kappa} \right] - \frac{\tilde{R}_{0,1}^*}{s^2} - \frac{\tilde{T}_{0,1}^* \ln s}{s^3} \right) \quad (42)$$

$$- \left(\frac{1}{s^3} \left[-\tilde{X}_0 + \mu \tilde{K}_{0,2} R_{2,1}^* - \tilde{C}_{0,2} \cdot R_{2,1}^* - \tilde{T}_{0,1}^{**} \right] \right), \quad (43)$$

$$= \tilde{q}_0 - \left(\tilde{\mathcal{A}}_0 \frac{\tilde{q}_2}{s} + \frac{\tilde{\mathcal{B}}_0}{s^2} + \frac{\tilde{\mathcal{C}}_0 \ln s}{s^3} + \frac{\tilde{\mathcal{D}}_0}{s^3} \right), \quad (44)$$

and

$$\begin{aligned}
Q_2 &= q_2 - \left(\frac{\tilde{q}_2}{s} \left[D_{2,2} - \mu(1 + \delta^2) + c_4 \tilde{D}_{4,2} + \frac{\Theta_{2,0}^* c_2}{\kappa} \right] + \frac{R_{2,1}^*}{s^2} \right) \\
&\quad - \left(\frac{\tilde{T}_{2,0}^* \ln s}{s^3} + \frac{1}{s^3} \left[X_2 + c_4 [\tilde{C}_{4,2} R_{2,1}^* + \tilde{R}_{4,2}^*] - D_{2,0} \tilde{R}_{0,1}^* + T_{2,0}^{**} \right] \right) \quad (45) \\
&= q_2 - \left(\mathcal{A}_2 \frac{\tilde{q}_2}{s} + \frac{\mathcal{B}_2}{s^2} + \frac{\mathcal{C}_2 \ln s}{s^3} + \frac{\mathcal{D}_2}{s^3} \right), \\
\tilde{Q}_2 &= \tilde{q}_2 - \left(\frac{\tilde{\mathcal{A}}_2 \ln s}{s^2} + \frac{\tilde{\mathcal{B}}_2}{s^2} \right), \quad (46)
\end{aligned}$$

and

$$\tilde{\mathcal{A}}_2 = -\frac{\delta b}{(p-1)^2} R_{0,1}^* + (\mu + \tilde{C}_{2,2}) R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2}, \quad (47)$$

$$\begin{aligned}
\tilde{\mathcal{B}}_2 &= -\frac{R_{0,1}^* - \eta \kappa}{c_2}, \text{ where } c_2 = 2\beta(1 + \delta^2) && \text{if } \beta \neq 0, \\
\tilde{\mathcal{B}}_2 &\text{ is arbitrary} && \text{if } \beta = 0,
\end{aligned} \quad (48)$$

and

$$X_2 = R_{2,2}^* + (C_{2,2} - \delta \mu) R_{2,1}^* + \frac{\Theta_{2,0}^* R_{0,1}^*}{\kappa} \text{ and } \tilde{X}_0 = \tilde{R}_{0,2}^* - (\delta \mu + \tilde{D}_{0,0}) \tilde{R}_{0,1}^*.$$

Using Definition 3.1, we claim to the following

Claim 3.2 (The size of $q \in \mathcal{V}_A$) For all $r \in \mathcal{V}_A(s)$, we have the following estimates

- (i) $\|r\|_{L^\infty(|y| < 2K\sqrt{s})} \leq C(K) \frac{A^{M+1}}{\sqrt{s}}$ and $\|r\|_{L^\infty} \leq C(K) \frac{A^{M+2}}{\sqrt{s}}$.
- (ii) For all $y \in \mathbb{R}$, $|r(y)| \leq C \frac{A^{M+1} \ln s}{s^2} (1 + |y|^{M+1})$.

Proof: The proof directly follows the definition of the shrinking set. ■

From item (i), our purpose is to control q to stay in $\mathcal{V}(A)$ for $s \geq s_{01}$. More over, the bounds in this set help us to conclude the result in the Theorems.

In the following, we aim to choose the initial data

Definition 3.3 (Choice of initial data) *Let us define, for $A \geq 1$, $s_0 = -\log T > 1$ and $d_0, d_1 \in \mathbb{R}$, the function*

$$\begin{aligned} \psi_{s_0, d_0, d_1}(y) = & \left[\left(\frac{A \ln^2 s}{s_0^4} \tilde{d}_0 + \frac{\tilde{\mathcal{B}}_0}{s_0^2} + \frac{(\tilde{\mathcal{A}}_0 \tilde{\mathcal{A}}_2 + \tilde{\mathcal{C}}_0) \ln s_0}{s_0^3} + \frac{\tilde{\mathcal{D}}_0 + \tilde{\mathcal{A}}_0 \tilde{\mathcal{B}}_2}{s_0^3} \right) \tilde{h}_0 \right. \\ & + \frac{A}{s_0^3} \tilde{d}_1 \tilde{h}_1(y) + d_0 h_0 \\ & + \left(\frac{\tilde{\mathcal{A}}_2 \ln s_0}{s_0^2} + \frac{\tilde{\mathcal{B}}_2}{s_0^2} \right) \tilde{h}_2 + \left(\frac{\mathcal{B}_2}{s_0^2} + \frac{\mathcal{D}_2 + \mathcal{A}_2 \tilde{\mathcal{B}}_2}{s_0^3} + \frac{(\mathcal{C}_2 + \mathcal{A}_2 \tilde{\mathcal{A}}_2) \ln s_0}{s_0^3} \right) h_2 \\ & \left. + \left(\frac{\tilde{\mathcal{B}}_4 + \tilde{\mathcal{A}}_4 \tilde{\mathcal{A}}_2}{s_0^3} + \frac{\tilde{\mathcal{A}}_4 \tilde{\mathcal{A}}_2 \ln s_0}{s_0^3} \right) \tilde{h}_4 + \left(\frac{\mathcal{B}_4 + \mathcal{A}_4 \tilde{\mathcal{B}}_2}{s_0^3} + \frac{\mathcal{A}_4 \tilde{\mathcal{A}}_2 \ln s_0}{s_0^3} \right) h_4 \right] \chi(2y, s_0), \end{aligned} \quad (49)$$

where $s_0 = -\log T$ and $h_i, \tilde{h}_i, i = 0, 1, 2, 3, 4$ are given in Lemma 2.2, χ is defined by (20) and $d_0 = d_0(\tilde{d}_0, \tilde{d}_1)$ will be fixed later in (i) of Proposition 3.6. The constants $\tilde{\mathcal{A}}_i, \mathcal{A}_i, \tilde{\mathcal{B}}_i, \mathcal{B}_i, \tilde{\mathcal{C}}_i, \mathcal{C}_i, \tilde{\mathcal{D}}_i, \mathcal{D}_i$ for $i = 0, 2, 4$ are given by (40–46).

Remark 3.4 *Let us recall that we will modulate the parameter θ to kill one of the neutral modes, see equation (50) below. It is natural that this condition must be satisfied for the initial data at $s = s_0$. Thus, it is necessary that we choose d_0 to satisfy condition (50), see (51) below.*

So far, the phase $\theta(s)$ introduced in (15) is arbitrary, in fact as we will show below in Proposition 3.7. We can use a modulation technique to choose $\theta(s)$ in such a way that we impose the condition

$$P_{0,M}(q(s)) = 0, \quad (50)$$

which allows us to kill the neutral direction of the operator $\tilde{\mathcal{L}}$ defined in (22). Reasonably, our aim is then reduced to the following proposition:

Proposition 3.5 (Existence of a solution trapped in $\mathcal{V}_A(s)$) *There exists $A_2 \geq 1$ such that for $A \geq A_2$ there exists $s_{02}(A)$ such that for all $s_0 \geq s_{02}(A)$, there exists $(\tilde{d}_0, \tilde{d}_1)$ such that if q is the solution of (22)-(50), with initial data given by (49) and (51), then $v \in \mathcal{V}_A(s)$, for all $s \geq s_0$.*

This proposition gives the stronger convergence to 0 in $L^\infty(\mathbb{R})$.

Let us first be sure that we can choose the initial data such that it starts in $\mathcal{V}_A(s_0)$. In other words, we will define a set where where will be selected the good parameters $(\tilde{d}_0, \tilde{d}_1)$ that will give the conclusion of Proposition 3.5. More precisely, we have the following result:

Proposition 3.6 (Properties of initial data) For each $A \geq 1$, there exists $s_{03}(A) > 1$ such that for all $s_0 \geq s_{03}$:

(i) $P_{0,M}(i\chi(2y, s_0)) \neq 0$ and the parameter $d_0(s_0, \tilde{d}_0, \tilde{d}_1)$ given by

$$\begin{aligned}
d_0(s_0, \tilde{d}_0, \tilde{d}_1) = & -\frac{A}{s_0^3} \tilde{d}_1 \frac{P_{0,M}(\tilde{h}_1\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\
& - \left(\frac{A \ln^2 s_0}{s_0^4} \tilde{d}_0 + \frac{\tilde{\mathcal{B}}_0}{s_0^2} + \frac{(\tilde{\mathcal{A}}_0 \tilde{\mathcal{A}}_2 + \tilde{\mathcal{C}}_0) \ln s_0}{s_0^3} + \frac{\tilde{\mathcal{D}}_0 + \tilde{\mathcal{A}}_0 \tilde{\mathcal{B}}_2}{s_0^3} \right) \frac{P_{0,M}(\tilde{h}_0\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\
& - \left(\frac{\tilde{\mathcal{A}}_2 \ln s_0}{s_0^2} + \frac{\tilde{\mathcal{B}}_2}{s_0^2} \right) \frac{P_{0,M}(\tilde{h}_2\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\
& - \left(\frac{\mathcal{B}_2}{s_0^2} + \frac{\mathcal{D}_2 + \mathcal{A}_2 \tilde{\mathcal{B}}_2}{s_0^3} + \frac{(\mathcal{C}_2 + \mathcal{A}_2 \tilde{\mathcal{A}}_2) \ln s_0}{s_0^3} \right) \frac{P_{0,M}(h_2\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\
& - \left(\frac{\tilde{\mathcal{B}}_4 + \tilde{\mathcal{A}}_4 \tilde{\mathcal{A}}_2}{s_0^3} + \frac{\tilde{\mathcal{A}}_4 \tilde{\mathcal{A}}_2 \ln s_0}{s_0^3} \right) \frac{P_{0,M}(\tilde{h}_4\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\
& - \left(\frac{\mathcal{B}_4 + \mathcal{A}_4 \tilde{\mathcal{B}}_2}{s_0^3} + \frac{\mathcal{A}_4 \tilde{\mathcal{A}}_2 \ln s_0}{s_0^3} \right) \frac{P_{0,M}(h_4\chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))}
\end{aligned} \tag{51}$$

is well defined, where χ defined in (20) and the constants $\tilde{\mathcal{A}}_i, \mathcal{A}_i, \tilde{\mathcal{B}}_i, \mathcal{B}_i, \tilde{\mathcal{C}}_i, \mathcal{C}_i$, for $i = 0, 2, 4$ are given by (40–46).

(ii) If ψ is given by (49) and (51) with d_0 defined by (51). Then, there exists a quadrilateral $\mathcal{D}_{s_0} \subset [-2, 2]^2$ such that the mapping

$$(\tilde{d}_0, \tilde{d}_1) \rightarrow \left(\tilde{\Psi}_0 = \tilde{\psi}_0 - \left(\frac{\tilde{\mathcal{B}}_0}{s_0^2} + \frac{(\tilde{\mathcal{A}}_0 \tilde{\mathcal{A}}_2 + \tilde{\mathcal{C}}_0) \ln s_0}{s_0^3} + \frac{\tilde{\mathcal{D}}_0 + \tilde{\mathcal{A}}_0 \tilde{\mathcal{B}}_2}{s_0^3} \right), \tilde{\psi}_1 \right)$$

(where ψ stands for $\psi_{s_0, \tilde{d}_0, \tilde{d}_1}$) is linear, one to one from \mathcal{D}_{s_0} onto $[-\frac{A \ln^2 s_0}{s_0^4}, \frac{A \ln^2 s_0}{s_0^4}] \times [-\frac{A}{s_0^3}, \frac{A}{s_0^3}]$. Moreover it is of degree 1 on the boundary.

(iii) For all $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, $\psi_e \equiv 0$, $\psi_0 = 0$, $|\tilde{\psi}_i| + |\psi_j| \leq CAe^{-\gamma s_0}$ for some $\gamma > 0$, for some $\gamma > 0$ and for all $3 \leq i \leq M$, $i \neq 4$ and $1 \leq j \leq M$, $j \neq 4$ and

$$|\tilde{\Psi}_i| + |\Psi_j| \leq CAe^{-\gamma s_0} \text{ for } i, j = \{2, 4\},$$

where $\tilde{\Psi}_i$ and Ψ_i are defined as in (40–46).

Moreover, $\| \frac{\psi_-(y)}{(1+|y|)^{M+1}} \|_{L^\infty(\mathbb{R})} \leq C \frac{A}{s_0^{\frac{M}{4}+1}}$.

(iv) For all $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, $\psi_{s_0, \tilde{d}_0, \tilde{d}_1} \in \mathcal{V}_A(s_0)$ with strict inequalities except for $(\tilde{\psi}_0, \tilde{\psi}_1)$.

Proof: The proof is the same as Proposition 4.2 in [MZ08] and Proposition 4.5 in [DNZ20]

In the following, we find a local in time solution for equation (22) coupled with the condition (50).

Proposition 3.7 (*Local in time solution and modulation for problem (22)-(50) with initial data (49)-(51)*) For all $A \geq 1$, there exists $T_3(A) \in (0, 1/e)$ such that for all $T \leq T_3$, the following holds: For all $(\tilde{d}_0, \tilde{d}_1) \in D_T$, there exists $s_{max} > s_0 = -\log T$ such that problem (22)-(50) with initial data at $s = s_0$,

$$(q(s_0), \theta(s_0)) = (\psi_{s_0, \tilde{d}_0, \tilde{d}_1}, 0),$$

where $\psi_{s_0, \tilde{d}_0, \tilde{d}_1}$ is given by (49) and (51), has a unique solution $q(s), \theta(s)$ satisfying $q(s) \in V_{A+1}(s)$ for all $s \in [s_0, s_{max}]$.

The proof is quite similar to Proposition 4.4 in [MZ08] and Proposition 4.6 in [DNZ20].

Let us now give the proof of Proposition 3.5.

Proof of Proposition 3.5: Let us consider $A \geq 1$, $s_0 \geq s_{03}$, $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, where s_{03} is given by Proposition 3.6. From the existence theory (which follows from the Cauchy problem for equation (1)), starting in $\mathcal{V}_A(s_0)$ which is in $\mathcal{V}_{A+1}(s_0)$, the solution stays in $\mathcal{V}_A(s)$ until some maximal time $s_* = s_*(\tilde{d}_0, \tilde{d}_1)$. Then, either:

- $s_*(\tilde{d}_0, \tilde{d}_1) = \infty$ for some $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, then the proof is complete.
- $s_*(\tilde{d}_0, \tilde{d}_1) < \infty$, for any $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, then we argue by contradiction. By continuity and the definition of s_* , the solution on s_* is in the boundary of $\mathcal{V}_A(s_*)$. Then, by definition of $\mathcal{V}_A(s_*)$, one at least of the inequalities in that definition is an equality. Owing to the following proposition, this can happen only for the first two components \tilde{q}_0, \tilde{q}_1 . Precisely we have the following result

Proposition 3.8 (Control of $q(s)$ by $(q_0(s), q_1(s))$ in $\mathcal{V}_A(s)$) There exists $A_4 \geq 1$ such that for each $A \geq A_4$, there exists $s_{04} \in \mathbb{R}$ such that for all $s_0 \geq s_{04}$. The following holds:

If q is a solution of (22) with initial data at $s = s_0$ given by (49) and (51) with $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, and $q(s) \in \mathcal{V}(A)(s)$ for all $s \in [s_0, s_1]$, with $q(s_1) \in \partial\mathcal{V}_A(s_1)$ for some $s_1 \geq s_0$, then:

(i) (**Smallness of the modulation parameter θ defined in (15)**) For all $s \in [s_0, s_1]$,

$$|\theta'(s)| \leq \frac{CA^{10} \ln^2 s}{s^3}.$$

(ii) (**Reduction to a finite dimensional problem**) We have:

$$\left(\tilde{Q}_0(s_1), \tilde{q}_1(s_1)\right) \in \partial \left(\left[-\frac{A \ln^2 s_1}{s_1^4}, \frac{A \ln^2 s_1}{s_1^4} \right] \times \left[-\frac{A}{s_1^3}, \frac{A}{s_1^3} \right] \right).$$

(iii) (**Transverse crossing**) There exists $\omega \in \{-1, 1\}$ such that

$$\omega \tilde{Q}_0(s_1) = \frac{A}{s_1^4} \text{ and } \omega \frac{d\tilde{Q}_0(s_1)}{ds}(s_1) > 0.$$

$$\omega \tilde{q}_1(s_1) = \frac{A}{s_1^3} \text{ and } \omega \frac{d\tilde{q}_1}{ds}(s_1) > 0.$$

(iii) (**Transverse crossing**) There exists $\omega \in \{-1, 1\}$ such that

$$\omega \tilde{Q}_0(s_1) = \frac{A}{s_1^4} \text{ and } \omega \frac{d\tilde{Q}_0(s_1)}{ds}(s_1) > 0.$$

$$\omega \tilde{q}_1(s_1) = \frac{A}{s_1^3} \text{ and } \omega \frac{d\tilde{q}_1}{ds}(s_1) > 0.$$

Proof See the proof in Page 23. ■

Assume the result of the previous proposition, for which the proof is given below in page 21, and continue the proof of Proposition 3.5. Let $A \geq A_4$ and $s_0 \geq s_{04}(A)$. It follows from Proposition 3.8, part (ii) that $\left(\tilde{Q}_0, \tilde{q}_1(s_*)\right) \in \partial \left(\left[-\frac{A}{s_1^4}, \frac{A}{s_1^4} \right] \times \left[-\frac{A}{s_1^3}, \frac{A}{s_1^3} \right] \right)$, and the following function

$$\begin{aligned} \phi &: \mathcal{D}_{s_0} \rightarrow \partial([-1, 1]^2) \\ (\tilde{d}_0, \tilde{d}_1) &\rightarrow \left(\frac{s_*^4}{A \ln^2 s_*} \tilde{Q}_0, \frac{s_*^3}{A} \tilde{q}_1 \right)_{(\tilde{d}_0, \tilde{d}_1)}(s_*), \text{ with } s_* = s_*(\tilde{d}_0, \tilde{d}_1), \end{aligned}$$

is well defined. Then, it follows from Proposition 3.8, part (iii) that ϕ is continuous. On the other hand, using Proposition 3.6 (ii)-(iv) together with the fact that $q(s_0) = \psi_{s_0, \tilde{d}_0, \tilde{d}_1}$, we see that when $(\tilde{d}_0, \tilde{d}_1)$ is in the boundary of the rectangle \mathcal{D}_{s_0} , we have strict inequalities for the other components.

Applying the transverse crossing property given by (iii) of Proposition 3.8, we see that $q(s)$ leaves $\mathcal{V}_A(s)$ at $s = s_0$, hence $s_*(\tilde{d}_0, \tilde{d}_1) = s_0$. Using Proposition 3.6, part (ii), we see that the restriction of ϕ to the boundary is of degree 1. A contradiction, then follows from the index theory. Thus there exists a value $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$ such that for all $s \geq s_0$, $q_{s_0, \tilde{d}_0, \tilde{d}_1}(s) \in \mathcal{V}_A(s)$. This concludes the proof of Proposition 3.5.

Using (i) of Proposition 3.8, we get the bound on $\theta'(s)$. This concludes the proof of (39). ■

3.1 Proof of the technical results

This section is devoted to the proof of the existence result given by Theorem 1. We proceed in 2 steps, each of them making a separate subsection.

- In the third subsection using the spectral properties of equation (22), we reduce our goal from the control of $q(s)$ (an infinite dimensional variable) in $\mathcal{V}_A(s)$ to control its two first components $(\tilde{Q}_0, \tilde{q}_1)$ a two variables in $\left[-\frac{A \ln^2 s}{s^4}, \frac{A \ln^2 s}{s^4}\right] \times \left[-\frac{A}{s^3}, \frac{A}{s^3}\right]$.
- In the fourth subsection, we solve the finite dimensional problem using the index theory and conclude the proof of Theorem 1 .

3.1.1 Properties of the shrinking set $\mathcal{V}_A(s)$ and preparation of initial data

In this subsection, we give some properties of the shrinking set defined in Definition 3.1.

In the following we give the proof of Local in time solution for problem (22)-(50). In fact, we impose some orthogonality condition given by (50), killing the one of the zero eigenfunction of the linearized operator of equation (22).

3.1.2 Reduction to a finite dimensional problem

In the following we give the proof of Proposition 3.8:

The idea of the proof is to project equation (22) on the different components of the decomposition (38). More precisely, we claim that Proposition 3.8 is a consequence of the following

Proposition 3.9 *There exists $A_5 \geq 1$ such that for all $A \geq A_5$, there exists $s_5(A)$ such that the following holds for all $s_0 \geq s_5$:
Assuming that for all $s \in [\tau, s_1]$ for some $s_1 \geq \tau \geq s_0$, $q(s) \in \mathcal{V}_A(s)$ and $q_0(s) = 0$, then the following holds for all $s \in [\tau, s_1]$:*

(i) *(Smallness of the modulation parameter):*

$$|\theta'(s)| \leq \frac{CA^{10} \ln^2 s}{s^3}.$$

(ii) (*ODE satisfied by the expanding mode*): For $m = 0$ and 1 , we have

$$\left| \tilde{Q}'_0(s) - Q_0(s) \right| \leq \frac{C \ln^2 s}{s^4},$$

and

$$\left| \tilde{q}'_1 - \frac{1}{2} \tilde{q}_1 \right| \leq \frac{C}{s^3}.$$

(iii) (*ODE satisfied by the null mode*):

$$\left| \tilde{Q}'_2(s) - \frac{2\tilde{Q}_2}{s} \right| \leq \frac{CA^8 \ln^2 s}{s^3}.$$

(iv) (*Control of negative modes*):

$$|q_1(s)| \leq e^{-\frac{(s-\tau)}{2}} |q_1(\tau)| + \frac{CA^3}{s^3},$$

$$|Q_2(s)| \leq e^{-(s-\tau)} |Q_2(\tau)| + \frac{CA^7 \ln^2 s}{s^4},$$

$$|q_3| \leq e^{-\frac{3}{2}(s-\tau)} |q_3(\tau)| + \frac{CA^2}{s^4},$$

$$|\tilde{q}_3| \leq e^{-\frac{s-\tau}{2}} |\tilde{q}_3(\tau)| + \frac{CA^2}{s^4},$$

$$|Q_4(s)| \leq e^{-2(s-\tau)} |Q_4(\tau)| + \frac{CA^6 \ln^2 s}{s^4},$$

$$\left| \tilde{Q}_4(s) \right| \leq e^{-(s-\tau)} \left| \tilde{Q}_4(\tau) \right| + \frac{CA^3 \ln^2 s}{s^4},$$

$$|q_j(s)| \leq e^{-j\frac{(s-\tau)}{2}} |q_j(\tau)| + \frac{CA^{j-1}}{s^{\frac{j+1}{2}}}, \text{ for all } 5 \leq j \leq M,$$

$$|\tilde{q}_j(s)| \leq e^{-(j-2)\frac{(s-\tau)}{2}} |\tilde{q}_j(\tau)| + \frac{CA^{j-1}}{s^{\frac{j+1}{2}}}, \text{ for all } 5 \leq j \leq M,$$

$$\left\| \frac{q_-(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty} \leq e^{-\frac{M+1}{4}(s-\tau)} \left\| \frac{q_-(\tau)}{1 + |y|^{M+1}} \right\|_{L^\infty} + C \frac{A^M}{s^{\frac{M+2}{2}}},$$

$$\|q_e(y, s)\|_{L^\infty} \leq e^{-\frac{(s-\tau)}{2(p-1)}} \|q_e(\tau)\|_{L^\infty} + \frac{CA^{M+1}}{\sqrt{\tau}} (1 + s - \tau),$$

where \tilde{Q}_0 , Q_2 , \tilde{Q}_2 , Q_4 and \tilde{Q}_4 are defined by (40-46).

Proof: Briefly speaking, the main idea of the proof of Proposition 3.9 is to project equations (18) and (22) according to the decomposition (38). Due to the length of the proof with a lot of technical computations, we will give the complete proof in Subsection 3.2. ■

The conclusion of Proposition 3.8

Let us now focus on the proof of Proposition 3.8 assuming Proposition 3.9 hold. Indeed, We will take $A_4 \geq A_5$. Hence, we can use the conclusion of Proposition 3.9.

(i) The proof follows from (i) of Proposition 3.9. Indeed by choosing T_4 small enough, we can make $s_0 = -\log T$ bigger than $s_5(A)$.

(ii) We notice that from Claim 3.2 and the fact that $q_0(s) = 0$, it is enough to prove that for all $s \in [s_0, s_1]$,

$$\left| \tilde{Q}_2(s) \right| = \left| \tilde{q}_2(s) - \left(\frac{\tilde{\mathcal{A}}_2 \ln s}{s^2} + \frac{\tilde{\mathcal{B}}_2}{s^2} \right) \right| < \frac{A^{10} \ln^2 s}{s^3}. \quad (52)$$

$$\begin{aligned} \|q_e\|_{L^\infty(\mathbb{R})} &\leq \frac{A^{M+2}}{2s^{\frac{1}{2}}}, & \left\| \frac{q_-(y,s)}{1+|y|^{M+1}} \right\|_{L^\infty} &\leq \frac{A^{M+1}}{2s^{\frac{M+2}{2}}}, \\ |q_j|, |\tilde{q}_j| &\leq \frac{A^j}{2s^{\frac{j+1}{2}}} \text{ for all } 5 \leq j \leq M, & |q_1| &\leq \frac{A^4}{2s^3}, \\ |Q_2| &\leq \frac{A^8 \ln^2 s}{2s^4}, & |q_3|, |\tilde{q}_3| &\leq \frac{A^3}{2s^4}, \\ |Q_4|, |\tilde{Q}_4| &\leq \frac{A^7 \ln s}{2s^4}. \end{aligned} \quad (53)$$

In fact, the estimates in (53) are quite the same as in Proposition 4.7 of [DNZ20]. For that reason, we only focus on the proof of (52): Indeed, we will use a contradictory argument, we assume that there exists $s_* \in [s_0, s_1]$ such that

$$\tilde{Q}_2(s_*) = \left(\tilde{q}_2(s_*) - \left(\frac{\tilde{\mathcal{A}}_2 \ln s_*}{s_*^2} + \frac{\tilde{\mathcal{B}}_2}{s_*^2} \right) \right) = \omega \frac{A^{10} \ln^2 s_*}{s_*^3},$$

for all and

$$s \in [s_0, s_*[$$

and

$$\left| \tilde{q}_2(s) - \left(\frac{\tilde{\mathcal{A}}_2 \ln s}{s^2} + \frac{\tilde{\mathcal{B}}_2}{s^2} \right) \right| < \frac{A^{10} \ln^2 s}{s^3},$$

where $\omega = \pm 1$. As a matter of fact, we can reduce to the case positive case where $\omega = 1$ (the case $\omega = -1$ also work by the same way). Note by item (iv) in Proposition 3.6 that

$$\left| \tilde{q}_2(s_0) - \left(\frac{\tilde{\mathcal{A}}_2 \ln s_0}{s_0^2} + \frac{\tilde{\mathcal{B}}_2}{s_0^2} \right) \right| < \frac{A^{10} \ln^2 s_0}{s_0^3},$$

thus $s_* > s_0$, and the interval $[s_0, s_*]$ is not empty.

Using the continuity of \tilde{Q}_2 and the definition of s_* , it is clearly that $\tilde{Q}_2(s_*)$ is the maximal value of \tilde{Q}_2 in $[s_* - \epsilon, s_*]$ with $\epsilon > 0$ and small enough in one hand, recalling, from (iii) Proposition 3.9 that

$$\left| \tilde{Q}'_2 + 2 \frac{\tilde{Q}_2}{s} \right| \leq \frac{CA^8 \ln^2 s}{s^4},$$

hence it follows

$$\tilde{Q}'_2(s_*) \leq -\frac{\tilde{Q}_2}{s} + \frac{CA^8 \ln^2 s}{s^4} \leq \frac{(-2A^{10} + CA^8) \ln^2 s}{s^4} < 0, \quad (54)$$

provided that A large enough. Then, \tilde{Q}_2 has to decrease in $[s_* - \epsilon_1, s_*]$ which implies a contradiction with the assumption that \tilde{Q}_2 admits maximum at s_* . In other word, (52) holds. Finally, it concludes the proof. ■

3.2 Proof of Proposition 3.9

In this section, we focus on the proof of Proposition 3.9. The idea mainly bases on the technique in [MZ08], [NZ18] and [DNZ20]. In fact, it involves to the projection equations (18) and (22) to get equations satisfied by the different coordinates of the decomposition (38). Let us summary the proof

- In the first subsection, we deal with equation (22) to write equations satisfied by \tilde{q}_j and q_j . Then, we prove (i), (ii), (iii) and (iv) (expect the two last identities) of Proposition 3.9.
- In the second subsection, we first derive from equation (22) an equation satisfied by q_- and prove the last but one identity in (iv) of Proposition 3.9.
- In the third subsection, we project equation (18) (which is simpler than (22)) to write an equation satisfied by q_e and prove the last identity in (iv) of Proposition 3.9.

3.2.1 The finite dimensional part q_+

We now divide the proof into two steps

- In Part 1, we give the details of projections of equation (22) to get ODEs, satisfied by modes \tilde{q}_j and q_j .
- In Part 2, we prove (i), (ii) and (iii) of Proposition 3.9, together with the estimates concerning \tilde{q}_j and q_j in (iv).

Part 1: The projection of equation (22) on the eigenfunction of the operator $\mathcal{L}_{\beta,\delta}$. In the following, we will find the main contribution in the projections $\tilde{P}_{n,M}$ and $P_{n,M}$ of the six terms appearing in equation (22): $\partial_s q$, $\mathcal{L}_{\beta,\delta} q$, $-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)q$, $V_1 q + V_2 \bar{q}$, $B(q, y, s)$ and $R^*(\theta', y, s)$.

First term: $\frac{\partial q}{\partial s}$. From (37), we directly derive

$$\tilde{P}_{n,M} \left(\frac{\partial q}{\partial s} \right) = \tilde{q}'_n \text{ and } P_{n,M} \left(\frac{\partial q}{\partial s} \right) = q'_n. \quad (55)$$

Second term: $\mathcal{L}_{\beta,\delta} q$, where $\mathcal{L}_{\beta,\delta}$ is defined as in (23). We will use the following Lemma from [MZ08]:

Lemma 3.10 (Projection of $\mathcal{L}_{\beta,\delta}$ on \tilde{h}_n and h_n for $n \leq M$)

a) If $n \leq M - 2$, then

$$\left| P_{n,M}(\mathcal{L}_{\beta,\delta} q) - \left(-\frac{n}{2} q_n(s) + c_{n+2} \tilde{q}_{n+2} \right) \right| \leq C \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty},$$

where c_n is given in Lemma 2.1. Moreover, we have the following

If $M - 1 \leq n \leq M$, then

$$\left| P_{n,M}(\mathcal{L}_{\beta,\delta} q) + \frac{n}{2} q_n(s) \right| \leq C \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty}.$$

(b) If $n \leq M$, then the projection of $\mathcal{L}_{\beta,\delta}$ on \tilde{h}_n satisfies

$$\left| \tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta} q) - \left(1 - \frac{n}{2} \right) \tilde{q}_n(s) \right| \leq C \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty}.$$

Proof: The proof is quiet the same as the proof of Lemma 5.1 in [MZ08].

Using Lemma 3.10 and the fact that $q(s) \in \mathcal{V}_A(s)$ (see Definition 3.1) we can improve the error by the following:

Corollary 3.1 For all $A \geq 1$, there exists $s_9 \geq 1$ such that for all $s \geq s_9(A)$, if $q(s) \in \mathcal{V}_A(s)$, then:

a) For $n = 0$, we have

$$|P_{0,M}(\mathcal{L}_{\beta,\delta}q) - c_2\tilde{q}_2| \leq C \frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

b) For $1 \leq n \leq M-1$, we have

$$\left| P_{n,M}(\mathcal{L}_{\beta,\delta}q) + \frac{n}{2}q_n(s) \right| \leq C \frac{A^{n+2}}{s^{\frac{n+3}{2}}}.$$

In particular, we have a smaller bound for $P_{2,M}(\mathcal{L}_{\beta,\delta}q)$:

$$|P_{2,M}(\mathcal{L}_{\beta,\delta}q) + q_2 - c_4\tilde{q}_4| \leq \frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

c) For $n = M$, we have

$$\left| P_{M,M}(\mathcal{L}_{\beta,\delta}q) + \frac{M}{2}q_M(s) \right| \leq C \frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

d) For $0 \leq n \leq M$, we have

$$\left| \tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta}q) - \left(1 - \frac{n}{2}\right)\tilde{q}_n(s) \right| \leq C \frac{A^{M+1}}{s^{\frac{M+1}{2}}}.$$

Third term: $-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q$. It is enough to project iq , from (37), we recall Lemma 5.3 from [MZ08]:

Lemma 3.11 (Projection of the term $-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q$ on h_n and \tilde{h}_n for $n \leq M$)
We have the following identities

$$\begin{aligned} & P_{n,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \\ &= - \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \left(\delta q_n + (1 + \delta^2)\tilde{q}_n + \sum_{j=n+1}^M K_{n,j}q_j + L_{n,j}\tilde{q}_j \right), \end{aligned}$$

where $K_{n,j}$ and $L_{n,j}$ defined by

$$K_{n,j} = P_{n,M}(ih_j), \tag{56}$$

$$L_{n,j} = P_{n,M}(i\tilde{h}_j). \tag{57}$$

Its projection on \tilde{h}_n is given by

$$\begin{aligned} & \tilde{P}_{n,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \\ &= - \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \left(-q_n - \delta \tilde{q}_n + \sum_{j=n+1}^M \tilde{K}_{n,j} q_j + \tilde{L}_{n,j} \tilde{q}_j \right), \end{aligned}$$

where $\tilde{K}_{n,j}$ and $\tilde{L}_{n,j}$ defined as follows

$$\tilde{K}_{n,j} = \tilde{P}_{n,M}(ih_j), \quad (58)$$

$$\tilde{L}_{n,j} = \tilde{P}_{n,M}(i\tilde{h}_j). \quad (59)$$

Using the fact that $q(s) \in \mathcal{V}_A(s)$ defined 3.1, the error estimates can be improved

Corollary 3.2 *For all $A \geq 1$, there exists $s_{10}(A) \geq 1$ such that for all $s \geq s_{10}(A)$, if $q \in \mathcal{V}_A(s)$ and $|\theta'(s)| \leq \frac{CA^{10}}{s^{\frac{5}{2}}}$, then:*

a) *For all $1 \leq n \leq M$, we have*

$$\left| P_{n,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| \leq C \frac{A^n}{s^{\frac{n+3}{2}}}.$$

b) *For $1 \leq n \leq M$, we have*

$$\left| \tilde{P}_{n,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| \leq C \frac{A^n}{s^{\frac{n+3}{2}}}.$$

In particular, when $n = 0, 2, 4$, we can get smaller bounds as follows:

c) *For $n = 0$, we have the following in particular*

$$\begin{aligned} & \left| P_{0,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right. \\ & \quad \left. + \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \{ \delta q_0 + (1 + \delta^2) \tilde{q}_0 + K_{0,2} q_2 + L_{0,2} \tilde{q}_2 \} \right| \\ & \leq C \frac{A^4 \ln s}{s^4}, \\ & \left| \tilde{P}_{0,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right. \\ & \quad \left. + \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \{ -q_0 - \delta \tilde{q}_0 + \tilde{K}_{0,2} q_2 + \tilde{L}_{0,2} \tilde{q}_2 \} \right| \\ & \leq C \frac{A^4 \ln s}{s^4}, \end{aligned}$$

d) For $n = 2$, we have

$$\begin{aligned}
& \left| P_{2,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right. \\
& \quad \left. + \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) [\delta q_2 + (1 + \delta^2) \tilde{q}_2] \right| \\
& \leq C \frac{A^4 \ln s}{s^4}, \\
& \left| \tilde{P}_{2,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right. \\
& \quad \left. + \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \left(-q_2 - \delta \tilde{q}_2 + \tilde{K}_{2,4} q_4 + \tilde{L}_{2,4} \tilde{q}_4 \right) \right| \\
& \leq C \frac{A^5}{s^4},
\end{aligned}$$

e) For $n = 3$, we have

$$\begin{aligned}
\left| P_{3,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| & \leq C \frac{A^2}{s^4}, \\
\left| \tilde{P}_{3,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| & \leq C \frac{A^2}{s^4},
\end{aligned}$$

f) For $n = 4$, we have

$$\begin{aligned}
\left| P_{4,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| & \leq C \frac{A^5}{s^4}, \\
\left| \tilde{P}_{4,M} \left(-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) \right| & \leq C \frac{A^5}{s^4},
\end{aligned}$$

Fourth term: $V_1 q + V_2 \tilde{q}$.

We recall Lemma 5.5 given in [MZ08]

Lemma 3.12 (Projection of $V_1 q$ and $V_2 \tilde{q}$) (i) It holds that

$$|V_i(y, s)| \leq C \frac{(1 + |y|^2)}{s}, \text{ for all } y \in \mathbb{R} \text{ and } s \geq 1, \quad (60)$$

and for all $k \in \mathbb{N}^*$

$$V_i(y, s) = \sum_{j=1}^k \frac{1}{s^j} W_{i,j}(y) + \tilde{W}_{i,k}(y, s), \quad (61)$$

where $W_{i,j}$ is an even polynomial of degree $2j$ and $\tilde{W}_{i,k}(y, s)$ satisfies

$$\text{for all } s \geq 1 \text{ and } |y| \leq \sqrt{s}, \quad \left| \tilde{W}_{i,k}(y, s) \right| \leq C \frac{(1 + |y|^{2k+2})}{s^{k+1}}. \quad (62)$$

(ii) The projection of V_1q and $V_2\bar{q}$ on $(1 + i\delta)h_n$ and ih_n , and we have

$$\begin{aligned} & |\tilde{P}_n(V_1q)| + |\hat{P}_n(V_1q)| \\ & \leq \frac{C}{s} \sum_{j=n-2}^M (|\tilde{q}_j| + |\hat{q}_j|) + \sum_{j=0}^{n-3} \frac{C}{s^{\frac{n-j}{2}}} (|\tilde{q}_j| + |\hat{q}_j|) + \frac{C}{s} \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty}, \end{aligned} \quad (63)$$

and the same holds for $V_2\bar{q}$

Remark 3.13 Note that, when $n \leq 2$, the first sum in (63) runs for $j = 0$ to M and the second sum doesn't exist.

By the fact that $q(s) \in \mathcal{V}_A(s)$, the error estimates can be bounded improved as follows

Corollary 3.3 For all $A \geq 1$, there exists $s_{11}(A) \geq 1$ such that for all $s \geq s_{11}(A)$, if $q \in \mathcal{V}_A(s)$, then for $3 \leq n \leq M$, we have

$$\left| \tilde{P}_n(V_1q + V_2\bar{q}) \right| + |P_n(V_1q + V_2\bar{q})| \leq \frac{CA^{n-2}}{s^{\frac{n+1}{2}}}.$$

Next, we study some asymptotic of $\tilde{P}_{2,M}(V_1q)$, $\tilde{P}_{2,M}(V_2\bar{q})$, $P_{0,M}(V_1q)$ and $P_{0,M}(V_2\bar{q})$:

Lemma 3.14 (Projection of V_1q and $V_2\bar{q}$ on $\hat{h}_0, \tilde{h}_0, \hat{h}_2, \tilde{h}_2, \hat{h}_4$ and h_4)

Using the definition of V_1, V_2 , the following hold:

(i) It holds that for $i = 1, 2$

$$\forall s \geq 1 \text{ and } |y| \leq s^{1/2}, \quad \left| V_i(y, s) - \frac{1}{s} W_{i,1}(y) \right| \leq \frac{C}{s^2} (1 + |y|^4), \quad (64)$$

where

$$\begin{aligned} W_{1,1}(y) &= -\frac{(p+1)b}{2(p-1)^2} (1 + i\delta)(y^2 - 2(1 - \delta\beta)), \\ W_{2,1}(y) &= -(1 + i\delta) \frac{b}{2(p-1)^2} (p - 1 + 2i\delta)(y^2 - 2(1 - \beta\delta)). \end{aligned} \quad (65)$$

(ii) The projection of V_1q and $V_2\bar{q}$ on \tilde{h}_2 satisfy

$$\left| \tilde{P}_n(V_1q + V_2\bar{q}) - \frac{1}{s} \sum_{j \geq 0} [\tilde{C}_{n,j}q_j + \tilde{D}_{n,j}\tilde{q}_j] \right| \leq \frac{C}{s^2} \sum_{j \geq 0} [|\hat{q}_j| + |\tilde{q}_j|] + \frac{1}{s} \left\| \frac{q_-(\cdot, s)}{1 + |y|^M} \right\|_{L^\infty} \quad (66)$$

and

$$\left| P_n(V_1q + V_2\bar{q}) - \frac{1}{s} \sum_{j \geq 0} [C_{n,j}q_j + D_{2,j}\tilde{q}_j] \right| \leq \frac{C}{s^2} \sum_{j \geq 0} [|\hat{q}_j| + |\tilde{q}_j|] + \frac{1}{s} \left\| \frac{q_-(\cdot, s)}{1 + |y|^M} \right\|_{L^\infty} \quad (67)$$

where for all $n, j \geq 0$, we have

$$C_{n,j} = P_{n,M}(W_{1,1}h_j + W_{2,1}\bar{h}_j) \quad \tilde{C}_{n,j} = \tilde{P}_{n,M}(W_{1,1}h_j + W_{2,1}\bar{h}_j), \quad (68)$$

$$D_{n,j} = P_{n,M}(W_{1,1}\tilde{h}_j + W_{2,1}\tilde{\bar{h}}_j) \quad \tilde{D}_{n,j} = \tilde{P}_{n,M}(W_{1,1}\tilde{h}_j + W_{2,1}\tilde{\bar{h}}_j). \quad (69)$$

In particular, using the fact that $q(s) \in \mathcal{V}_A(s)$, the error estimates can be improved as follows:

Corollary 3.4 *For all $A \geq 1$, there exists $s_{12}(A) \geq 1$ such that for all $s \geq s_{12}(A)$, if $q(s) \in \mathcal{V}_A(s)$, then*

$$\begin{aligned} & \left| P_{0,M}(V_1q + V_2\bar{q}) - \left(C_{0,0} \frac{q_0}{s} + D_{0,0} \frac{\tilde{q}_0}{s} + C_{0,2} \frac{q_2}{s} + D_{0,2} \frac{\tilde{q}_2}{s} \right) \right| \leq C \frac{\ln s}{s^4}, \\ & \left| \tilde{P}_{0,M}(V_1q + V_2\bar{q}) - \left(\tilde{D}_{0,0} \frac{\tilde{q}_0}{s} + \tilde{C}_{0,2} \frac{q_2}{s} + \tilde{D}_{0,2} \frac{\tilde{q}_2}{s} \right) \right| \leq C \frac{\ln s}{s^4}, \\ & \left| P_{2,M}(V_1q + V_2\bar{q}) - \left(\frac{D_{2,0}\tilde{q}_0}{s} + \frac{C_{2,2}q_2}{s} + \frac{D_{2,2}\tilde{q}_2}{s} \right) \right| \leq C \frac{\ln s}{s^4}, \\ & \left| \tilde{P}_{2,M}(V_1q + V_2\bar{q}) - \frac{1}{s} \left\{ \tilde{q}_0 \tilde{D}_{2,0} + q_2 \tilde{C}_{2,2} + \tilde{q}_2 \tilde{D}_{2,2} \right\} \right| \leq C \frac{\ln s}{s^4}, \\ & \left| P_{4,M}(V_1q + V_2\bar{q}) - \left(C_{4,2} \frac{q_2}{s} + D_{4,2} \frac{\tilde{q}_2}{s} \right) \right| \leq C \frac{\ln s}{s^4}, \\ & \left| \tilde{P}_{4,M}(V_1q + V_2\bar{q}) - \left(\tilde{C}_{4,2} \frac{q_2}{s} + \tilde{D}_{4,2} \frac{\tilde{q}_2}{s} \right) \right| \leq C \frac{\ln s}{s^4}. \end{aligned}$$

and

$$\begin{aligned} |P_{3,M}(V_1q + V_2\bar{q})| & \leq \frac{CA^2}{s^4}, \\ |\tilde{P}_{3,M}(V_1q + V_2\bar{q})| & \leq \frac{CA^2}{s^4}. \end{aligned}$$

Fifth term: $B(q, y, s)$ Let us recall from (23) that:

$$B(q, y, s) = (1+i\delta) \left(|\varphi + q|^{p-1}(\varphi + q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q} + \bar{\varphi}q) \right).$$

We have the following

Lemma 3.15 *The function $B = B(q, y, s)$ can be decomposed for all $s \geq 1$ and $|q| \leq 1$ as*

$$\sup_{|y| \leq s^{1/2}} \left| B - \sum_{l=0}^M \sum_{\substack{0 \leq j, k \leq M+1 \\ 2 \leq j+k \leq M+1}} \frac{1}{s^l} \left[B_{j,k}^l\left(\frac{y}{s^{1/2}}\right) q^j \bar{q}^k + \tilde{B}_{j,k}^l(y, s) q^j \bar{q}^k \right] \right| \leq C|q|^{M+2} + \frac{C}{s^{\frac{M+1}{2}}},$$

where $B_{j,k}^l\left(\frac{y}{s^{1/2}}\right)$ is an even polynomial of degree less or equal to M and the rest $\tilde{B}_{j,k}^l(y, s)$ satisfies

$$\forall s \geq 1 \text{ and } |y| < s^{1/2}, \left| \tilde{B}_{j,k}^l(y, s) \right| \leq C \frac{1 + |y|^{M+1}}{s^{\frac{M+1}{2}}}.$$

Moreover,

$$\forall s \geq 1 \text{ and } |y| < s^{1/2}, \left| B_{j,k}^l\left(\frac{y}{s^{1/2}}\right) + \tilde{B}_{j,k}^l(y, s) \right| \leq C.$$

On the other hand, in the region $|y| \geq s^{1/2}$, we have

$$|B(q, y, s)| \leq C|q|^{\bar{p}}, \quad (70)$$

for some constant C where $\bar{p} = \min(p, 2)$.

Proof: See the proof of Lemma 5.9, page 1646 in [MZ08]. ■

Lemma 3.16 (The quadratic term $B(q, y, s)$) *For all $A \geq 1$, there exists $s_{13} \geq 1$ such that for all $s \geq s_{13}$, if $q(s) \in \mathcal{V}_A(s)$, then:*

a) *the projection of $B(q, y, s)$ on h_n and on \tilde{h}_n , for $n \geq 3$ satisfy*

$$\left| \tilde{P}_{n,M}(B(q, y, s)) \right| + |P_{n,M}(B(q, y, s))| \leq C \frac{A^n}{s^{\frac{n+2}{2}}}. \quad (71)$$

b) *For $n = 0, 1, 2, 3, 4$, we have*

$$\left| \tilde{P}_{n,M}(B(q, y, s)) \right| + |P_{n,M}(B(q, y, s))| \leq \frac{C \ln^2 s}{s^4}. \quad (72)$$

Proof : See Lemma 5.10 given in [MZ08] . ■

Sixth term: $R^*(\theta', y, s)$

In the following, we expand R^* as a power series of $\frac{1}{s}$ as $s \rightarrow \infty$, uniformly for $|y| \leq s^{1/2}$.

Lemma 3.17 (Power series of R^* as $s \rightarrow \infty$) For all $n \in \mathbb{N}$,

$$R^*(\theta', y, s) = \Pi_n(\theta', y, s) + \tilde{\Pi}_n(\theta', y, s), \quad (73)$$

where,

$$\Pi_n(\theta', y, s) = \sum_{k=0}^{n-1} \frac{1}{s^{k+1}} P_k(y) - i \left(-\eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \left(\frac{a}{s} (1 + i\delta) + \sum_{k=0}^{n-1} e_k \frac{y^{2k}}{s^k} \right), \quad (74)$$

and

$$\forall |y| < s^{1/2}, \quad \left| \tilde{\Pi}_n(\theta', y, s) \right| \leq C(1 + s|\theta'(s)|) \frac{(1 + |y|^{2n})}{s^{n+1}}, \quad (75)$$

where P_k is a polynomial of order $2k$ for all $k \geq 1$ and $e_k \in \mathbb{R}$.

In particular,

$$\begin{aligned} & \sup_{|y| \leq s^{1/2}} \left| R^*(\theta', y, s) - \sum_{k=0}^1 \frac{1}{s^{k+1}} P_k(y) + i \left(-\eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta' \right) \left[\kappa + \frac{(1 + i\delta)}{s} \left(a - \frac{b\kappa y^2}{(p-1)^2} \right) \right] \right| \\ & \leq C \left(\frac{1 + |y|^4}{s^3} + C \left(\frac{\ln s}{s^2} + |\theta'| \right) \frac{|y|^4}{s^2} \right). \end{aligned} \quad (76)$$

Proof: The proof is quite the same Lemma 5.11 in [MZ08] ■

In the following, we introduce $F_j(R^*)(\theta, s)$ as the projection of the rest term $R^*(\theta', y, s)$ on the standard Hermite polynomial, introduced in Lemma 2.1.

Lemma 3.18 (Projection of R^* on the eigenfunction of \mathcal{L}) It holds that $F_j(R^*)(\theta', s) \equiv 0$ when j is odd, and $|F_j(R^*)(\theta', s)| \leq C \frac{1 + s|\theta'(s)|}{s^{\frac{j}{2}+1}}$, when j is even and $j \geq 4$.

Proof: See Lemma 5.12 in [MZ08]. ■

More precisely, we can describe the projection of R^* as follows:

Lemma 3.19 (Projection of R^* on the eigenfunction \tilde{h} and h_n) Let us consider R^* defined as in the above, then the following hold:

(i) For $j \geq 4$ which is even, then $\tilde{P}_j(R^*)(\theta', s)$ and $P_j(R^*)(\theta', s)$ are $O\left(\frac{1+s|\theta'|}{s^{\frac{j}{2}+1}}\right)$.

(ii) For all j odd, we have $\tilde{P}_j(R^*)(\theta', s) = P_j(R^*)(\theta', s) = 0$.

(iii) For $j = 0$, we have

$$\begin{aligned} P_{0,M}(R^*(\theta'(s), s)) &= \frac{R_{0,0}^*}{s} + \frac{R_{0,1}^*}{s^2} + \frac{R_{0,2}^*}{s^3} + \theta'(s) \left(-\kappa + \frac{\Theta_{0,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) \\ &+ \frac{\ln s}{s^2} \left(\eta\kappa + \frac{T_{0,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) + \frac{1}{s^2} \left(-\eta\kappa + \frac{T_{0,0}^{**}}{s} + O\left(\frac{1}{s^2}\right) \right) \\ &+ O\left(\frac{1}{s^4}\right), \\ \tilde{P}_{0,M}(R^*(\theta'(s), s)) &= \frac{\tilde{R}_{0,0}^*}{s} + \frac{\tilde{R}_{0,1}^*}{s^2} + \frac{\tilde{R}_{0,2}^*}{s^3} + \theta'(s) \left(\frac{\tilde{\Theta}_{0,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) \\ &+ \frac{\ln s}{s^2} \left(\frac{\tilde{T}_{0,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) + \frac{1}{s^2} \left(\frac{\tilde{T}_{0,0}^{**}}{s} + O\left(\frac{1}{s^2}\right) \right) + O\left(\frac{1}{s^4}\right). \end{aligned}$$

(iv) For $j = 2$, we have

$$\begin{aligned} P_{2,M}(R^*(\theta'(s), s)) &= \frac{R_{2,1}^*}{s} + \frac{R_{2,2}^*}{s^3} + \theta'(s) \left(\frac{\Theta_{2,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) \\ &+ \frac{\ln s}{s^2} \left(\frac{T_{2,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) + \frac{1}{s^2} \left(\frac{T_{2,0}^{**}}{s} + O\left(\frac{1}{s^2}\right) \right) + O\left(\frac{1}{s^4}\right), \\ \tilde{P}_{2,M}(R^*(\theta'(s), s)) &= \frac{\tilde{R}_{2,1}^*}{s^2} + \frac{\tilde{R}_{2,1}^*}{s^3} + \theta'(s) \left(\frac{\tilde{\Theta}_{2,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) \\ &+ \frac{\ln s}{s^2} \left(\frac{\tilde{T}_{2,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right) + \frac{1}{s^2} \left(\frac{\tilde{T}_{2,0}^{**}}{s} + O\left(\frac{1}{s^2}\right) \right) + O\left(\frac{1}{s^4}\right). \end{aligned}$$

where $R_{j,k}^*$, $\tilde{R}_{j,k}^*$, $\Theta_{j,k}^*$, $\tilde{\Theta}_{j,k}^*$ are constants, depending on p, δ, β only. For more details see page 49 and equation (98).

(v) In particular, we choose

$$\begin{cases} a &= \frac{2(1-\delta\beta)b}{(p-1)^2}, \\ \mu &= -\frac{2\beta b}{(p-1)^2}(1+\delta^2), \\ b &= \frac{(p-1)^2}{4(p-\delta^2-(p+1)\delta\beta)}. \end{cases}$$

Then, we have

$$R_{0,0}^* = \tilde{R}_{0,0}^* = \tilde{R}_{2,1}^* = 0. \quad (77)$$

Proof: For the details, we kindly refer the readers to Appendix C. \square

Part 2: Proof of Proposition 3.9

In this part, we consider $A \geq 1$ and take s large enough so that Part 1 is satisfied.

+ The proof of item (i): We control $\theta'(s)$, from the projection of (22) on $h_0(y) = i$, we obtain

$$\begin{aligned} q'_0 &= c_2 \tilde{q}_2 + P_{0,M} \left(-i \left(\frac{\mu}{s} - \frac{\eta \ln s}{s^2} + \frac{\eta}{s^2} + \theta' \right) q \right) \\ &\quad + P_{0,M}(V_1 q + V_2 \bar{q}) + P_{0,M}(B) + P_{0,M}(R^*(\theta'(s), s)), \end{aligned} \quad (78)$$

where $c_2 = 2\beta(1 + \delta^2)$, defined as in Lemma 2.1. In addition to that, from the fact that $q_0 \equiv 0$ by the modulation, we also obtain that

$$q'_0 \equiv 0.$$

Using the fact that $q \in \mathcal{V}_A(s)$ given Definition 3.1; Corollaries 3.1 and 3.4; Lemmas 3.10, 3.16 3.19, we obtain the following:

$$\begin{aligned} P_{0,M}(\mathcal{L}_{\delta,\beta} q) &= c_2 \tilde{q}_2 + O\left(\frac{1}{s^{\frac{M+2}{2}}}\right) \\ P_{0,M} \left(-i \left(\frac{\mu}{s} - \frac{\eta \log s}{s^2} + \frac{\eta}{s^2} + \theta' \right) q \right) \\ &= - \left(\frac{\mu}{s} - \frac{\eta \log s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \{ \delta q_0 + (1 + \delta^2) \tilde{q}_0 + K_{0,2} q_2 + L_{0,2} \tilde{q}_2 \} + O\left(\frac{\ln s}{s^4}\right), \end{aligned}$$

and

$$\begin{aligned} P_{0,M}(V_1 q + V_2 \bar{q}) &= C_{0,0} \frac{q_0}{s} + D_{0,0} \frac{\tilde{q}_0}{s} + C_{0,2} \frac{q_2}{s} + D_{0,2} \frac{\tilde{q}_2}{s} + O\left(\frac{\ln s}{s^4}\right) \\ P_{0,M}(B(q)) &= O\left(\frac{\ln^2 s}{s^4}\right) \\ P_{0,M}(R^*) &= \left\{ -\kappa + \frac{\Theta_{0,0}^*}{s} + O\left(\frac{1}{s^2}\right) \right\} \theta'(s) + \frac{(R_{0,1}^* - \eta\kappa)}{s^2} + \frac{R_{0,2}^*}{s^3} + \frac{\eta\kappa \ln s}{s^2} + T_{0,1}^* \frac{\ln s}{s^3} \\ &\quad + O\left(\frac{\ln s}{s^4}\right), \end{aligned}$$

$$\begin{aligned}
& \left| -\kappa\theta'(s) + c_2\tilde{q}_2 - \frac{\mu}{s}\{(1+\delta^2)\tilde{q}_0 + K_{0,2}q_2 + L_{0,2}\tilde{q}_2\} + D_{0,0}\frac{\tilde{q}_0}{s} \right. \\
& + C_{0,2}\frac{q_2}{s} + D_{0,2}\frac{\tilde{q}_2}{s} + (\eta\kappa)\frac{\ln s}{s^2} + \frac{(R_{0,1}^* - \eta\kappa)}{s^2} + \frac{(R_{0,2}^* - T_{0,0}^{**})}{s^3} + T_{0,0}^*\frac{\ln s}{s^3} \\
& \left. + \frac{c_2\Theta_{0,0}^*\tilde{q}_2}{\kappa s} + \frac{\Theta_{0,0}^*(R_{0,1}^* - \eta\kappa)}{\kappa s^3} + \frac{\Theta_{0,0}^*(\eta\kappa)\ln s}{\kappa s^3} \right| \leq C\frac{\ln^2 s}{s^4}. \quad (79)
\end{aligned}$$

In particular, we use again the fact that $q \in \mathcal{V}_A$ that

$$\left| c_2\tilde{q}_2(s) + \frac{(R_{0,1}^* - \eta\kappa)}{s^2} + \frac{\eta\kappa \ln s}{s^2} \right| \leq \frac{A^{10} \ln^2 s}{s^3}.$$

which can be written

$$\left| \tilde{q}_2(s) - \frac{\tilde{\mathcal{A}}_2 \ln s}{s^2} - \frac{\tilde{\mathcal{B}}_2}{s^2} \right| \leq \frac{A^{10} \ln^2 s}{s^3},$$

where

$$\tilde{\mathcal{A}}_2 = -\frac{\eta\kappa}{c_2} \text{ and } \tilde{\mathcal{B}}_2 = -\frac{(R_{0,1}^* - \eta\kappa)}{c_2}.$$

Thus, we obtain

$$|\theta'(s)| \leq \frac{CA^{10} \ln^2 s}{s^3}, \quad (80)$$

and

$$\left| -\kappa\theta'(s) + c_2\tilde{q}_2(s) + \frac{(R_{0,1}^* - \eta\kappa)}{s^2} + \frac{\eta\kappa \ln s}{s^2} \right| \leq \frac{C \ln s}{s^3}, \quad (81)$$

which concludes item (i) of Proposition 3.9.

+ The proof of item (iii): Let us project equation (22) on \tilde{h}_2 , we get

$$\begin{aligned}
\tilde{q}'_2 &= \tilde{P}_{2,M}(\mathcal{L}_{\beta,\delta}q) + \tilde{P}_{2,M}\left(-i\left(\frac{\mu}{s} - \frac{\eta \ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s)\right)q\right) + \tilde{P}_{2,M}(V_1q + V_2\bar{q}) \\
&+ \tilde{P}_{2,M}(B(q)) + \tilde{P}_{2,M}(R^*(\theta'(s), s)). \quad (82)
\end{aligned}$$

Using the fact that $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, s_1]$, we repeat the same process as for \tilde{q}_0 by Corollaries 3.1 and 3.4; Lemmas 3.10, 3.16 3.19, we can obtain some bounds for the terms in the right hand side of (82):

$$\tilde{P}_{2,M}(\partial_s q) = \partial_s \tilde{q}_2 \quad (83)$$

$$\left| \tilde{P}_{2,M}(\mathcal{L}_{\beta,\delta} q) \right| \leq \frac{A^{M+1}}{s^{\frac{M+2}{2}}}, \quad (84)$$

In particular, we also have the following expansion:

Terms coming from $\tilde{P}_{2,M} \left(-i \left(\frac{\mu}{s} - \frac{\eta \ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right)$ We have

$$\tilde{P}_{2,M} \left(-i \left(\frac{\mu}{s} - \frac{\eta \ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right) = -\frac{\mu}{s} (-q_2 - \delta \tilde{q}_2) + O \left(\frac{\ln s}{s^4} \right).$$

Terms coming from $\tilde{P}_{2,M}(V_1 q + V_2 \bar{q})$

$$\begin{aligned} & \tilde{P}_{2,M}(V_1 q + V_2 \bar{q}) \\ &= \frac{1}{s} \left\{ \tilde{q}_0 \tilde{D}_{2,0} + q_2 \tilde{C}_{2,2} + \tilde{q}_2 \tilde{D}_{2,2} \right\} + O \left(\frac{\ln s}{s^4} \right). \end{aligned}$$

This yields the following,

$$\begin{aligned} & \tilde{P}_2(V_1 q + V_2 \bar{q}) \\ &= \frac{\tilde{D}_{2,2}}{s} \tilde{q}_2 + \frac{1}{s^3} \left\{ \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* \right\} + O \left(\frac{\ln s}{s^4} \right). \end{aligned}$$

Terms coming from $\tilde{P}_{2,M}(B(q))$:

$$|\tilde{P}_{2,M}(B)| \leq \frac{C \ln^2 s}{s^4}.$$

Terms coming from $\tilde{P}_{2,M}(R^*)$

$$\begin{aligned} \tilde{P}_{2,M}(R^*) &= \tilde{T}_{2,0}^* \frac{\ln s}{s^3} + \frac{(\tilde{R}_{2,2}^* + \tilde{T}_{2,0}^{**})}{s^3} + \frac{\theta'(s)\kappa}{s} \frac{-\delta b}{(p-1)^2} + O \left(\frac{\ln s}{s^4} \right) \\ &= \tilde{T}_{2,0}^* \frac{\ln s}{s^3} + \frac{(\tilde{R}_{2,2}^* + \tilde{T}_{2,0}^{**})}{s^3} + \left(c_2 \tilde{q}_2(s) + \frac{(R_{0,1}^* - \eta\kappa)}{s^2} + \frac{\eta\kappa \ln s}{s^2} \right) \frac{-\delta b}{s(p-1)^2} + O \left(\frac{\ln s}{s^4} \right) \\ &= -\frac{c_2 \delta b}{(p-1)^2 s} \tilde{q}_2 - \frac{\delta b}{(p-1)^2 s^3} R_{0,1}^* + \frac{\tilde{R}_{2,2}^*}{s^3} + O \left(\frac{\ln s}{s^4} \right). \end{aligned}$$

Note that we combined the facts given (80) and (81); and μ, b and a given as in (77). Finally, by adding these estimates, we obtain the following

$$\begin{aligned} \tilde{q}'_2 &= \frac{\tilde{q}_2}{s} \left\{ \delta\mu + \tilde{D}_{2,2} - \frac{c_2 \delta b}{(p-1)^2} \right\} \\ &+ \frac{1}{s^3} \left\{ \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^* \right\} \end{aligned} \quad (85)$$

$$+ O\left(\frac{\ln^2 s}{s^4}\right). \quad (86)$$

Let us remark that even though there exists the order of $\frac{\ln s}{s^3}$ in the ODE of \tilde{q}_2 , it will be cancelled when we add all terms in the ODE. From the explicit formulas of μ, b, c_2 and $\tilde{D}_{2,2}$, we can compute

$$\delta\mu + \tilde{D}_{2,2} - \frac{c_2 \delta b}{(p-1)^2} = -2.$$

In addition to that, using the definition of \tilde{Q}_2 given as in (46), we establish the following

$$\begin{aligned} \tilde{Q}'_2 &= -2 \frac{\tilde{Q}_2}{s} + \frac{1}{s^3} \left\{ -\tilde{\mathcal{A}}_2 + \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^* \right\} \\ &+ O\left(\frac{\ln^2 s}{s^4}\right). \end{aligned}$$

In fact, we now prove that there exists η such that the order $\frac{1}{s^3}$ is canceled. Indeed, we choose η such that

$$-\tilde{\mathcal{A}}_2 + \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^* = 0.$$

Using the fact that

$$\tilde{\mathcal{A}}_2 = -\frac{\eta \kappa}{c_2},$$

we derive

$$\eta = -\frac{c_2}{\kappa} \left\{ \left(-\frac{\delta b}{(p-1)^2} \right) R_{0,1}^* + (\mu + \tilde{C}_{2,2}) R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} \right\}. \quad (87)$$

In addition to that, the explicit formula of η will be given in the Appendix.

$$\tilde{Q}'_2 = -\frac{2}{s} \tilde{Q}_2 + O\left(\frac{A^8 \ln s^2}{s^4}\right),$$

which implies item (iii) of Proposition 3.9.

For the other estimates, we kindly refer the readers to find in Proposition 4.6 in [MZ08] and Proposition 4.10 in [DNZ20]. For that reason, we finish our proof here. ■

3.2.2 The infinite dimensional part: q_-

The proof is quite the same as Section 5.2 in [MZ08]. So, we will sketch the main step and the readers can find the details in [MZ08]. Using the definition of the projection P_- , defined in (36), we apply to equation (22)

$$\begin{aligned} P_- \left(\frac{\partial q}{\partial s} \right) &= P_- \left(\frac{\partial q}{\partial s} \right) + P_- \left[-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right] \\ &\quad + P_-(V_1 q + V_2 \bar{q}) + P_-(B(q, y, s)) + P_-(R^*(\theta', y, s)). \end{aligned} \quad (88)$$

In particular, we get

First term: $\frac{\partial q}{\partial s}$

From (37), its projection is

$$P_- \left(\frac{\partial q}{\partial s} \right) = \frac{\partial q_-}{\partial s}.$$

Second term: $\tilde{\mathcal{L}}_{\beta, \delta} q$

From (22), we have the following,

$$P_-(\mathcal{L}_{\beta, \delta} q) = \mathcal{L}_{\beta} q_- + P_-[(1 + i\delta)\mathfrak{R}q_-].$$

Third term: $-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q$

Since P_- commutes with the multiplication by i , we deduce that

$$P_- \left[-i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q \right] = -i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) q_-.$$

Fourth term: $V_1 q$ and $V_2 \bar{q}$

$$\left\| \frac{P_-(V_1 q)}{1 + |y|^{M+1}} \right\|_{L^\infty} \leq \|V_1\|_{L^\infty} \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty} + C \frac{A^M}{s^{\frac{M+2}{2}}},$$

and

$$\left\| \frac{P_-(V_2 \bar{q})}{1 + |y|^{M+1}} \right\|_{L^\infty} \leq \|V_1\|_{L^\infty} \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty} + C \frac{A^M}{s^{\frac{M+2}{2}}},$$

Fifth term: $B(q, y, s)$: Using (24), we have the following estimate from Lemmas A.3 and 3.15.

$$\left\| \frac{P_-(B(q, y, s))}{1 + |y|^{M+1}} \right\|_{L^\infty} \leq C(M) \left[\left(\frac{A^{M+2}}{s^{\frac{1}{2}}} \right)^{\bar{p}} + \frac{A^{5+(M+1)^2}}{s} \right] \frac{1}{s^{\frac{M+1}{2}}}, \quad (89)$$

where $\bar{p} = \min(p, 2)$.

Sixth term: $R^*(\theta', y, s)$: Using the fact that $\theta'(s) \leq \frac{CA^{10} \ln^2 s}{s^4}$, then the following holds

$$\left\| \frac{P_-(R^*(\theta', y, s))}{1 + |y|^{M+1}} \right\| \leq \frac{C}{s^{\frac{M+3}{2}}}.$$

Using (88) and Duhamel's integral equation, we get for all $s \in [\tau, s_1]$,

$$\begin{aligned} q_-(s) &= e^{(s-\tau)\mathcal{L}_\beta} q_-(\tau) \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_- [(1 + i\delta)\Re q_-] ds' \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_- \left[-i \left(\frac{\mu'}{s} - \frac{\eta \ln s'}{(s')^2} + \frac{\eta}{s^2} + \theta'(s') \right) q \right] ds' \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_- [V_1 q + V_2 \bar{q} + B(q, y, s') + R^*(\theta', y, s')] ds'. \end{aligned}$$

Using Lemma A.2, we get

$$\begin{aligned} \left\| \frac{q_-(s)}{1 + |y|^{M+1}} \right\|_{L^\infty} &\leq e^{-\frac{M+1}{2}(s-\tau)} \left\| \frac{q_-(\tau)}{1 + |y|^{M+1}} \right\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')} \sqrt{1 + \delta^2} \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty} ds' \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')} \left\| \frac{P_- \left[-i \left(\frac{\mu'}{s} - \frac{\eta \ln s'}{(s')^2} + \frac{\eta}{s^2} + \theta'(s') \right) q \right]}{1 + |y|^{M+1}} \right\|_{L^\infty} ds' \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')} \left\| \frac{P_- [V_1 q + V_2 \bar{q} + B(q, y, s') + R^*(\theta', y, s')]}{1 + |y|^{M+1}} \right\|_{L^\infty} ds'. \end{aligned}$$

By using the above estimates, we derive

$$\begin{aligned} \left\| \frac{q_-(s)}{1 + |y|^{M+1}} \right\|_{L^\infty} &\leq e^{-\frac{M+1}{2}(s-\tau)} \left\| \frac{q_-(\tau)}{1 + |y|^{M+1}} \right\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')} \left(\sqrt{1 + \delta^2} + \| |V_1| + |V_2| \|_{L^\infty} \right) \left\| \frac{q_-}{1 + |y|^{M+1}} \right\|_{L^\infty} ds' \\ &+ C(M) \int_\tau^s e^{-\frac{M+1}{2}(s-s')} \left[\frac{A^{(M+1)^2+5}}{(s')^{\frac{M+3}{2}}} + \frac{A^{(M+2)\bar{p}}}{(s')^{\frac{\bar{p}-1}{2}}} \frac{1}{(s')^{\frac{M+2}{2}}} + \frac{A^M}{(s')^{\frac{M+2}{2}}} \right] ds'. \end{aligned}$$

Since we have already fixed M in (34) such that

$$M \geq 4 \left(\sqrt{1 + \delta^2} + 1 + 2 \max_{i=1,2, y \in \mathbb{R}, s \geq 1} |V_i(y, s)| \right),$$

using Gronwall's lemma, we deduce that

$$e^{\frac{M+1}{2}s} \left\| \frac{q_-(s)}{1+|y|^{M+1}} \right\|_{L^\infty} \leq e^{\frac{M+1}{4}(s-\tau)} e^{\frac{M+1}{2}\tau} \left\| \frac{q_-(\tau)}{1+|y|^{M+1}} \right\|_{L^\infty} + e^{\frac{M+1}{2}s} 2^{\frac{M+3}{4}} \left[\frac{A^{(M+1)^2+5}}{s^{\frac{M+3}{4}}} + \frac{A^{(M+2)\bar{p}}}{s^{\frac{\bar{p}-1}{2}}} \frac{1}{(s')^{\frac{M+2}{2}}} + \frac{A^M}{s^{\frac{M+2}{2}}} \right]$$

which concludes the proof of the last but one identity in (iv) of Proposition 3.9.

3.2.3 The outer region: q_e

As a matter of fact, our shrinking set, $\mathcal{V}_A(s)$ is quiet the same as [MZ08]. In particular, the estimate of q_e is exactly the same. For that reason, we also ignore the detail computation. The below is the main idea, for more details, the readers can find in that work.

In fact, using the fact that $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, s_1]$ and item (i) of Proposition 3.9, we derive the following rough estimates

$$\|q(s)\|_{L^\infty(|y| \leq 2Ks^{1/2})} \leq C \frac{A^{M+1}}{s^{1/2}} \text{ and } |\theta'(s)| \leq \frac{CA^{10} \ln^2 s}{s^4}. \quad (90)$$

In particular, using the definition of q_e , given as in (21), q_e reads

$$\begin{aligned} \frac{\partial q_e}{\partial s} &= \mathcal{L}_\beta q_e - \frac{1}{p-1} q_e + (1-\chi) e^{\frac{i\delta}{p-1}s} \{L(q, \theta', y, s) + R^*(\theta', y, s)\} \\ &\quad - e^{\frac{i\delta}{p-1}s} q(s) \left(\partial_s \chi + (1+i\beta) \Delta \chi + \frac{1}{2} y \cdot \nabla \chi \right) + 2e^{\frac{i\delta}{p-1}s} (1+i\beta) \operatorname{div}(q(s) \nabla \chi). \end{aligned} \quad (91)$$

In addition to that, we can write (91) under Duhamel's integral equation and take a L^∞ estimate

$$\begin{aligned} \|q_e(s)\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty}, \\ &\quad + \int_\tau^s e^{-\frac{s-s'}{p-1}} (\|(1-\chi)L(q, \theta', y, s')\|_{L^\infty} + \|(1-\chi)R^*(\theta', y, s')\|_{L^\infty}) ds' \\ &\quad + \int_\tau^s e^{-\frac{s-s'}{p-1}} \left\| q(s') \left(\partial_s \chi + (1+i\beta) \Delta \chi + \frac{1}{2} y \cdot \nabla \chi \right) \right\|_{L^\infty} ds' \\ &\quad + \int_\tau^s e^{-\frac{s-s'}{p-1}} \frac{1}{\sqrt{1-e^{-(s-s')}}} \|q(s') \nabla \chi\|_{L^\infty} ds'. \end{aligned}$$

Thanks to a detail computation given in Subsection 5.3 of [MZ08], we obtain

$$\begin{aligned} \|q_e(s)\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{s-s'}{p-1}} \left(\frac{1}{2(p-1)} \|q_e(s')\|_{L^\infty} + C \frac{A^{M+1}}{\sqrt{s'}} + C \frac{A^{M+1}}{s'} \frac{1}{\sqrt{1-e^{-(s-s')}}} \right) ds'. \end{aligned}$$

BY using Gronwall's inequality, we derive

$$\|q_e(s)\|_{L^\infty} \leq e^{-\frac{(s-\tau)}{2(p-1)}} \|q_e(\tau)\|_{L^\infty} + \frac{CA^{M+1}}{\tau^{\frac{1}{4}}} (s-\tau + \sqrt{s-\tau}),$$

which yields the proof of item (iv) of Proposition 3.9.

3.3 The case $\beta = 0$

We here give an argument to the special case where $\beta = 0$. The main reason comes from Definition (3.1) for $\mathcal{V}_A(s)$, in particular, (47). In particular, there is only one bound that becomes singular $\tilde{A}_2 = -\frac{R_{0,1}^*}{c_2}$. Naturally, we change this bound by the new one

$$\left| \tilde{Q}_2 \right| = \left| \tilde{q}_2 - \left(\frac{\tilde{A}_2 \ln s}{s^2} + \frac{\tilde{B}_2}{s^2} \right) \right| \leq \frac{A^{10} \ln^2 s}{s^3},$$

where \tilde{A}_2 is defined by (47) and \tilde{B}_2 can be chosen arbitrary. In addition to that, we also denote the new shrinking set by $\mathcal{V}_A(s)$. In particular, Proposition 3.9 remains valid, except the ODEs of $\theta'(s)$ and \tilde{Q}_2 .

- For $\theta'(s)$: Repeat the process for the case $\beta \neq 0$, we derive

$$\left| \kappa \theta'(s) - \left(\frac{R_{0,1}^*}{s^2} + \frac{T_{0,1}^* \ln s}{s^2} \right) \right| \leq \frac{C \ln s}{s^3}.$$

When $\beta = 0$, $R_{0,1}^* = 0$. However, the leading order $\frac{T_{0,1}^* \ln s}{s^2}$, will generate

$$\theta(s) \sim \frac{\theta_0 \ln s}{s}.$$

This violates our purpose that

$$\theta(s) \ll \frac{\ln s}{s}.$$

Hence, it imposes

$$\eta = 0.$$

Note that constants $T_{i,j}^*, \tilde{T}_{i,j}^* = 0$. Thus, we get the following

$$\left| \kappa \theta'(s) - \frac{R_{0,1}^*}{s^2} \right| \leq \frac{C \ln s}{s^3}. \quad (92)$$

It is sufficient to prove item (iii) in Proposition 3.9. Indeed, taking projection of equation (22) of $\tilde{P}_{2,M}$ as in page 35. In particular, plugging (92) into

$$\tilde{P}_{2,M}(R^*) = \frac{\tilde{R}_{2,2}^*}{s^3} + \frac{\theta'(s)\kappa}{s} \frac{-\delta b}{(p-1)^2} + O\left(\frac{\ln s}{s^4}\right),$$

we obtain

$$\begin{aligned} \tilde{P}_{2,M}(R^*) &= \frac{\tilde{R}_{2,2}^*}{s^3} + \left(\frac{R_{0,1}^*}{s^2}\right) \frac{-\delta b}{s(p-1)^2} + O\left(\frac{\ln s}{s^4}\right) \\ &= \left(\tilde{R}_{2,2}^* - \frac{\delta b R_{0,1}^*}{(p-1)^2}\right) \frac{1}{s^3} + O\left(\frac{\ln s}{s^4}\right). \end{aligned}$$

Note that, when $\beta = 0$, we have

$$\delta\mu + \tilde{D}_{2,2} = -2,$$

and

$$\tilde{T}_{2,2}^* - \frac{T_{0,1}^* \delta b}{(p-1)^2} = 0.$$

Hence, we have

$$\tilde{q}'_2 = -\frac{2\tilde{q}_2}{s} + \frac{1}{s^3} \left\{ \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^* \right\} + O\left(\frac{\ln^2 s}{s^4}\right).$$

Using the decomposition $\tilde{Q}_2 = \tilde{q}_2 - \left(\frac{\tilde{A}_2(\beta=0) \ln s}{s^2} + \frac{\tilde{B}_2(\beta=0)}{s^2}\right)$, then \tilde{Q}_2 reads

$$\begin{aligned} \tilde{Q}_2 &= -\frac{2}{s} \tilde{Q}_2 + \left(-\tilde{A}_2 + \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^*\right) \frac{1}{s^3} \\ &+ O\left(\frac{\ln^2 s}{s^4}\right). \end{aligned}$$

Note that, it is not similar to the case $\beta \neq 0$, the role of η vanishes. The order $\frac{1}{s^3}$ is cancelled by

$$\mathcal{A}_2 = \mu R_{2,1}^* + \tilde{C}_{2,2} R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} - \frac{\delta b}{(p-1)^2} R_{0,1}^*. \quad (93)$$

In particular, when $\beta = \delta = 0$, we can explicitly compute

$$\tilde{A}_2 = -\tilde{C}_{2,2}R_{2,1}^* + \tilde{R}_{2,2}^* = -\left(-\frac{1}{4}\right)\frac{5\kappa}{8p} - \frac{5}{32}\kappa\frac{(5p-4)}{p^2} = -\frac{5\kappa(p-1)}{8p^2}.$$

This constant exactly matches to the formal approach given by Velázquez, Galaktionov, and Herrero [VGH91].

A Spectral properties of \mathcal{L}_β

In this Appendix, we recall from Appendix A of [MZ08] some properties associated to the operator \mathcal{L}_β , defined in (19). We recall that:

$$\mathcal{L}_\beta v = (1 + i\beta)\Delta v - \frac{1}{2}y \cdot \nabla v = \frac{1}{\rho_\beta} \operatorname{div}(\rho_\beta \nabla w).$$

where

$$\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{N/2}}.$$

Moreover, the operator \mathcal{L}_β is self adjoint with respect to the weight ρ_β in the sense that

$$\int_{\mathbb{R}^N} u(y)\mathcal{L}_\beta w(y)\rho_\beta(y)dy = \int_{\mathbb{R}^N} w(y)\mathcal{L}_\beta u(y)\rho_\beta(y)dy. \quad (94)$$

In one space dimension ($N = 1$), the eigenfunction f_n of \mathcal{L}_β are dilations of standard Hermite polynomials $H_n(y)$:

$$f_n(y) = H_n\left(\frac{y}{2\sqrt{1+i\beta}}\right), \text{ where } \mathcal{L}_\beta H_n = -\frac{n}{2}H_n.$$

If $N \geq 2$, its eigenfunction $f_\alpha(y_1, \dots, y_N)$ where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ is a multi-indic are given by

$$f_\alpha(y) = \prod_{i=1}^N f_{\alpha_i}(y_i) = \prod_{i=1}^N H_{\alpha_i}\left(\frac{y_i}{2\sqrt{1+i\beta}}\right).$$

The family f_α is orthogonal in the sense that for all α and $\xi \in \mathbb{N}^N$,

$$\int f_\alpha f_\xi \rho_\beta dy = \delta_{\alpha,\xi} \int f_\alpha^2 \rho_\beta dy.$$

The semigroup generated by \mathcal{L}_β is well defined and has the following kernel:

$$e^{s\mathcal{L}_\beta}(y, x) = \frac{1}{[4\pi(1+i\beta)(1-e^{-s})]^{N/2}} \exp\left[-\frac{|x - ye^{-\frac{s}{2}}|^2}{4(1+i\beta)(1-e^{-s})}\right]. \quad (95)$$

In the following, we give some properties associated to the kernel.

Lemma A.1 a) *The semigroup associated to \mathcal{L}_β satisfies the maximum principle:*

$$\|e^{s\mathcal{L}_\beta}\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty}.$$

b) *Moreover, we have*

$$\|e^{s\mathcal{L}_\beta} \operatorname{div}(\varphi)\|_{L^\infty} \leq \frac{C}{\sqrt{1-e^{-s}}} \|\varphi\|_{L^\infty},$$

where C only depends on β .

Proof: a) It follows directly by part, this also follows from the definition of the semigroup (95).

b) Using an integration by part, this also follows from the definition of the semigroup (95).

Lemma A.2 *There exists a constant C such that if ϕ satisfies*

$$\forall x \in \mathbb{R} \quad |\phi(x)| \leq (1 + |x|^{M+1})$$

then for all $y \in \mathbb{R}$, we have

$$|e^{s\mathcal{L}_\beta} P_-(\phi(y))| \leq C e^{-\frac{M+1}{2}s} (1 + |y|^{M+1})$$

Proof: This also follows directly from the semigroup's definition, through an integration by part, for a similar case see page 556-558 from [BK94].

Moreover, we have the following useful lemma about P_- .

Lemma A.3 *For all $k \geq 0$, we have*

$$\left\| \frac{P_-(\phi)}{1 + |y|^{M+k}} \right\|_{L^\infty} \leq C \left\| \frac{\phi}{1 + |y|^{M+k}} \right\|.$$

Proof: Using (35), we have

$$|\phi_n| \leq C \left\| \frac{\phi}{1 + |y|^{M+k}} \right\|_{L^\infty}.$$

Since for all $m \leq M$, $|h_m(y)| \leq C(1 + |y|^{m+k})$ and

$$|\phi| \leq C \left\| \frac{\phi}{1 + |y|^{M+k}} \right\|_{L^\infty} (1 + |y|^{m+k}),$$

the result follows from definition (33) of ϕ .

B Details of expansions of the potential terms: V_1 and V_2

In this section, we aim at giving some expansions of V_1 and V_2 in order to give the conclusion of item (i) from Lemma 3.14 and some related constants. Indeed, we recall the definition of V_1 and V_2

$$\begin{aligned} V_1(y, s) &= (1 + i\delta) \frac{p+1}{2} \left(|\varphi|^{p-1} - \frac{1}{p-1} \right), \\ V_2(y, s) &= (1 + i\delta) \frac{p-1}{2} \left(|\varphi|^{p-3} \varphi^2 - \frac{1}{p-1} \right), \end{aligned}$$

where

$$\varphi(y, s) = \varphi_0(y, s) + \frac{(1 + i\delta)a}{s} = \kappa \left(1 + \frac{b}{p-1} \frac{|y|^2}{s} \right)^{-\frac{1+i\delta}{p-1}} + \frac{(1 + i\delta)a}{s},$$

and

$$a = \frac{2\kappa b(1 - \delta\beta)}{(p-1)^2}.$$

Then, using Taylor expansion, we claim to the following asymptotic behaviors

$$V_1(y, s) = \frac{1}{s} W_{1,1}(y) + \frac{1}{s^2} W_{1,2} + O\left(\frac{1 + |y|^6}{s^3}\right), \quad (96)$$

and

$$V_2(y, s) = \frac{1}{s} W_{2,1}(y) + \frac{1}{s^2} W_{2,2}(y) + O\left(\frac{1 + |y|^6}{s^3}\right), \quad (97)$$

where

$$\begin{aligned}
W_{1,1}(y) &= (1+i\delta)\frac{(p+1)}{2}\frac{b}{(p-1)^2}(-y^2+2(1-\delta\beta)), \\
W_{1,2}(y) &= (1+i\delta)\frac{(p+1)}{2}\frac{b^2}{(p-1)^3}\left\{y^4-\frac{(2(1-\delta\beta)(p-2+\delta^2))}{p-1}y^2\right. \\
&\quad \left.+\frac{(p-1)(1+\delta^2)(1-\delta\beta)^2+(p-3)(1-\delta^2)(1-\delta\beta)^2}{p-1}\right\} \\
&= (1+i\delta)\frac{(p+1)}{2}\frac{b^2}{(p-1)^4}\{(p-1)y^4-[2(1-\delta\beta)(p-2+\delta^2)]y^2 \\
&\quad +2(p-2+\delta^2)(1-\delta\beta)^2\}
\end{aligned}$$

and

$$\begin{aligned}
W_{2,1}(y) &= (1+i\delta)\frac{b}{2(p-1)^2}\{(p-1+2i\delta)(-y^2+2(1-\delta\beta))\}, \\
W_{2,2}(y) &= (1+i\delta)\frac{b^2}{2(p-1)^4}\{(p-1+2i\delta)(p-1+i\delta)y^4 \\
&\quad - (2(p-1)(p-2)+(2p-10)\delta^2+(8p-16)\delta i)(1-\delta\beta)y^2 \\
&\quad + (1-\delta\beta)^2\left[\frac{(p+1)(p-1)(1+i\delta)^2}{2}+(p+1)(p-3)(1+\delta^2)+\frac{(p-3)(p-5)(1-i\delta)^2}{2}\right]\}.
\end{aligned}$$

For the proof of (96) and (97), we kindly refer the readers to see the Appendix *B*.

In addition to that, we aim at determining the constants given in item (ii) of Lemma 3.14:

$$\begin{aligned}
\tilde{D}_{4,2} &= \tilde{P}_{4,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \frac{b(\delta^2 - p)}{(p-1)^2}, \\
D_{2,2} &= P_{2,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) \\
&= -\frac{b}{2(p-1)^2} \{-24p\delta + 56\delta^3 + 64\delta^2\beta + 32\delta + 24p\delta^2\beta + 40\delta^4\beta\}, \\
\tilde{L}_{2,4} &= \tilde{P}_{2,M}(i\tilde{h}_4) = 6\delta^2\beta - 12\delta - 6\beta, \\
D_{4,2} &= P_{4,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \frac{b}{(p-1)^2} \{-2\delta(1 + \delta^2)\} \\
\tilde{D}_{2,0} &= \tilde{P}_{2,M}(W_{1,1}\tilde{h}_0 + W_{2,1}\bar{\tilde{h}}_0) = -\frac{b}{2(p-1)^2}(2p - 2\delta^2) \\
\tilde{L}_{0,2} &= \tilde{P}_{0,M}(i\tilde{h}_2) = -2\delta + \delta^2\beta - \beta, \\
\tilde{D}_{0,2} &= \tilde{P}_{0,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) \\
&= -\frac{b}{2(p-1)^2} \{-32\delta\beta - 12p\beta^2 + 12\delta^2\beta^2 - 16\delta^2 + 16p - 4\delta^4\beta^2 + 4p\delta^2\beta^2 - 32p\delta\beta\}, \\
\tilde{C}_{2,2} &= \tilde{P}_{2,M}(W_{1,1}h_2 + W_{2,1}\bar{h}_2) \\
&= -\frac{b}{2(p-1)^2} \{-14\delta^2\beta + 2p\beta - 12\beta\}, \\
\tilde{C}_{2,4} &= \tilde{P}_{2,M}(W_{1,1}h_4 + W_{2,1}\bar{h}_4) \\
&= -\frac{b}{2(p-1)^2} \{96p\beta + 224\delta^3\beta^2 - 288\delta^2\beta - 128p\delta\beta^2 - 192\beta + 96\delta\beta^2\} \\
\tilde{D}_{2,4} &= \tilde{P}_{2,M}(W_{1,1}\tilde{h}_4 + W_{2,1}\bar{\tilde{h}}_4) \\
&= -\frac{b}{2(p-1)^2} \{-96p\delta^2\beta^2 - 168p\delta\beta + 96p - 528\delta\beta - 96\delta^2 + 216\delta^2\beta^2 - 168p\beta^2 + 144\delta^4\beta^2 - 360\delta^3\beta\} \\
\tilde{F}_{2,2} &= \tilde{P}_{2,M}(W_{1,2}\tilde{h}_2 + W_{2,2}\bar{\tilde{h}}_2) \\
&= \frac{b^2}{2(p-1)^4} \{-240p + 276p^2 - 312p\delta^2 - 204\delta^4 + (-288p - 552p^2 + 696)\delta\beta \\
&\quad + (432 - 144p)\delta^3\beta + 144\delta^5\beta + (180p - 180p^2)\beta^2 + (96p^2 + 288p - 96)\delta^2\beta^2 + (108 + 36p)\delta^4\beta^2\} \\
D_{0,2} &= P_{0,M}(W_{1,1}\tilde{h}_2 + W_{1,2}\bar{\tilde{h}}_2) \\
&= -\frac{b}{2(p-1)^2} \{32\delta + 24\delta^5\beta^2 + 64\delta^2\beta + 48\delta^3\beta^2 + 64\delta^4\beta + 32\delta^3 + 24\delta\beta^2 + 96p\delta^3\beta^2 + 96p\delta\beta^2\} \\
L_{0,2} &= P_{0,M}(i\tilde{h}_2) = 4\delta\beta + 4\delta^3\beta.
\end{aligned}$$

C Details of expansions of $R^*(y, s, \theta'(s))$

Using the definition of φ , the fact that φ_0 satisfies (17) and (22), we see that R^* is defined

$$\begin{aligned}
R^* &= \frac{(1+i\beta)}{s} \Delta_z \varphi_0(z) + \frac{1}{2s} z \cdot \nabla \varphi_0 + \frac{a}{s^2} (1+i\delta) - \frac{(1+i\delta)^2 a}{(p-1)s} \\
&+ (1+i\delta) \left(F \left(\varphi_0(z) + \frac{a}{s} (1+i\delta) \right) - F(\varphi_0(z)) \right) \\
&- i \left(\frac{\mu}{s} - \eta \frac{\ln s}{s^2} + \frac{\eta}{s^2} + \theta'(s) \right) \left(\varphi_0(z) + \frac{a}{s} (1+i\delta) \right), \\
&= R_1^*(y, s) + \frac{\ln s}{s^2} T_1 + \frac{1}{s^2} T_2 + \theta'(s) \Theta(y, s),
\end{aligned}$$

where $F(w) = |w|^{p-1} w$, $\Theta(y, s) = -i \left(\varphi_0(y, s) + \frac{a(1+i\delta)}{s} \right)$, $T_1^* = -\eta \Theta$ and $T_2^* = \eta \Theta$.

Expansion of $R_1^*(y, s)$ in terms of h_j and \tilde{h}_j

As a matter of fact, we can expand R_1^* in series of $\frac{1}{s^j}$ as follows

$$R_1^*(y, s) = \frac{1}{s} \mathcal{R}_0(y) + \frac{1}{s^2} \mathcal{R}_1(y) + \frac{1}{s^3} \mathcal{R}_2(y) + \tilde{\mathcal{R}}(y, s),$$

where $\tilde{\mathcal{R}}$ satisfies

$$|\tilde{\mathcal{R}}(y, s)| \leq \frac{C(1+|y|^6)}{s^4},$$

which implies that

$$|P_{j,M}(\tilde{\mathcal{R}})| + |\tilde{P}_{j,M}(\tilde{\mathcal{R}})| \leq \frac{C}{s^4}.$$

In addition to that, we can write $\mathcal{R}_j(y)$ in the basis generated by h_k , and \tilde{h}_k as follows

$$\mathcal{R}_j(y) = \sum_{k=0}^j (R_{j,k}^* h_k + \tilde{R}_{j,k}^* \tilde{h}_k).$$

Replaying the method given at Section D in [DNZ20], we can explicitly find the formulas of the constants $R_{i,j}^*$ and $\tilde{R}_{i,j}^*$. Here we only give the results:

$$\begin{aligned}
R_{0,0}^* &= -\kappa \left(\mu + \frac{2b\beta(1+\delta^2)}{(p-1)^2} \right), \\
\tilde{R}_{0,0}^* &= a - \frac{2\kappa b(1-\delta\beta)}{(p-1)^2}, \\
\tilde{R}_{2,1}^* &= \frac{4\kappa(p+(p+1)\delta\beta-\delta^2)b^2}{(p-1)^4} - \frac{\kappa b}{(p-1)^2}, \\
R_{2,1}^* &= \frac{2\kappa b^2(\delta+3p\beta+3p\delta^2\beta-\beta+\delta^3+\delta^4\beta)}{(p-1)^4} \\
\tilde{R}_{0,1}^* &= \frac{2\kappa b^2}{(p-1)^4} (3\delta^3\beta + (2p\beta^2 + 6\beta^2 - 5)\delta^2 + (-7\beta - 10p\beta)\delta + 5p - 3p\beta^2 + \beta^2) \\
R_{0,1}^* &= -\frac{4\beta\kappa b^2}{(p-1)^4} (2\delta^4 + \beta\delta^3 + 3p\delta^2 + \beta\delta + 3p - 2) \\
\tilde{R}_{2,2}^* &= \frac{6\kappa b^2}{(p-1)^4} \{ \delta^3\beta - 2\delta^2 - (2p+1)\delta\beta + 2p \} \\
&\quad - \frac{2\kappa b^3}{(p-1)^6} \{ 3\beta^2\delta^6 - 12\beta\delta^5 + (9 - 12\beta^2 - 6p\beta^2)\delta^4 + (42p\beta + 42\beta)\delta^3 \\
&\quad + (70p\beta^2 + 19p^2\beta^2 - 78p - 6\beta^2)\delta^2 \\
&\quad + (-98p^2\beta + 36\beta - 74p\beta)\delta - 20p + 49p^2 + 18p\beta^2 - 30p^2\beta^2 \}.
\end{aligned}$$

Besides that, we don't need to formulate to the other constants.

Expansion of $\theta'(s)\Theta(y)$

We introduce

$$\Theta(y, s) = -i \left(\varphi_0(y, s) + \frac{a(1+i\delta)}{\sqrt{s}} \right),$$

where φ_0 and a defined as in (16) and (15), respectively. Using Taylor expansions, we write

$$\begin{aligned}
\Theta(y, s) &= -i\kappa + \kappa(\delta - i) \frac{y^2}{s} \frac{b}{(p-1)^2} + a(\delta - i) \frac{1}{s} \\
&\quad + \kappa(1 - i\beta)\delta(p+1) \frac{y^4}{s^2} \frac{b^2}{2(p-1)^4} + \tilde{\Theta}(y, s),
\end{aligned}$$

where $\tilde{\Theta}(y, s)$ satisfies the following

$$\left| \tilde{\Theta}(y, s) \right| \leq \frac{C(1+|y|^6)}{s^3},$$

which yields

$$|P_{j,M}(\tilde{\Theta})| + |\tilde{P}_{j,M}(\tilde{\Theta})| \leq \frac{C}{s^3}.$$

and

$$\begin{aligned} & -i\kappa + \kappa(\delta - i) \frac{y^2}{s} \frac{b}{(p-1)^2} + a(\delta - i) \frac{1}{s} + \kappa(1 - i\beta)\delta(p+1) \frac{y^4}{s^2} \frac{b^2}{2(p-1)^4} \\ = & \left(-\kappa + \frac{\Theta_{0,0}^*}{s} \right) h_0 + \frac{\tilde{\Theta}_{0,0}^*}{s} \tilde{h}_0 + \frac{\Theta_{2,0}^*}{s} h_2 + \frac{\tilde{\Theta}_{2,0}^*}{s} \tilde{h}_2 + O\left(\frac{1 + |y|^4}{s^2}\right). \end{aligned} \quad (98)$$

In addition to that, we can calculate these constants and we obtain

$$\begin{aligned} \Theta_{0,0}^* &= 4(1 + \delta^2)\delta\beta \frac{\kappa b}{(p-1)^2}, \\ \tilde{\Theta}_{0,0}^* &= -\beta(1 + \delta^2) \frac{\kappa b}{(p-1)^2}, \\ \tilde{\Theta}_{2,0}^* &= -\delta \frac{\kappa b}{(p-1)^2}, \\ \Theta_{2,0}^* &= (1 + \delta^2) \frac{\kappa b}{(p-1)^2}, \\ \tilde{\Theta}_{2,1}^* &= -3\delta(p+1)(-\beta^2 + \beta\delta - 2) \frac{\kappa b^2}{(p-1)^4}. \end{aligned}$$

In particular, we also have the following expansions of $T^* = -\eta\Theta$ and $T^{**} = \eta\Theta$ as follows

$$\begin{aligned} T^* &= \left(\eta\kappa + \frac{T_{0,0}^*}{s} \right) h_0 + \frac{\tilde{T}_{0,0}^*}{s} \tilde{h}_0 + \frac{T_{2,0}^*}{s} h_2 + \frac{\tilde{T}_{2,0}^*}{s} \tilde{h}_2 + O\left(\frac{1 + |y|^4}{s^2}\right), \\ T^{**} &= \left(-\eta\kappa + \frac{T_{0,0}^*}{s} \right) h_0 + \frac{\tilde{T}_{0,0}^*}{s} \tilde{h}_0 + \frac{T_{2,0}^*}{s} h_2 + \frac{\tilde{T}_{2,0}^*}{s} \tilde{h}_2 + O\left(\frac{1 + |y|^4}{s^2}\right) \end{aligned}$$

where

$$(T_{i,j}^*, \tilde{T}_{i,j}^*) = -\eta(\Theta_{i,j}^*, \tilde{\Theta}_{i,j}^*) \text{ and } (T_{i,j}^{**}, \tilde{T}_{i,j}^{**}) = \eta(\Theta_{i,j}^*, \tilde{\Theta}_{i,j}^*).$$

Finally, we aim at giving the explicit of η here: Indeed, we have the following formula from (87)

$$\eta = -\frac{c_2}{\kappa} \left\{ \left(-\frac{\delta b}{(p-1)^2} \right) R_{0,1}^* + (\mu + \tilde{C}_{2,2}) R_{2,1}^* - \tilde{D}_{2,0} \tilde{R}_{0,1}^* + \tilde{R}_{2,2} \right\}.$$

Using the formulas of the constant in η , we obtain AREVOIR

$$\begin{aligned}
\eta &= -\frac{\beta(1 + \delta^2)}{8(p - (p + 1)\delta\beta - \delta^2)^3} \times & (99) \\
&\quad \{\delta^6\beta^2 + 3\delta^5\beta + (3\beta^2p + 10)\delta^4 + (5\beta + 18p\beta)\delta^3 \\
&\quad + (2\beta^2p^2 + 7\beta^2 + 10p + \beta^2p)\delta^2 + (-18\beta + 18p\beta + 20p^2\beta)\delta \\
&\quad + 10p - 2\beta^2 + 12\beta^2p^2 - 2\beta^2p - 10p^2\}.
\end{aligned}$$

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